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Research

Accuracy Analysis of the Estimated Process Yield Based on S_{pk}

W. L. Pearn[1,*,†] and C. C. Chuang[2]
[1]*Department of Industrial Engineering and Management, National Chiao Tung University, Taiwan, Republic of China*
[2]*R&D Center for SCM, Pouchen International Group, Taiwan, Republic of China*

Process yield has been the most basic and common criterion used in the manufacturing industry as a base for measuring process performance. Boyles considered a measurement formula called S_{pk}, which establishes the relationship between the manufacturing specification and the actual process performance, providing an exact (rather than approximate) measure of process yield. Unfortunately, the sampling distribution and the associated statistical properties of S_{pk} are analytically intractable. In this paper, we consider the natural estimator of the measure S_{pk}. We investigate the accuracy of the natural estimator of S_{pk} computationally, using a simulation technique to find the relative bias and the relative mean square error for some commonly used quality requirements. Extensive simulation results are provided and analyzed, which are useful to the engineers for factory applications in measuring process performance. ***Copyright © 2004 John Wiley & Sons, Ltd.***

KEY WORDS: process yield measure; relative bias; relative mean square error

1. INTRODUCTION

Process yield has long been a standard criterion used in the manufacturing industry as a common measure on process performance. Process yield is currently defined as the percentage of processed product unit passing inspection. That is, the product characteristic must fall within the manufacturing tolerance. For product units rejected (non-conformities), additional costs would be incurred to the factory for scrapping or repairing the product. All passed product units are equally accepted by the producer, which incurs the factory no additional cost. For processes with two-sided manufacturing specifications, the process yield can be calculated as Yield $= F(USL) - F(LSL)$, where USL and LSL are the upper and the lower specification limits, respectively, and $F(\cdot)$ is the cumulative distribution function of the process characteristic. If the process characteristic is normally distributed, then the process yield can be alternatively expressed as Yield $= \Phi[(USL - \mu)/\sigma] - \Phi[(\mu - LSL)/\sigma]$, where μ is the process mean, σ is the process standard deviation and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution $N(0, 1)$.

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[*] Correspondence to: W. L. Pearn, Department of Industrial Engineering and Management, National Chiao Tung University, 1001 TA Hsueh Road, Hsinchu, Taiwan 30050, Republic of China.
[†] E-mail: roller@cc.nctu.edu.tw

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Table I. Some S_{pk} values and the corresponding non-conformities

S_{pk}	Yield	Non-conformities (ppm)
1.00	0.997 300 2039	2699.796
1.10	0.999 033 1517	966.848
1.20	0.999 681 7828	318.217
1.30	0.999 903 8073	96.193
1.33	0.999 933 9267	66.073
1.40	0.999 973 3085	26.691
1.50	0.999 993 2047	6.795
1.60	0.999 998 4133	1.587
1.67	0.999 999 4557	0.544
1.70	0.999 999 6603	0.340
1.80	0.999 999 9334	0.067
1.90	0.999 999 9880	0.012
2.00	0.999 999 9980	0.002

Numerous process capability indices have been proposed to measure the process yield, particularly for processes with normally distributed characteristics. Those include C_{pk} (Kane[1]), C_{pm} (Chan *et al.*[2]), and C_{pmk} (Pearn *et al.*[3]). These indices are defined as

$$C_{pk} = \min\left\{\frac{USL-\mu}{3\sigma}, \frac{\mu-LSL}{3\sigma}\right\}$$

$$C_{pm} = \frac{USL-LSL}{6\sqrt{\sigma^2+(\mu-T)^2}}$$

$$C_{pmk} = \min\left\{\frac{USL-\mu}{3\sqrt{\sigma^2+(\mu-T)^2}}, \frac{\mu-LSL}{3\sqrt{\sigma^2+(\mu-T)^2}}\right\}$$

where T is the target value. These indices establish the relationship between the manufacturing specifications and the actual process performance, which provide some lower bounds on the process yield. For example, we may establish the relationship of $\text{Yield} = F[(USL-\mu)/\sigma] - F[(LSL-\mu)/\sigma] \geq 2\Phi(3C_{pk}) - 1$ or, equivalently, an upper bound on the fraction of the non-conformities $P(NC) = 1 - F[(USL-\mu)/\sigma] + F[(LSL-\mu)/\sigma] \leq 2\Phi(-3C_{pk})$. Thus, for a process with $C_{pk} \geq 1.00$, we can assure that the process yield is greater than or equal to 0.9973.

The statistical properties of the estimators of those indices under various process conditions have been investigated extensively by authors including Chan *et al.*[2], Pearn *et al.*[3], Bordignon and Scagliarini[4], Borges and Ho[5], Chang *et al.*[6], Hoffman[7], Nahar *et al.*[8], Noorossana[9], Pearn *et al.*[10], Pearn and Lin[11] and Zimmer *et al.*[12]. Kotz and Johnson[13] presented a review for the development of process capability indices in the past ten years. Based on the above expression of process yield, Boyles[14] considered the yield measure S_{pk}, as defined in the following:

$$S_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL-\mu}{\sigma}\right) + \frac{1}{2}\Phi\left(\frac{\mu-LSL}{\sigma}\right)\right\}$$

The measure S_{pk} establishes the relationship between the manufacturing specifications and the actual process performance, which provides an exact measure on the process yield. If $S_{pk} = c$, then the process yield can be expressed as $\text{Yield} = 2\Phi(3c) - 1$. Obviously, there is a one-to-one correspondence between S_{pk} and the process yield. Thus, S_{pk} provides an exact (rather than approximate) measure of the process yield. Table I summarizes the process yield, non-conformities (in parts per million (ppm)) as a function of the measure S_{pk}, for $S_{pk} = 1.00(0.1)2.00$, including the most commonly-used performance requirements 1.00, 1.33, 1.50, 1.67 and 2.00. For example, if for a particular process the yield measure $S_{pk} = 1.33$, then the corresponding value of non-conformities is roughly 66 ppm.

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2. ESTIMATION OF S_{pk}

In practice, the process parameters μ and σ are unknown and have to be estimated from the sampled data. To estimate the yield measure S_{pk}, we consider the following natural estimator $\hat{S}_{pk}$, where the statistics

$$\bar{X} = \left(\sum_{i=1}^{n} X_i\right)\bigg/ n \quad \text{and} \quad S = \left[(n-1)^{-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right]^{1/2}$$

are the sample mean and the sample standard deviation of the conventional estimators of μ and σ, respectively, which may be obtained from a well-controlled (demonstrably in statistical control) process. So

$$\hat{S}_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL-\bar{X}}{S}\right) + \frac{1}{2}\Phi\left(\frac{\bar{X}-LSL}{S}\right)\right\}$$

The exact distribution of $\hat{S}_{pk}$ is mathematically intractable. On the other hand, Lee *et al.*[15] obtained an approximate distribution of $\hat{S}_{pk}$ using a Taylor expansion technique. By taking the first order of the Taylor expansion, it is shown that the estimator $\hat{S}_{pk}$ can be expressed approximately as

$$\hat{S}_{pk} \approx S_{pk} + \frac{1}{6\sqrt{n}}[\phi(3S_{pk})]^{-1}W$$

$$W = \frac{d}{2\sigma^3}[\sqrt{n}(S^2-\sigma^2)]\left[\left(1+\frac{\mu-m}{d}\right)\phi\left(\frac{1+(\mu-m)/d}{\sigma/d}\right) + \left(1-\frac{\mu-m}{d}\right)\phi\left(\frac{1-(\mu-m)/d}{\sigma/d}\right)\right]$$
$$- \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}\left[\phi\left(\frac{1-(\mu-m)/d}{\sigma/d}\right) - \phi\left(\frac{1+(\mu-m)/d}{\sigma/d}\right)\right]$$

where $d = (USL - LSL)/2$, and ϕ is the probability density function of the standard normal distribution $N(0, 1)$. It is easy to show that the statistic W is distributed as a normal distribution with mean 0 and variance $a^2 + b^2$, where a and b are functions of μ and σ, defined as

$$a = \frac{d}{\sqrt{2}\sigma}\left[\left(1-\frac{\mu-m}{d}\right)\phi\left(\frac{1-(\mu-m)/d}{\sigma/d}\right) + \left(1+\frac{\mu-m}{d}\right)\phi\left(\frac{1+(\mu-m)/d}{\sigma/d}\right)\right]$$
$$b = \phi\left(\frac{1-(\mu-m)/d}{\sigma/d}\right) - \phi\left(\frac{1+(\mu-m)/d}{\sigma/d}\right)$$

Thus, the estimator $\hat{S}_{pk}$ is approximately (asymptotically) distributed as $N(S_{pk}, [a^2+b^2]\{36n[\phi(3S_{pk})]^2\}^{-1})$ and the estimator $\hat{S}_{pk}$ is asymptotically unbiased.

Through a rather complicated and tedious development (Pearn *et al.*[16]), we can further obtain the following, where Z and Y are distributed as the joint bivariate normal distribution and D_1, D_2, D_3, D_4 and D_5 are functions of $(\mu - m)/d$, and σ/d,

$$\hat{S}_{pk} = S_{pk} + D_1Z + D_2Y + D_3Z^2 + D_4ZY + D_5Y^2 + O_p\left(\frac{1}{n\sqrt{n}}\right)$$

Therefore, the distribution of $\hat{S}_{pk}$ may be approximated, alternatively, by the following polynomial combination of the distributions of Z and Y:

$$S_{pk} + D_1Z + D_2Y + D_3Z^2 + D_4ZY + D_5Y^2 \quad (Z, Y) \xrightarrow{d} N((0,0), \Sigma_2)$$
$$\Sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

with the bias approximated by the term

$$D_1Z + D_2Y + D_3Z^2 + D_4ZY + D_5Y^2$$

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The formulae for both approximations to the distribution of $\hat{S}_{pk}$ are rather complicated and the calculation is cumbersome to deal with. Since the parameters a and b in the approximate formula must also be estimated in real applications, then a great uncertainty may be introduced into the performance assessments due to the additional sampling errors from the estimation of a and b. Further, the accuracy of the approximation has not been investigated. Thus, the approximation would not be practically useful before those issues are resolved. For practical purpose, in the following we investigate the accuracy of the natural estimator $\hat{S}_{pk}$ computationally, using the simulation technique to find the relative bias and relative mean square error for some commonly used performance requirements. The simulation results obtained are useful for practitioners/engineers in measuring process performance for their factory applications, particularly if their processes are controlled/monitored on a routine basis.

3. THE SIMULATION PARAMETERS

We note that the natural estimator S_{pk} can be rewritten and expressed as a function of the parameters C_p and C_a. The parameter C_p is defined as $C_p = (USL - LSL)/6\sigma$, which measures the overall process variation relative to the specification tolerance and therefore only reflects process precision (consistency). The parameter C_a is defined as $C_a = 1 - |\mu - m|/d$ (see Pearn *et al.*[11]), which measures the degree of process centering, where $m = (USL + LSL)/2$ is the mid-point between the upper and the lower specification limits and $d = (USL - LSL)/2$ is half of the length of the specification interval. Thus, the parameter C_a alerts the user if the process mean deviates from its target value. In fact, a mathematical relationship among the three measurements can be established as $\Phi(3S_{pk}) = \{\Phi(3C_pC_a) + \Phi[3C_p(2 - C_a)]\}/2$.

$$\begin{aligned} S_{pk} &= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL-\mu}{\sigma}\right)+\frac{1}{2}\Phi\left(\frac{\mu-LSL}{\sigma}\right)\right\} \\ &= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{d-|\mu-m|}{\sigma}\right)+\frac{1}{2}\Phi\left(\frac{d+|\mu-m|}{\sigma}\right)\right\} \\ &= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{1-|\mu-m|/d}{\sigma/d}\right)+\frac{1}{2}\Phi\left(\frac{1+|\mu-m|/d}{\sigma/d}\right)\right\} \\ &= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left[3C_pC_a\right]+\frac{1}{2}\Phi\left[3C_p\left(2-C_a\right)\right]\right\} \end{aligned}$$

Table II displays the simulation parameters of the process used in the simulation, covering the most commonly used performance requirements $S_{pk} = 1.00$, 1.33, 1.50, 1.67 and 2.00. Table II(a) summarizes the precision measure $C_p = 1.0(0.1)2.0$, the corresponding accuracy measure C_a and (μ, σ) for $S_{pk} = 1.00$. Table II(b) summarizes the precision measure $C_p = 1.33$, $1.4(0.1)2.3$, the corresponding accuracy measure C_a and (μ, σ) for $S_{pk} = 1.33$. Table II(c) summarizes the precision measure $C_p = 1.5(0.1)2.5$, the corresponding accuracy measure C_a and (μ, σ) for $S_{pk} = 1.50$. Table II(d) summarizes the precision measure $C_p = 1.67$, $1.7(0.1)2.6$, the corresponding accuracy measure C_a and (μ, σ) for $S_{pk} = 1.67$. Table II(e) summarizes the precision measure $C_p = 2.0(0.1)3.0$, the corresponding accuracy measure C_a and (μ, σ) for $S_{pk} = 2.00$. Those combinations of (C_p, C_a), or (μ, σ), cover a wide range of underlying distributions resulting in the same value of S_{pk}, providing critical information regarding the sensitivity of the estimation error.

To analyze the accuracy of the natural estimator $\hat{S}_{pk}$, we investigate the relative bias defined as

$$BIAS_R(\hat{S}_{pk}) = [E(\hat{S}_{pk}) - S_{pk}]/S_{pk} = E(\hat{S}_{pk}/S_{pk}) - 1$$

which measures the average relative (percentage) deviation of $\hat{S}_{pk}$ from the true S_{pk}. We also investigate the relative mean square error defined as $MSE_R(\hat{S}_{pk}) = E[(\hat{S}_{pk} - S_{pk})/S_{pk}]^2 = E[(\hat{S}_{pk}/S_{pk}) - 1]^2$ which measures the average of the squared relative deviation of $\hat{S}_{pk}$ from the true S_{pk}. We further consider the statistic $[MSE_R(\hat{S}_{pk})]^{1/2}$, the square root of the relative mean square error, which is a more direct measurement of

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Table II. Various combinations of C_p and C_a for (a) $S_{pk} = 1.00$, (b) $S_{pk} = 1.33$; (c) $S_{pk} = 1.50$; (d) $S_{pk} = 1.67$; (e) $S_{pk} = 2.00$

C_p	C_a	μ	σ	C_p	C_a	μ	σ
(a)				(d)			
1.0	1.000 000 000	15.000 000 00	1.666 666667	1.67	1.000 000 000	15.000 000 00	1.000 000 000
1.1	0.845 650 984	15.771 745 08	1.515 151515	1.7	0.960 124 663	15.199 376 69	0.980 392 157
1.2	0.772 993 431	16.135 032 85	1.388 888889	1.8	0.902 865 766	15.485 671 17	0.925 925 926
1.3	0.713 386 252	16.433 068 74	1.282 051282	1.9	0.855 248 895	15.723 755 53	0.877 192 982
1.4	0.662 422 888	16.687 885 56	1.190 476190	2.0	0.812 484 428	15.937 577 86	0.833 333 333
1.5	0.618 261 111	16.908 694 45	1.111 111111	2.1	0.773 794 664	16.131 026 68	0.793 650 794
1.6	0.579 619 785	17.101 901 08	1.041 666667	2.2	0.738 622 179	16.306 889 11	0.757 575 758
1.7	0.545 524 504	17.272 377 48	0.980 392157	2.3	0.706 508 171	16.467 459 15	0.724 637 681
1.8	0.515 217 587	17.423 912 07	0.925 925926	2.4	0.677 070 331	16.614 648 35	0.694 444 444
1.9	0.488 100 872	17.559 495 64	0.877 192982	2.5	0.649 987 517	16.750 062 42	0.666 666 667
2.0	0.463 695 828	17.681 520 86	0.833 333333	2.6	0.624 987 997	16.875 060 02	0.641 025 641
(b)				(e)			
1.33	1.000 000 000	15.000 000 00	1.250 000 000	2.0	1.000 000 000	15.000 000 00	0.833 333 333
1.4	0.912 324 580	15.438 377 10	1.190 476 190	2.1	0.934 480 725	15.327 596 38	0.793 650 794
1.5	0.849 520 868	15.752 395 66	1.111 111 111	2.2	0.891 884 461	15.540 5777	0.757 575 758
1.6	0.796 341 133	16.018 294 34	1.041 666 667	2.3	0.853 105 312	15.734 473 44	0.724 637 681
1.7	0.749 494 830	16.252 525 85	0.980 392 157	2.4	0.817 559 242	15.912 203 79	0.694 444 444
1.8	0.707 856 167	16.460 719 17	0.925 925 926	2.5	0.784 856 872	16.075 715 64	0.666 666 667
1.9	0.670 600 579	16.646 997 11	0.877 192 982	2.6	0.754 670 069	16.226 649 66	0.641 025 641
2.0	0.637 070 550	16.814 647 25	0.833 333 333	2.7	0.726 719 326	16.366 403 37	0.617 283 951
2.1	0.606 733 857	16.966 330 72	0.793 650 794	2.8	0.700 765 064	16.496 174 68	0.595 238 095
2.2	0.579 155 045	17.104 224 78	0.757 575 758	2.9	0.676 600 752	16.616 996 24	0.574 712 644
2.3	0.553 974 391	17.230 128 05	0.724 637 681	3.0	0.654 047 393	16.729 763 04	0.555 555 556
(c)							
1.5	1.000 000 000	15.000 000 00	1.111 111 111				
1.6	0.906 849 563	15.465 752 19	1.041 666 667				
1.7	0.853 029 665	15.734 851 68	0.980 392 157				
1.8	0.805 624 734	15.971 876 33	0.925 925 926				
1.9	0.763 223 120	16.183 884 40	0.877 192 982				
2.0	0.725 061 959	16.374 690 21	0.833 333 333				
2.1	0.690 535 199	16.547 324 01	0.793 650 794				
2.2	0.659 147 235	16.704 263 83	0.757 575 758				
2.3	0.630 488 660	16.847 556 70	0.724 637 681				
2.4	0.604 218 299	16.978 908 51	0.694 444 444				
2.5	0.580 049 567	17.099 752 17	0.666 666 667				

the relative deviation (percentage of the deviation). Note that either explicit or implicit mathematical formulae for both $BIAS_R(\hat{S}_{pk})$ and $MSE_R(\hat{S}_{pk})$ are analytically intractable. The simulation approach seems to be the best alternative for the accuracy study. The simulation was carried out using SAS programming software, for the commonly used performance requirements $S_{pk} = 1.00$, 1.33, 1.50, 1.67 and 2.00. Each combination of the precision measure C_p, the accuracy measure C_a and the corresponding (μ, σ) pair is first set in the SAS program to generate random normal samples of size n. The sample data is then calculated to obtain the estimator $\hat{S}_{pk}$. A total of $N = 10\,000$ replications are carried out for each sample size of $n = 5(5)100$, then the average value $E(\hat{S}_{pk})$ is computed and compared with the preset true S_{pk} to obtain the relative bias. The simulation error is also examined, showing no greater value than 5×10^{-3}.

4. SIMULATION RESULTS

For the case of $S_{pk} = 1.00$, the simulation results indicate that for a sample size $n = 85$, the relative bias of the estimator is 0.3% for $C_p = 1.0$, 0.6% for $C_p = 1.1$, 1.3, 0.7% for $C_p = 1.2$, 1.4, 1.5, 1.6, 1.7, 1.9, 2.0 and 0.8%

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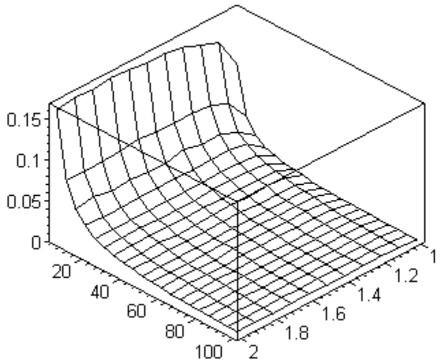

Figure 1. Surface plot of $BIAS_{\mathrm{R}}(\hat{S}_{\mathrm{p}k})$ for $S_{\mathrm{p}k} = 1.00$, with $C_{\mathrm{p}} = 1.0(0.1)2.0$ and sample size $n = 5(5)100$

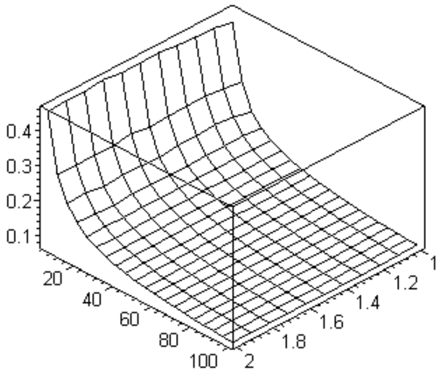

Figure 2. Surface plot of $[MSE_{\mathrm{R}}]^{1/2}$ for $S_{\mathrm{p}k} = 1.00$, with $C_{\mathrm{p}} = 1.0(0.1)2.0$ and sample size $n = 5(5)100$

for $C_{\mathrm{p}} = 1.8$. For a sample size $n = 100$ the relative bias of the estimator is 0.3% for $C_{\mathrm{p}} = 1.0$ and 0.5–0.6% for all other values of C_{p}. We note that for $n = 100$, $[MSE_{\mathrm{R}}(\hat{S}_{\mathrm{p}k})]^{1/2}$ is 7.0–7.1% for all values of C_{p} except for $C_{\mathrm{p}} = 1.0$ where $[MSE_{\mathrm{R}}(\hat{S}_{\mathrm{p}k})]^{1/2} = 7.3\%$. Thus, for the case of $S_{\mathrm{p}k} = 1.00$, the estimation error of $\hat{S}_{\mathrm{p}k}$ is stable for sample sizes $n \geq 100$. Figure 1 presents a surface plot of $BIAS_{\mathrm{R}}(\hat{S}_{\mathrm{p}k})$ for $S_{\mathrm{p}k} = 1.00$, as a function of C_{p} and the sample size n. Figure 2 presents the surface plot of $[MSE_{\mathrm{R}}(\hat{S}_{\mathrm{p}k})]^{1/2}$ for $S_{\mathrm{p}k} = 1.00$ as a function of C_{p} and the sample size n.

For the case of $S_{\mathrm{p}k} = 1.33$, the simulation results indicate that for a sample size $n = 85$, the relative bias of the estimator is 0.3% for $C_{\mathrm{p}} = 1.33$, 0.4% for $C_{\mathrm{p}} = 1.4$, 0.5% for $C_{\mathrm{p}} = 1.5$, 1.6, 1.7, 1.8, 1.9, 2.0 and 0.6% for $C_{\mathrm{p}} = 2.2$, 2.3. For a sample size $n = 100$, the relative bias of the estimator is 0.2% for $C_{\mathrm{p}} = 1.33$ and 0.3–0.4% for all other values of C_{p}, except $C_{\mathrm{p}} = 2.1$ where the relative bias is 0.6%. We note that for $n = 100$, $[MSE_{\mathrm{R}}(\hat{S}_{\mathrm{p}k})]^{1/2}$ is 7.0–7.2% for all values of C_{p}. Thus, for the case of $S_{\mathrm{p}k} = 1.33$, the estimation error of $\hat{S}_{\mathrm{p}k}$ is stable for sample sizes $n \geq 100$. Figure 3 presents a surface plot of $BIAS_{\mathrm{R}}(\hat{S}_{\mathrm{p}k})$ for $S_{\mathrm{p}k} = 1.33$, as a function of C_{p} and the sample size n. Figure 4 presents the surface plot of $[MSE_{\mathrm{R}}(\hat{S}_{\mathrm{p}k})]^{1/2}$ for $S_{\mathrm{p}k} = 1.33$, as a function of C_{p} and the sample size n.

For the case of $S_{\mathrm{p}k} = 1.50$, the simulation results indicate that for a sample size $n = 85$, the relative bias of the estimator is 0.4% for $C_{\mathrm{p}} = 1.5$, 0.7% for $C_{\mathrm{p}} = 2.5$, 0.8% for $C_{\mathrm{p}} = 1.6$, 1.7, 2.1, 2.3, 2.4 and 0.9% for $C_{\mathrm{p}} = 1.9$, 2.2. For a sample size $n = 100$ the relative bias of the estimator is 0.3% for $C_{\mathrm{p}} = 1.5$ and 0.6–0.8% for all other values of C_{p}. We note that for $n = 100$, $[MSE_{\mathrm{R}}(\hat{S}_{\mathrm{p}k})]^{1/2}$ is 7.1–7.2% for all values of C_{p}. Thus, for the case of $S_{\mathrm{p}k} = 1.50$, the estimation error of $\hat{S}_{\mathrm{p}k}$ is stable for sample size $n \geq 100$. Figure 5 presents a surface

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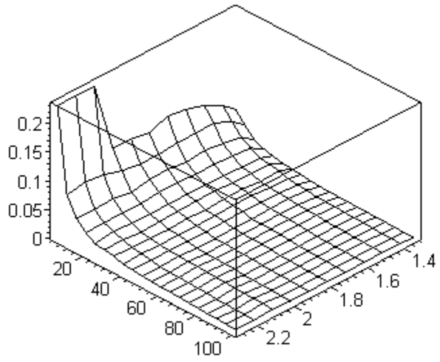

Figure 3. Surface plot of $BIAS_R(\hat{S}_{pk})$ for $S_{pk} = 1.33$, with $C_p = 1.33$, 1.4(0.1)2.3 and sample size $n = 5(5)100$

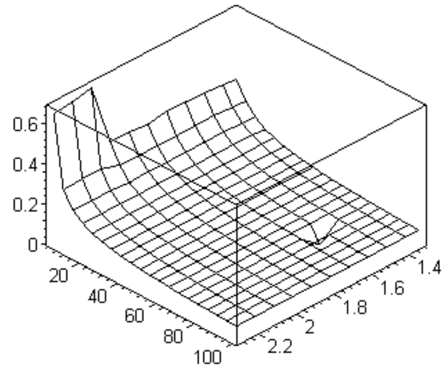

Figure 4. Surface plot of $[MSE_R]^{1/2}$ for $S_{pk} = 1.33$, with $C_p = 1.33$, 1.4(0.1)2.3 and sample size $n = 5(5)100$

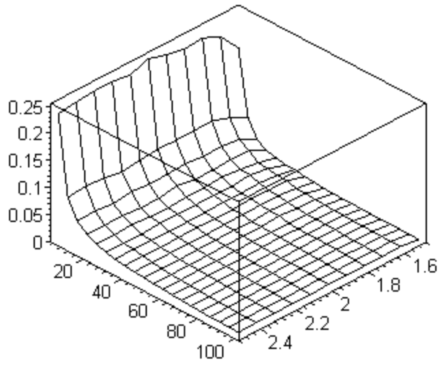

Figure 5. Surface plot of $BIAS_R(\hat{S}_{pk})$ for $S_{pk} = 1.50$, with $C_p = 1.5(0.1)2.5$ and sample size $n = 5(5)100$

plot of $BIAS_R(\hat{S}_{pk})$ for $S_{pk} = 1.50$, as a function of C_p and the sample size n. Figure 6 presents the surface plot of $[MSE_R(\hat{S}_{pk})]^{1/2}$ for $S_{pk} = 1.50$, as a function of C_p and the sample size n.

For the case of $S_{pk} = 1.67$, the simulation results indicate that for a sample size $n = 85$, the relative bias of the estimator is 0.3% for $C_p = 1.67$, 0.6% for $C_p = 1.7$ and 0.8–0.9% for all other values of C_p, except $C_p = 2.4$ where the relative bias is 1%. For a sample size $n = 100$, the relative bias of the estimator is 0.2% for $C_p = 1.67$, 0.6% for $C_p = 1.7$ and 0.8–0.9% for all other values of C_p. We note that for $n = 100$, $[MSE_R(\hat{S}_{pk})]^{1/2}$ is 7.1–7.2% for all values of C_p, except for $C_p = 1.7$, 1.8 where the relative bias is 7.3%. Thus, for the case $S_{pk} = 1.67$,

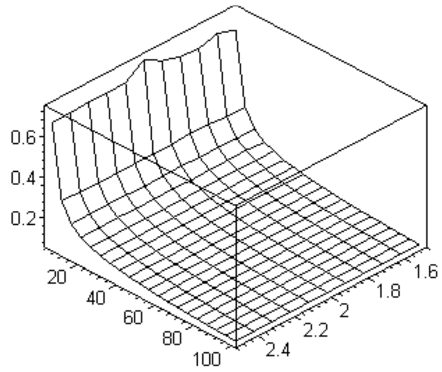

Figure 6. Surface plot of $[MSE_{\mathrm{R}}]^{1/2}$ for $S_{\mathrm{p}k} = 1.50$, with $C_{\mathrm{p}} = 1.5(0.1)2.5$ and sample size $n = 5(5)100$

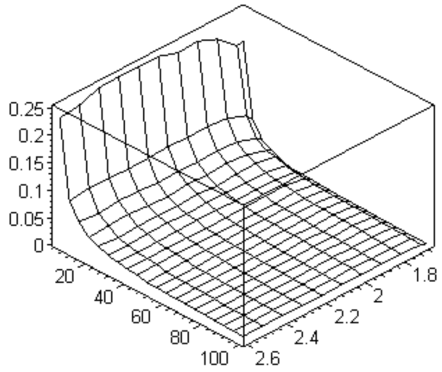

Figure 7. Surface plot of $BIAS_{\mathrm{R}}(\hat{S}_{\mathrm{p}k})$ for $S_{\mathrm{p}k} = 1.67$, with $C_{\mathrm{p}} = 1.67$, 1.7(0.1)2.6 and sample size $n = 5(5)100$

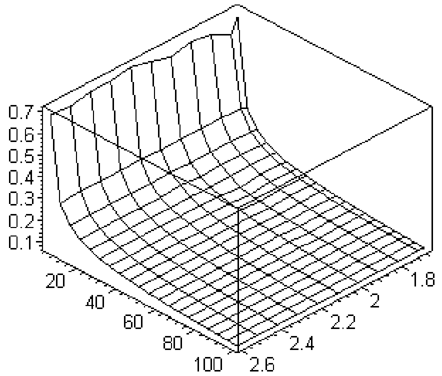

Figure 8. Surface plot of $[MSE_{\mathrm{R}}]^{1/2}$ for $S_{\mathrm{p}k} = 1.67$, with $C_{\mathrm{p}} = 1.67$, 1.7(0.1)2.6 and sample size $n = 5(5)100$

the estimation error of $\hat{S}_{\mathrm{p}k}$ is stable for sample sizes $n \geq 100$. Figure 7 presents a surface plot of $BIAS_{\mathrm{R}}(\hat{S}_{\mathrm{p}k})$ for $S_{\mathrm{p}k} = 1.67$, as a function of C_{p} and the sample size n. Figure 8 presents the surface plot of $[MSE_{\mathrm{R}}(\hat{S}_{\mathrm{p}k})]^{1/2}$ for $S_{\mathrm{p}k} = 1.67$, as a function of C_{p} and the sample size n.

For the case of $S_{\mathrm{p}k} = 2.00$, the simulation results indicate that for a sample size $n = 85$, the relative bias of the estimator is 0.5% for $C_{\mathrm{p}} = 2.0$ and 0.8–0.9% for all other values of C_{p}, except $C_{\mathrm{p}} = 2.5$ where the relative bias is 1%. For a sample size $n = 100$, the relative bias of the estimator is 0.3% for $C_{\mathrm{p}} = 2.0$ and is

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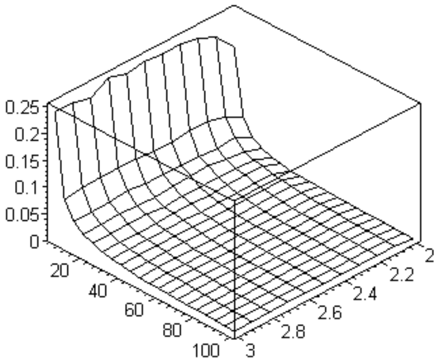

Figure 9. Surface plot of $BIAS_R(\hat{S}_{pk})$ for $S_{pk} = 2.00$, with $C_p = 2.0(0.1)3.0$ and sample size $n = 5(5)100$

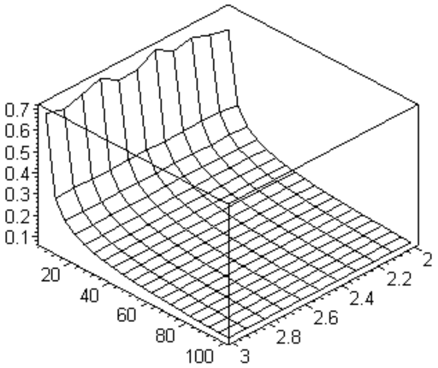

Figure 10. Surface plot of $[MSE_R]^{1/2}$ for $S_{pk} = 2.00$, with $C_p = 2.0(0.1)3.0$ and sample size $n = 5(5)100$

0.6–0.7% for all other values of C_p, except $C_p = 2.7$ where the relative bias is 0.8%. We note that for $n = 100$, $[MSE_R(\hat{S}_{pk})]^{1/2}$ is 7.1–7.3% for all values of C_p. Thus, for the case of $S_{pk} = 2.00$, the estimation error of $\hat{S}_{pk}$ is stable for sample sizes $n \geq 100$. Figure 9 presents a surface plot of $BIAS_R(\hat{S}_{pk})$ for $S_{pk} = 2.00$, as a function of C_p and the sample size n. Figure 10 presents the surface plot of $[MSE_R(\hat{S}_{pk})]^{1/2}$ for $S_{pk} = 2.00$, as a function of C_p and the sample size n.

The simulation results clearly indicate that the estimator $\hat{S}_{pk}$ overestimates the true value of S_{pk} in all the cases we investigated. The magnitude of the overestimation, in terms of the relative bias, $BIAS_R(\hat{S}_{pk})$, appears to be increasing in C_p at the beginning then remains stable roughly after $C_p > S_{pk} + 0.2$. This is true, in particular, for $n > 15$ in all cases. After the sample size $n > 15$, the fluctuation of $BIAS_R(\hat{S}_{pk})$ is no greater than 0.5% and is no greater than 0.1–0.2% for $n > 60$. The pattern is also apparent in $[MSE_R(\hat{S}_{pk})]^{1/2}$. In most cases, the magnitude of the deviation is increasing in C_p at the beginning, then becomes stable roughly after $C_p > S_{pk} + 0.1$, in particular, for $n > 20$.

For practical purposes, we may take the maximal values of $BIAS_R(\hat{S}_{pk})$ and $[MSE_R(\hat{S}_{pk})]^{1/2}$ to obtain bounds (fairly close to the actual values) on the error of the estimation for reliability purpose. Table III displays $\max\{BIAS_R(\hat{S}_{pk})\}$ and $\max\{[MSE_R]^{1/2}\}$ of $\hat{S}_{pk}$ for $S_{pk} = 1.00, 1.33, 1.50, 1.67, 2.00$ and $n = 5(5)100$. Figure 11 plots $\max\{BIAS_R(\hat{S}_{pk})\}$ for $S_{pk} = 1.00, 1.33, 1.50, 1.67, 2.00$ versus the sample size n. Figure 12 plots $\max\{BIAS_R(\hat{S}_{pk})\}$ for $S_{pk} = 1.00, 1.33, 1.50, 1.67, 2.00$ versus the sample size n. Thus, for an in-control process which runs under stable conditions, for $S_{pk} = 1.33$ and a sample size $n = 80$, we expect that the relative bias of $\hat{S}_{pk}$ calculated from the sample, on average, would not exceed 0.9% and the average relative error of $\hat{S}_{pk}$ would not exceed 8.1% of the true S_{pk}.

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Table III. $\max\{BIAS_{\mathrm{R}}\}$ and $\max\{[MSE_{\mathrm{R}}]^{1/2}\}$ of $\hat{S}_{\mathrm{p}k}$ for $S_{\mathrm{p}k} = 1.00, 1.33, 1.50, 1.67, 2.00$ and $n = 5(5)100$

n	1.00		1.33		1.50		1.67		2.00	
5	0.165	0.462	0.232	0.678	0.250	0.737	0.250	0.711	0.252	0.709
10	0.085	0.295	0.090	0.302	0.094	0.311	0.094	0.310	0.092	0.307
15	0.056	0.224	0.054	0.220	0.058	0.223	0.061	0.224	0.059	0.223
20	0.039	0.181	0.039	0.182	0.041	0.183	0.046	0.189	0.045	0.184
25	0.029	0.156	0.030	0.157	0.034	0.159	0.035	0.160	0.033	0.159
30	0.025	0.140	0.025	0.141	0.029	0.142	0.030	0.143	0.028	0.143
35	0.021	0.128	0.021	0.129	0.023	0.130	0.025	0.130	0.023	0.130
40	0.019	0.118	0.017	0.119	0.021	0.121	0.022	0.120	0.020	0.119
45	0.016	0.111	0.015	0.112	0.018	0.112	0.020	0.113	0.018	0.112
50	0.014	0.104	0.013	0.104	0.017	0.105	0.018	0.107	0.016	0.106
55	0.013	0.099	0.012	0.099	0.017	0.100	0.017	0.101	0.014	0.100
60	0.013	0.094	0.012	0.096	0.013	0.096	0.016	0.097	0.014	0.096
65	0.011	0.091	0.011	0.091	0.013	0.091	0.015	0.092	0.013	0.092
70	0.010	0.087	0.010	0.087	0.011	0.088	0.015	0.089	0.011	0.088
75	0.009	0.083	0.010	0.084	0.011	0.085	0.014	0.086	0.010	0.085
80	0.008	0.081	0.009	0.081	0.010	0.082	0.013	0.082	0.010	0.082
85	0.008	0.079	0.008	0.079	0.010	0.079	0.012	0.080	0.010	0.079
90	0.007	0.077	0.008	0.076	0.009	0.077	0.011	0.077	0.009	0.077
95	0.007	0.073	0.008	0.075	0.008	0.075	0.011	0.074	0.008	0.074
100	0.006	0.071	0.006	0.072	0.008	0.072	0.010	0.073	0.008	0.073

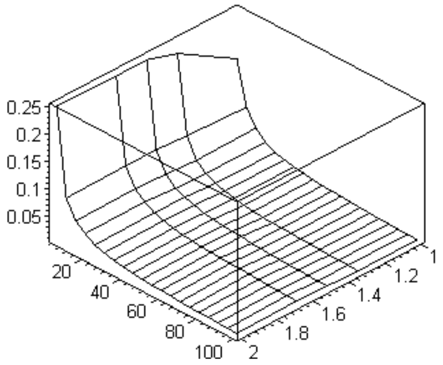

Figure 11. Surface plot of $\max\{BIAS_{\mathrm{R}}\}$ for $S_{\mathrm{p}k} = 1.00$, 1,33, 1.50, 1.67, 2.00 and sample size $n = 5(5)100$

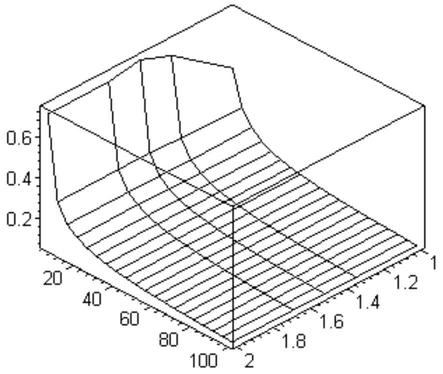

Figure 12. Surface plot of $\max\{[MSE_{\mathrm{R}}]^{1/2}\}$ for $S_{\mathrm{p}k} = 1.00$, 1,33, 1.50, 1.67, 2.00 and sample size $n = 5(5)100$

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Table IV. A total of 80 sample observations

13.23	13.19	13.22	13.19	13.18
13.20	13.20	13.22	13.19	13.17
13.21	13.21	13.20	13.20	13.20
13.19	13.19	13.22	13.20	13.21
13.23	13.19	13.19	13.21	13.21
13.19	13.19	13.19	13.19	13.20
13.21	13.21	13.21	13.22	13.21
13.20	13.20	13.21	13.20	13.19
13.22	13.20	13.19	13.19	13.20
13.19	13.22	13.20	13.19	13.21
13.21	13.20	13.21	13.19	13.21
13.20	13.22	13.21	13.21	13.18
13.19	13.19	13.19	13.19	13.21
13.20	13.20	13.20	13.20	13.20
13.19	13.20	13.21	13.22	13.19
13.20	13.19	13.21	13.19	13.17

Table V. The 20 consecutive days $\hat{S}_{pk}$

1.33	1.34	1.38	1.23
1.25	1.20	1.25	1.41
1.31	1.29	1.33	1.20
1.26	1.17	1.29	1.45
1.33	1.21	1.39	1.19

5. AN APPLICATION EXAMPLE

Consider the following example involving a factory manufacturing pistons, which are one of the most critical components for the oil-hydraulic cylinders. When the oil goes through the oil-hydraulic cylinder, it exerts pressure making the piston move. Two grooves on the piston must fit with the U-shaped oil seal to prevent the oil from leaking when the piston is in motion. If the oil leaks, it affects the efficiency and performance of the oil-hydraulic cylinder. The prominent parts of the piston hold the two U-shaped oil seals to make them assume the pressure from the oil-hydraulic cylinder. It is the U-shaped oil seals, not the main body of the piston, which is in direct contact with the tube of the oil-hydraulic cylinder. Thus, it is essential to make the piston grooves comply with the required manufacturing specifications.

The manufacturing specifications for the grooves of a particular type of piston are: $USL = 13.25$ mm, $LSL = 13.15$ mm, target $T = 13.2$ mm. Historical data based on routine process monitoring shows that the process is under statistical control and the process distribution is justified and is shown to be fairly close to the normal distribution. A sample data collection procedure is implemented in the factory on a daily basis to monitor/control process quality. The factory production resource and schedule allows the data collection plan be implemented with a sample size $n \leq 80$. The collected sample data (a total of 80 observations), in a specific day, are displayed in Table IV.

The calculated estimation $\hat{S}_{pk}$ is 1.36. A simple approach to determine the true value (rather than a lower confidence bound) of S_{pk} is to perform the sampling on a routine basis consecutively for a number of, say, 20 days. The calculated values of single-day $\hat{S}_{pk}$ for 20 consecutive days are displayed in Table V. The average $\hat{S}_{pk}$ value for the 20 days is obtained as $E(\hat{S}_{pk}) = 1.23$. Checking Table III for $\max\{BIAS_R(\hat{S}_{pk})\}$, we find the upper bound on the error in relative bias, for sample size $n = 80$ is $\max\{|E(\hat{S}_{pk}) - S_{pk}|\} = 1.3\%$. Therefore, the true value of S_{pk} can be determined as $1.23/(1 + 1.3\%) = 1.21$. The error of the approximation becomes negligibly small over time.

6. CONCLUSION

Process yield is the most common and standard criteria used in the manufacturing industry for measuring process performance. Boyles[1] considered a measure, called S_{pk} to calculate the yield for processes with normal distributions. The capability measure S_{pk} establishes the relationship between the manufacturing specifications

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and the actual process performance, which provides an exact measure for process yield. The statistical properties of the natural estimator of S_{pk} are mathematically intractable and the existing approximations are rather complicated and difficult to apply. In this paper, we investigated the accuracy of the natural estimator of S_{pk} computationally, using the standard simulation technique to find the relative bias and the relative mean square error for some commonly used quality requirements. Extensive simulation results are tabulated and analyzed to provide the practitioners/engineers with critical information regarding the true value of S_{pk}, which is useful in determining the process performance.

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REFERENCES

1. Kane VE. Process capability indices. *Journal of Quality Technology* 1986; **18**(1):41–52.
2. Chan LK, Cheng SW, Spiring FA. A new measure of process capability: C_{pm}. *Journal of Quality Technology* 1988; **20**:162–173.
3. Pearn WL, Kotz S, Johnson NL. Distributional and inferential properties of process capability indices. *Journal of Quality Technology* 1992; **24**:216–231.
4. Bordignon S, Scagliarini M. Statistical analysis of process capability indices with measurement errors. *Quality and Reliability Engineering International* 2002; **18**(4):321–332.
5. Borges WS, Ho LL. A fraction defective based capability index. *Quality and Reliability Engineering International* 2001; **17**(6):447–458.
6. Chang YS, Choi IS, Bai DS. Process capability indices for skewed populations. *Quality and Reliability Engineering International* 2002; **18**(5):383–393.
7. Hoffman LL. Obtaining confidence intervals for C_{pk} using percentiles of the distribution of C_p. *Quality and Reliability Engineering International* 2001; **17**(2):113–118.
8. Nahar PC, Hubele NF, Zimmer LS. Assessment of a capability index sensitive to skewness. *Quality and Reliability Engineering International* 2001; **17**(4):233–241.
9. Noorossana R. Process capability analysis in the presence of autocorrelation. *Quality and Reliability Engineering International* 2002; **18**(1):75–77.
10. Pearn WL, Lin GH, Chen KS. Distributional and inferential properties of the process accuracy and process precision indices. *Communications in Statistics: Theory & Methods* 1998; **27**(4):985–1000.
11. Pearn WL, Lin PC. Computer program for calculating the p-value in testing process capability index C_{pmk}. *Quality and Reliability Engineering International* 2002; **18**(4):333–342.
12. Zimmer LS, Hubele NF, Zimmer WJ. Confidence intervals and sample size determination for C_{pm}. *Quality and Reliability Engineering International* 2001; **17**(1):51–68.
13. Kotz S, Johnson NL. Process capability indices—a review, 1992–2000. *Journal of Quality Technology* 2002; **34**(1): 1–19.
14. Boyles RA. Process capability with asymmetric tolerances. *Communication in Statistics: Simulation and Computation* 1994; **23**(3):615–643.
15. Lee JC, Hung HN, Pearn WL, Kueng TL. On the distribution of the estimated process yield index S_{pk}. *Quality & Reliability Engineering International* 2002; **18**(2):111–116.
16. Pearn WL, Hung HN, Lin GH. Distributional and inferential properties of the estimated process yield index S_{pk}. *Working Paper*, National Chiao Tung University. 2002.

Authors' biographies

W. L. Pearn is a professor of operations research and quality management at the Department of Industrial Engineering and Management, National Chiao Tung University, Taiwan, Republic of China. His research areas include process capability analysis, network optimization, queuing service management and semiconductor manufacturing scheduling.

C. C. Chuang received her MS degree from the Department of Industrial Engineering and Management, National Chiao Tung University, Taiwan, Republic of China. Currently, she is a project analyst at the R&D Center for SCM, Pouchen International Group, Taiwan, Republic of China.

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