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Note

Cyclically decomposing the complete graph into cycles

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Abstract

Let $m_1, m_2, \ldots, m_k$ be positive integers not less than 3 and let $n = \sum_{i=1}^{k} m_i$. Then, it is proved that the complete graph of order $2n+1$ can be cyclically decomposed into $k(2n+1)$ cycles such that, for each $i = 1, 2, \ldots, k$, the cycle of length m_i occurs exactly $2n+1$ times.

Keywords: Complete graph; Cycle system; Skolem sequence; Hooked Skolem sequence; Near Skolem sequence

1. Introduction

A Steiner triple system (STS) is an ordered pair (V, B), where V is a finite nonempty set of elements, and B is a collection of 3-element subsets of V called triples, such that each pair of distinct elements of V occurs together in exactly one triple of B. The order of a Steiner triple system (V, B) is the size of V, denoted by $|V|$.

From "graph decomposition" point of view, the existence of a Steiner triple system of order v (STS(v)) is equivalent to the existence of a decomposition of the complete graph K_v of order v into edge-disjoint triangles, denoted by C_3. It is not difficult to see the necessary condition for such a decomposition to exist is that $v \equiv 1$ or $3 \pmod 6$. In fact, this condition was proved to be sufficient around 150 years ago by Kirkman [4]. An automorphism of a STS (V, B) is a bijection $\alpha : V \to V$ such that $\{x, y, z\} \in B$ if and only if $\{\alpha(x), \alpha(y), \alpha(z)\} \in B$. A STS($v$) is *cyclic* if it has an automorphism that is a permutation consisting of a single cycle of length v, for example $(1, 2, 3, \ldots v)$.

Cyclic Steiner triple systems do exist. In 1939, Peltesohn used the so-called difference method to settle the existence problem.

Theorem 1.1 (Peltesohn [7]). *For all $v \equiv 1$ or $3 \pmod 6$ except $v = 9$, there exists a cyclic* STS(v).

We move on to consider an analog of Steiner triple systems. An m-cycle system of order v is a pair (V, C), where $V = V(K_v)$ and C is a collection of edge-disjoint m-cycles which partition the edge set of K_v. Let Π be an automorphism group of the m-cycle system (V, C) (i.e., a group of permutations on v vertices leaving the collection C of cycles invariant). If there is an automorphism $\pi \in \Pi$ of order v, then the m-cycle system (V, C) is said to be *cyclic*. For an m-cycle system of K_v, the vertex set V can be identified with Z_v. It is easy to see the necessary conditions for such a decomposition are (i) v is odd and (ii) $m \mid \binom{v}{2}$.

The study of "existence problem" of m-cycle systems started around 40 years ago. Recently, Alspach and Gavlas [2] and Šajna [10] proved that an m-cycle system exists as long as the above conditions are met. Thus, we have all the m-cycle systems for each $m \geqslant 3$.

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Similar to a cyclic Steiner triple system, we can also consider the existence of cyclic m-cycle systems. Actually, the earlier works on the existence of m-cycle systems give cyclic systems. The case when $m \equiv 0 \pmod 4$ and $v \equiv 1 \pmod{2m}$ was obtained by Kotzig [5] and the case when $m \equiv 2 \pmod 4$ and $v \equiv 1 \pmod{2m}$ was due to Rosa [8]. Furthermore, Rosa [9] proved that if m is odd and $v \equiv 1 \pmod{2m}$ or if m is an odd prime and $v \equiv m \pmod{2m}$, then K_v can be decomposed into closed trails of length m. In the case when $m = 5$ or 7, Rosa proved that the closed trials were indeed cycles. Therefore, cyclic 5-cycle systems and cyclic 7-cycle systems are obtained. Recently, Buratti and Del Fra [3] proved that for each odd prime p, cyclic p-cycle system exists.

In 1981, the following problem was posed by Alspach [1].

Conjecture. Let $m_1, m_2, \ldots, m_h$ be positive integers not less than 3 such that $\sum_{i=1}^{h} m_i = \binom{n}{2}$ for odd n (respectively, $\sum_{i=1}^{h} m_i = \binom{n}{2} - n/2$ for even n). Then K_n (respectively, $K_n - F$) can be decomposed into cycles $C_1, C_2, \ldots, C_h$ such that the length of C_i is m_i for $i = 1, 2, \ldots, h$.

In this paper, we prove a special case of the conjecture, namely, we prove that if $m_1, m_2, \ldots, m_k$ are positive integers all at least 3, then the complete graph K_{2n+1}, where $n = \sum_{i=1}^{k} m_i$, has a cyclic decomposition into $k(2n+1)$ cycles such that for each $i = 1, 2, \ldots, k$, there are exactly $2n+1$ cycles of length m_i.

2. The main results

Throughout this paper, we shall use difference methods. The *difference* between two vertices x and y in the complete graph K_n with $V(K_n) = Z_n$ is $|x - y|$ or $n - |x - y|$, whichever is smaller. We will say that the edge xy has difference $\min\{|x - y|, n - |x - y|\}$. Thus, the set of differences possible in K_n is $\{1, 2, \ldots, \lfloor n/2 \rfloor\}$ and each difference induces a 2-factor except the difference $n/2$ induces a 1-factor whenever n is even. For convenience, we shall use $G[D]$ to denote the subgraph of G induced by the set of differences $D \subseteq \{1, 2, \ldots, \ell\}$. It is easy to check that $K_{2\ell+1}[i]$ is a disjoint union of cycles of length $(2\ell+1)/(2\ell+1, i)$, where $(2\ell+1, i)$ denotes the greatest common divisor of $2\ell+1$ and i. Clearly, if $(2\ell+1, i) = 1$, then $K_{2\ell+1}[i]$ is a Hamiltonian cycle in $K_{2\ell+1}$. It should be mentioned that if cycles C_i $(1 \leqslant i \leqslant k)$ have difference sets A_i which partition $\{1, 2, \ldots, \ell\}$, then there exists a cyclic decomposition of $K_{2\ell+1}$ into cycles C_i.

Notice that if H is a subgraph of $K_{2\ell+1}$ such that each edge of H has a distinct difference, then the graph $H + i$ obtained from H by adding $i \pmod{2\ell+1}$ to each vertex of H is an isomorphic copy of H. The following results are given in [13] and will be used in the proof of Theorem 2.5.

Lemma 2.1 (Wu [13]). *For positive integers b and s, there exists a cycle C of length $4s$ with difference set*

$$\{b, b+1, \ldots, b+4s-1\}$$

in K_n where n is odd with $n \geqslant 2(b+4s-1)+1$.

Lemma 2.2 (Wu [13]). *Let b and s be positive integers.*

(1) *There exists a cycle C of length $4s+2$ with difference set*

$$\{b, b+1, \ldots, b+4s, b+4s+2\}$$

in K_n where n is odd with $n \geqslant 2(b+4s+2)+1$.

(2) *There exists a cycle C of length $4s+2$ with difference set*

$$\{b, b+2, b+3, \ldots, b+4s+2\}$$

in K_n where n is odd with $n \geqslant 2(b+4s+2)+1$.

Note that one may use a consecutive block of integers to construct cycles of length congruent to 0 modulo 4 and/or an even number of cycles of length congruent to 2 modulo 4. For example, if $m_1 = 4s+2$ and $m_2 = 4t+2$, then applying (1) and (2) of Lemma 2.2 give cycles C_1 and C_2 of lengths m_1 and m_2 with difference sets $\{b, b+1, \ldots, b+4s, b+4s+2\}$ and $\{b+4s+1, b+4s+3, \ldots, b+4s+4t+3\}$, respectively, for any positive integer b.

For convenience, in the following lemmas, we use a typical odd cycle as in Fig. 1.

Lemma 2.3. *For positive integers a, b, c, and r, with $c = a+b$ and $r > c$, and a nonnegative integer s, there exists a cycle C of length $4s+3$ with difference set $\{a, b, c, r, r+1, \ldots, r+4s-1\}$ in K_n where n is odd and $n \geqslant 2(r+4s-1)+1$.*

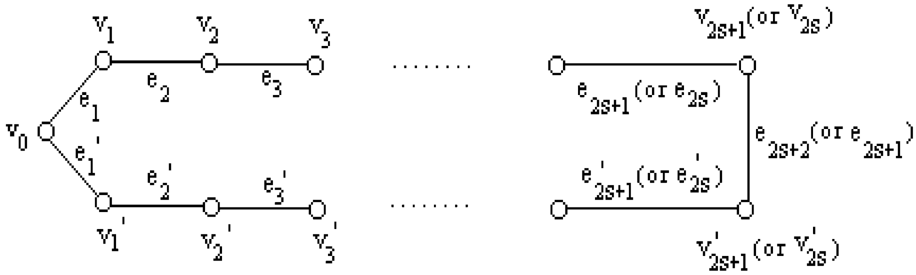
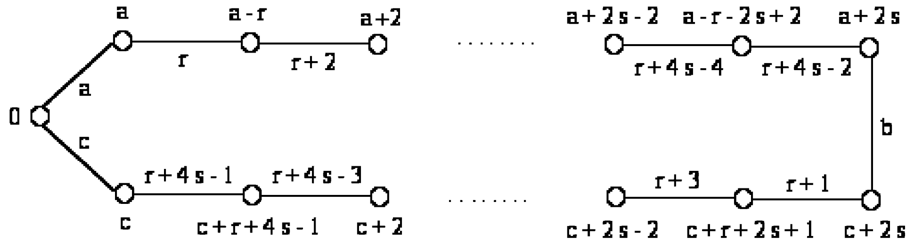


Fig. 1.

Proof. The proof is divided into two cases.

Case 1: Either a or b is odd, say b.

The cycle C of length $4s+3$ is defined as the following:



An easy verification shows that the vertices of the cycle C are: for $i=0,1,\ldots,s$, $v_{2i+1}=a+2i$, $v'_{2i+1}=c+2i$, $v_{2i}=a-r-2(i-1)$, $v'_{2i}=c+r+4s-2i+1$, where all indices are taken modulo n, and the difference set is $\{a,b,c,r,r+1,\ldots,r+4s-1\}$. Observe that since $c=a+b$ and b is odd, it follows that a and c have opposite parity. Thus $a,a+2,\ldots,a+2s$ and $c,c+2,\ldots,c+2s$ have opposite parity and hence are distinct. Also, $c+r+4s-1$, $c+r+4s-2,\ldots,c+r+2s+1$ and $a-r,a-r-2,\ldots,a-r-2s-2$ have opposite parity when considered modulo n and thus are distinct. Therefore, the vertices of C are distinct.

Case 2: Both a and b are even.

(i) r is even: Let $e_1=a$, $e'_1=r+4s-2$, $e_2=r+4s-1$, $e'_2=c$, $e_{2s+2}=b$, and for $i=3,4,\ldots,2s+1$, let $e_i=r+4s-2i+3$, $e'_i=r+2i-6$. Now, we define the vertices accordingly. Let $v_0=0$, $v_1=a$, $v'_1=r+4s-2$ and for $i=1,2,\ldots,s, v_{2i}=a+r+4s-2i+1$, $v'_{2i}=c+r+4s+2i-4$, $v_{2i+1}=a+2i$, and $v'_{2i+1}=c+4s-2i$.

(ii) r is odd: In this subcase, we let $e_1=a$, $e'_1=r+4s-1$, $e_2=r+4s-2$, $e'_2=c, e_{2s+2}=b$, and for $i=3,4,\ldots,2s+1, e_i=r+4s-2i+2$, $e'_i=r+2i-5$. Then according to the differences, we define the vertices for the cycle. Let $v_0=0$, $v_1=a$, $v'_1=r+4s-1$, and for $i=1,2,\ldots,s$, $v_{2i}=a+r+4s-2i$, $v_{2i+1}=a+2i$, $v'_{2i}=c+r+4s+2i-3$, and $v'_{2i+1}=c+4s-2i$. $\square$

Remark. In Lemma 2.3, if $c=a+b\pm 2$, then we can use a similar method mentioned above to construct a cycle of length $4s+1$ with difference set $\{a,b,c,r,r+1,\ldots,r+4s-4,r+4s-2\}$. Notice that this construction will be used in Theorem 2.5. See, for example, Fig. 2, where $a=2, b=3, c=7, s=3$, and $r=9$.

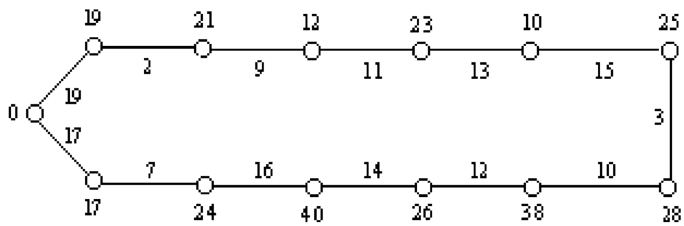
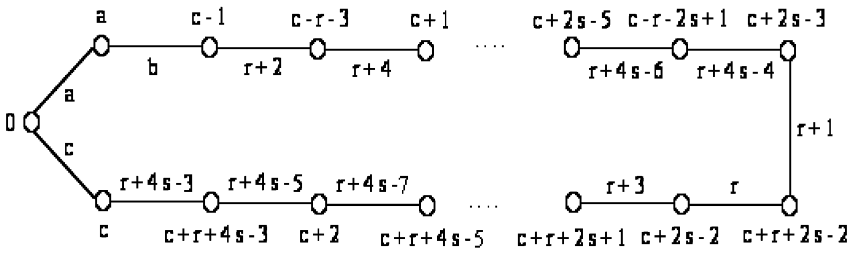


Fig. 2.

Next, we consider cycles of length $4s+1$.

Lemma 2.4. *For positive integers a,b,c, and r, with $c=a+b\pm 1$ and $r>c$, and a nonnegative integer s, there exists a cycle C of length $4s+1$ with difference set $\{a,b,c,r,r+1,\ldots,r+4s-3\}$ in K_n where n is odd and $n\geqslant 2(r+4s-3)+1$.*

Proof. Let us define the cycle C of length $4s+1$ as



Notice that the value of $a+b$ is even (odd) and c is odd (even). By routine computation, it follows that the difference set is $\{a,b,c,r,r+1,\ldots,r+4s-3\}$, and the distinct vertices of C are: $v_1=a$, $v'_1=c$, $v_{2s}=c+2s-3$, $v'_{2s}=c+r+2s-2$, and for $i=1,2,\ldots,s-1$, $v_{2i}=c+2i-3$, $v'_{2i}=c+r+4s-1-2i$, $v_{2i+1}=c-r-2i-1$, and $v'_{2i+1}=c+2i$. $\square$

In order to prove the main theorem, we also need to use Skolem sequences, hooked Skolem sequences, and near Skolem sequences.

A *Skolem sequence* of order n is a sequence $(s_1,s_2,\ldots,s_{2n})$ such that for each $j\in\{1,2,\ldots,n\}$, there exists a unique $i\in\{1,2,\ldots,2n\}$ such that $s_i=s_{i+j}=j$. It is proved by Skolem [12] that such a sequence exists if and only if $n\equiv 0$ or $1 \pmod 4$.

A *hooked Skolem sequence* of order n is a sequence $(s_1,s_2,\ldots,s_{2n+1})$ such that $s_{2n}=0$ and for each $j\in\{1,2,\ldots,n\}$, there exists a unique $i\in\{1,2,\ldots,2n-1,2n+1\}$ such that $s_i=s_{i+j}=j$. These sequences are known to exist if and only if $n\equiv 2, 3 \pmod 4$ (see [6]).

An *m-near Skolem sequence* of order n ($m\leqslant n$) is a sequence $(s_1,s_2,\ldots,s_{2n-2})$ of $2n-2$ integers which satisfies for every $j\in\{1,2,\ldots,n\}\setminus\{m\}$, there exists a unique $i\in\{1,2,\ldots,2n-2\}$ such that $s_i=s_{i+j}=j$. It is proved by Shalaby [11] that an m-near Skolem sequence of order n exists if and only if either (1) m is odd and $n\equiv 0$ or $1 \pmod 4$ or (2) m is even and $n\equiv 2$ or $3 \pmod 4$.

Remark that a Skolem sequence $(s_1,s_2,\ldots,s_{2n})$ of order n gives a partition of $\{1,2,\ldots,3n\}$ into triples $\{j,n+i,n+i+j | s_i=s_{i+j}=j\}$ for $j=1,2,\ldots,n$. Similarly, a hooked Skolem sequence gives a partition of $\{1,2,\ldots,3n-1,3n+1\}$ into triples $\{j,s_j,t_j\}$ satisfying $j+s_j=t_j$ ($1\leqslant j\leqslant n$) and an m-near Skolem sequence gives a partition of $\{1,2,\ldots,3n-2\}\setminus\{m\}$ into triples $\{j,s_j,t_j\}$ satisfying $j+s_j=t_j$ for each $j\in\{1,2,\ldots,n\}\setminus\{m\}$.

Now, we are ready for the proof of our main result.

Theorem 2.5. *Let $m_1,m_2,\ldots,m_k$ be positive integers not less than 3 such that $n=\sum_{i=1}^{k} m_i$. Then there exists a cyclic $(m_1,m_2,\ldots,m_k)$-cycle system of order $2n+1$.*

Proof. For convenience, let $m_1,m_2,\ldots,m_{i_1}$ denote the integers which are congruent to 3 modulo 4, $m_{i_1+1},m_{i_1+2},\ldots,m_{i_2}$ denote the integers which are congruent to 1 modulo 4, $m_{i_2+1},m_{i_2+2},\ldots,m_{i_3}$ denote the integers which are congruent to 0 modulo 4, and thus $m_{i_3+1},m_{i_3+2},\ldots,m_k$ will be the integers which are congruent to 2 modulo 4. It suffices to partition the set $\{1,2,\ldots,n\}$ into sets $A_1,A_2,\ldots,A_k$ such that:

- $|A_i|=m_i$ for each i with $1\leqslant i\leqslant k$;
- each of the sets $A_1,A_2,\ldots,A_{i_1}$ satisfies the conditions for the difference set given in Lemma 2.3;
- each of the sets $A_{i_1+1},A_{i_1+2},\ldots,A_{i_2}$ satisfies the conditions for the difference set given in Lemma 2.4 or the remark after Lemma 2.3;
- each of the sets $A_{i_2+1},A_{i_2+2},\ldots,A_{i_3}$ satisfies the conditions for the difference set given in Lemma 2.1; and
- each of the sets $A_{i_3+1},A_{i_3+2},\ldots,A_k$ satisfies the conditions for the difference set given in either (1) or (2) in Lemma 2.2.

Case 1: *Suppose that* $i_2 \equiv 0$, 1 (mod 4). Clearly, if $i_2 = 0$, then it is easy to define the sets $A_1, A_2, \ldots, A_k$ by choosing the differences for the cycles of length congruent to 0 modulo 4 first followed by choosing the differences for those cycles of length congruent to 2 modulo 4 last, using $n+1$ for n as necessary. In fact, after defining the sets $A_1, A_2, \ldots, A_{i_2}$, if we left with a set $\{b, b+1, \ldots, n\}$ for some positive integer b, then we can easily choose the differences for sets $A_{i_2+1}, A_{i_2+2}, \ldots, A_k$.

Since $i_2 \equiv 0$, 1 (mod 4), there exists a Skolem sequence of order i_2 such that the set $\{1, 2, \ldots, 3i_2\}$ can be partitioned into triples $\{i, s_i, t_i\}$ with $i + s_i = t_i$ for $i = 1, 2, \ldots, i_2$. Suppose first that $i_2 = i_1$ so that there are no cycles of length congruent to 1 modulo 4. Then, the sets $A_1, A_2, \ldots, A_{i_1}$ are defined as follows:

- 1, $s_1, t_1 \in A_1$,
- 2, $s_2, t_2 \in A_2$,

$\vdots$

- i_1, s_{i_1}, $t_{i_1} \in A_{i_1}$, and
- starting with $3i_1 + 1$, assign the next $m_1 - 3$ consecutive integers to A_1, the next $m_2 - 3$ consecutive integers to A_2 and so on until assigning $m_{i_1} - 3$ consecutive integers to A_{i_1}.

Observe that the differences left are $\sum_{i=1}^{i_1} m_i + 1$, $\sum_{i=1}^{i_1} m_i + 2, \ldots, n$, and as remarked earlier, the sets $A_{i_2+1}, A_{i_2+2}, \ldots, A_k$ are easily found.

Now suppose that $i_2 > i_1$. Suppose first that $i_2 - i_1$ is even. Define the sets $A_1, A_2, \ldots, A_{i_1}$ as follows:

- 1, s_1, $t_1 \in A_1$,
- 2, s_2, $t_2 \in A_2$,

$\vdots$

- i_1, s_{i_1}, $t_{i_1} \in A_{i_1}$,
- $i_1 + 2$, s_{i_1+1}, $t_{i_1+1} \in A_{i_1+1}$,
- $i_1 + 1$, s_{i_1+2}, $t_{i_1+2} \in A_{i_1+2}$,

$\vdots$

- i_2, s_{i_2-1}, $t_{i_2-1} \in A_{i_2-1}$,
- $i_2 - 1$, s_{i_2}, $t_{i_2} \in A_{i_2}$, and
- starting with $3i_2 + 1$, assign the next $m_1 - 3$ consecutive integers to A_1, the next $m_2 - 3$ consecutive integers to A_2 and so on until assigning $m_{i_2} - 3$ consecutive integers to A_{i_2}.

The differences remaining are $\sum_{i=1}^{i_2} m_i + 1$, $\sum_{i=1}^{i_2} m_i + 2, \ldots, n$ and the sets $A_{i_2+1}, A_{i_2+2}, \ldots, A_k$ are easily found.

Now assume that $i_2 - i_1$ is odd. Then, by [11], there exists a 1-near Skolem sequence of order i_2 so that the set $\{2, 3, \ldots, 3i_2 - 2\}$ can be partitioned into triples $\{i, s_i, t_i\}$ with $i + s_i = t_i$ for $i = 2, \ldots, i_2$. If there are no cycles of length congruent to 2 modulo 4, then we define $A_1, A_2, \ldots, A_{i_2}$ as follows:

- 2, s_2, $t_2 \in A_1$,
- 3, s_3, $t_3 \in A_2$,

$\vdots$

- $i_1 + 1$, s_{i_1+1}, $t_{i_1+1} \in A_{i_1}$,
- $i_1 + 3$, s_{i_1+2}, $t_{i_1+2} \in A_{i_1+1}$,
- $i_1 + 2$, s_{i_1+3}, $t_{i_1+3} \in A_{i_1+2}$,

$\vdots$

- $i_2 - 1$, s_{i_2}, $t_{i_2} \in A_{i_2-1}$,
- 1, $n-1$, $n+1 \in A_{i_2}$, and
- starting with $3i_2 - 1$, assign the next $m_1 - 3$ consecutive integers to A_1, the next $m_2 - 3$ consecutive integers to A_2 and so on until assigning $m_{i_2} - 3$ consecutive integers to A_{i_2}.

(Note that if $i_1+1=i_2$, then 1, $n-1, n+1 \in A_{i_2}$.) If there are cycles of length congruent to 2 modulo 4, then we define the sets $A_1, A_2, \ldots, A_{i_2}, A_{i_3+1}$ as follows:

- 2, s_2, $t_2 \in A_1$,
- 3, s_3, $t_3 \in A_2$,

$\vdots$

- i_1+1, s_{i_1+1}, $t_{i_1+1} \in A_{i_1}$,
- i_1+3, s_{i_1+2}, $t_{i_1+2} \in A_{i_1+1}$,
- i_1+2, s_{i_1+3}, $t_{i_1+3} \in A_{i_1+2}$,

$\vdots$

- i_2-1, s_{i_2}, $t_{i_2} \in A_{i_2-1}$,
- 1, $3i_2-1$, $3i_2+1 \in A_{i_2}$,
- $A_{i_3+1} = \{3i_2, 3i_2+2, 3i_2+3, \ldots, 3i_2+mi_3+1\}$, and
- starting with $3i_2+m_{i_3+1}+1$, assign the next m_1-3 consecutive integers to A_1, the next m_2-3 consecutive integers to A_2 and so on until assigning $m_{i_2}-3$ consecutive integers to A_{i_2}.

In either case, the differences remaining form a set of consecutive integers, and the sets $A_{i_2+1}, A_{i_2+2}, \ldots, A_{i_3-1}, A_{i_3+1}, \ldots, A_k$ are easily found.

Now, we have completed the partition of $\{1,2,\ldots,n\}$ into the sets $A_1, A_2, \ldots, A_k$. By Lemmas 2.1–2.4, we are able to obtain a cyclic cycle decomposition for K_{2n+1}. This concludes the proof of the case when $i_2 \equiv 0$ or $1 \pmod 4$.

Case 2: *Suppose that* $i_2 \equiv 2, 3 \pmod 4$. The proof can be obtained by using a procedure similar to those in Case 1, and so we omit the details. □

For clearness, we present two examples to show the idea of partition.

Example 1. $n=51$. $(m_1, m_2, \ldots, m_{10}) = (3,3,7,5,9,4,4,4,6,6)$, $i_2=5$. $A_1=\{1,6,7\}$, $A_2=\{2,12,14\}$, $A_3=\{3,8,11,16,17,18,19\}$, $A_4=\{5,9,13,20,21\}$, $A_5=\{4,10,15,22,23,24,25,26,27\}$, $A_6=\{28,29,30,31\}$, $A_7=\{32,33,34,35\}$, $A_8=\{36,37,38,39\}$, $A_9=\{40,41,42,43,44,46\}$, and $A_{10}=\{45,47,48,49,50,51\}$.

Example 2. $n=51$. $(m_1, m_2, \ldots, m_{10}) = (3,3,3,7,5,5,9,4,6,6)$, $i_2=7$. $A_1=\{1,8,9\}$, $A_2=\{3,14,17\}$, $A_3=\{4,11,15\}$, $A_4=\{5,13,18,22,23,24,25\}$, $A_5=\{7,10,16,26,27\}$, $A_6=\{6,12,19,28,29\}$, $A_7=\{2,20,21,30,31,32,33,34,35\}$, $A_8=\{36,37,38,39\}$, $A_9=\{40,41,42,43,44,46\}$, and $A_{10}=\{45,47,48,49,50,51\}$.

With the theorem we proved, the following known result can be obtained easily.

Corollary 2.6. *For each* $m \geqslant 3$ *there exists a cyclic* m*-cycle system of order* $v \equiv 1 \pmod{2m}$.

Proof. Since $v=2km+1$ for some integer $k \geqslant 1$, by Theorem 2.5 where we choose $n=mk$, we conclude the proof. □

Remark. By an independent effort, we can also obtain the consequence, that is, for each odd prime p, there exists a cyclic p-cycle system.

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