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On the Maximum Benefit Chinese Postman Problem

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Abstract

The Maximum Benefit Chinese Postman Problem (MBCPP) is a practical generalization of the classical Chinese Postman Problem (CPP), which has many real-world applications. In this paper, we consider the MBCPP on undirected networks, and show that the MBCPP is more complex than the Rural Postman Problem (RPP). We present a sufficient condition for the MBCPP solution to cover the whole network, and provide an upper bound. Based on the upper bound, we propose an efficient solution procedure to solve the MBCPP approximately. The proposed algorithm applies the minimal spanning tree and the minimal-cost matching algorithms, which performs well on problems satisfying the sufficient condition.
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Keywords: Chinese Postman Problem; Rural Postman Problem; Upper bound

1. Introduction

There are many generalizations of the well-known Chinese Postman Problem (CPP). Examples include the Rural Postman Problem (RPP) [1], the Hierarchical Postman Problem (HPP) [2], the k-person Chinese Postman Problem (k-CPP) [5], the Capacitated Arc Routing Problem (CARP) [3], and many others.

The Maximum Benefit Chinese Postman Problem (MBCPP) is another interesting generalization of the CPP, in which each edge on the network is associated with a service cost for the traversal with service, a deadhead cost for the traversal with no service, and a set of benefits. Each time an edge is traversed a benefit is generated. The objective of the MBCPP is to find a postman tour traversing a selected set of edges with the total net benefit maximized. Such a generalization reflects the real-world situations more closely than the classical CPP. Applications directly related to the MBCPP include routing of street sweepers, snowplows, spraying roads with salt, inspection of streets for maintenance, and reading of electric meters.

The problem may be briefly defined as follows. Given an undirected network $G(V,E)$, with V representing the set of nodes, and E representing the set of edges. For each edge $(i,j)\in E$, we are given a non-negative service cost c_{ij}^{s} for the edge traversal with service, and a non-negative deadhead cost c_{ij}^{d} for the edge traversal with no service, which we expect $c_{ij}^{\mathrm{s}} \geqslant c_{ij}^{\mathrm{d}}$. We are also given a set of non-negative benefits $b_{ijr_{ij}}$ from node i to node j for the r_{ij}th traversal, where $r_{ij} = 1,2,\ldots,n_{ij}$.

To reflect real situations more closely, we assume that the benefit $b_{ijr_{ij}}$ is non-increasing in r_{ij}. The net cost of the r_{ij}th traversal of the edge (i,j), therefore, can be explicitly expressed as $c_{ijr_{ij}} = c_{ij}^{\mathrm{s}} - b_{ijr_{ij}}$ for $r_{ij} = 1,2,\ldots,n_{ij}$, with the index $n_{ij} = \max\{r_{ij} \mid b_{ijr_{ij}} < c_{ij}^{\mathrm{d}}\}$, and $b_{ij(n_{ij}+1)} = 0$. The net cost for $r_{ij} \geqslant n_{ij}+1$, is therefore $c_{ijr_{ij}} = c_{ij}^{\mathrm{d}}$. That is, for the traversal of the deadhead edges, no benefit is generated. Then, the MBCPP is to find a postman tour, starting from the depot, traversing a set of edges in E, and returning to the same depot with total net cost minimized (or total net benefit is maximized).

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Malandraki and Daskin [4] have investigated the MBCPP on directed networks. Their cost/benefit structure, however, is more restricted than one we consider here.

2. Problem complexity

In the following, we present a linear transformation, which converts the Rural Postman Problem (RPP) into a special case of the MBCPP. Hence, the MBCPP is more complex than the RPP.

2.1. The transformation

Consider a (totally) undirected RPP network $G(V,E,E_R)$ with E_R representing the set of required edges. For each edge (i,j) with distance d_{ij}, (1) if $(i,j)\in E_R$, define the service cost and the deadhead cost as $c^s_{ij}=c^d_{ij}=d_{ij}$, benefit $b_{ij1}=2d_{ij}+\varepsilon$, $b_{ij2}=d_{ij}-\varepsilon$, and $b_{ijr_{ij}}=0$, for $r_{ij}\geqslant 3$, where ε is a sufficiently small number less than one, and (2) if $(i,j)\in E-E_R$, define the service cost and the deadhead cost as $c^s_{ij}=c^d_{ij}=d_{ij}$, benefit $b_{ij1}=d_{ij}-\varepsilon/(4m)$, $b_{ij2}=d_{ij}-\varepsilon/(2m)$, and $b_{ijr_{ij}}=0$ for $r_{ij}\geqslant 3$, where $m=|E|$ = the number of edges in E.

Clearly, for $(i,j)\in E_R$ the net cost can be calculated as $c_{ij1}=-(d_{ij}+\varepsilon)$, $c_{ij2}=\varepsilon$, and $c_{ijr_{ij}}=d_{ij}$, for $r_{ij}\geqslant 3$. On the other hand, for $(i,j)\in E-E_R$, the net cost can be calculated as $c_{ij1}=\varepsilon/(4m)$, $c_{ij2}=\varepsilon/(2m)$, and $c_{ijr_{ij}}=d_{ij}$ for $r_{ij}\geqslant 3$. If we denote the transformed network as $G'(V,E,E_R)$, then, we can show the following theorem.

Theorem. *The optimal RPP solution over the original network $G(V,E,E_R)$ is equivalent to that of the MBCPP over the transformed network $G'(V,E,E_R)$ with equal solution values.*

We first note that minimizing the total net cost $c_{ijr_{ij}}=c^s_{ij}-b_{ijr_{ij}}$ is obviously equivalent to maximizing the total net benefit $-c_{ijr_{ij}}=b_{ijr_{ij}}-c^s_{ij}$. Let $|E_R|$ be the number of edges in E_R. If the MBCPP solution over the transformed network $G'(V,E,E_R)$ traverses all the edges in E_R exactly twice without traversing any edge in $E-E_R$, then the total net benefit is $\sum_{(i,j)} d(i,j)+|E_R|\varepsilon$ for $(i,j)\in E_R$. If the MBCPP solution over the transformed network $G'(V,E,E_R)$ traverses all the edges in E_R exactly twice without traversing any edge in $E-E_R$, then the total net benefit is $\sum_{(i,j)} d(i,j)+|E_R|\varepsilon-|E_R|\varepsilon=\sum_{(i,j)} d(i,j)$ for $(i,j)\in E_R$. Since traversing one edge in E_R for the second time reduces the total net benefits by ε, any MBCPP solution would attempt not to traverse the edges in E_R for more than once.

Further, traversing over any edge in $E-E_R$ would only reduce the total net benefit by either $\varepsilon/(4m)$ for the first time traversal, or $\varepsilon/(2m)$ for the second time traversal, any MBCPP solution would attempt to minimize the cost incurred from traversing the edges in $E-E_R$ in order to maximize the total net benefit. Noting that the sum of the edge cost (corresponding to the first time traversal) from E_R is a constant, it is then clear that the optimal MBCPP solution over the transformed network $G'(V,E,E_R)$ is an Euler cycle covering the edges in E_R with minimal distance (cost), hence is equivalent to the optimal RPP solution over the original network $G(V,E,E_R)$.

3. A special case CPP

Consider a special case of MBCPP with the traversal cost of edge (i,j) defined as $c^s_{ij}=c^d_{ij}=d_{ij}$ (length of the edge (i,j)), and benefit defined as $b_{ij1}=d_{ij}+c_0$, $b_{ij2}=d_{ij}-c_1$, and $b_{ijr_{ij}}=0$, for $r_{ij}\geqslant 3$, where c_0, c_1 are constants with $c_0>c_1>0$. The net benefit for traversing each edge for the first time is c_0, and the second time is $-c_1$.

If the postman traverses each edge on the network $G(V,E)$ exactly once, then the total net benefit can be calculated as $|E|c_0$. On the other hand, if the postman traverses all the edges in the network exactly twice, then the total net benefit is $|E|(c_0-c_1)$. Since traversing any edge in E for the second time reduces the total net benefit, the MBCPP solution would attempt not to traverse the edges in E for the second time. Consequently, minimizing the cost from traversing the edges for the second time would certainly maximize the total net benefit. As a result, the MBCPP with the defined cost/benefit structure reduces to the classical CPP.

4. Solution properties

We note that in the special case of MBCPP presented above, the condition of $b_{ij1}+b_{ij2}>c_{ij1}+c_{ij2}$ is satisfied for all edges. Such condition is sufficient for the MBCPP solution to cover all the edges on $G(V,E)$, but not vice versa. In fact, if the MBCPP solution does not cover the whole network, then adding two copies of the edges currently not in the solution would certainly increase the total net benefit.

If the condition of $b_{ij1}+b_{ij2}>c_{ij1}+c_{ij2}$ is not satisfied, the MBCPP solution may or may not cover the whole network, as shown in the following examples. Fig. 1 depicts an MBCPP with solution covering the whole network. Fig. 2 depicts an MBCPP with solution not covering the whole network. In both cases, the condition of $b_{ij1}+b_{ij2}>c_{ij1}+c_{ij2}$ is not satisfied for some edges.

For the special case considered in Malandraki and Daskin [4], the condition assumed (with benefit $b_{ij2}=0$ for the second time traversal) does not guarantee the MBCPP solution to cover all the edges in the network (see Example 2). Hence, the MBCPP does not reduce to the CPP in that setting.

Example 1. Consider the MBCPP depicted in Fig. 1 with six nodes and six edges. The edge traversal cost is defined as $c^s_{ij}=c^d_{ij}=d_{ij}$, and benefit $b_{ijr_{ij}}=d_{ij}+1$, $r_{ij}<3$, $b_{ijr_{ij}}=$

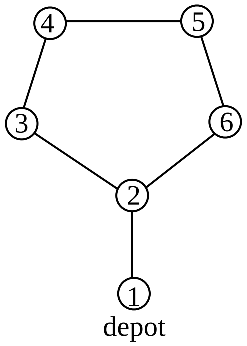

Fig. 1. An MBCPP example with solution covering the whole network.

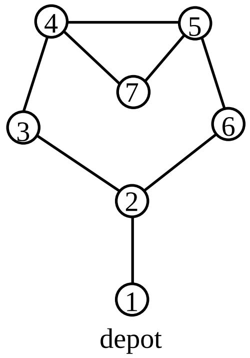

Fig. 2. An MBCPP example with solution not covering the whole network.

d_{ij}, $r_{ij} \geqslant 3$, for all $(i,j) \in \{(2,3),(3,4),(4,5),(5,6),(6,2)\}$ except for the edge (1, 2) with edge traversal cost defined as $c^{\rm s}_{12} = c^{\rm d}_{12} = 1$, and benefit $b_{12r_{ij}} = 0$ for all r_{ij}. The optimal solution covers the whole network with total net benefit 8. We note that the condition $b_{ij1} + b_{ij2} > c_{ij1} + c_{ij2}$ is not satisfied for edge (1, 2).

Example 2. Consider the MBCPP depicted in Fig. 2 with seven nodes and eight edges. The edge traversal cost is defined as $c^{\rm s}_{ij} = c^{\rm d}_{ij} = 1$, and benefit $b_{ij1} = 2$, $b_{ij2} = 0$, for $(i,j) \in \{(2,3),(3,4),(4,5),(5,6),(6,2)\}$, and $b_{ij1} = 2$, $b_{ij2} = 1$, for edge (1, 2). For edges (4, 7) and (5, 7), the traversal cost is $c^{\rm s}_{12} = c^{\rm d}_{12} = 1$, and the benefit is $b_{ijr_{ij}} = 5/4$ for $r_{ij} < 2$, and $b_{ijr_{ij}} = 0$ for $r_{ij} \geqslant 2$. The optimal postman tour is (1, 2, 3, 4, 5, 6, 2, 1), with a total net benefit of 6, while the edges (4, 7) and (5, 7) are not traversed in the postman tour.

We note that the condition of $b_{ij1} + b_{ij2} > c_{ij1} + c_{ij2}$ is not satisfied for edges (4, 7) and (5, 7) in Example 2. If all the edges in the network must be covered, then the postman tour becomes (1, 2, 3, 4, 7, 5, 4, 5, 6, 2, 1), with a total net benefit of $6 + 1/4 + 1/4 - 1 = 6 - 1/2$. The total net benefit obtained in this case, is obviously not maximal.

4.1. An upper bound

If the postman only services the edges (traversals) with positive net benefit, $b_{ijr_{ij}} - c^{\rm s}_{ij} > 0$, without traversing any other edges with traversal yielding $b_{ijr_{ij}} - c^{\rm s}_{ij} < 0$, then the total net benefit obtained is maximal. Those traversals may not form a complete postman tour, but the total net benefit generated certainly provides an upper bound on the solution. It is straightforward to verify that for the MBCPP described in Example 1, an upper bound on the total net benefit can be found as 10. For the MBCPP described in Example 2, an upper bound can be found as $6 + 1/2$.

To form a complete postman tour (Euler cycle), additional edge traversals may be required, which results in a reduction on the total net benefit. If such net benefit reduction is significant, the MBCPP solution may choose not to service some of those edge traversals with $b_{ijr_{ij}} - c^{\rm s}_{ij} > 0$, in order to maximizes the total net benefit. This is true, in particular, for cases where those edges (traversals) with positive net benefit do not form a connected network.

5. A solution algorithm

Since the MBCPP is more complex than the RPP which is NP-complete, then it must be difficult to solve the problem exactly. Based on the solution properties discussed above, we present an efficient algorithm, which solves the MBCPP approximately. The algorithm expands the original network by replacing each edge with a set of edges of positive net benefit. Minimal spanning tree (MST) and matching algorithms are then applied to generate a complete postman tour.

5.1. The algorithm

Step 1: (Network expansion) Replace each edge (i,j) by a set of new edges with net cost $c_{ijr_{ij}} = c^{\rm s}_{ij} - b_{ijr_{ij}}$, where $r_{ij} = 1, 2, \ldots, q_{ij}$, to $q_{ij} = \max\{r_{ij} | c_{ijr_{ij}} = c^{\rm s}_{ij} - b_{ijr_{ij}} < 0\}$, to obtain an expanded network G^*.

Step 2: (Minimal spanning tree) If G^* is connected, then proceed to Step 3. Otherwise, let G^* be a set of disconnected components $\{C_t\}$. Define the distance between every pair of components as $D(C_{t(i)}, C_{t(j)}) = \min_{x,y}\{\mathrm{spl}(x,y) | x \in C_{t(i)},\ y \in C_{t(i)}, \}$, where spl$(x, y)$ is the least-cost path over the network $G_{\rm T}$ consisting of edges with traversal cost $c_{ij(q_{ij}+1)}$ for edges in G^* and $c_{ijr_{ij}} = c^{\rm d}_{ij}$ for edges not in G^*. Let $E_{\rm T}$ be the minimum spanning tree (MST) solution over $G_{\rm T}$. Note that if G^* is connected, then $E_{\rm T} = \varnothing$.

Step 3: (Minimal cost matching) Identify the set of odd-degree nodes, S, on the $G^* \cup E_{\rm T}$, and construct a matching network $G_{\rm M}$ on S with distance between the nodes defined as the least-cost path over the network with edge traversal cost $c_{ij(q_{ij}+1)}$ for edges in $G^* - E_{\rm T}$, $c_{ij(q_{ij}+2)}$ for edges in $G^* - E_{\rm T}$, and $c_{ijr_{ij}} = c^{\rm d}_{ij}$ for edges not in G^*.

Find the minimal-cost matching solution E_M over G_M. The resulting network, $G^* \cup E_T \cup E_M$, is the MBCPP solution.

Step 4: (Benefit maximization) Find cycles with negative net benefit if they exist. Remove those cycles from $G^* \cup E_T \cup E_M$ if the removal does not separate the remaining graph into disconnected components.

If we apply the algorithm to the two MBCPP examples described above, both optimal solutions can be obtained, which is 8 for the problem in Example 1, and 6 for the problem in Example 2. Note that in Example 2, the cycle (7, 5, 4) with negative net benefit −1/2 is found and removed from the solution (1, 2, 3, 4, 7, 5, 4, 5, 6, 2, 1) generated from Step 3. For problems with cost/benefit satisfying the sufficient condition of $b_{ij1} + b_{ij2} > c_{ij1} + c_{ij2}$, the algorithm proceeds with Step 1 and Step 3 (without the MST segment), which generates a solution covering the whole network. Hence, the algorithm is expected to work well on problems with cost/benefit satisfying the sufficient condition. In particular, if the expanded network G^* is an even graph, the solution found is optimal.

5.2. Discussion

The cycles with negative net benefit are removed from the solution $G^* \cup E_T \cup E_M$ generated in Step 3, to maximize the total net benefit. Those cycles can be found by repeatedly proceeding with (1) identifying a path in $E_T \cup E_M$, and (2) finding a path in G^* with minimal net benefit, between the two end nodes of the identified path in $E_T \cup E_M$, to form closed cycles.

In general, if the net benefit reduction in traversing the edges in $E_T \cup E_M$ is significant, then the improvement made by removing the cycles (Step 4) would be significant. The MBCPP solution, in that case, chooses not to service some of the edges in G^* (with positive net benefit) in order to maximize the total net benefit. Effective strategies in identifying paths in $E_T \cup E_M$, and choosing the corresponding paths in G^* to form closed cycles, are essential to the maximization of the total net benefit, and will be investigated further.

6. Computational examples

Consider the MBCPP depicted in Fig. 3 with 15 nodes and 26 edges. The depot is at node 1. Table 1 displays the edge traversal costs for the edges. For simplicity of the computation, we assume that $c^s_{ij} = c^d_{ij} = c_{ij}$. We proceed with Step 1 of the algorithm to obtain the expanded network G^*, with the edge traversal net benefits shown in Table 2. Since the expanded network G^* is connected, we proceed with Step 3 to identify the set of odd-degree nodes S on G^* by checking Table 2 to obtain $S = \{6, 7, 11, 12, 13, 15\}$, and find the minimal-cost matching solution. The network $G^* \cup E_M$ obtained in Step 3 is shown in Fig. 4.

Following Step 4 we find three cycles with negative net benefit, including the cycle (7, 4, 6, 4, 7) with a net benefit

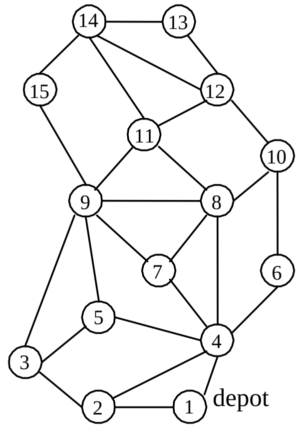

Fig. 3. The 15-node example.

Table 1
Traversal costs of the edges on the network

Edge	c_{ij}	Edge	c_{ij}
(1, 2)	18	(7, 9)	10
(1, 4)	15	(8, 10)	13
(2, 3)	16	(8, 9)	4
(2, 4)	7	(8, 11)	10
(3, 5)	11	(9, 11)	10
(3, 9)	15	(9, 15)	5
(4, 5)	17	(10, 12)	8
(4, 6)	18	(11, 12)	18
(4, 7)	8	(11, 14)	8
(4, 8)	9	(12, 13)	11
(5, 9)	9	(12, 14)	13
(6, 10)	11	(13, 14)	6
(7, 8)	10	(14, 15)	18

Table 2
The edge traversal net benefits of the edges on G^*

Edge	$b_{ijr_{ij}} - c^s_{ij}$	Edge	$b_{ijr_{ij}} - c^s_{ij}$
(1, 2)	9, 7, 4, 1	(7, 9)	11, 8, 4
(1, 4)	8, 6, 4, 2	(8, 10)	14, 7, 3
(2, 3)	9, 6, 3, 2	(8, 9)	4, 3
(2, 4)	3, 2	(8, 11)	9, 4, 1
(3, 5)	6, 3, 1	(9, 11)	8, 5, 2
(3, 9)	12, 10, 5	(9, 15)	5, 3
(4, 5)	11, 7, 4, 2	(10, 12)	3, 1
(4, 6)	10, 6, 3, 2	(11, 12)	15, 10, 5
(4, 7)	7, 3, 2	(11, 14)	7, 4
(4, 8)	6, 3, 1	(12, 13)	11, 8, 3
(5, 9)	10, 8, 4	(12, 14)	9, 6, 2
(6, 10)	10, 7, 3	(13, 14)	6, 3
(7, 8)	6, 5, 2	(14, 15)	3, 2, 1

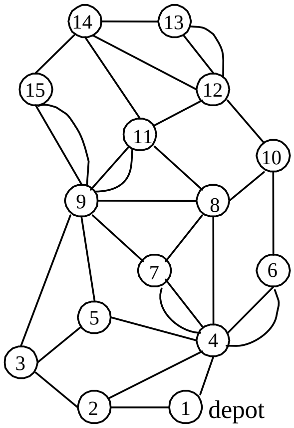

Fig. 4. The network $G^* \cup E_M$.

Table 3
Number of edge traversals in the solution

Edge	Traversal	Edge	Traversal
(1, 2)	4	(7, 9)	3
(1, 4)	4	(8, 10)	3
(2, 3)	4	(8, 9)	2
(2, 4)	2	(8, 11)	3
(3, 5)	3	(9, 11)	2
(3, 9)	3	(9, 15)	1
(4, 5)	3	(10, 12)	2
(4, 6)	3	(11, 12)	3
(4, 7)	3	(11, 14)	2
(4, 8)	3	(12, 13)	2
(5, 9)	3	(12, 14)	3
(6, 10)	3	(13, 14)	2
(7, 8)	3	(14, 15)	3

of −22, the cycle (12, 13, 12) with a net benefit of −8, and the cycle (11, 9, 15, 9, 11) with a net benefit of −10. The resulting network obtained by removing the three cycles from the network $G^* \cup E_M$ is the desired solution, which has a total net benefit of 403.

The number of traversals on each edge is tabulated in Table 3, and the total net benefit generated on each edge is displayed in Table 4. We note that the solution does not cover the edge traversals (4, 6), (4, 7), (11, 12), (12, 13), and (14, 15) with positive net benefits 2, 2, 5, 3, and 1, respectively. The corresponding solution covering those edge traversals has a total net benefit of 363.

Table 4
Total net benefits on each edge in the solution

Edge	Benefit	Edge	Benefit
(1, 2)	21	(7, 9)	23
(1, 4)	20	(8, 10)	24
(2, 3)	20	(8, 9)	7
(2, 4)	5	(8, 11)	14
(3, 5)	10	(9, 11)	13
(3, 9)	27	(9, 15)	5
(4, 5)	22	(10, 12)	4
(4, 6)	19	(11, 12)	30
(4, 7)	12	(11, 14)	11
(4, 8)	10	(12, 13)	19
(5, 9)	22	(12, 14)	17
(6, 10)	20	(13, 14)	9
(7, 8)	13	(14, 15)	6

7. Conclusions

In this note, we considered the MBCPP, an interesting generalization of the classical Chinese postman problem, on the undirected networks. We showed that the MBCPP is more complex than the RPP by presenting a linear transformation converting the RPP into a special case of the MBCPP. Hence, the MBCPP includes the CPP and the TSP as special cases.

We also discussed some solution properties, and presented a sufficient condition for the MBCPP solution to cover all the edges on the network. Based on those solution properties, we proposed an efficient algorithm to solve the MBCPP approximately, and presented an example to illustrate the proposed algorithm.

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