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Displacements and stresses due to a vertical point load in an inhomogeneous transversely isotropic half-space

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Abstract

We present the solutions for displacements and stresses subjected to a vertical point load in a continuously inhomogeneous transversely isotropic half-space with Young's and shear moduli varying exponentially with depth. Planes of transverse isotropy are assumed to be parallel to the horizontal surface. The solutions for the half-space are obtained by superposing the solutions of two full spaces, one with a point load in its interior and the other with opposite traction of the first full space along the $z = 0$ plane. The Hankel transform in a cylindrical co-ordinate system is employed for deriving the solutions. However, the resulting integrals for displacements and stresses involve polynomial, exponential function, and Bessel function that cannot be given in closed form; hence, numerical techniques are adopted in this work. In order to check the accuracy of numerical procedures, the comparisons are carried out with the homogeneous solutions of Liao and Wang, and the calculated results agree with those to nine decimal places. Furthermore, two illustrative examples are presented to elucidate the effect of inhomogeneity, and the type and degree of rock anisotropy on the vertical surface displacement and vertical normal stress in the inhomogeneous isotropic/transversely isotropic rocks subjected to a vertical concentrated force acting on the surface. The calculated results show that the induced displacement and stress are decisively influenced by the inhomogeneity, and the degree and type of material anisotropy. The proposed solutions can more realistically simulate the actual stratum of loading problem in many areas of engineering practice.
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Keywords: Displacements and stresses; Inhomogeneous transversely isotropic half-space; Moduli vary exponentially with depth; Hankel transform; Inhomogeneity; Rock anisotropy

1. Introduction

In general, the magnitude and distribution of the displacements and stresses in rock are predicted by using solutions that model rock as a linearly elastic, homogeneous and isotropic continuum. However, for rock masses cut by discontinuities, these solutions should account for anisotropy. From the standpoint of practical considerations in engineering, anisotropy rocks are often modeled as orthotropic or transversely isotropic medium. Besides, the effects of deposition, overburden, desiccation, etc., can lead geological media, which exhibit both inhomogeneity and anisotropic deformability characteristics. The type of elastic inhomogeneity is a useful approximation for modeling certain problems of geotechnical interest [1]. In this work, an elastostatic loading problem for a continuously inhomogeneous transversely isotropic half-space with Young's and shear moduli varying exponentially with depth is relevant.

The solutions of displacements and stresses for various types of applied loads to homogeneous and inhomogeneous isotropic/anisotropic full/half-spaces have played an important role in the design of foundations. However, it is well known that a point load solution is the basis of complex loading problems

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Nomenclature

C_{ij} $(i,j = 1-6)$ elastic moduli or elasticity constants (Pa)
E, E', v, v', G' engineering elastic constants of transversely isotropic materials (Pa)
h the buried depth, as seen in Fig. 2 (m)
$J_n(\ )$ Bessel function of the first kind of order n (dimensionless)
k the inhomogeneity parameter (1/m)
P_z a vertical point load in a cylindrical co-ordinate system (N)
r, θ, z a cylindrical co-ordinate system (m, radian, m)
R, Θ, Z body force components in a cylindrical co-ordinate system (N/m^3)
u_1, u_2 roots of the characteristic equation (dimensionless)
U_r, U_z displacement components (m)

Greek symbols

ε_{rr}, $\varepsilon_{\theta\theta}$, ε_{zz} normal strain components (dimensionless)
$\gamma_{r\theta}$, $\gamma_{\theta z}$, γ_{rz} shear strain components (dimensionless)
σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} normal stress components (Pa)
$\tau_{r\theta}$, $\tau_{\theta z}$, τ_{rz} shear stress components (Pa)

for all constituted materials. A large body of the literature was devoted to the calculation of displacements/stresses in isotropic media with the Young's or shear modulus varying with depth according to the power law, the linear law, and the exponential law, etc. The related works prior to 1960 can be found in Griffith [2], Fröhlich [3], Holl [4], Borowicka [5], Mikhlin [6], Ohde [7], Klein [8], Koronev [9,10], Mossakovskii [11], Popov [12], and Olszak [13], etc.; a more recent survey of the existing solutions for an inhomogeneous isotropic half-space is summarized in Table 1. Table 1 indicates the types of inhomogeneity, analytical or numerical solutions presented, and possible restrictions on Poisson's ratio in the solutions. Corresponding to the isotropic solutions, the literature contributions to the inhomogeneous transversely isotropic half-space are very limited. The lack of analytical/numerical solutions is primarily because of the mathematical difficulties involved. A summary of the available solutions for an inhomogeneous anisotropic material is given in Table 2. To the best of the authors' knowledge, no solutions for displacements and stresses in a transversely isotropic half-space subjected to a point load with Young's and shear moduli varying exponentially with depth have been presented. Utilizing the approaches proposed by Liao and Wang [94], the solutions of displacements and stresses in the Hankel domain for the continuously inhomogeneous transversely isotropic full and half-spaces subjected to a vertical point load are derived, respectively. These solutions indicate that the displacements and stresses in an inhomogeneous transversely isotropic full/half-space induced by a point load are affected by the inhomogeneity, and the type and degree of material anisotropy. The actual expressions for displacements and stresses can be obtained by taking the numerical inversion of the Hankel transforms. However, the resulting integrals involve products of Bessel functions of the first kind, an exponential function, and a polynomial, which cannot be given in closed form; hence, the numerical integrations are required. The numerical techniques are adopted from Longman's [95,96] as well as Davis and Rabinowitz's [97] methods. In order to check the accuracy of numerical procedure, the presented solutions are then simplified as the homogeneous solutions [94] by approaching the inhomogeneity parameter k to zero. The calculated results agree with those [94] to nine decimal places. Two illustrative examples, a point load acting on the surface of an inhomogeneous isotropic/transversely isotropic half-space are given to show the effect of inhomogeneity, and the type and degree of rock anisotropy on the vertical surface displacement and vertical normal stress.

2. Displacements and stresses in an inhomogeneous transversely isotropic full space

To solve the displacements and stresses in an inhomogeneous transversely isotropic full space induced by a single concentrated force, we follow the approach of Liao and Wang [94] for the corresponding homogeneous full space. Fig. 1 depicts that a cylindrical co-ordinate system (r, θ, z) is chosen such that z-axis is normal to the free surface of the inhomogeneous transversely isotropic material. The X–Y plane of a Cartesian co-ordinate system is parallel to the $r-\theta$ plane. The anisotropic medium possesses inhomogeneous elastic properties, and can be assumed to vary from point to point along the z-axis within the solid [77]. Then, the expression of stress–strain for a continuously inhomogeneous transversely isotropic medium in a cylindrical co-ordinate system is given

Table 1
Existing analytical/numerical solutions for inhomogeneous isotropic media

Types of inhomogeneity	Author	Analytical or numerical solutions	Poisson's ratio
$E = m_E z^\alpha$ or $G = m_G z^\alpha$ $(0 \leqslant \alpha \leqslant 1)$	Rostovtsev [14]	Settlement due to an elliptical, a circular, and a paraboloid of revolution load	$v = 1/(2+\alpha)$
	Lekhnitskii [15]	Radial stress for plane strain and generalized plane stress	General
	Popov [16]	Surface displacement due to a circular load	General
	Rostovtsev [17]	Stresses and displacements for plane and axisymmetric problems	General
	Zaretsky and Tsytovich [18]	Contact stress beneath a rigid strip	$v = 1/2$
	Kassir [19]	Stresses and displacement due to axisymmetric twisting deformation (Reissner–Sagoci problem)	General
	Rostovtsev and Khranevskaia [20]	Stresses and displacements for plane and axisymmetric problems	General
	Kassir [21]	Surface displacements and stresses due to a general-shaped cylindrical rigid punch	$v = 1/(2+\alpha)$
	Carrier and Christian [22]	Displacements and stresses due to a circular load by FEM	$v = 1/2$
	Puro [23]	Displacements due to axisymmetric loads	General
	Popov [24]	Displacements due to vertical/horizontal circular punches	General
	Booker et al. [25]	Displacements and stresses due to vertical/horizontal surface line and point loads	General
	Booker et al. [26]	Surface displacement due to strip, ring, and circular loads	General
	Oner [27]	Displacements due to vertical/horizontal point, circular and rectangular loads	General
	Booker [28]	Surface displacement due to a rectangular load	General
	Giannakopoulos and Suresh [29]	Analytical and numerical solutions for stresses and displacements due to a vertical point load	$v = 1/(\alpha+2)$
	Giannakopoulos and Suresh [30]	Analytical and numerical solutions for stresses and displacements due to rigid axisymmetric indentors	$v = 1/(\alpha+2)$
	Stark and Booker [31]	Surface displacements due to a uniform vertical/horizontal load on an arbitrarily shaped area by numerical technique	General
	Stark and Booker [32]	Surface displacements due to a uniform vertical/horizontal load on a rectangular area by numerical technique	General
	Yue et al. [33]	Displacements and stresses due to vertical/horizontal point loads for a layered half-space by backward transfer matrix method	$v = 1/(\alpha+2)$
	Holzlöhner [34]	Displacements and stresses for a non-linear half-space due to strip loads	General
$E = E_0(a+bz)^c$ or $G = G_0(a+bz)^c$	Plevako [35]	Displacements for plane problems	$v = 1/(1+c)$
	Plevako [36]	Displacements due to vertical/horizontal point loads	General
	Chuaprasert and Kassir [37]	Surface displacement and stresses for Reissner–Sagoci problem	General
	Chuaprasert and Kassir [38]	Displacements and stresses due to a uniform circular load	$v = 1/(2+c)$
	Kassir and Chuaprasert [39]	Stresses and displacements due to the axisymmetric problem of a rigid punch	General
	Dhaliwal and Singh [40]	Stresses and displacements for Reissner–Sagoci problem	General
	Harnpattanapanich and Vardoulakis [41]	Numerical surface displacements of a rectangular footing for consolidation problems	General

Table 1 (*continued*)

Types of inhomogeneity	Author	Analytical or numerical solutions	Poisson's ratio
	Rajapakse and Selvadurai [42]	Stresses and displacement due to rigid circular and cylindrical foundations	General
	Jeng and Lin [43]	Water wave-induced pore pressure on a pipeline problem by FEM	General
$E = E_0 + \lambda z$ or $G = G_0 + \lambda z$	Gibson [44]	Displacements and stresses due to strip and circular loads	$v = 1/2$
	Gibson et al. [45]	Stresses and displacements for plane strain and axisymmetric problems	$v = 1/2$
	Brown and Gibson [46]	Surface displacement due to a strip or circular load	General
	Awojobi and Gibson [47]	Stresses and displacements for plane strain and axisymmetric problems	General
	Brown and Gibson [48]	Surface displacement due to a rectangular load	General
	Carrier and Christian [49]	Settlement and stresses due to a rigid circular plate by FEM	General
	Gibson [50]	Surface displacement of uniformly circular loads	$v = 1/2$
	Alexander [51]	Vertical displacement due to a circular load	$v = 1/2$
	Calladine and Greenwood [52]	Displacements and stresses for plain strain and axisymmetric problems	$v = 1/2$
	Brown and Gibson [53]	Surface displacement for a layer of finite depth due to a rectangular load	$v = 0$, 1/3, 1/2
	Rajapakse [54]	Stresses and displacements due to an interior arbitrarily axisymmetric vertical load	$v = 1/2$
	Rajapakse [55]	Stresses and displacements due to a partially or fully embedded axially loaded rigid axisymmetric inclusion	$v = 1/2$
	Chow [56]	Vertical displacement of the smooth, rigid rectangular foundation by FEM	General
	Rajapakse and Selvadurai [57]	Axisymmetric elastic response of circular footings and anchor plates	$v = 1/2$
	Dempsey and Li [58]	Surface displacement of rectangular and strip footings by numerical approach	$v = 1/3$
	Yue et al. [33]	Displacements and stresses due to a circular load for a layered half-space by backward transfer matrix method	$v = 1/2$
$E = E_0 + E_1 e^{\xi z}$ or $G = G_0 + G_1 e^{\xi z}$	Ter-Mkrtich'ian [59]	Stresses and displacements due to a circular load	General
	Rowe and Booker [60]	Settlements due to strip footings by finite layer method	General
	Rowe and Booker [61]	Settlements due to circular footings by finite layer method	General
	Row and Booker [62]	Displacements and stresses by finite layer method	General
	Selvadurai et al. [63]	Displacement for Reissner–Sagoci problem	General
	Vrettos [64]	Displacements for SV/P surface wave problem	General
	Vrettos [65]	Stresses and displacements due to an interior time-harmonic vertical point load	General
	Vrettos [66]	Displacements due to vertical/horizontal time-harmonic surface line loads	General
	Selvadurai [67]	Settlement due to a rigid circular foundation	General
	Giannakopoulos and Suresh [29]	Analytical and numerical solutions for stresses and displacements due to a vertical point load	$v = 1/(\xi + 2)$
	Giannakopoulos and Suresh [30]	Analytical and numerical solutions for stresses and displacements due to rigid axisymmetric indentors	$v = 1/(\xi + 2)$

Table 1 (*continued*)

Types of inhomogeneity	Author	Analytical or numerical solutions	Poisson's ratio
	Vrettos [68]	Stresses and displacements due to a static vertical surface point load	General
	Jeng and Lin [43]	Water wave-induced pore pressure on a pipeline problem by FEM	General
$G = G_0 e^{-\xi r}$	George [69]	Stress field due to torsional loads	General
$G = G_0 r^{\alpha} z^{\beta}$	Singh [70]	Stress and displacement for Reissner–Sagoci problem	General
	Dhaliwal and Singh [71]	Griffith crack problem in an infinite solid under shear	General
$G = G_0 * h/(h-z)$	Awojobi [72]	Settlement of a circular foundation	General
	Awojobi [73]	Stresses and displacements for plane strain problem	General
G=constant	Gibson and Sills [74]	Stresses and displacements due to point and circular loads	$v = f(z)$

as follows:

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \tau_{\theta z} \\ \tau_{rz} \\ \tau_{r\theta} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{11}-2C_{66} & C_{13} & 0 & 0 & 0 \\ C_{11}-2C_{66} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix} e^{-kz} \quad (1)$$

where k is referred to as the inhomogeneity parameter; C_{ij} $(i, j = 1-6)$ are the elastic moduli or elasticity constants of the medium, and can be in terms of five independent elastic constants E, E', v, v' and G' as

$$C_{11} = \frac{E(1-(E/E')\upsilon'^2)}{(1+\upsilon)(1-\upsilon-(2E/E')\upsilon'^2)}, \quad C_{13} = \frac{E\upsilon'}{1-\upsilon-(2E/E')\upsilon'^2},$$
$$C_{33} = \frac{E'(1-\upsilon)}{1-\upsilon-(2E/E')\upsilon'^2}, \quad C_{44} = G', \quad C_{66} = \frac{E}{2(1+\upsilon)}, \quad (2)$$

where

1. E and E' are Young's moduli in the plane of transverse isotropy and in a direction normal to it, respectively;
2. v and v' are Poisson's ratios characterizing the lateral strain response in the plane of transverse isotropy to a stress acting parallel or normal to it, respectively;
3. G' is the shear modulus in planes normal to the plane of transverse isotropy.

The differences between the homogeneous transversely isotropic elastic constants [94] and inhomogeneous ones adopted in this paper can be summarized in Table 3. It is clear that, for the inhomogeneous transversely isotropic medium described by Eq. (1), only three (E, E', and G') of five engineering elastic constants exponentially depend on the inhomogeneity parameter k; the two Poisson's ratios are constants. Furthermore, depending upon the parameter k, we have the following three different situations:

(1) $k > 0$, denotes a hardened surface, whereas E, E', and G' decrease with the increase of depth.
(2) $k = 0$, is referred to as the conventional homogeneous condition [94].
(3) $k < 0$, denotes a soft surface, whereas E, E', and G' increase with the increase of depth.

The expressions of strain–displacement relations for small strain conditions in a cylindrical co-ordinate system are:

$$\varepsilon_{rr} = -\frac{\partial U_r}{\partial r}, \quad (3)$$

$$\varepsilon_{\theta\theta} = -\frac{U_r}{r} - \frac{1}{r}\frac{\partial U_\theta}{\partial \theta}, \quad (4)$$

$$\varepsilon_{zz} = -\frac{\partial U_z}{\partial z}, \quad (5)$$

$$\gamma_{r\theta} = -\frac{1}{r}\frac{\partial U_r}{\partial \theta} - \frac{\partial U_\theta}{\partial r} + \frac{U_\theta}{r}, \quad (6)$$

Table 2
Existing analytical/numerical solutions for inhomogeneous anisotropic media

Types of inhomogeneity	Author	Analytical or numerical solutions	Poisson's ratio
$G' = mz$, $E = \alpha G'$, $E' = E/\lambda$	Gibson and Kalsi [75]	Surface displacement for axisymmetric problem (transverse isotropy)	Incompressible
$E = \alpha z$, $E' = \beta z$, $G' = \gamma z$	Gibson [50]	Surface displacement due to general surface pressure (orthotropy)	Incompressible
$E = \alpha z$, $E' = \beta z$, $G' = \gamma z$	Gibson and Sills [76]	Surface displacement for plane strain problem (orthotropy)	Incompressible
$G = mz\cos\theta$	Calladine and Greenwood [52]	Displacements and stresses for plain strain problem (transverse isotropy)	Incompressible
$E' = E_0 - \rho_0 z$, $z < z_c$; $E' = E_0 - \rho_0 z_c + \rho(z - z_c)$, $z > z_c$	Rowe and Booker [61]	Parametric study of settlements due to a circular footings by finite layer method (transverse isotropy)	General
$C_{ii} = G_{ii} r^{\beta}$ $(i = 4, 6)$	Erguven [77]	Displacement, stresses, and torque for Reissner–Sagoci problem (transverse isotropy)	General
Cylindrical anisotropy	Tarn and Wang [78]	Fundamental solutions for torsional problems	General
$C_{ii} = G_{ii} r^{\alpha} e^{\lambda z}$ and $C_{ii} = G_{ii} r^{\alpha}(z + c)^{\beta}$ $(i = 4, 6)$	Tarn and Wang [79]	Fundamental solutions for torsional problems (transverse isotropy)	General
$C_{ii} = G_{ii} r^{\beta}(z + c)^{\alpha}$ $(i = 4, 6)$	Erguven [80]	Deformation and shear stresses in the semi-infinite solid for an axisymmetric torsional problem (transverse isotropy)	General
$C_{ii} = G_{ii}\cosh^2 kz$ $(i = 4, 6)$	Erguven [81]	Deformation and shear stresses in the semi-infinite solid for an axisymmetric torsional problem (transverse isotropy)	General
$C_{44} = G_0 e^{2kz}$, $C_{66} = G_0' e^{2kz}$	Erguven [82]	Displacement and stresses for axisymmetric fundamental solutions (transverse isotropy)	General
$C_{ii} = G_{ii} r^{\beta}\cosh^2 kz$	Erguven [83]	Surface displacement and stress for Reissner–Sagoci problem (transverse isotropy)	General
$E = m_0 z^m$	Kumar [84]	Surface displacement for plane strain opening problem by finite/infinite element method (transverse isotropy)	General
$C_{ii} = G_{ii} r^{\beta}$ $(i = 4, 6)$	Erguven [85]	Displacement and stresses for the Reissner–Sagoci problem (transverse isotropy)	General
$E = m_0 z^m$	Kumar [86]	Displacements and stresses due to point and circular loads by finite/infinite element method (transverse isotropy)	General
$C_{44} = G_0(1 + mz)^{\alpha}$ or $C_{44} = G_0 e^{\beta z}$, $C_{66} = \gamma C_{44}$	Rajapakse [87]	Displacement for an axisymmetric torsion problem (transverse isotropy)	General
Transversely isotropic and layered half-space	Yue and Wang [88]	The transfer matrix approach to solve the fundamental solutions	General
Transversely isotropic and layered half-space	Pan [89]	The vector functions and propagator matrix methods to solve the deformations by point dislocations	General
Transversely isotropic and layered half-space	Pan [90]	The vector functions and propagator matrix methods to solve the deformations by general surface loads	General
Transversely isotropic and layered half-space	Pan [91]	The vector functions and propagator matrix methods to solve the deformations by point loads	General
General anisotropic and layered half-space	Yang and Pan [92]	Fourier transforms and Stroh formalism	General
General anisotropic inhomogeneous full-space	Martin et al. [93]	Analytical method	General

$$\gamma_{\theta z} = -\frac{\partial U_\theta}{\partial z} - \frac{1}{r}\frac{\partial U_z}{\partial \theta}, \tag{7}$$

$$\gamma_{rz} = -\frac{\partial U_r}{\partial z} - \frac{\partial U_z}{\partial r}, \tag{8}$$

where U_r, U_θ and U_z are radial, tangential and vertical displacement, respectively.

Also, the partial differential forms of equilibrium equations are

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = R, \tag{9}$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = \Theta, \tag{10}$$

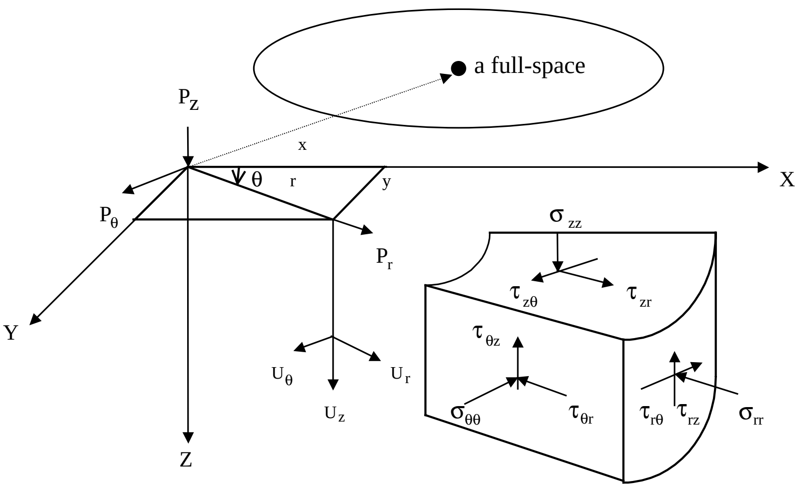

Fig. 1. (P_r, P_θ, P_z) acting in a full space.

Table 3
The differences between the homogeneous and inhomogeneous transversely isotropic elastic constants

Homogeneous [94]	Inhomogeneous
E	Ee^{-kz}
E'	$E'e^{-kz}$
ν	ν
ν'	ν'
G'	$G'e^{-kz}$

$$\frac{\partial \tau_{rz}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}+\frac{\partial \sigma_{zz}}{\partial z}+\frac{\tau_{rz}}{r}=Z, \tag{11}$$

where R, Θ, Z are the components of the body forces per unit volume on the co-ordinate directions, r, θ and z, respectively. A static point load with components (P_r, P_θ, P_z), acting at the origin of the co-ordinate for a full space can be expressed as the form of body forces:

$$R=\frac{P_r}{r}\,\delta(r)\delta(\theta)\delta(z), \tag{12}$$

$$\Theta=\frac{P_\theta}{r}\,\delta(r)\delta(\theta)\delta(z), \tag{13}$$

$$Z=\frac{P_z}{r}\,\delta(r)\delta(\theta)\delta(z), \tag{14}$$

where $\delta(\ )$ is the Dirac-delta function.

Substituting Eq. (1) and Eqs. (12)–(14) into Eqs. (9)–(11), and adopting the strain–displacement relations (Eqs. (3)–(8)), then Eqs. (9)–(11) can be regrouped as the Navier–Cauchy equations for an inhomogeneous transversely isotropic material.

$$\begin{aligned}&\left[C_{11}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)+\frac{C_{66}}{r^2}\frac{\partial^2}{\partial \theta^2}+C_{44}\frac{\partial^2}{\partial z^2}\right]U_r\\&\quad+\left[\frac{(C_{11}-C_{66})}{r}\frac{\partial^2}{\partial r\partial\theta}-\frac{(C_{11}+C_{66})}{r^2}\frac{\partial}{\partial\theta}\right]U_\theta\\&\quad+(C_{13}+C_{44})\frac{\partial^2 U_z}{\partial r\,\partial z}-kC_{44}\left(\frac{\partial U_r}{\partial z}+\frac{\partial U_z}{\partial r}\right)\\&\quad=-\frac{P_r}{r}\,\delta(r)\,\delta(\theta)\,\delta(z),\end{aligned} \tag{15}$$

$$\begin{aligned}&\left[\frac{(C_{11}-C_{66})}{r}\frac{\partial^2}{\partial r\,\partial\theta}+\frac{(C_{11}+C_{66})}{r^2}\frac{\partial}{\partial\theta}\right]U_r\\&\quad+\left[C_{66}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)+\frac{C_{11}}{r^2}\frac{\partial^2}{\partial\theta^2}+C_{44}\frac{\partial^2}{\partial z^2}\right]U_\theta\\&\quad+\frac{(C_{13}+C_{44})}{r}\frac{\partial^2 U_z}{\partial\theta\,\partial z}-kC_{44}\left(\frac{\partial U_\theta}{\partial z}+\frac{1}{r}\frac{\partial U_z}{\partial\theta}\right)\\&\quad=-\frac{P_\theta}{r}\,\delta(r)\,\delta(\theta)\,\delta(z),\end{aligned} \tag{16}$$

$$\begin{aligned}&(C_{13}+C_{44})\left(\frac{\partial^2}{\partial r\,\partial z}+\frac{1}{r}\frac{\partial}{\partial z}\right)U_r+\frac{(C_{13}+C_{44})}{r}\frac{\partial^2 U_\theta}{\partial\theta\,\partial z}\\&\quad+\left[C_{44}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)+C_{33}\frac{\partial^2}{\partial z^2}\right]U_z\\&\quad-k\left[C_{13}\frac{\partial U_r}{\partial r}+C_{13}\left(\frac{U_r}{r}+\frac{1}{r}\frac{\partial U_\theta}{\partial\theta}\right)+C_{33}\frac{\partial U_z}{\partial z}\right]\\&\quad=-\frac{P_z}{r}\,\delta(r)\,\delta(\theta)\,\delta(z).\end{aligned} \tag{17}$$

In this work, considerations are given to

(1) axial symmetry about the z-axis, and the displacements and stresses are independent of tangential co-ordinate θ;
(2) only P_z acting in the medium ($P_r=P_\theta=0$).

Then, Eqs. (15)–(17) will reduce to

$$\begin{aligned}&C_{11}\left(\frac{\partial^2 U_r}{\partial r^2}+\frac{1}{r}\frac{\partial U_r}{\partial r}-\frac{U_r}{r^2}\right)+C_{44}\frac{\partial^2 U_r}{\partial z^2}\\&\quad+(C_{13}+C_{44})\frac{\partial^2 U_z}{\partial r\partial z}-kC_{44}\left(\frac{\partial U_r}{\partial z}+\frac{\partial U_z}{\partial r}\right)=0,\end{aligned} \tag{18}$$

$$(C_{13}+C_{44})\left(\frac{\partial^2 U_r}{\partial r\,\partial z}+\frac{1}{r}\frac{\partial U_r}{\partial z}\right)+C_{44}\left(\frac{\partial^2 U_z}{\partial r^2}+\frac{1}{r}\frac{\partial U_z}{\partial r}\right)+C_{33}\frac{\partial^2 U_z}{\partial z^2}-k\left[C_{13}\left(\frac{\partial U_r}{\partial r}+\frac{U_r}{r}\right)+C_{33}\frac{\partial U_z}{\partial z}\right]=-\frac{P_z}{2\pi r}\delta(r)\,\delta(z). \quad (19)$$

In particular, the Hankel transform has found a wide usage for solving the solutions of axisymmetric half-space as the radial co-ordinate r ranges from 0 to ∞ [98]. Hence, the displacements U_r and U_z in Eqs. (18) and (19) are transformed by a system of proper Hankel transformations [99,100] with respect to r of order 1 and 0, respectively, in the following:

$$\begin{Bmatrix} U_r^* \\ U_z^* \end{Bmatrix}=\int_0^\infty r\begin{Bmatrix} U_r J_1(\xi r) \\ U_z J_0(\xi r) \end{Bmatrix}\mathrm{d}r, \quad (20)$$

where $J_n(\xi r)$ denotes a Bessel function of the first kind of order n $(n=0,1)$, and ξ is the transform parameter.

Then, Eqs. (18) and (19) are rewritten by a system of ordinary differential equations as follows:

$$\left[-C_{11}\xi^2-kC_{44}\frac{\mathrm{d}}{\mathrm{d}z}+C_{44}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right]U_r^*-\left[(C_{13}+C_{44})\frac{\mathrm{d}}{\mathrm{d}z}-kC_{44}\right]\xi U_z^*=0, \quad (21)$$

$$\left[-kC_{13}+(C_{13}+C_{44})\frac{\mathrm{d}}{\mathrm{d}z}\right]\xi U_r^*+\left[-C_{44}\xi^2-kC_{33}\frac{\mathrm{d}}{\mathrm{d}z}+C_{33}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right]U_z^*=-\frac{P_z}{2\pi}\delta(z). \quad (22)$$

The homogeneous solutions of Eqs. (21)–(22) are obtained by solving the simultaneous ordinary differential equations as

$$U_r^*(H)=A_1\mathrm{e}^{-u_1\xi z}+A_2\mathrm{e}^{u_1\xi z}+A_3\mathrm{e}^{(k+u_2\xi)z}+A_4\mathrm{e}^{(k-u_2\xi)z}, \quad (23)$$

$$U_z^*(H)=B_1\mathrm{e}^{-u_1\xi z}+B_2\mathrm{e}^{u_1\xi z}+B_3\mathrm{e}^{(k+u_2\xi)z}+B_4\mathrm{e}^{(k-u_2\xi)z}, \quad (24)$$

where

$$u_1=\sqrt{\frac{S+\sqrt{S^2-4Q}}{2}},\quad u_2=\sqrt{\frac{S-\sqrt{S^2-4Q}}{2}},$$
$$S=\frac{C_{11}C_{33}-C_{13}(C_{13}+2C_{44})}{C_{33}C_{44}},$$
$$Q=\frac{C_{11}}{C_{33}}.$$

A_i and B_i $(i=1\sim4)$ are determined by substituting Eqs. (23) and (24) into Eqs. (21) and (22). Then, Eqs. (23) and (24) can be expressed in terms of B_i $(i=1\sim4)$ as follows:

$$U_r^*(H)=S_1B_1\mathrm{e}^{-u_1\xi z}+S_2B_2\mathrm{e}^{u_1\xi z}+S_3B_3\mathrm{e}^{(k+u_2\xi)z}+S_4B_4\mathrm{e}^{(k-u_2\xi)z}, \quad (25)$$

$$U_z^*(H)=B_1\mathrm{e}^{-u_1\xi z}+B_2\mathrm{e}^{u_1\xi z}+B_3\mathrm{e}^{(k+u_2\xi)z}+B_4\mathrm{e}^{(k-u_2\xi)z}, \quad (26)$$

where

$$S_1=\frac{-[(C_{13}+C_{44})u_1\xi^2+kC_{44}\xi]}{[(-C_{11}+C_{44}u_1^2)\xi^2+kC_{44}u_1\xi]},$$
$$S_2=\frac{[(C_{13}+C_{44})u_1\xi^2-kC_{44}\xi]}{[(-C_{11}+C_{44}u_1^2)\xi^2-kC_{44}u_1\xi]},$$
$$S_3=\frac{[(C_{13}+C_{44})u_2\xi^2+kC_{13}\xi]}{[(-C_{11}+C_{44}u_2^2)\xi^2+kC_{44}u_2\xi]},$$
$$S_4=\frac{-[(C_{13}+C_{44})u_2\xi^2-kC_{13}\xi]}{[(-C_{11}+C_{44}u_2^2)\xi^2+kC_{44}u_2\xi]}.$$

In order to derive the particular solutions of Eqs. (21) and (22), defining two displacement functions as follows (for $z>0$, the sign of z is downward positive):

$$U_r^*(P)=C_1\mathrm{e}^{-u_1\xi z}+C_2\mathrm{e}^{u_1\xi z}+C_3\mathrm{e}^{(k+u_2\xi)z}+C_4\mathrm{e}^{(k-u_2\xi)z}, \quad (27)$$

$$U_z^*(P)=D_1\mathrm{e}^{-u_1\xi z}+D_2\mathrm{e}^{u_1\xi z}+D_3\mathrm{e}^{(k+u_2\xi)z}+D_4\mathrm{e}^{(k-u_2\xi)z}. \quad (28)$$

The undetermined coefficients C_i and D_i $(i=1-4)$ can be obtained by the method of variation of parameters [101]. The general solutions are the sum of the homogeneous and the particular solutions. The constants B_i $(i=1-4)$ can be determined by the condition that the effect of the point load must vanish at infinity. Therefore, the final resulting expressions of general solutions for U_r^* and U_z^* are

$$U_r^*=\frac{-P_z}{4\pi C_{33}C_{44}}\left\{\frac{[(C_{13}+C_{44})u_1\xi+kC_{44}]}{u_1[(k+u_1\xi)^2-u_2^2\xi^2]}\mathrm{e}^{-u_1\xi z}-\frac{[(C_{13}+C_{44})(k-u_2\xi)-kC_{44}]}{u_2[(k-u_2\xi)^2-u_1^2\xi^2]}\mathrm{e}^{(k-u_2\xi)z}\right\}, \quad (29)$$

$$U_z^*=\frac{P_z}{4\pi C_{33}C_{44}}\left\{\frac{(C_{44}u_1^2\xi+kC_{44}u_1-C_{11}\xi)}{u_1[(k+u_1\xi)^2-u_2^2\xi^2]}\mathrm{e}^{-u_1\xi z}+\frac{(C_{44}u_2^2\xi-kC_{44}u_2-C_{11}\xi)}{u_2[(k-u_2\xi)^2-u_1^2]}\mathrm{e}^{(k-u_2\xi)z}\right\}. \quad (30)$$

The desired solutions for the displacements U_r and U_z in the inhomogeneous transversely isotropic full space can be obtained by taking the inverse Hankel transform with

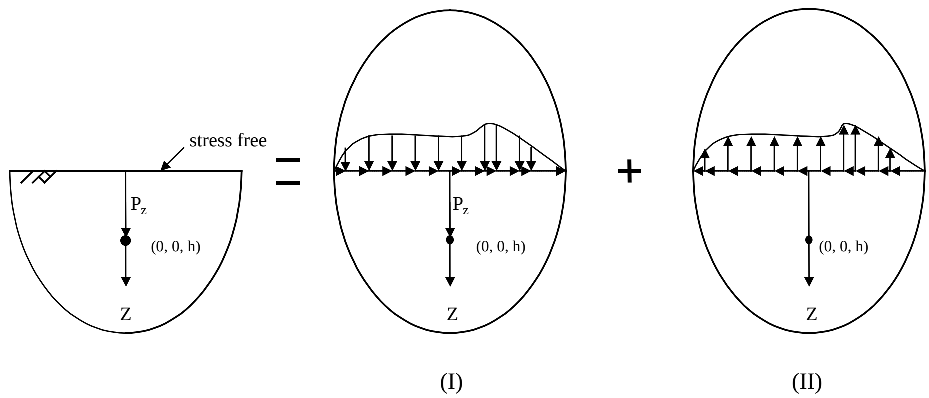

Fig. 2. P_z acting in the interior of a half-space.

respect to ξ in the following:

$$\begin{Bmatrix} U_r \\ U_z \end{Bmatrix} = \int_0^\infty \xi \begin{Bmatrix} U_r^* J_1(\xi r) \\ U_z^* J_0(\xi r) \end{Bmatrix} \mathrm{d}\xi. \tag{31}$$

From Eqs. (29), (30), (3)–(8) and (1), the vertical and shear stresses for axisymmetric problem in the Hankel domain are expressed as

$$\sigma_{zz}^* = -\left(C_{13}\xi U_r^* + C_{33}\frac{\mathrm{d}U_z^*}{\mathrm{d}z} \right). \tag{32}$$

$$\tau_{rz}^* = -C_{44}\left(\frac{\mathrm{d}U_r^*}{\mathrm{d}z} - \xi U_z^* \right). \tag{33}$$

The resulting expressions for the vertical normal and shear stresses are given as follows:

$$\sigma_{zz}^* = \frac{P_z}{4\pi C_{33}C_{44}} \left\{ \frac{[C_{33}C_{44}u_1^3 + (C_{13}^2 + C_{13}C_{44} - C_{11}C_{33})u_1]\xi^2 + kC_{44}(C_{33}u_1^2 + C_{13})\xi}{u_1[(k+u_1\xi)^2 - u_2^2\xi^2]} \mathrm{e}^{-u_1\xi z} + \frac{[C_{33}C_{44}u_2^3 + (C_{13}^2 + C_{13}C_{44} - C_{11}C_{33})u_2]\xi^2 - k(2C_{33}C_{44}u_2^2 + C_{13}^2 - C_{11}C_{33})\xi + k^2C_{33}C_{44}u_2}{u_2[(k-u_2\xi)^2 - u_1^2\xi^2]} \mathrm{e}^{(k-u_2\xi)z} \right\}, \tag{34}$$

$$\tau_{rz}^* = \frac{-P_z}{4\pi C_{33}} \left\{ \frac{(C_{13}u_1^2 + C_{11})\xi^2}{u_1[(k+u_1\xi)^2 - u_2^2\xi^2]} \mathrm{e}^{-u_1\xi z} + \frac{(C_{13}u_2^2 + C_{11})\xi^2 - 2kC_{13}u_2\xi + k^2C_{13}}{u_2[(k-u_2\xi)^2 - u_1^2\xi^2]} \mathrm{e}^{(k-u_2\xi)z} \right\}. \tag{35}$$

The expressions for displacements U_r^* (Eq. (29)), U_z^* (Eq. (30)) and stresses σ_{zz}^* (Eq. (34)), τ_{rz}^* (Eq. (35)) in the inhomogeneous transversely isotropic full space include integrals, which have to be obtained by taking the numerical Hankel inversion theorem with respect to ξ of order 1, 0, 0, and 1, respectively. In this study, the inverse Hankel transforms were evaluated by means of 68 points Gauss quadrature formula [102]. The detailed numerical integrations to estimate the displacements and stresses will be elucidated in Section 4.

3. Displacements and stresses in an inhomogeneous transversely isotropic half-space

Since the half-space problem is of particular interest to the geotechnical engineering, a point load acting in the interior (including on the surface) of a half-space is considered in this section. The solutions for displacements and stresses in an inhomogeneous transversely isotropic half-space are derived by the principle of superposition as shown in Fig. 2. Fig. 2 depicts that a half-space is composed of two full spaces, one with a point load in its interior and the other with opposite traction of the first full space along $z = 0$. The traction in the first full space along $z = 0$ is due to the point load. The solutions for the half-space are thus obtained by superposing the solutions of the two full spaces. That is, the solutions can be derived from the governing equations for a full space (including the general solutions (I) and homogeneous solutions (II)) by satisfying the free traction on the surface of the half-space. The solutions for displacements U_r^* and U_z^* in the half-space can be directly obtained by the principle of superposition of general solutions (Eqs. (29) and (30)) by shifting $|z|$ to $|z-h|$ and being denoted by $U_r'{*}(G)$ and $U_z'{*}(G)$, and homogeneous solutions (Eqs. (25) and (26)) in which B_i $(i = 1-4)$ are denoted by B_i' $(i = 1-4)$, and z is replaced by $(z-h)$ as follows [94]:

$$U_r = U_r'{*}(G) + S_1B_1'\mathrm{e}^{-u_1\xi|z-h|} + S_2B_2'\mathrm{e}^{u_1\xi|z-h|} + S_3B_3'\mathrm{e}^{(k+u_2\xi)|z-h|} + S_4B_4'\mathrm{e}^{(k-u_2\xi)|z-h|}, \tag{36}$$

$$U_z = U_z'{*}(G) + B_1'\mathrm{e}^{-u_1\xi|z-h|} + B_2'\mathrm{e}^{u_1\xi|z-h|} + B_3'\mathrm{e}^{(k+u_2\xi)|z-h|} + B_4'\mathrm{e}^{(k-u_2\xi)|z-h|}. \tag{37}$$

For a half-space with free traction on the bounding plane, the boundary conditions in the Hankel domain can be expressed in the forms of Eqs. (32) and (33) as

$$\sigma^*_{zz}|_{z=0} = \left[-C_{13}\xi U^*_r - C_{33}\frac{\partial U^*_z}{\partial z} \right]\Bigg|_{z=0} = 0, \tag{38}$$

$$\tau^*_{rz}|_{z=0} = \left[-C_{44}\left(\frac{\partial U^*_r}{\partial z} - \xi U^*_z \right) \right]\Bigg|_{z=0} = 0. \tag{39}$$

For solving Eqs. (36) and (37), the coefficients $B'_i\,(i = 1-4)$ can be determined by assuming the displacements U_r and U_z tend to zero as z tend to infinity; hence, B'_2 and B'_3 are both zero. However, the remaining coefficients B'_1 and B'_4 are obtained by the transformed boundary conditions (Eqs. (38) and (39)) as follows:

$$B'_1 = \frac{-P_z}{4\pi C_{33}C_{44}}\left\{ \frac{[(-C_{11}+C_{44}u_1^2)\xi + kC_{44}u_1]\Delta_1}{u_1[(k+u_1\xi)^2 - u_2^2\xi^2]\Delta} e^{-2u_1\xi h} + \frac{[(-C_{11}+C_{44}u_1^2)\xi + kC_{44}u_1]\Delta_2}{u_2[(k-u_2\xi)^2 - u_1^2\xi^2]\Delta} e^{[k-(u_1+u_2)\xi h]} \right\}, \tag{40}$$

$$B'_4 = \frac{-P_z}{4\pi C_{33}C_{44}}\left\{ \frac{[(-C_{11}+C_{44}u_2^2)\xi - kC_{44}u_2]\Delta_3}{u_1[(k+u_1\xi)^2 - u_2^2\xi^2]\Delta} e^{[k-(u_1+u_2)\xi]h} + \frac{[(-C_{11}+C_{44}u_2^2)\xi - kC_{44}u_2]\Delta_4}{u_2[(k-u_2\xi)^2 - u_1^2\xi^2]\Delta} e^{2(k-u_2\xi)h} \right\}, \tag{41}$$

where

$$\begin{aligned} \Delta = {} & \{C_{13}[(C_{13}+C_{44})u_1\xi + kC_{44}] \\ & + C_{33}u_1[(-C_{11}+C_{44}u_1^2)\xi + kC_{44}u_1]\} \\ & \times \{(k-u_2\xi)[(C_{13}+C_{44})u_2\xi - kC_{13}] \\ & + [(-C_{11}+C_{44}u_2^2)\xi^2 - kC_{44}u_2\xi]\} \\ & + \{u_1[(C_{13}+C_{44})u_1\xi + kC_{44}] \\ & - [(-C_{11}+C_{44}u_1^2)\xi + kC_{44}u_1]\} \\ & \times \{C_{13}[(C_{13}+C_{44})u_2\xi^2 - kC_{13}\xi] \\ & - C_{33}(k-u_2\xi)[(-C_{11}+C_{44}u_2^2)\xi - kC_{44}u_2]\}, \end{aligned} \tag{42}$$

$$\begin{aligned} \Delta_1 = {} & \{C_{13}[(C_{13}+C_{44})u_1\xi + kC_{44}] \\ & - C_{33}u_1[(-C_{11}+C_{44}u_1^2)\xi + kC_{44}u_1]\} \\ & \times \{(k-u_2\xi)[(C_{13}+C_{44})u_2\xi - kC_{13}] \\ & + [(-C_{11}+C_{44}u_2^2)\xi^2 - kC_{44}u_2\xi]\} \\ & - \{u_1[(C_{13}+C_{44})u_1\xi + kC_{44}] \\ & + [(-C_{11}+C_{44}u_1^2)\xi + kC_{44}u_1]\} \\ & \times \{C_{13}[(C_{13}+C_{44})u_2\xi^2 - kC_{13}\xi] \\ & - C_{33}(k-u_2\xi)[(-C_{11}+C_{44}u_2^2)\xi - kC_{44}u_2]\}, \end{aligned} \tag{43}$$

$$\begin{aligned} \Delta_2 = {} & 2(k-u_2\xi)\{C_{13}[(C_{13}+C_{44})u_2\xi - kC_{13}]^2 \\ & + C_{33}[(-C_{11}+C_{44}u_2^2)\xi - kC_{44}u_2]^2\}, \end{aligned} \tag{44}$$

$$\begin{aligned} \Delta_3 = {} & 2u_1\xi\{C_{13}[(C_{13}+C_{44})u_1\xi + kC_{44}]^2 \\ & + C_{33}[(-C_{11}+C_{44}u_1^2)\xi + kC_{44}u_1]^2\}, \end{aligned} \tag{45}$$

$$\begin{aligned} \Delta_4 = {} & -\{C_{13}[(C_{13}+C_{44})u_1\xi + kC_{44}] \\ & + C_{33}u_1[(-C_{11}+C_{44}u_1^2)\xi + kC_{44}u_1]\} \\ & \times \{(k-u_2\xi)[(C_{13}+C_{44})u_2\xi - kC_{13}] \\ & - [(-C_{11}+C_{44}u_2^2)\xi^2 - kC_{44}u_2\xi]\} \\ & + \{u_1[(C_{13}+C_{44})u_1\xi + kC_{44}] \\ & - [(-C_{11}+C_{44}u_1^2)\xi + kC_{44}u_1]\} \\ & \times \{C_{13}[(C_{13}+C_{44})u_2\xi^2 - kC_{13}\xi] \\ & + C_{33}(k-u_2\xi)[(-C_{11}+C_{44}u_2^2)\xi - kC_{44}u_2]\}. \end{aligned} \tag{46}$$

Finally, the displacements in Hankel domain for an inhomogeneous transversely isotropic half-space subjected to a vertical point load P_z that acts at $z = h$ (measured from the surface) are expressed as follows:

$$\begin{aligned} U^*_r = \frac{-P_z}{4\pi C_{33}C_{44}} \Bigg\{ & \frac{[(C_{13}+C_{44})u_1\xi + kC_{44}]}{u_1[(k+u_1\xi)^2 - u_2^2\xi^2]} e^{-u_1\xi|z-h|} \\ & + \frac{[(C_{13}+C_{44})u_2\xi - kC_{13}]}{u_2[(k-u_2\xi)^2 - u_1^2\xi^2]} e^{(k-u_2\xi)|z-h|} \\ & - \frac{[(C_{13}+C_{44})u_1\xi + kC_{44}]\Delta_1}{u_1[(k+u_1\xi)^2 - u_2^2\xi^2]\Delta} e^{-u_1\xi(z+h)} \\ & - \frac{[(C_{13}+C_{44})u_1\xi + kC_{44}]\Delta_2}{u_2[(k-u_2\xi)^2 - u_1^2\xi^2]\Delta} e^{(k-u_1\xi)z} e^{-u_2\xi h} \\ & - \frac{[(C_{13}+C_{44})u_2\xi - kC_{13}]\Delta_3}{u_1[(k+u_1\xi)^2 - u_2^2\xi^2]\Delta} e^{(k-u_2\xi)z} e^{-u_1\xi h} \\ & - \frac{[(C_{13}+C_{44})u_2\xi - kC_{13}]\Delta_4}{u_2[(k-u_2\xi)^2 - u_1^2\xi^2]\Delta} e^{(k-u_2\xi)(z+h)} \Bigg\}, \end{aligned} \tag{47}$$

$$\begin{aligned} U^*_z = \frac{P_z}{4\pi C_{33}C_{44}} \Bigg\{ & \frac{[(-C_{11}+C_{44}u_1^2)\xi + kC_{44}u_1]}{u_1[(k+u_1\xi)^2 - u_2^2\xi^2]} e^{-u_1\xi|z-h|} \\ & + \frac{[(-C_{11}+C_{44}u_2^2)\xi - kC_{44}u_2]}{u_2[(k-u_2\xi)^2 - u_1^2\xi^2]} e^{(k-u_2\xi)|z-h|} \\ & - \frac{[(-C_{11}+C_{44}u_1^2)\xi + kC_{44}u_1]\Delta_1}{u_1[(k+u_1\xi)^2 - u_2^2\xi^2]\Delta} e^{-u_1\xi(z+h)} \\ & - \frac{[(-C_{11}+C_{44}u_1^2)\xi + kC_{44}u_1]\Delta_2}{u_2[(k-u_2\xi)^2 - u_1^2\xi^2]\Delta} e^{(k-u_1\xi)z} e^{-u_2\xi h} \\ & - \frac{[(-C_{11}+C_{44}u_2^2)\xi - kC_{44}u_2]\Delta_3}{u_1[(k+u_1\xi)^2 - u_2^2\xi^2]\Delta} e^{(k-u_2\xi)z} e^{-u_1\xi h} \\ & - \frac{[(-C_{11}+C_{44}u_2^2)\xi - kC_{44}u_2]\Delta_4}{u_2[(k-u_2\xi)^2 - u_1^2\xi^2]\Delta} e^{(k-u_2\xi)(z+h)} \Bigg\} \end{aligned} \tag{48}$$

From Eqs. (47), (48), (3)–(8) and (1), the vertical normal and shear stresses in Hankel domain for the half-space

Table 4
Terms that require numerical integrations in the presented solutions

Bessel function of first kind of order n ($n = 0, 1$)	
$\int_0^\infty \frac{1}{k+(u_1+u_2)\xi} e^{-u_1\xi z} J_n(\xi r)\,d\xi$	$\int_0^\infty \frac{1}{k-(u_1+u_2)\xi} e^{(k-u_2\xi)z} J_n(\xi r)\,d\xi$
$\int_0^\infty \frac{1}{k+(u_1-u_2)\xi} e^{-u_1\xi z} J_n(\xi r)\,d\xi$	$\int_0^\infty \frac{1}{k+(u_1-u_2)\xi} e^{(k-u_2\xi)z} J_n(\xi r)\,d\xi$
$\int_0^\infty \frac{\xi}{k+(u_1+u_2)\xi} e^{-u_1\xi z} J_n(\xi r)\,d\xi$	$\int_0^\infty \frac{\xi}{k-(u_1+u_2)\xi} e^{(k-u_2\xi)z} J_n(\xi r)\,d\xi$
$\int_0^\infty \frac{\xi}{k+(u_1-u_2)\xi} e^{-u_1\xi z} J_n(\xi r)\,d\xi$	$\int_0^\infty \frac{\xi}{k+(u_1-u_2)\xi} e^{(k-u_2\xi)z} J_n(\xi r)\,d\xi$
$\int_0^\infty \frac{\xi^2}{k+(u_1+u_2)\xi} e^{-u_1\xi z} J_n(\xi r)\,d\xi$	$\int_0^\infty \frac{\xi^2}{k-(u_1+u_2)\xi} e^{(k-u_2\xi)z} J_n(\xi r)\,d\xi$
$\int_0^\infty \frac{\xi^2}{k+(u_1-u_2)\xi} e^{-u_1\xi z} J_n(\xi r)\,d\xi$	$\int_0^\infty \frac{\xi^2}{k+(u_1-u_2)\xi} e^{(k-u_2\xi)z} J_n(\xi r)\,d\xi$

can also be expressed as

$$\begin{aligned}
\sigma_{zz}^* = \frac{P_z}{4\pi C_{33}C_{44}} \Bigg\{ & \frac{[(C_{33}C_{44}u_1^2 + C_{13}^2 + C_{13}C_{44} - C_{11}C_{33})u_1\xi^2 + kC_{44}(C_{33}u_1^2 + C_{13})\xi]}{u_1[(k+u_1\xi)^2 - u_2^2\xi^2]} e^{-u_1\xi|z-h|} \\
& + \frac{[(C_{33}C_{44}u_2^2 + C_{13}^2 + C_{13}C_{44} - C_{11}C_{33})u_2\xi^2 - k(2C_{33}C_{44}u_2^2 + C_{13}^2 - C_{11}C_{33})\xi + k^2C_{33}C_{44}u_2]}{u_2[(k-u_2\xi)^2 - u_1^2\xi^2]} e^{(k-u_2\xi)|z-h|} \\
& - \frac{[(C_{33}C_{44}u_1^2 + C_{13}^2 + C_{13}C_{44} - C_{11}C_{33})u_1\xi^2 + kC_{44}(C_{33}u_1^2 + C_{13})\xi]}{u_1[(k+u_1\xi)^2 - u_2^2\xi^2]} \frac{\Delta_1}{\Delta} e^{-u_1\xi(z+h)} \\
& - \frac{[(C_{33}C_{44}u_1^2 + C_{13}^2 + C_{13}C_{44} - C_{11}C_{33})u_1\xi^2 + k(C_{13}C_{44} + C_{11}C_{33})\xi - k^2C_{33}C_{44}u_1]}{u_2[(k-u_2\xi)^2 - u_1^2\xi^2]} \frac{\Delta_2}{\Delta} e^{(k-u_1\xi)z - u_2\xi h} \\
& - \frac{[(C_{33}C_{44}u_2^2 + C_{13}^2 + C_{13}C_{44} - C_{11}C_{33})u_2\xi^2 - k(2C_{33}C_{44}u_2^2 + C_{13}^2 - C_{11}C_{33})\xi + k^2C_{33}C_{44}u_2]}{u_1[(k-u_1\xi)^2 - u_2^2\xi^2]} \frac{\Delta_3}{\Delta} e^{(k-u_2\xi)z - u_1\xi h} \\
& - \frac{[(C_{33}C_{44}u_2^2 + C_{13}^2 + C_{13}C_{44} - C_{11}C_{33})u_2\xi^2 - k(2C_{33}C_{44}u_2^2 + C_{13}^2 - C_{11}C_{33})\xi + k^2C_{33}C_{44}u_2]}{u_2[(k-u_2\xi)^2 - u_1^2\xi^2]} \frac{\Delta_4}{\Delta} e^{(k-u_2\xi)(z+h)} \Bigg\},
\end{aligned} \tag{49}$$

$$\begin{aligned}
\tau_{rz}^* = \frac{-P_z}{4\pi C_{33}} \Bigg\{ & \frac{(C_{11} + C_{13}u_1^2)\xi^2}{u_1[(k+u_1\xi)^2 - u_2^2\xi^2]} e^{-u_1\xi|z-h|} \\
& + \frac{[(C_{11} + C_{13}u_2^2)\xi^2 - 2kC_{13}u_2\xi + k^2C_{13}]}{u_2[(k-u_2\xi)^2 - u_1^2\xi^2]} e^{(k-u_2\xi)|z-h|} \\
& + \frac{(C_{11} + C_{13}u_1^2)\xi^2}{u_1[(k+u_1\xi)^2 - u_2^2\xi^2]} \frac{\Delta_1}{\Delta} e^{-u_1\xi(z+h)} \\
& + \frac{[(C_{11} + C_{13}u_1^2)\xi^2 - k(C_{13} + C_{44})u_1\xi - k^2C_{13}]}{u_2[(k-u_2\xi)^2 - u_1^2\xi^2]} \times \frac{\Delta_2}{\Delta} e^{(k-u_1\xi)z - u_2\xi h} \\
& + \frac{[(C_{11} + C_{13}u_2^2)\xi^2 - 2kC_{13}u_2\xi + k^2C_{13}]}{u_1[(k-u_1\xi)^2 - u_2^2\xi^2]} \frac{\Delta_3}{\Delta} e^{(k-u_2\xi)z - u_1\xi h} \\
& + \frac{[(C_{11} + C_{13}u_2^2)\xi^2 - 2kC_{13}u_2\xi + k^2C_{13}]}{u_2[(k-u_2\xi)^2 - u_1^2\xi^2]} \frac{\Delta_4}{\Delta} e^{(k-u_2\xi)(z+h)} \Bigg\}.
\end{aligned} \tag{50}$$

The displacements U_r^* (Eq. (47)), U_z^* (Eq. (48)) and stresses σ_{zz}^* (Eq. (49)), τ_{rz}^* (Eq. (50)) in the inhomogeneous transversely isotropic half-space can also be obtained by taking the numerical inversion of Hankel theorem with respect to ξ of order 1, 0, 0, and 1, respectively. The infinite integrals involving products of Bessel function of the first kind of order n ($n = 0, 1$), an exponential function, and a polynomial, in Eqs. (29), (30), (34), (35) for the full space, and in Eqs. (47)–(50) for the half-space, are listed in Table 4. It seems that several terms in Table 4 cannot be given in closed form that requires numerical integrations. The related numerical procedures to calculate the induced displacement and stress in an inhomogeneous transversely isotropic half-space by a vertical point load will be illustrated in Section 5.

4. Numerical integrations

The integrals in Table 4 involve polynomial, exponential function, and Bessel function of the first kind of order n ($n = 0, 1$). It seems that several terms cannot be presented in closed form because those where the integrand includes Bessel functions are perhaps the most difficult to evaluate analytically in engineering analysis. Besides, the Bessel function is an oscillation function; numerical integration is often far from straightforward as an adequate level of computational accuracy can be difficult to achieve [98]. According to the aforementioned reasons, various algorithms, like numerical quadrature, logarithmic change of variables, asymptotic expansion of the Bessel function, and projection-based methods have been published [103]. Early attempts at numerical evaluation of the Hankel transform were made by Longman [95,96]. He formulated a method based on Euler's transformation of slowly convergent alternating series for the numerical evaluation of integrals. Blackmore et al. [104] divided the infinite range of oscillatory integrals into several terms using the zero points of Bessel function. Namely, such treatment enabled to keep the Bessel function always positive or negative, so that after integrating individually and summing up all contributions, it could improve the numerical accuracy. Davis and Rabinowitz [97] and Evans [105] believed that both using the methods of Longman [95,96] and Blackmore et al. [104] was the most efficient way to solve this problem. However, Cree and Bones [103] reviewed a number of algorithms, and found that the projection-based methods could provide acceptable accuracy. Recently, Lu and Perng [106] considered that a point heat source induced thermo-consolidation problem for an elastic isotropic medium. They evaluated the inverse Hankel transforms by means of 68 points Gauss quadrature formula, and concluded that only the calculation of the first six terms of Bessel function was accurate enough for engineering practices.

The method employed in this work is performing the integration over each of the first 20 half-cycles of Bessel functions. Euler's transformation was applied to this series to speed up rapidly the convergence [95]. The first 20 terms of zeros of Bessel function of the first kind of order n ($n = 0$, 1) are quoted from Watson [107]. Also, the Gauss quadrature formula was utilized for 68 points of subdivision of each interval in order to obtain high accuracy of numerical values. The method proposed by Longman [95] can be expressed as follows:

$$\int_0^\infty J_0(x)\,dx \cong \sum_{n=0}^{20} \int_{x_n}^{x_{n+1}} J_0(x)\,dx. \tag{51}$$

In each division, the 16 points of Gaussian quadrature are adopted, and x can be transferred by

$$x = \frac{(x_{n+1} - x_n)t + x_{n+1} + x_n}{2}. \tag{52}$$

Then, revising Eq. (51) with Eq. (52) yields the following expression:

$$\sum_{n=0}^{20} \int_{x_n}^{x_{n+1}} J_0(x)\,dx \cong \sum_{n=0}^{20} \frac{(x_{n+1} - x_n)}{2} \times \left[\sum_{i=1}^{16} w_i J_0\left(\frac{(x_{n+1} - x_n)t_i + x_{n+1} + x_n}{2}\right)\right]. \tag{53}$$

In order to speed up the convergence, summing up the values of the former ten terms, and introducing the Euler's transformation into the latter ten ones as

$$\int_0^\infty J_0(x)\,dx \cong \sum_{n=0}^{10} \int_{x_n}^{x_{n+1}} J_0(x)\,dx + \sum_{n=11}^{20} \frac{\Lambda^{n-10}}{2^{n-10}} = 0.999999992, \tag{54}$$

where Λ is the first advancing row of differences [95], and n is a constant ranging from 0 to 20. Thus, the approximate value (0.999999992) calculated by Eq. (54) is very close to the exact result (is equal to 1).

Regarding the singularities involved in Table 4, they can be solved by means of the Taylor's theorem expansion as [97]

$$\begin{aligned} f(x) &= \int_a^b \frac{f(t)}{t-x}\,dt = \int_a^b \frac{f(t)-f(x)}{t-x}\,dt + \int_a^b \frac{f(x)}{t-x}\,dt \\ &= \int_a^b \frac{f(t)-f(x)}{t-x}\,dt + f(x)\log\frac{b-x}{x-a} \\ &= \int_a^{x-\varepsilon} \frac{f(t)-f(x)}{t-x}\,dt + \int_{x-\varepsilon}^b \frac{f(t)-f(x)}{t-x}\,dt \\ &\quad + f(x)\log\frac{b-x}{x-a} + 2\varepsilon f'(x) + \frac{\varepsilon^3}{9} f'''(x) + \cdots, \end{aligned} \tag{55}$$

where x is a singular point, ε is a tiny parameter, and a, b are the lower and upper limits, respectively.

5. Illustrative example

This section presents a parametric study to confirm the derived solutions and elucidate the effect of inhomogeneity, and the type and degree of material anisotropy on the displacements and stresses. Two illustrative examples related to a vertical point load acting on the surface of an inhomogeneous transversely isotropic half-space are given to show the effect of various parameters on the vertical surface displacement and vertical normal stress, respectively. The effect of the inhomogeneity parameter k, and the degree of anisotropy, specified by the ratios G/G', E/E' and ν/ν' on the displacement and stress is considered. Several types of isotropic and transversely isotropic rocks are considered as foundation materials. Table 5 lists their elastic properties, with G/G' and E/E' ranging between 1 and 3, and ν/ν' varying between 0.75 and 1.5. The values of E and ν adopted in Table 5 are 50 GPa and 0.25, respectively. The selected domains of anisotropic variation follow the suggestions of Gerrard [108] and Amadei et al. [109]. The variation of the proposed solutions for the inhomogeneity parameter k varies between 0 (homogeneous) and -0.5. The calculated results by aforementioned numerical approaches, as depicted in Figs. 3–6, are presented.

Based on Eqs. (29), (30), (34), (35) for the full space, and Eqs. (47)–(50) for the half-space, a FORTRAN program was written to calculate the displacement and stress components due to a point load in an inhomogeneous transversely isotropic medium. The presented solutions indicate that the displacements and stresses are affected by the inhomogeneity parameter k, and the degree and type of material anisotropy. In order to check the accuracy of numerical procedures, the comparisons are carried out with the homogeneous solutions [94] (when $k = 0$), and the calculated results agree with those to nine decimal places.

Table 5
Rock types and their elastic properties

Rock type	G/G'	E/E'	ν/ν'
Rock 1: Isotropy	1.0	1.0	1.0
Rock 2: Transverse isotropy	2.0	1.0	1.0
Rock 3: Transverse isotropy	3.0	1.0	1.0
Rock 4: Transverse isotropy	2.0	2.0	1.0
Rock 5: Transverse isotropy	2.0	3.0	1.0
Rock 6: Transverse isotropy	2.0	1.0	0.75
Rock 7: Transverse isotropy	2.0	1.0	1.5

In this study, firstly, the influence of inhomogeneity, and the degree and type of rock anisotropy on the vertical surface displacement in the half-space is investigated. Figs. 3a and b present the effect of inhomogeneity parameter k on the normalized vertical surface displacement (U_z/P_z) for Rock 1 and Rock 5, as listed in Table 5, respectively. These figures reveal that as the degree of inhomogeneity of a rock increases (from $k = 0$ to -0.5 (since $k < 0$ denotes a soft surface, whereas E, E', and G' increase with the increase of depth)), the magnitude of the vertical displacement on the surface decreases (Figs. 3a and b). Figs. 4a–c plot the rock anisotropic ratios of G/G', E/E', and ν/ν' on the displacement. It is evident that the magnitude of surface displacement is decisively influenced by rock anisotropy. Figs. 4a and b show that, for the fixed parameter k ($k = 0, -0.1, -0.5$), the vertical displacement on the surface increases with the increase of G/G' ($E/E' = \nu/\nu' = 1$, for Rocks 1, 2, 3), and E/E' ($\nu/\nu' = 1$, $G/G' = 2$, for Rocks 2, 4, 5). It reflects that the displacement increases with the increase of deformability in the direction parallel to the applied load. However, the ratio ν/ν' ($E/E' = 1$, $G/G' = 2$, for Rocks 2, 6, 7) has little effect on the vertical surface displacement.

Secondly, the effect of inhomogeneity, and rock anisotropy on the vertical normal stress in the transversely isotropic half-space is studied. In order to investigate the variation of σ_{zz} point by point in the $r - z$ plane, the relation of two non-dimensional factors, r/z and $z^2\sigma_{zz}/P_z$ is presented in Figs. 5a–d. In these figures, increasing the value of k (from $k = 0$ to -0.5) for each rock reduces the magnitude of vertical stress

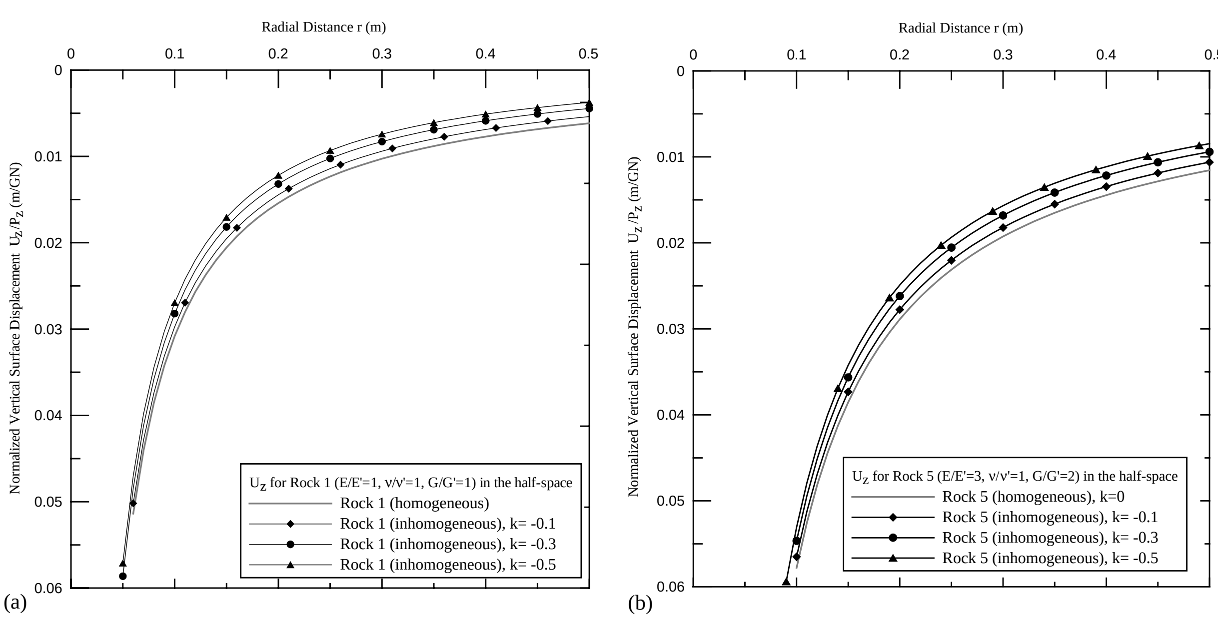

Fig. 3. (a) Effect of inhomogeneity parameter k on the normalized vertical surface displacement (Rock 1: isotropy). (b) Effect of inhomogeneity parameter k on the normalized vertical surface displacement (Rock 5: transverse isotropy).

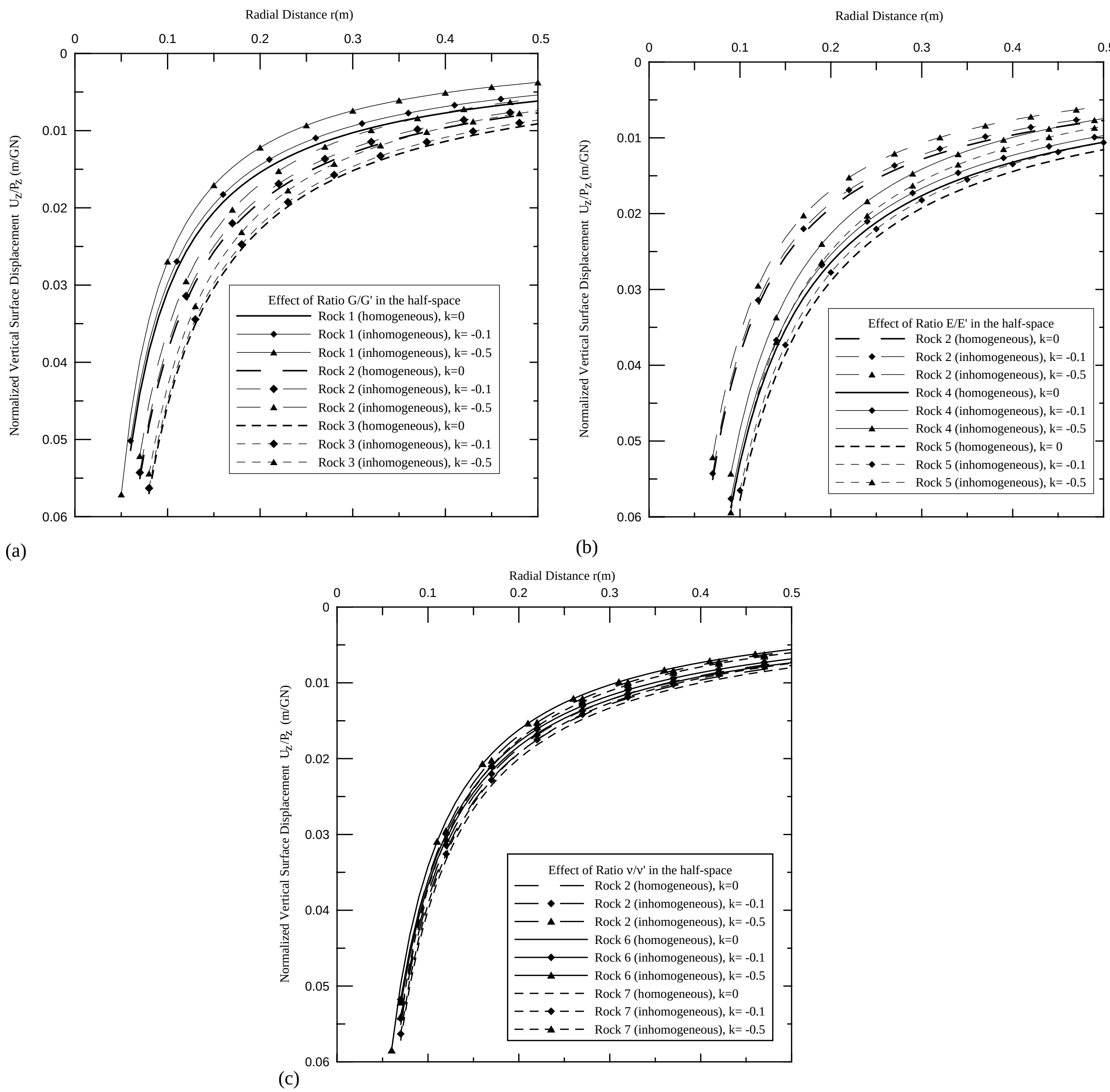

Fig. 4. (a) Effect of ratio G/G' on the normalized vertical surface displacement (Rocks 1–3). (b) Effect of ratio E/E' on the normalized vertical surface displacement (Rocks 2, 4, 5). (c) Effect of ratio ν/ν' on the normalized vertical surface displacement (Rocks 2, 6, 7).

considerably. Notably, the normalized stress in some regions could be larger than one unit when $k = 0$ and -0.1 (Fig. 5b). It means that the excessive compressive-stress may appear in these media. Figs. 6a–c plot the rock anisotropic ratios of G/G', E/E', and ν/ν' on the stress. From these figures, the vertical normal stress increases with increasing G/G' (Fig. 6a, $E/E' = \nu/\nu' = 1$, for Rocks 1–3), but decreases with increasing E/E' (Fig. 6b, $\nu/\nu' = 1$, $G/G' = 2$, for Rocks 2, 4, 5). Again, the ratio ν/ν' has nearly no influence on the stress (Fig. 6c, $E/E' = 1$, $G/G' = 2$, for Rocks 2, 6, 7).

The above examples were presented to elucidate the solutions and clarify how the inhomogeneity, and the type and degree of rock anisotropy affect the vertical surface displacement and vertical normal stress in the medium. The results show that the displacement and stress in the continuously inhomogeneous transversely isotropic half-space subjected to a point load (on the surface or in the interior) are easily calculated by the presented solutions. The magnitude and distribution of vertical surface displacement and stress are both evidently sensitive to the inhomogeneity parameter k (Figs. 3a and b, Figs. 5a–d), the anisotropic ratios of G/G' (Fig. 4a and 6a), and E/E' (Fig. 4b and 6b); however, the ratio of ν/ν' has little effect on the displacement (Fig. 4c) and stress (Fig. 6c). Hence, both the inhomogeneity and anisotropic deformability must be considered when estimating the displacements and stresses in a transversely isotropic full/half-space subjected to applied loads.

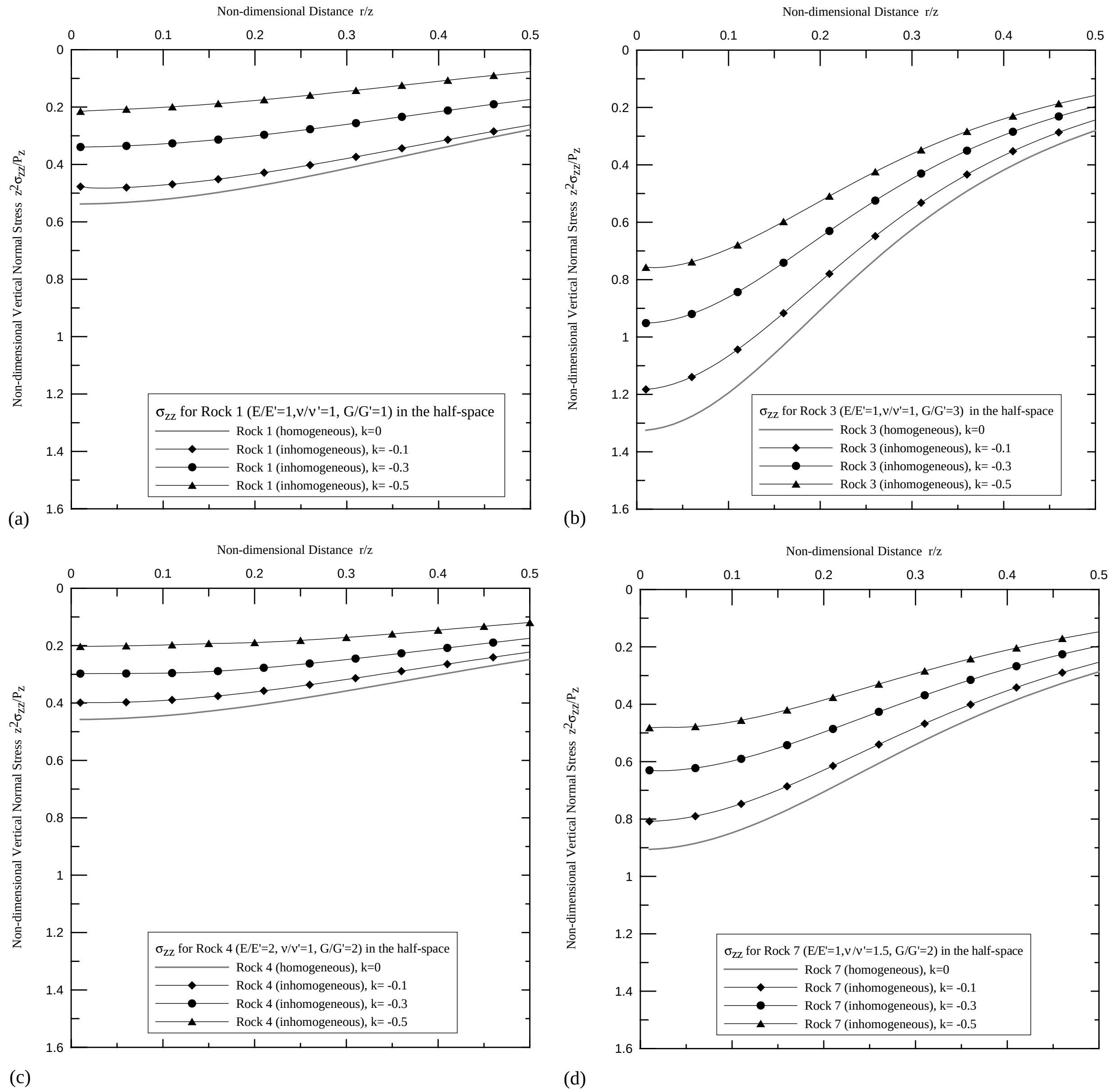

Fig. 5. (a) Effect of inhomogeneity parameter k on the non-dimensional vertical normal stress (Rock 1: isotropy). (b) Effect of inhomogeneity parameter k on the non-dimensional vertical normal stress (Rock 3: transverse isotropy). (c) Effect of inhomogeneity parameter k on the non-dimensional vertical normal stress (Rock 4: transverse isotropy). (d) Effect of inhomogeneity parameter k on the non-dimensional vertical normal stress (Rock 7: transverse isotropy).

6. Conclusions

The solutions for displacements and stresses in a continuously inhomogeneous transversely isotropic full/half-space with Young's and shear moduli varying exponentially with depth subjected to a vertical point load are proposed in this work. These solutions are limited to planes of transverse isotropy that are parallel to the horizontal surface of the spaces. The Hankel transform is employed for solving this problem, and the desired solutions can be obtained from the governing equations for a full space by satisfying the free traction on the surface of the half-space. The resulting integrals for displacements and stresses associate with polynomial, exponential function, and Bessel function of the first kind of order n ($n = 0$, 1) that cannot be given in closed form; hence, numerical integrations are required. The solutions are the same as the Liao and Wang's solutions [94] when the inhomogeneity parameter k is approaching zero. It is shown that the presented solutions are prominently influenced by the inhomogeneity, and the degree and type of material anisotropy. In particular, a parametric study has been carried out for two

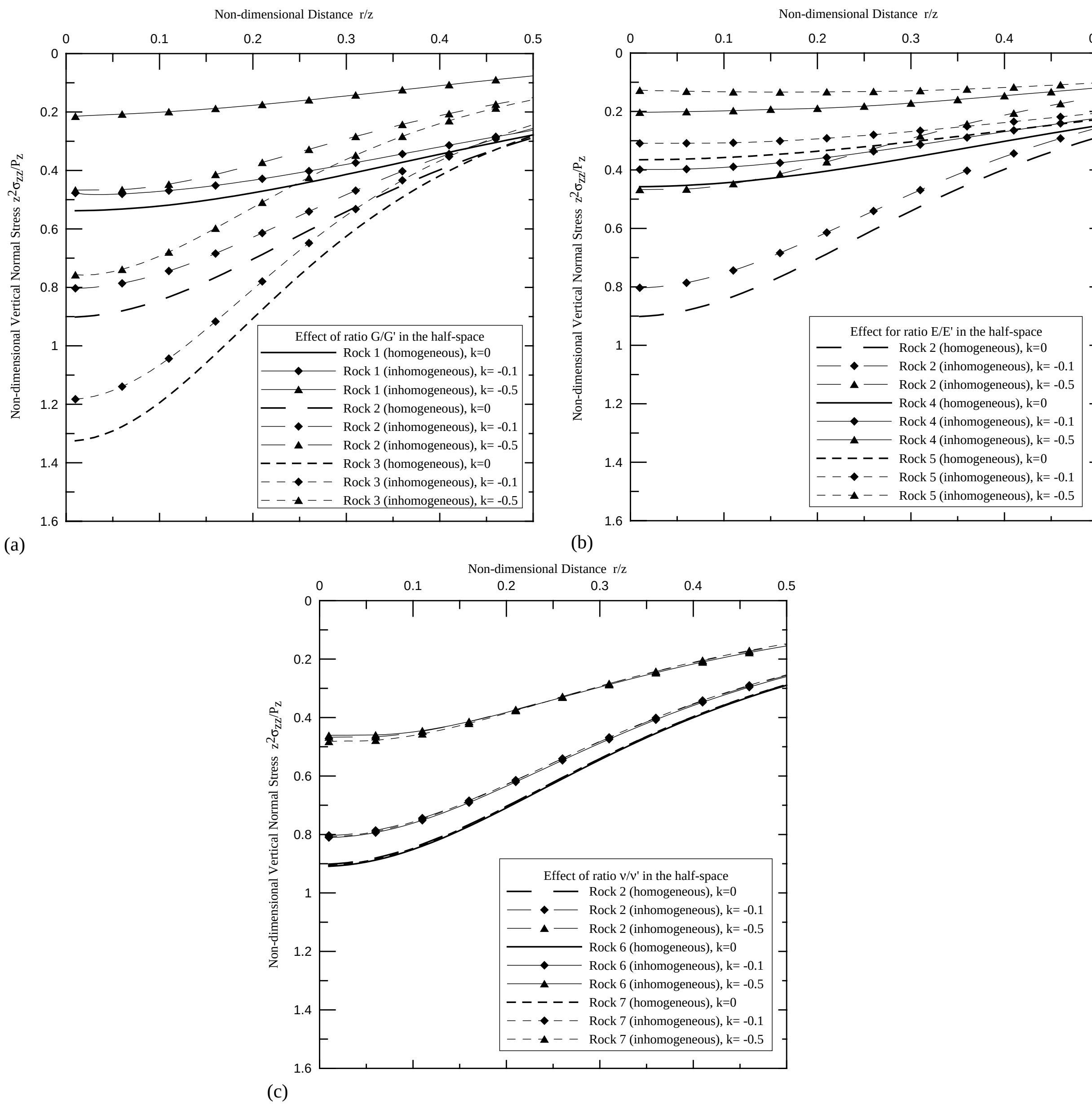

Fig. 6. (a) Effect of ratio G/G' on the non-dimensional vertical normal stress (Rocks 1–3). (b) Effect of ratio E/E' on the non-dimensional vertical normal stress (Rocks 2, 4, 5). (c) Effect of ratio v/v' on the non-dimensional vertical normal stress (Rocks 2, 6, 7).

illustrative examples, which has yielded the following interesting conclusions:

(1) The inhomogeneity considered has a great influence on the vertical surface displacement and vertical normal stress. As the degree of inhomogeneity of a rock increases, there is a decrease in the effect of loading at some distance from the point where displacement (Figs. 3a and b) and stress (Figs. 5a–d) are measured.
(2) The vertical surface displacement increases with increasing deformability in the direction parallel to the applied point load (Figs. 4a and b).
(3) The vertical normal stress increases with increase in G/G' (Fig. 6a), but decreases with increase in E/E' (Fig. 6b). Furthermore, the ratio v/v' has little influence on the stress (Fig. 6c).
(4) With increase in G/G', the non-dimensional vertical normal stress could become larger than one unit (Fig. 5b, when $k = 0$ and -0.1). It means that the excessive compressive stress may appear in these media.
(5) The magnitude and distribution of vertical surface displacement and vertical normal stress are both notably sensitive to the ratios G/G' (Fig. 4a and 6a), and E/E' (Fig. 4b and 6b); however, the ratio v/v'

has little effect on the displacement (Fig. 4c) and stress (Fig. 6c).

The calculation of displacements and stresses due to a point load in an inhomogeneous transversely isotropic full/half-space is fast and correct by the presented solutions. These solutions can more realistically simulate the actual stratum of loading problem in many areas of engineering practice. Also, they can be applied to estimate the displacements and stresses in the media due to an embedded point load for an end-bearing pile, uniform skin friction, linear variation of skin friction, and non-linear variation of skin friction for a friction pile, respectively. Besides, these solutions based on the assumption of axis symmetry can be extended to solve the displacements and stresses subjected to circular, and ring loads, etc. The results will be presented in forthcoming papers.

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