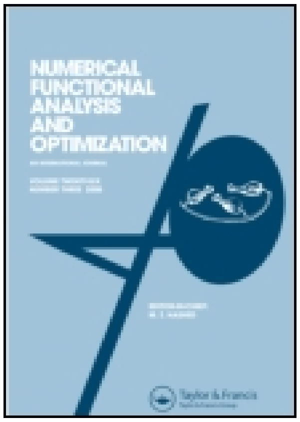

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Jang Jou [a] & Jinn-Liang Liu [b]
[a] Department of Applied Statistics and Information Science , Ming Chuan University , Kwei Shan, Taoyuan, Taiwan
[b] Department of Applied Mathematics , National Chiao Tung University , Hsinchu, Taiwan
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Jang Jou[1] and Jinn-Liang Liu[2,*]

[1]Department of Applied Statistics and Information Science, Ming Chuan University, Kwei Shan, Taoyuan, Taiwan
[2]Department of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan

ABSTRACT

A residual type a posteriori error estimator is presented for the least-squares finite element solution of stationary incompressible Navier–Stokes equations based on the velocity–vorticity–pressure formulation with nonstandard and standard boundary conditions. Using the coerciveness of the corresponding Stokes operator and the special feature of the nonlineariry of the formulation, it is shown that the error estimator is exact for the Stokes problem and is asymptotically exact for the Navier–Stokes problem in an energy-like norm. The resulting adaptive method is highly parallel because it does not require to assemble the global matrix and the error estimation can be completely localized without using any information from neighboring elements.

1. INTRODUCTION

A posteriori error estimation is now a standard component in adaptive methods (Oden et al., 1989; Verfürth, 1996; Zienkiewicz, 1992). The least-squares finite

*Correspondence: Jinn-Liang Liu, Department of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan; E-mail: jinnliu@math.nctu.edu.tw.

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element method (LSFEM) has been recognized as an attractive method in many applications (Bochev, 1997; Cai et al., 1994, 1997; Jiang, 1998; Jiang and Chang, 1990). It is shown here that, for the Stokes problem in first-order least-squares formulation, the residual type error estimator is locally as well as globally equal to the exact error in the norm induced by the least-squares functional. For Navier–Stokes equations, the error estimator is proved to be asymptotically exact. In other words, the error estimator is perfectly reliable for the LSFE approximation of the Stokes problem and very reliable for the Navier–Stokes problem. Moreover, the computation of the estimator is completely localized without any restriction on the approximation order and without requiring any information from the neighboring elements and therefore very efficient for parallel computations. These advantageous properties can be regarded as an additional appealing feature of LSFEM.

We consider the steady, incompressible Navier–Stokes equations

$$-\frac{1}{\mathrm{Re}}\,\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega, \tag{1}$$

$$\nabla\cdot\,\mathbf{u} = 0 \quad \text{in} \quad \Omega, \tag{2}$$

$$\mathcal{B}(\mathbf{u}, p) = \mathbf{0} \quad \text{on} \quad \partial\Omega. \tag{3}$$

where the symbols Δ, ∇, and $\nabla\cdot$ stand for the Laplacian, gradient, and divergence operators, respectively, Ω is an open bounded connected domain in R^2 with the boundary $\partial\Omega$, $\mathrm{Re} > 0$ the Reynolds number, $\mathbf{u} = (u_1, u_2)^T \in [H^1(\Omega)]^2$ the velocity field, and $\mathbf{f} = (f_1, f_2)^T \in [L^2(\Omega)]^2$ the body force. The pressure $p \in L^2(\Omega)$ if the admissible homogeneous boundary operator $\mathcal{B}$ describes the pressure on $\partial\Omega$, otherwise $p \in L_0^2(\Omega)$. Here $H^s(\Omega)$, $s \in R$, denotes a usual Sobolev space equipped with the norm $\|\cdot\|_s$ and $L_0^2(\Omega) = \{q \in L^2(\Omega)\,|\,(q,1)_0 = 0\}$, where $(u,v)_0 := \int_\Omega uv\,d\Omega$. We denote $\widetilde{H}^s(\Omega) = H^s(\Omega) \cap L_0^2(\Omega)$.

For least-squares formulation, one usually reduces the second-order PDE to a first-order system by introducing some suitable new variables. The standard velocity–vorticity–pressure formulation is given in Sec. 2. We are interested in the coercivity of the linear operator obtained from this particular formulation. A priori error analysis of the LSFE approximation based on this formulation has been thoroughly studied by Bochev, Gunzburger, and Jiang, see e.g., Bochev, 1997; Bochev and Gunzburger, 1998; Jiang, 1998. We are concerned here with the a posteriori error analysis which is given in Sec. 3. The analysis is mainly based on the coerciveness of the first-order Stokes operator and the special feature of the nonlinear term in the formulation.

2. VELOCITY–VORTICITY–PRESSURE FORMULATION

In two space dimensions, with the vorticity $\omega = \nabla\times\mathbf{u} = \partial u_2/\partial x - \partial u_1/\partial y$ and the Bernoulli pressure or the total pressure $r = p + (1/2)|\mathbf{u}|^2$, the Navier–Stokes Eqs. (1)–(3) can be reduced to the first-order system (Bochev, 1997; Jiang and Chang, 1990)

$$\mathcal{N}\mathbf{U} = \mathbf{F} \quad \text{in} \quad \Omega, \tag{4}$$

with the boundary condition

$$\mathcal{R}\mathbf{U} = \mathbf{0} \quad \text{on} \quad \partial\Omega, \tag{5}$$

where

$$\mathcal{N}\mathbf{U} = \begin{cases} -\mathbf{u}\times\omega + \dfrac{1}{\mathrm{Re}}\ \nabla\times\omega + \nabla r \\ \omega - \nabla\times\ \mathbf{u} \\ \nabla\cdot\ \mathbf{u} \end{cases} \tag{6}$$

$\mathbf{U} = (\mathbf{u}, \omega, r)^T$ and $\mathbf{F} = (\mathbf{f}, 0, 0)^T$. Note that the cross product $\mathbf{u}\times\omega$ is defined by embedding $\mathbf{u}$ and ω into three-dimensional vectors $(u_1, u_2, 0)^T$ and $(0, 0, \omega)^T$, i.e., $\mathbf{u}\times\omega = (u_2\omega,\ -u_1\omega)^T$. The corresponding first-order Stokes operator is

$$\mathcal{L}\mathbf{V} = \begin{cases} \dfrac{1}{\mathrm{Re}}\ \nabla\times\xi + \nabla\eta \\ \xi - \nabla\times\ \mathbf{v} \\ \nabla\cdot\ \mathbf{v}. \end{cases} \tag{7}$$

Let

$$\mathcal{V} = \{\mathbf{V} = (\mathbf{v}, \xi, \eta) \in [H^1(\Omega)]^2 \times H^1(\Omega) \times \overline{H}^1(\Omega);\ \mathcal{R}\mathbf{V} = \mathbf{0} \text{ on } \partial\Omega\}, \tag{8}$$

where $\overline{H}^1(\Omega)$ denotes the space $H^1(\Omega)$ whenever the boundary operator $\mathcal{R}$ prescribes the pressure r on $\partial\Omega$, and $\widetilde{H}^1(\Omega)$ otherwise. We shall consider five different types of the boundary condition $\mathcal{R}\mathbf{U}$ described as in Lemma 1.

The nonlinear term $\mathbf{u}\times\omega$ in Eq. (6) is of zero order and thus is not related to any derivatives while the rest of the terms constitute the linear Stokes operator. Therefore, the nonlinear term has no effect on the classification of the Navier–Stokes equations and the boundary conditions for the Stokes equations are valid for the Navier–Stokes equations. And it does not matter how large the Reynolds number is, the whole system is elliptic. For this reason, the permissible boundary conditions for Navier–Stokes equations are those for Stokes equations.

The coercivity of the Stokes operator $\mathcal{L}$ on function space $\mathcal{V}$ for a large number of boundary operators $\mathcal{R}$ was studied by Bochev (1997) and Jiang (1998). In Bochev (1997), according to the elliptic regularity theory of Agmon et al. (1964), Bochev examined the complementing condition of Agmon (1964), which is both necessary and sufficient for such coercivity to hold for the operators $\mathcal{L}$ and $\mathcal{R}$. Jiang proved the same coercivity based on the bounded inverse theorem and the Friedrichs inequalities related to grad, div, and curl operators. For the sake of simplicity, we consider only homogeneous boundary conditions. These results can be extended to mixed and inhomogeneous boundary conditions without difficulty. Following Jiang (1998), we summarize these results as follows.

Lemma 1. *For the first-order Stokes operator* $\mathcal{L}$ *of Eq.* (7), *let the boundary operator* $\mathcal{R}$ *be of the following five types:*

$$\mathcal{R}\mathbf{V} = \begin{pmatrix} \mathbf{v}\cdot\mathbf{n} \\ \xi \end{pmatrix} \qquad \mathcal{R}\mathbf{V} = \begin{pmatrix} \mathbf{v}\cdot\mathbf{n} \\ \eta \end{pmatrix}$$
$$\mathcal{R}\mathbf{V} = \begin{pmatrix} \mathbf{v}\cdot\mathbf{t} \\ \xi \end{pmatrix} \qquad \mathcal{R}\mathbf{V} = \begin{pmatrix} \mathbf{v}\cdot\mathbf{t} \\ \eta \end{pmatrix} \tag{9}$$
$$\mathcal{R}\mathbf{V} = \begin{pmatrix} \mathbf{v}\cdot\mathbf{n} \\ \mathbf{v}\cdot\mathbf{t} \end{pmatrix}$$

where **n** *and* **t** *are the outward normal and tangential unit vectors to* $\partial\Omega$, *respectively. Then there exists a positive constant C depending on the Reynolds number such that, for types* (i)–(iv),

$$\|\mathcal{L}\mathbf{V}\|_0^2 \geq C\left(\|\mathbf{v}\|_1^2 + \|\xi\|_1^2 + \|\eta\|_1^2\right) \qquad \forall\, \mathbf{V} \in \mathcal{V} \tag{10}$$

and that, for type (*v*),

$$\|\mathcal{L}\mathbf{V}\|_0^2 \geq C\left(\|\mathbf{v}\|_1^2 + \|\xi\|_0^2 + \|\eta\|_0^2\right) \qquad \forall\, \mathbf{V} \in \mathcal{V}. \tag{11}$$

For $\mathbf{V} \in \mathcal{V}$, define the functional:

$$J(\mathbf{V}) = \frac{1}{2}\left(\left\|-\mathbf{v}\times\xi + \frac{1}{\mathrm{Re}}\ \nabla\times\xi + \nabla\eta - \mathbf{f}\right\|_0^2 + \|\xi - \nabla\times\ \mathbf{v}\|_0^2 + \|\nabla\cdot\ \mathbf{v}\|_0^2\right).$$

A necessary condition that the solution $\mathbf{U}_n \in \mathcal{V}$ of Eq. (4) be a minimizer of the functional J is

$$\lim_{t\to 0}\frac{d}{dt}J(\mathbf{U}_n + t\mathbf{V}) = 0 \qquad \forall \mathbf{V} \in \mathcal{V},$$

which is equivalent to

$$B_n(\mathbf{U}_n, \mathbf{V}) = 0 \qquad \forall \mathbf{V} \in \mathcal{V}, \tag{12}$$

where

$$B_n(\mathbf{U}_n, \mathbf{V}) = B_s(\mathbf{U}_n, \mathbf{V}) - F(\mathbf{V}) + N(\mathbf{U}_n, \mathbf{V}) \tag{13}$$
$$B_s(\mathbf{U}_n, \mathbf{V}) = (\mathcal{L}\mathbf{U}_n, \mathcal{L}\mathbf{V})_0 \tag{14}$$
$$F(\mathbf{V}) = (\mathbf{F},\ \mathcal{L}\mathbf{V})_0 = (\mathbf{f},\ \frac{1}{\mathrm{Re}}\ \nabla\times\xi + \nabla\eta)_0 \tag{15}$$
$$\begin{aligned} N(\mathbf{U}_n, \mathbf{V}) &= (-\mathbf{u}\times\omega + \frac{1}{\mathrm{Re}}\ \nabla\times\omega + \nabla r - \mathbf{f},\ \ -\mathbf{v}\times\omega - \mathbf{u}\times\xi)_0 \\ &\quad + (-\mathbf{u}\times\omega,\ \frac{1}{\mathrm{Re}}\ \nabla\times\xi + \nabla\eta)_0. \end{aligned} \tag{16}$$

Similarly, corresponding to the Stokes problem, we have the variational formulation

$$B_s(\mathbf{U}_s, \mathbf{V}) = F(\mathbf{V}), \tag{17}$$

where $\mathbf{U}_s$ is the solution of Eq. (4) in which the operator $\mathcal{N}$ is replaced by $\mathcal{L}$.

With respect to Eqs. (17) and (12), the finite element problems are to seek the solutions $\mathbf{U}_{s,h} \in S_h$ and $\mathbf{U}_{n,h} \in S_h$ such that

$$B_s(\mathbf{U}_{s,h}, \mathbf{V}_h) = F(\mathbf{V}_h) \qquad \forall \mathbf{V}_h \in S_h \tag{18}$$

$$B_n(\mathbf{U}_{n,h}, \mathbf{V}_h) = 0 \qquad \forall \mathbf{V}_h \in S_h \tag{19}$$

where S_h is a finite element subspace of $\mathcal{V}$ parametrized by the mesh size h of some triangulation (denoted by T_h) on the domain Ω. The abstract approximation theory for branches of nonsingular solutions developed by Brezzi et al. (1980) allows us to address existence, uniqueness, and a priori error estimates for LSFEM solutions of the Navier–Stokes equations by using the results established in the context of the linear Stokes equations (Bochev, 1997). The subspace S_h can be constructed by the standard finite elements for all variables in the vector-valued function $\mathbf{U}$. For example, the velocity components u_1, u_2, vorticity ω, and total pressure r can all be approximated by the same piecewise linear polynomials. Newton's iteration on Eq. (19) always results in symmetric positive definite systems of linear algebraic equations independent of the Reynolds number provided that the initial guess of the iteration is sufficiently close to the solution.

3. ERROR ESTIMATION

Once an approximate solution $\mathbf{U}_{s,h}$ or $\mathbf{U}_{n,h}$ is available, one of the major concerns in practice is to assess the reliability of the approximation, i.e., to estimate the exact error $\mathbf{E}_s = \mathbf{U}_s - \mathbf{U}_{s,h}$ or $\mathbf{E}_n = \mathbf{U}_n - \mathbf{U}_{n,h}$ in some suitable norm for which, following the a priori estimates Eqs. (10) and (11), we choose the norm $\|\mathcal{L}\mathbf{V}\|_0$, $\forall \mathbf{V} \in [H^1(\Omega)]^4$. For LSFE approximation, residual type of error estimation is a natural choice. Define the local residual norms

$$\mathcal{E}_{s,i} = \|\mathbf{F} - \mathcal{L}\mathbf{U}_{s,h}\|_{0,t_i}, \quad \mathcal{E}_{n,i} = \|\mathbf{F} - \mathcal{N}\mathbf{U}_{n,h}\|_{0,t_i} \tag{20}$$

on each element $t_i \in T_h$ and the estimators

$$\mathcal{E}_s = \left(\sum_{t_i \in T_h} \mathcal{E}_{s,i}^2 \right)^{1/2}, \qquad \mathcal{E}_n = \left(\sum_{t_i \in T_h} \mathcal{E}_{n,i}^2 \right)^{1/2}, \tag{21}$$

where the norm $\|\cdot\|_{0,t_i}$ is the L^2 norm restricted to the element t_i.

The error indicators $\mathcal{E}_{s,i}$ and $\mathcal{E}_{n,i}$ are readily computable without any jump conditions across inter-element boundaries and hence the resulting computations are highly efficient and very suitable for parallel implementation. Together with the symmetric property of the algebraic system, the resulting adaptive procedure of LSFE computations can be completely parallel if a conjugate gradient solver is used because there is no need for a global assembly of the system and the iterative process can be done locally (Jiang and Carey, 1987). Moreover, for the Stokes problem, the error estimator and error indicators are perfectly reliable and effective.

Theorem 1. *Let* $\mathbf{E}_s = \mathbf{U}_s - \mathbf{U}_{s,h}$ *where* $\mathbf{U}_s$ *and* $\mathbf{U}_{s,h}$ *are the solutions of problems* (17) *and* (18), *respectively. Then*

$$\mathcal{E}_{s,i} = \left\| \mathcal{L}\mathbf{E}_s \right\|_{0,t_i}, \quad \forall t_i \in T_h, \tag{22}$$

$$\mathcal{E}_s = \left\| \mathcal{L}\mathbf{E}_s \right\|_0. \tag{23}$$

Proof.

$$\begin{aligned}\mathcal{E}_{s,i}^2 &= \left\| \mathbf{F} - \mathcal{L}\mathbf{U}_{s,h} \right\|_{0,t_i}^2 \\ &= \left\| \mathcal{L}\mathbf{U}_s - \mathcal{L}\mathbf{U}_{s,h} \right\|_{0,t_i}^2.\end{aligned}$$

Hence, we have Eqs. (22) and (23).

We now study the estimator for the Navier–Stokes equations. For this we make the following assumption

$$\|\mathbf{u} - \mathbf{u}_h\|_0 + \|\omega - \omega_h\|_0 \le Ch^{\alpha}\|\mathbf{u} - \mathbf{u}_h\|_1, \quad \alpha > 0, \tag{24}$$

where C is a generic positive constant independent of h. The assumption essentially states that the convergence rate of the approximate velocity $\mathbf{u}_h$ and vorticity ω_h in L^2 norm is of order $\alpha > 0$ which is higher than that in H^1 norm. This kind of convergence is commonly observed in numerical experiments on finite element computations of second-order partial differential equations (see Bochev (1997), Bochev and Gunzburger (1998), Jiang (1998) for Navier–Stokes equations).

Theorem 2. *Let* $\mathbf{E}_n = \mathbf{U}_n - \mathbf{U}_{n,h}$ *where* $\mathbf{U}_n$ *and* $\mathbf{U}_{n,h}$ *are the solutions of Problems* (12) *and* (19), *respectively. If assumption* (24) *holds, then the error estimator* $\mathcal{E}_n$ *is asymptotically exact, i.e.*,

$$(1 - O(h^{\alpha}))\left\| \mathcal{L}\mathbf{E}_n \right\|_0 \le \mathcal{E}_n \le (1 + O(h^{\alpha}))\left\| \mathcal{L}\mathbf{E}_n \right\|_0. \tag{25}$$

Proof. Let

$$\mathcal{K}\mathbf{U}_n = \mathcal{N}\mathbf{U}_n - \mathcal{L}\mathbf{U}_n \qquad \forall \mathbf{U}_n \in \mathcal{V}.$$

Then,

$$\begin{aligned}\left\| \mathcal{K}\mathbf{U}_n - \mathcal{K}\mathbf{U}_{n,h} \right\|_0^2 &= \left\| \begin{pmatrix} -\mathbf{u} \times \omega \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -\mathbf{u}_h \times \omega_h \\ 0 \\ 0 \end{pmatrix} \right\|_0^2 \\ &= \left\| \begin{pmatrix} -u_2\omega \\ u_1\omega \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -u_{2,h}\omega_h) \\ u_{1,h}\omega_h \\ 0 \\ 0 \end{pmatrix} \right\|_0^2 \\ &= \left\| u_2\omega - u_{2,h}\omega_h \right\|_0^2 + \left\| u_1\omega - u_{1,h}\omega_h \right\|_0^2 \\ &= \left\| u_2(\omega - \omega_h) + \omega_h(u_2 - u_{2,h}) \right\|_0^2 + \left\| u_1(\omega - \omega_h) + \omega_h(u_1 - u_{1,h}) \right\|_0^2 \\ &\le 4\left(\|\mathbf{u}\|_0^2 \left\| \omega - \omega_h \right\|_0^2 + \left\| \omega_h \right\|_0^2 \left\| \mathbf{u} - \mathbf{u}_h \right\|_0^2 \right)\end{aligned}$$

The convergence assumption implies that $\|\omega_h\|_0$ is bounded independently of h and hence

$$\|\mathbf{u}\|_0^2 \left\|\omega - \omega_h\right\|_0^2 + \left\|\omega_h\right\|_0^2 \left\|\mathbf{u} - \mathbf{u}_h\right\|_0^2 \le C\left(\left\|\mathbf{u} - \mathbf{u}_h\right\|_0^2 + \left\|\omega - \omega_h\right\|_0^2\right)$$

It follows Eq. (24) that

$$\begin{aligned}\left\|\mathcal{K}\mathbf{U}_n - \mathcal{K}\mathbf{U}_{n,h}\right\|_0 &\le C(\left\|\mathbf{u} - \mathbf{u}_h\right\|_0 + \left\|\omega - \omega_h\right\|_0)\\ &\le Ch^\alpha \left\|\mathbf{u} - \mathbf{u}_h\right\|_1\\ &\le Ch^\alpha \left\|\mathcal{L}(\mathbf{U}_n - \mathbf{U}_{n,h})\right\|_0.\end{aligned}$$

Since

$$\begin{aligned}\mathcal{E}_n &= \|\mathbf{F} - \mathcal{N}\mathbf{U}_{n,h}\|_0\\ &= \left\|\mathcal{N}\mathbf{U}_n - \mathcal{N}\mathbf{U}_{n,h}\right\|_0\\ &= \left\|\mathcal{L}\mathbf{U}_n - \mathcal{L}\mathbf{U}_{n,h} + \mathcal{K}\mathbf{U}_n - \mathcal{K}\mathbf{U}_{n,h}\right\|_0\\ &= \left\|\mathcal{L}\mathbf{E}_n + (\mathcal{K}\mathbf{U}_n - \mathcal{K}\mathbf{U}_{n,h})\right\|_0\end{aligned}$$

we have

$$\left\|\mathcal{L}\mathbf{E}_n\right\|_0 - \left\|\mathcal{K}\mathbf{U}_n - \mathcal{K}\mathbf{U}_{n,h}\right\|_0 \le \mathcal{E}_n \le \left\|\mathcal{L}\mathbf{E}_n\right\|_0 + \left\|\mathcal{K}\mathbf{U}_n - \mathcal{K}\mathbf{U}_{n,h}\right\|_0$$

and therefore Eq. (25).

4. CONCLUSION

It is well known that the LSFEM provides very attractive properties in applications. For example, a single piecewise polynomial finite element space may be used for all test and trial functions, it always leads to symmetric positive definite systems, and it does not require the inf–sup condition when compared with the mixed finite element method. However, it usually results in more degrees of freedom in the systems due to extra state variables. Adaptive methods with effective mesh refinement can dramatically reduce DOFs especially for the singular problems.

The a posteriori error analysis presented in this article shows that the error indicator is perfectly reliable for the guidance of mesh refinement at least for the Stokes problems and is very effective for the Navier–Stokes problem. And the error estimator is also highly reliable for feedback error control in self-adaptive automatic computations. The implementation of the residual estimator is very simple. The error indicators can be computed strictly within each element without using any information from neighboring elements because they do not involve jump conditions across element boundaries and local boundary conditions. Therefore, together with the symmetric property of the algebraic system in a neighborhood of a solution (Bochev and Gunzburger, 1993), the adaptive procedure of least squares computations for the Navier–Stokes equations can be completely parallel on an element-by-element basis if a conjugate gradient solver is used (Jiang and Carey, 1987). For more numerical results of adaptive LSFE computations, we also refer to Hsieh et al. (1999).

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