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Maximum entropy analysis to the *N* policy M/G/1 queueing system with a removable server

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Abstract

We study a single removable server in an M/G/1 queueing system operating under the *N* policy in steady-state. The server may be turned on at arrival epochs or off at departure epochs. Using the maximum entropy principle with several well-known constraints, we develop the approximate formulae for the probability distributions of the number of customers and the expected waiting time in the queue. We perform a comparative analysis between the approximate results with exact analytic results for three different service time distributions, exponential, 2-stage Erlang, and 2-stage hyper-exponential. The maximum entropy approximation approach is accurate enough for practical purposes. We demonstrate, through the maximum entropy principle results, that the *N* policy M/G/1 queueing system is sufficiently robust to the variations of service time distribution functions.

Keywords: Control; Maximum entropy; Lagrange's method; M/G/1 queue

1. Introduction

This paper deals with a single removable server queueing system with Poisson arrivals and general distribution service times using the maximum entropy principle. The decision-maker can turn a single server on at any arrival epoch or off at any service completion (departure) epoch. The term 'removable server' is just an abbreviation for the system of turning on and turning off the server, depending on the number of customers in the system. The total number of arriving customers and the system capacity are infinite.

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The M/G/1 queueing system whose arrivals of customers follow a Poisson process with parameter λ and service times are independent and identically distributed (i.i.d.) random variables obeying a general distribution $S(u)(u \geqslant 0)$ with a mean service time $1/\mu$ and a finite variance σ^2. If one customer is in service, then arriving customers have to wait in the queue until the server is available. We assume that customers arrive at the server form a single waiting line and are served in the order of their arrivals; that is, the first-come, first-served discipline. Suppose that the server can serve only one customer at a time, and that the service is independent of the arrival of the customers.

The maximum entropy principle is a probability inference method which has been widely applied in fields such as statistical mechanics (see [1]), queueing theory (see [2,3]), and computer performance analysis (see [4]). The maximum entropy principle is utilised to analyse the ordinary queueing system by several researchers such as Shore [2,3], Ferdinand [4], Arizono, et al. [5], Wu and Chan [6], El-Affendi and Kouvatsos [7], Kouvatsos [8], and others. Ferdinand [4] used the maximum entropy method to derive the steady-state solutions for the ordinary M/M/1 queueing system. Shore [2] derive the steady-state and time-dependent distributions for the ordinary M/M/∞/N and M/M/∞ queueing systems by means of entropy maximisation. In the ordinary M/G/1 and G/G/1 queueing systems, Shore [3] used the maximum entropy method to derive the steady-state characteristics of the system, such as the expected number of customers in the system, the expected waiting time in the system, and etc. Arizono et al. [5] proposed an exact entropy model in order to develop the steady-state probability distributions of the number of customers in the ordinary M/M/S queueing system. Applying the method of maximum entropy to the ordinary GI/G/C queueing system, Wu and Chan [6] derived the approximate formulae for the steady-state probability distributions of the number of customers and the expected waiting time. El-Affendi and Kouvatsos [7] provided the maximum entropy formalism to analyse the ordinary M/G/1 and G/M/1 queueing systems. Based on the principle of maximum entropy, Kouvatsos [8] developed a closed form expression for the queue length distribution of the ordinary G/G/1 queueing system.

Yadin and Naor [9] first introduced the concept of an N policy which turns the server on whenever $N\,(N \geqslant 1)$ or more customers are present, turns the server off when there are no customers present. After the server is turned off, the server may not operate until N customers are present in the system. The N policy M/G/1 queueing system was first studied by Heyman [10] and was investigated by several researchers such as Bell [11,12], Kimura [13], Tijms [14], and others. Wang and Huang [15] developed exact steady-state solutions for the N policy M/E_k/1 queueing system. Recently, Wang and Ke [16] developed a recursive method, using the supplementary variable technique, to obtain the exact steady-state solutions for the N policy M/G/1 queueing system.

The purpose of this paper is:

(a) to construct the maximum entropy formalism for the N policy M/G/1 queueing system;
(b) to derive the maximum entropy approximate solutions for the N policy M/G/1 queueing system by using Lagrange's method;
(c) to obtain the approximate expected waiting time in the queue for the N policy M/G/1 queueing system;
(d) to perform a comparative analysis between the approximate results obtained through maximum entropy principle and the exact results obtained from Sivazlian and Stanfel [17], Wang and Huang [15], and Wang et al. [18].

1.1. Notations and probabilities

In this paper the following notations and probabilities are used.

N	threshold
$S(u)$	service time distribution function
$P_0(0)$	steady-state probability of no customers in the system when the server is turned off
$P_0(n)$	steady-state probability of n customers in the system when the server is turned off
$P_1(n)$	steady-state probability of n customers in the system when the server is turned on and working
λ	mean arrival rate
$1/\mu$	mean service time
ρ	traffic intensity, where $\rho = \lambda/\mu$. In the steady state $\rho < 1$
$E(S)$	first moment of the service time distribution
$E(S^2)$	second moment of the service time distribution
L_{on}	expected number of customers in the system when the server is turned on and working
L_{off}	expected number of customers in the system when the server is turned off
L_N	expected number of customers in the system
$E(W)$	exact expected waiting time in the queue
$E(W^*)$	approximate expected waiting time in the queue

2. The maximum entropy formalism

We consider a system Q which has a finite or countable infinite set B of all possible discrete states $B_0, B_1, B_2, \ldots, B_n, \ldots$ Let $P(B_n)$ represent the probability that the system Q is in state B_n. Following El-Affendi and Kouvatsos [7], we obtain the entropy function as follows:

$$H = -\sum_{B_n \in Q} P(B_n) \ln\{P(B_n)\}, \tag{1}$$

which is maximised subject to the following two constraints

$$\sum_{B_n \in Q} P(B_n) = 1, \tag{2}$$

and

$$\sum_{B_n \in Q} f_k P(B_n) = F_k, \quad k = 1, 2, \ldots, m \tag{3}$$

where $\{F_k\}$ denotes that the expected values defined on the set of several suitable functions $\{f_1(B_n), f_2(B_n), \ldots, f_m(B_n)\}$.

The maximisation of (1) subject to constraints (2) and (3) can be obtained using Lagrange's method of undetermined multipliers leading to the solution

$$P(B_n) = \exp\left[-\beta_0 - \sum_{k=1}^{m} \beta_k f_k(B_n)\right],$$

where β_0 is a Lagrangian multiplier determined by the normalisation constraint (2) and $\{\beta_k\}$, $k = 1, 2 \ldots, m$ are the Lagrangian multipliers determined from the set of constraints (3).

3. The maximum entropy solution

In this section, we will develop the maximum entropy solution for the steady-state probabilities of the N policy M/G/1 queueing system. There are three basic known results from the literature (see [10,14,16]) that facilitate the application of the maximum entropy formalism to study the N policy M/G/1 queueing system.

3.1. The three basic known constraints

For the N policy M/G/1 queueing system, three well-known results are stated as follows:

The first result is the probability of turning the server off given by

$$P_0(0) = P_0(1) = \cdots = P_0(N-1), \tag{4}$$

(see [16]).

The second result is the probability that the server is busy given by

$$\sum_{n=1}^{\infty} P_1(n) = \rho, \tag{5}$$

(see [10]).

The third result is the well-known formula for the expected number of customers in the system given by

$$L_N = \frac{N-1}{2} + \lambda E(S) + \frac{\lambda^2 E(S^2)}{2[1 - \lambda E(S)]}, \tag{6}$$

(see [14]).

3.2. The maximum entropy model

In order to develop the steady-state probabilities $P_0(n)$ and $P_1(n)$ by using maximum entropy method, we formulate the maximum entropy model as follows. According to (1) and (4)–(6), the entropy function H of the N policy M/G/1 queueing system is formed as

$$H = -\sum_{n=0}^{N-1} P_0(n) \ln P_0(n) - \sum_{n=1}^{\infty} P_1(n) \ln P_1(n),$$

or equivalently

$$H = -NP_0(0) \ln P_0(0) - \sum_{n=1}^{\infty} P_1(n) \ln P_1(n). \tag{7}$$

The maximum entropy solution for the N policy M/G/1 queueing system is obtained by maximising (7) subject to the following three constraints, written as,

(i) normalising condition:

$$\sum_{n=0}^{N-1} P_0(n) + \sum_{n=1}^{\infty} P_1(n) = NP_0(0) + \sum_{n=1}^{\infty} P_1(n) = 1, \tag{8}$$

(ii) the probability that the server is busy:

$$\sum_{n=1}^{N} P_1(n) = \rho \quad (\rho = \lambda/\mu), \tag{9}$$

(iii) the expected number of customers in the system:

$$\sum_{n=0}^{N-1} nP_0(n) + \sum_{n=1}^{\infty} nP_1(n) = \frac{N(N-1)}{2} P_0(0) + \sum_{n=1}^{\infty} nP_1(n) = L_N, \tag{10}$$

where L_N is given by (6).

In (8)–(10), (8) is multiplied by ω, (9) is multiplied by θ, and (10) is multiplied by ϕ. Thus the Lagrangian function h is given by

$$\begin{aligned} h = {} & -NP_0(0) \ln P_0(0) - \sum_{n=1}^{\infty} P_1(n) \ln P_1(n) - \omega \left[NP_0(0) + \sum_{n=1}^{\infty} P_1(n) - 1 \right] \\ & - \theta \left[\sum_{n=1}^{\infty} P_1(n) - \rho \right] - \phi \left[\frac{N(N-1)}{2} P_0(0) + \sum_{n=1}^{\infty} P_1(n) - L_N \right], \end{aligned} \tag{11}$$

where ω, θ, and ϕ are the Lagrangian multipliers corresponding to constraints (8)–(10), respectively.

To find the maximum entropy solutions $P_0(n)$ and $P_1(n)$, maximising in (7) subject to constraints (8)–(10) is equivalent to maximising (11).

The maximum entropy solutions are obtained by taking the partial derivatives of h with respect to $P_0(0)$ and $P_1(n)$, and setting the results equal to zero, namely,

$$\frac{\partial h}{\partial P_0(0)} = -N \ln P_0(0) - N - \omega N - \phi \frac{N(N-1)}{2} = 0, \tag{12}$$

and

$$\frac{\partial h}{\partial P_1(n)} = - \ln P_1(n) - 1 - \omega - \theta - \phi n = 0. \tag{13}$$

It follows from (12) and (13) that we obtain

$$P_0(0) = \mathrm{e}^{-(1+\omega)} \mathrm{e}^{\frac{-(N-1)\phi}{2}}, \tag{14}$$

and

$$P_1(n) = \mathrm{e}^{-(1+\omega)} \mathrm{e}^{\phi n} \mathrm{e}^{-\theta}, \quad n = 1, 2, \ldots \tag{15}$$

Let

$$\alpha = \mathrm{e}^{-(1+\omega)}, \quad \beta = \mathrm{e}^{-\phi}, \quad \text{and} \quad \gamma = \mathrm{e}^{-\theta}.$$

We transform (14) and (15) in terms of α, β, and γ given by

$$P_0(0) = \alpha\beta^{\frac{(N-1)}{2}}, \tag{16}$$

and

$$P_1(n) = \alpha\beta^n\gamma, \quad n = 1, 2, \ldots. \tag{17}$$

Substituting (16) and (17) into (8)–(10), respectively, yields

$$N\alpha\beta^{\frac{(N-1)}{2}} = 1 - \rho, \tag{18}$$

and

$$\sum_{n=1}^{\infty} \alpha\beta^n\gamma = \frac{\alpha\beta\gamma}{1-\beta} = \rho. \tag{19}$$

From (18), we get

$$\alpha = \frac{1-\rho}{N}\beta^{\frac{-(N-1)}{2}}. \tag{20}$$

After the algebraic manipulations, we obtain γ from (19) given by

$$\gamma = \frac{\rho(1-\beta)}{\alpha\beta}. \tag{21}$$

Substituting (20) into (21) finally gives

$$\gamma = \frac{N\rho}{1-\rho}\beta^{\frac{N-3}{2}}(1-\beta). \tag{22}$$

We substitute (20) and (22) into (16) and (17), respectively, yielding

$$P_0(0) = \frac{1-\rho}{N}, \tag{23}$$

and

$$P_1(n) = \rho(1-\beta)\beta^{n-1}, \quad n = 1, 2, \ldots \tag{24}$$

The expected number of customers in the system when the server is turned off is given

$$L_{\text{off}} = \sum_{n=0}^{N-1} nP_0(n) = \frac{N(N-1)}{2}P_0(0) = \frac{(N-1)(1-\rho)}{2}. \tag{25}$$

Note that $L_N = L_{\text{on}} + L_{\text{off}}$. From (25), the expected number of customers in the system when the server is turned on and working, L_{on}, is given by

$$L_{\text{on}} = \sum_{n=1}^{\infty} nP_1(n) = L_N - L_{\text{off}} = L_N - \frac{(N-1)(1-\rho)}{2}. \tag{26}$$

To determine the unknown Lagrangian multiplier β, substituting (24) into (26) finally gets

$$\beta = 1 - \frac{\rho}{L_{\text{on}}}. \tag{27}$$

We substitute (27) into (20) to determine the other unknown Lagrangian multiplier α as

$$\alpha = \frac{1-\rho}{N}\left(1-\frac{\rho}{L_{\text{on}}}\right)^{\frac{-(N-1)}{2}}. \tag{28}$$

Substituting (27) into (24), we finally obtain

$$P_1(n) = \frac{\rho^2}{L_{\text{on}}}\left(1-\frac{\rho}{L_{\text{on}}}\right)^{n-1}, \quad n = 1,2,\ldots \tag{29}$$

We first note that the results for the ordinary M/G/1 queueing system are obtained by setting $N = 1$. When $N = 1$ (that is, $L_{\text{on}} = L_N$), expression (29) for $P_1(n)$ reduces to a special case of expression (3.14) of El-Affendi and Kouvatsos [7] (p. 344). Next, we should note that the derived solutions $P_0(0)$ and $P_1(n)$ $(n \geqslant 1)$ satisfied constraints (8)–(10).

4. The exact and the approximate expected waiting time in the queue

Here we derive the exact and the approximate formulae for the expected waiting time in the queue for the N policy M/G/1 queueing system.

4.1. The exact expected waiting time in the queue

Let $E(W)$ denote the exact expected waiting time in the queue for the N policy M/G/1 queueing system. Using Little's formula, it follows that

$$E(W) = \frac{1}{\lambda}(L_N - \rho), \tag{30}$$

where L_N is given in (6).

4.2. The approximate expected waiting time in the queue

We define the idle state and the busy state as follows:

(i) Idle state denoted by I: the server is turned off and the number of customers waiting in the system is $\leqslant N-1$.
(ii) Busy state denoted by B: the server is busy and provides service to a customer.

We wish to find the mean arriving time and the mean service time for both states I and B. Let W denote that the time of a customer C waits in the queue until he starts his service. Suppose that the customer C finds n customers waiting in the queue for service in front of him, while the system is at any one of the states I and B are described, respectively, as follows:

(i) In idle state I: The server will be turned on after $(N-n-1)$ customers arrive in the system. Thus customer C will be served until $(N-n-1)$ customers arrive and n customers in front of him waiting for service. The mean arriving time of $(N-n-1)$ customers and the mean service time of n customers is given by $(N-n-1)/\lambda$ and n/μ, respectively.

(ii) In busy state B: Since the server is turned on, customer C only waits n customers in front of him to be served. The mean service time of n customers is n/μ.

Thus we obtain the approximate expected waiting time in the queue given by

$$E(W^*) = \sum_{n=0}^{N-1}\left[\frac{n}{\mu}+\frac{N-n-1}{\lambda}\right]P_0(0)+\sum_{n=1}^{\infty}\frac{n}{\mu}P_1(n), \tag{31}$$

where $P_0(0)$ and $P_1(n)$ are given in (23) and (29), respectively.

5. Comparative analysis between the exact and the approximate results

In this section, we present specific numerical comparisons between the exact results and the approximate results. This section includes the following three subsections:

(1) Comparative analysis between $E(W)$ and $E(W^*)$ to the N policy M/M/1 queueing system;
(2) Comparative analysis between $E(W)$ and $E(W^*)$ to the N policy M/E_k/1 queueing system;
(3) Comparative analysis between $E(W)$ and $E(W^*)$ to the N policy M/H_2/1 queueing systems.

5.1. Comparative analysis between $E(W)$ and $E(W^*)$ to the N policy M/M/1 queueing system

Here, we perform a comparative analysis between $E(W)$ and $E(W^*)$ for the N policy M/M/1 queueing system.

From Sivazlian and Stanfel [17], we obtain

$$L_N = \frac{N-1}{2}+\frac{\rho}{1-\rho}. \tag{32}$$

It implies from (30) that

$$E(W) = \frac{1}{\lambda}\left[\frac{N-1}{2}+\frac{\rho^2}{1-\rho}\right]. \tag{33}$$

Substituting (26) and (32) into (29), we finally get

$$P_1(n) = \rho\left[\frac{1}{\frac{N-1}{2}+\frac{1}{1-\rho}}\right]\left[1-\frac{1}{\frac{N-1}{2}+\frac{1}{1-\rho}}\right]^{n-1}, \quad n=1,2,\ldots \tag{34}$$

Substituting (23) and (34) into (31) yields

$$E(W^*) = \frac{1}{\lambda}\left[\frac{N-1}{2}+\frac{\rho^2}{1-\rho}\right]. \tag{35}$$

It is interesting to note that the approximate result $E(W^*)$ obtained in (35) is identical to the exact result $E(W)$ obtained in (33).

5.2. Comparative analysis between $E(W)$ and $E(W^*)$ to the N policy $M/E_k/1$ queueing system

Here, we perform a comparative analysis between $E(W)$ and $E(W^*)$ for the N policy $M/E_k/1$ queueing system.

From Wang and Huang [15], we have

$$L_N = \frac{N-1}{2} + \frac{\rho(\rho - k\rho + 2k)}{2k(1-\rho)}. \tag{36}$$

Thus we get

$$L_{on} = L_N - L_{off} = L_N - \frac{(N-1)(1-\rho)}{2}. \tag{37}$$

From (30), we have

$$E(W) = \frac{1}{\lambda}\left[\frac{N-1}{2} + \frac{(1+k)\rho^2}{2k(1-\rho)}\right]. \tag{38}$$

Substituting (23) and (29) into (31), it finally gets

$$E(W^*) = \frac{1+\rho}{\lambda}L_N - \frac{1}{\lambda}L_{on} = \frac{\rho}{\lambda}L_N + \frac{1}{\lambda}L_{off}. \tag{39}$$

We choose $\mu = 1.0$, and varying the values of $\lambda\ (= \rho)$ for two cases (i) $N = 5$ and (ii) $N = 10$. We perform a comparative analysis for the expected waiting time in the queue for the N policy $M/E_2/1$ queueing system between the approximate results obtained through maximum entropy principle, and the exact analytic results obtained from Wang and Huang [15]. Comparison between the approximate results and the exact results are shown in Tables 1 and 2, for cases (i) and (ii), respectively. The relative error percentages are very small (0–4%).

5.3. Comparative analysis between $E(W)$ and $E(W^*)$ to the N policy $M/H_2/1$ queueing system

Here, we perform a comparative analysis between $E(W)$ and $E(W^*)$ for the N policy $M/H_2/1$ queueing system.

From Wang et al. [18], we get

$$L_N = \frac{N-1}{2} + \rho + \frac{q_1\rho_1^2 + q_2\rho_2^2}{1-\rho}, \tag{40}$$

where $q_1 + q_2 = 1$, $\rho_1 = \lambda/\mu_1$, $\rho_2 = \lambda/\mu_2$, and $\rho = q_1\rho_1 + q_2\rho_2$.

Table 1
Comparison between the approximate results and the exact results for the N policy $M/E_2/1$ queueing system ($N = 5, \mu = 1$)

$\lambda(\rho)$	$E(W^*)$	$E(W)$	% Error
0.1	20.0833	20.1083	0.12
0.2	10.1875	10.2375	0.49
0.4	5.5000	5.6000	1.82
0.6	4.4583	4.6083	3.36
0.8	5.5000	5.7000	2.51
0.9	8.9722	9.1972	2.86

Table 2
Comparison between the approximate results and the exact results for the N policy M/E_2/1 queueing system ($N = 10$, $\mu = 1$)

$\lambda(\rho)$	$E(W^*)$	$E(W)$	% Error
0.1	45.0833	45.1083	0.05
0.2	22.6875	22.7375	0.22
0.4	11.7500	11.8500	0.85
0.6	8.6250	8.7750	1.74
0.8	8.6250	8.8250	2.32
0.9	11.7500	11.9750	1.91

From (30) and (40), the exact result $E(W)$ can be calculated.

Substituting (23) and (29) into (31) again, we have

$$E(W^*) = \sum_{n=0}^{N-1}\left[\frac{n}{\mu}+\frac{N-n-1}{\lambda}\right]\frac{1-\rho}{N}+\sum_{n=1}^{\infty}\frac{n}{\mu}\frac{\rho^2}{L_{\text{on}}}\left(1-\frac{\rho}{L_{\text{on}}}\right)^{n-1}, \tag{41}$$

where $1/\mu = q_1/\mu_1 + q_2/\mu_2$, and L_{on} is given in (26).

We choose $q_1 = 0.4$, $q_2 = 0.6$, $\mu_1 = 0.8$, $\mu_2 = 1.0$, and varying the values of λ for two cases (i) $N = 5$ and (ii) $N = 10$. We perform a comparative analysis for the expected waiting time in the queue for the N policy M/H_2/1 queueing system between the approximate results obtained

Table 3
Comparison between the approximate results and the exact results for the N policy M/H_2/1 queueing system ($N = 5$, $q_1 = 0.4$, $q_2 = 0.6$, $\mu_1 = 0.8$, $\mu_2 = 1$)

λ	ρ	$E(W^*)$	$E(W)$	% Error
0.1	0.11	20.1376	20.1361	0.007
0.2	0.22	10.3141	10.3111	0.029
0.4	0.44	5.875	5.8690	0.100
0.6	0.66	5.4951	5.4861	0.160
0.8	0.88	10.6667	10.6547	0.111
0.9	0.99	112.4722	112.4587	0.012

Table 4
Comparison between the approximate results and the exact results for the N policy M/H_2/1 queueing system ($N = 10$, $q_1 = 0.4$, $q_2 = 0.6$, $\mu_1 = 0.8$, $\mu_2 = 1$)

λ	ρ	$E(W^*)$	$E(W)$	% Error
0.1	0.11	45.1376	45.1361	0.003
0.2	0.22	22.8141	22.8111	0.013
0.4	0.44	12.1250	12.1190	0.049
0.6	0.66	9.6618	9.6528	0.093
0.8	0.88	13.7917	13.7797	0.087
0.9	0.99	115.250	115.2365	0.012

through maximum entropy principle, and the exact results obtained from Wang et al. [18]. Comparison between the approximate results and the exact results are shown in Tables 3 and 4, for cases (i) and (ii), respectively. Again, the relative error percentages are very small (0–0.2%).

6. Conclusions

In this paper, we have developed approximate steady-state solutions for the N policy M/G/1 queueing system by using maximum entropy principle. A comparative analysis of approximate results with exact results has shown that the relative error percentages are very small (below 4%). The numerical results indicate that the use of maximum entropy principle is accurate enough for practical purposes. One observes from Tables 1–4 that the maximum entropy approximation provides very good results for the N policy M/M/1, the N policy M/E_2/1, and the N policy M/H_2/1 queueing systems. Through the maximum entropy principle study, we demonstrate that the N policy M/G/1 queueing system is sufficiently robust to the variations of service time distribution functions.

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