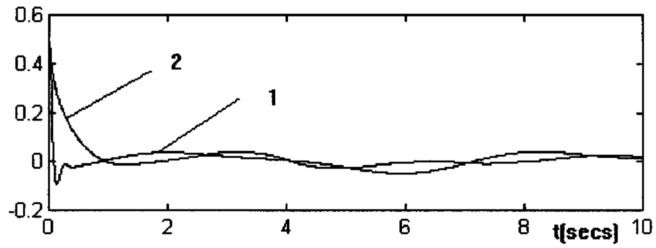


Fig. 1. The tracking error $y(t) - y^*(t)$. Curve 1 … our controller; Curve 2 … Kanellakopoulous' adaptive controller.

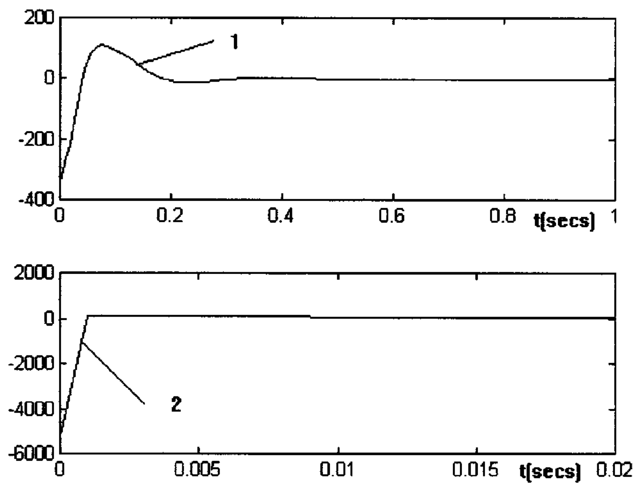


Fig. 2. The control $u$. Curve 1…our controller; Curve 2…Kanellakopoulous' adaptive controller.

that, by choosing $\beta$, $\beta_1$, $\beta_2$ properly, our robust controller seems to be more promising than Kanellakopoulous' adaptive one although it can no longer achieve asymptotic tracking.

## VI. Conclusion

We have designed a robust output-feedback controller with a very simple structure for a class of nonlinear uncertain systems. The global boundness of all closed-loop signals can be guaranteed and the output tracking error can be made arbitrarily small if the controller constants are chosen large enough. We find in our simulations, for a second order system, besides its simplicity, our robust controller is also superior to Kanellakopoulous' adaptive one in a sense that it achieves similar tracking performance with much less effort in the transient period. We have also found in our simulations that we can achieve similar performance with much less control effort if we choose a better choice of the positive design functions. Therefore, it is very important to find a suitable choice of the positive design functions we introduced. More effort will be made in this direction in our future work.

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# Optimal Minimal-Order Least-Squares Estimators Via the General Two-Stage Kalman Filter

Chien-Shu Hsieh and Fu-Chuang Chen

***Abstract*—A direct derivation of the optimal minimal-order least-squares estimator (OMOLSE) is presented using the recently developed general two-stage Kalman filter (GTSKF). Using this new result, the reduced-order estimators of O'Reilly and Fairman are readily shown to be equivalent. A practical implementation issue to consider these two estimators is also addressed.**

***Index Terms*—Least-squares estimator, minimal-order estimator, reduced-order estimator, two-stage Kalman filter.**

## I. Introduction

The problem of estimating the system state in a discrete-time linear stochastic system having partially noise free measurements is considered. It is solved by the well-known Kalman filter (KF). However, to reduce the computational burden of the KF, researchers have tried to incorporate Luenberger observers (LO) [1]. Aoki and Huddle [2] and Brammer [3] first suggested the use of a LO to solve this problem. Other researchers have also contributed to this area, e.g., Tse and Athans [4], Tse [5], Yoshikawa [6], Leondes and Novak [7], Fairman [8], [9], Fogel and Huang [10], and O'Reilly [11]. Among these, two approaches are considered in this note. Fairman [8], [9] proposed to solve this problem by way of a so-called "hybrid estimator" consists of first introducing the LO, and then using the KF to estimate the states of the observer. It was claimed suboptimal by O'Reilly [11], [12]. Although this misleading result has been clarified by Fairman [13], however the derivation is somewhat complex. On the other hand, Yoshikawa [6] and O'Reilly [11] presented a structurally simple minimal-order estimator. This approach as noted in [13] is to introduce the KF first and then modify it by using a LO. However, the result did not conform to standard KF forms. Furthermore, the gain

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computation involves the inversion of an $m \times m$ matrix, where $m$ is the measurement dimension.

The main goal of this note is to derive a new optimal minimal-order least-squares estimator (OMOLSE) directly via the recently developed general two-stage Kalman filter (GTSKF) [16]. The GTSKF consists of two reduced-order filters and is equivalent to the KF. By a suitable preprocessing of the measurement matrix, it is shown that one reduced-order filter of the GTSKF generates the noise-free measurements, which represents part of the filtered state, and the other filter generates the remaining state. Unlike the derivations in [8] and [11], the GTSKF is obtained by directly applying a two-stage $U$–$V$ transformation to the KF. Since this two-stage transformation is nonsingular, the optimality of the proposed OMOLSE is self-evident. Furthermore, it is shown that the above-mentioned two minimal-order estimators are equivalent under the suitable preprocessing of the measurement matrix. Thus, a second goal is to show that the Fairman's hybrid estimator [13] is equivalent to the minimal-order estimator of O'Reilly [11]. To this end, the classical sequential measurement update equations [14] is used to establish this connection.

This note is organized as follows. The problem and the previous proposed GTSKF are stated in Section II. In Section III, we derive the OMOLSE directly from the GTSKF. In Section IV, it is shown that the OMOLSE is equivalent to the Yoshikawa–O'Reilly estimator [11]. In Section V, the Fairman's hybrid estimator [13] is derived directly from the OMOLSE by using the classical sequential measurement update equations [14]. Section VI is the conclusion.

## II. The General Two-Stage Kalman Filter

Consider the following discrete-time system:

$$x_{k+1} = A_k x_k + B_k u_k + w_k \tag{1}$$

$$y_k = \begin{bmatrix} y_k^1 \\ y_k^2 \end{bmatrix} = C_k x_k + \begin{bmatrix} 0 \\ \eta_k \end{bmatrix} \tag{2}$$

where

| | |
|---|---|
| $x_k \in R^n$ | system state; |
| $u_k \in R^q$ | control input; |
| $y_k \in R^m$, $y_k^1 \in R^{m_1}$, and $y_k^2 \in R^{m_2}$ | measurement vectors. |

Matrices $A_k$, $B_k$, and $C_k$ have the appropriate dimensions (rank of $C_k$ is $m < n$). The process noise $w_k$ and the measurement noise $\eta_k$ are zero-mean white Gaussian sequences with the following covariances: $E[w_k(w_l)'] = Q_k\,\delta_{kl}$, $E[\eta_k(\eta_l)'] = R_k^{22}\,\delta_{kl}$, and $E[w_k(\eta_l)'] = 0$, where $'$ denotes transpose and $\delta_{kl}$ denotes the Kronecker delta function. The initial state $x_0$ is assumed to be uncorrelated with the white noise sequences $w_k$ and $\eta_k$, and is assumed to be Gaussian random variables with $E\{x_0\} = \overline{x}_0$ and $Cov\{x_0\} = \overline{P}_0$.

It is well-known that the KF can be used to produce the optimal state estimate. However, the computational cost and the numerical errors of the KF increase drastically with the state dimension. Hence, the KF may be impractical to implement. In such cases, reduced-order filters are preferable since there is no need to estimate those states which are known exactly. In this paper, we show that the recently proposed GTSKF [16] may serve as an alternative to solve this problem. The GTSKF is obtained by applying the following two-stage transformation:

$$T(M) = \begin{bmatrix} I_p & M \\ 0 & I_{n-p} \end{bmatrix} \tag{3}$$

where $p$ is chosen appropriately, to a standard KF. Then, the transformed filter becomes

$$\overline{x}_{k|k-1} = T(-U_k)x_{k|k-1} \tag{4}$$

$$\overline{x}_{k|k} = T(-V_k)x_{k|k} \tag{5}$$

$$\overline{P}_{k|k-1} = T(-U_k)P_{k|k-1}T'(-U_k) \tag{6}$$

$$\overline{K}_k = T(-V_k)K_k \tag{7}$$

$$\overline{P}_{k|k} = T(-V_k)P_{k|k}T'(-V_k) \tag{8}$$

where $\overline{x} = [(\overline{x}^1)' \quad (\overline{x}^2)']'$, $\overline{K} = [(\overline{K}^1)' \quad (\overline{K}^2)']'$, and $\overline{P} = \mathrm{diag}\{\overline{P}^1, \overline{P}^2\}$. The blending matrices $U_k$ and $V_k$ are left to be determined to make the predicted and the filtered covariances block diagonal, respectively.

Next, based on the *two-step iterative substitution* method of [15] and the following notations:

$$[T_k \quad E_k] = T(U_k), \quad [T_k \quad \overline{E}_k] = T(V_k)$$
$$\begin{bmatrix} H_k & N_k \\ L_k & M_k \end{bmatrix} = A_k T(V_k) \tag{9}$$

the transformed filter expressed by (4)–(8) can be recursively calculated by the following *two-stage decoupled subfilter one* (TSDSO):

$$\overline{x}_{k|k-1}^1 = H_{k-1}\overline{x}_{k-1|k-1}^1 + N_{k-1}\overline{x}_{k-1|k-1}^2 + B_{k-1}^1 u_{k-1} - U_k \overline{x}_{k|k-1}^2 \tag{10}$$

$$\overline{x}_{k|k}^1 = \overline{x}_{k|k-1}^1 + \overline{K}_k^1\left(y_k - C_k T_k \overline{x}_{k|k-1}^1\right) \tag{11}$$

$$\overline{P}_{k|k-1}^1 = H_{k-1}\overline{P}_{k-1|k-1}^1 H_{k-1}' + N_{k-1}\overline{P}_{k-1|k-1}^2 N_{k-1}' + Q_{k-1}^{11} - U_k\left(P_{k|k-1}^{12}\right)' \tag{12}$$

$$\overline{K}_k^1 = \overline{P}_{k|k-1}^1 (C_k T_k)'\left\{C_k T_k \overline{P}_{k|k-1}^1 (C_k T_k)' + R_k\right\}^{-1} \tag{13}$$

$$\overline{P}_{k|k}^1 = (I - \overline{K}_k^1 C_k T_k)\overline{P}_{k|k-1}^1 \tag{14}$$

and the following *two-stage decoupled subfilter two* (TSDST):

$$\overline{x}_{k|k-1}^2 = M_{k-1}\overline{x}_{k-1|k-1}^2 + L_{k-1}\overline{x}_{k-1|k-1}^1 + B_{k-1}^2 u_{k-1} \tag{15}$$

$$\overline{x}_{k|k}^2 = \overline{x}_{k|k-1}^2 + \overline{K}_k^2\left(y_k - C_k T_k \overline{x}_{k|k-1}^1 - C_k E_k \overline{x}_{k|k-1}^2\right) \tag{16}$$

$$\overline{P}_{k|k-1}^2 = M_{k-1}\overline{P}_{k-1|k-1}^2 M_{k-1}' + L_{k-1} \cdot \overline{P}_{k-1|k-1}^1 L_{k-1}' + Q_{k-1}^{22} \tag{17}$$

$$\overline{K}_k^2 = \overline{P}_{k|k-1}^2 (C_k E_k)'\left\{C_k E_k \overline{P}_{k|k-1}^2 (C_k E_k)' + C_k T_k \cdot \overline{P}_{k|k-1}^1 (C_k T_k)' + R_k\right\}^{-1} \tag{18}$$

$$\overline{P}_{k|k}^2 = (I - \overline{K}_k^2 C_k E_k)\overline{P}_{k|k-1}^2 \tag{19}$$

where

$$P_{k|k-1}^{12} = H_{k-1}\overline{P}_{k-1|k-1}^1 L_{k-1}' + N_{k-1} \cdot \overline{P}_{k-1|k-1}^2 M_{k-1}' + Q_{k-1}^{12} \tag{20}$$

$$R_k = \begin{bmatrix} 0 & 0 \\ 0 & R_k^{22} \end{bmatrix}. \tag{21}$$

The blending matrices are then given by

$$U_k = P_{k|k-1}^{12}\left(\overline{P}_{k|k-1}^2\right)^{-1} \tag{22}$$

$$V_k = U_k - \overline{K}_k^1 C_k E_k. \tag{23}$$

Based on the above two-stage covariance-decoupled subfilters, i.e., (10)–(19), it can be shown (see [16]) that the Kalman estimates can be reconstructed by the following GTSKF:

$$\hat{x}_{k|k} = T_k \overline{x}_{k|k}^1 + \overline{E}_k \overline{x}_{k|k}^2 \tag{24}$$

$$\hat{P}_{k|k} = T_k \overline{P}_{k|k}^1 T_k' + \overline{E}_k \overline{P}_{k|k}^2 \overline{E}_k'. \tag{25}$$

The initial condition and the efficiency of the GTSKF are given in [16].

The implication of the above result is that a standard KF can be decomposed into two covariance-decoupled reduced-order KFs by the two-stage decoupling technique. The advantage of using this new result is that the optimality of the obtained filter is guaranteed. One application of applying the GTSKF is to obtain the optimal modified stochastic Luenberger observer (OMSLO) [17], which can be used to derive the optimal minimal-order observer of Leondes and Novak [18]. In the following section, we show another application to apply the GTSKF to derive a new optimal minimal-order least-squares estimator, which is shown later to be equivalent to the Yoshikawa–O'Reilly estimator [11] and the Fairman's hybrid estimator [13].

## III. The Optimal Minimal-Order Least-Squares Estimator

It is well-known that the reduced-order filter is obtained by replacing part of state estimates directly by noise-free measurements. Thus, the reduced-order filter is readily obtained from the TSDST filter presented in the preceding section if we can verify that the output of the TSDSO filter generates the following:

$$\overline{x}^1_{k|k} = y^1_k \quad \overline{P}^1_{k|k} = 0. \tag{26}$$

This is achieved if $C_k$ and $p$ of the GTSKF are chosen by

$$C_k = \begin{bmatrix} I_{m_1} & 0 \\ C^{21}_k & C^{22}_k \end{bmatrix} \qquad p = m_1. \tag{27}$$

The first equation of (27) is always promised (see [11]). Then, one has

$$C_k T_k = \begin{bmatrix} I_{m_1} \\ C^{21}_k \end{bmatrix}, \qquad C_k E_k = \begin{bmatrix} U_k \\ C^{21}_k U_k + C^{22}_k \end{bmatrix}. \tag{28}$$

Using (22) and the first equation of (28), the TSDSO filter, i.e., (10)–(14), becomes

$$\begin{aligned}\overline{x}^1_{k|k-1} =& H_{k-1} y^1_{k-1} + N_{k-1} \overline{x}^2_{k-1|k-1} \\ &+ B^1_{k-1} u_{k-1} - U_k \overline{x}^2_{k|k-1}\end{aligned} \tag{29}$$

$$\overline{x}^1_{k|k} = y^1_k \tag{30}$$

$$\overline{P}^1_{k|k-1} = N_{k-1} \overline{P}^2_{k-1|k-1} N'_{k-1} + Q^{11}_{k-1} - U_k \overline{P}^2_{k|k-1} U'_k \tag{31}$$

$$\overline{K}^1_k = [\, I_{m_1} \quad 0 \,] \tag{32}$$

$$\overline{P}^1_{k|k} = 0. \tag{33}$$

The blending matrix $V_k$ in (23) is given, by using the second equation of (28) and (32), as

$$V_k = 0. \tag{34}$$

Using (34) in (9) yields

$$\begin{bmatrix} H_k & N_k \\ L_k & M_k \end{bmatrix} = A_k = \begin{bmatrix} A^{11}_k & A^{12}_k \\ A^{21}_k & A^{22}_k \end{bmatrix}. \tag{35}$$

Using the above results (28)–(35), the following notations:

$$F_k = \begin{bmatrix} I_{m1} \\ 0 \end{bmatrix}, \qquad D_k = \begin{bmatrix} 0 \\ I_{n-m1} \end{bmatrix}$$

$$\breve{S}_k = C_k E_k, \qquad z = \overline{x}^2 \quad P^z = \overline{P}^2 \quad \breve{K}^z = \overline{K}^2$$

the TSDST filter, i.e., (15)–(19), and (24)–(25), we propose the following optimal minimal-order least-squares estimator (OMOLSE):

$$\hat{x}_{k|k} = F_k y^1_k + D_k z_{k|k} \tag{36}$$

$$\hat{P}^x_{k|k} = D_k P^z_{k|k} D'_k \tag{37}$$

where

$$z_{k|k-1} = A^{22}_{k-1} z_{k-1|k-1} + A^{21}_{k-1} y^1_{k-1} + B^2_{k-1} u_{k-1} \tag{38}$$

$$z_{k|k} = z_{k|k-1} + \breve{K}^z_k \left( \breve{y}^z_k - \breve{S}_k z_{k|k-1} \right) \tag{39}$$

$$P^z_{k|k-1} = A^{22}_{k-1} P^z_{k-1|k-1} (A^{22}_{k-1})' + Q^{22}_{k-1} \tag{40}$$

$$\breve{K}^z_k = P^z_{k|k-1} \breve{S}'_k \left\{ \breve{S}_k P^z_{k|k-1} \breve{S}'_k + \breve{R}^z_k \right\}^{-1} \tag{41}$$

$$P^z_{k|k} = (I - \breve{K}^z_k \breve{S}_k) P^z_{k|k-1} \tag{42}$$

in which

$$\breve{y}^z_k = y_k - C_k T_k \overline{x}^1_{k|k-1} \tag{43}$$

$$\breve{R}^z_k = C_k T_k \overline{P}^1_{k|k-1} (C_k T_k)' + R_k. \tag{44}$$

To facilitate the following derivation, we show that the above OMOLSE can be further simplified. First, we rewrite the gain equation of the OMOLSE, i.e., (41), by using (42) as follows:

$$\breve{K}^z_k = P^z_{k|k} \breve{S}'_k (\breve{R}^z_k)^{-1}. \tag{45}$$

Next, let the LDU decomposition of the measurement noise covariance $\breve{R}^z_k$ be given as

$$\breve{R}^z_k = \begin{bmatrix} I & 0 \\ C^{21}_k & I \end{bmatrix} \begin{bmatrix} \overline{P}^1_{k|k-1} & 0 \\ 0 & R^{22}_k \end{bmatrix} \begin{bmatrix} I & (C^{21}_k)' \\ 0 & I \end{bmatrix}. \tag{46}$$

Substituting (46) into (45), we obtain

$$\breve{K}^z_k = K^z_k \begin{bmatrix} I & 0 \\ -C^{21}_k & I \end{bmatrix} \tag{47}$$

where

$$K^z_k = P^z_{k|k} S'_k \bullet \text{diag} \left\{ \left( \overline{P}^1_{k|k-1} \right)^{-1}, \, (R^{22}_k)^{-1} \right\} \tag{48}$$

$$S_k = [\, U'_k \quad (C^{22}_k)' \,]' \tag{49}$$

$$U_k = \left( A^{12}_{k-1} P^z_{k-1|k-1} (A^{22}_{k-1})' + Q^{12}_{k-1} \right) \left( P^z_{k|k-1} \right)^{-1}. \tag{50}$$

Substituting (47) into (39) and (42) yields

$$z_{k|k} = z_{k|k-1} + K^z_k \left( y^z_k - S_k z_{k|k-1} \right) \tag{51}$$

$$P^z_{k|k} = (I - K^z_k S_k) P^z_{k|k-1} \tag{52}$$

where

$$\begin{aligned} y^z_k = y_k &+ \begin{bmatrix} U_k z_{k|k-1} - A^{12}_{k-1} z_{k-1|k-1} \\ 0 \end{bmatrix} \\ &- \begin{bmatrix} A^{11}_{k-1} y^1_{k-1} + B^1_{k-1} u_{k-1} \\ 0 \end{bmatrix}. \end{aligned} \tag{53}$$

Using (52), (48) can be rewritten as follows:

$$K^z_k = P^z_{k|k-1} S'_k \left\{ S_k P^z_{k|k-1} S'_k + R^z_k \right\}^{-1} \tag{54}$$

where

$$\begin{aligned} R^z_k = \text{diag} \big\{ & A^{12}_{k-1} P^z_{k-1|k-1} (A^{12}_{k-1})' + Q^{11}_{k-1} \\ & - U_k P^z_{k|k-1} U'_k, \; R^{22}_k \big\}. \end{aligned} \tag{55}$$

At last, through (51), (52), and (54), we can obtain the following compact OMOLSE:

$$\hat{x}_{k|k} = F_k y^1_k + D_k z_{k|k} \tag{56}$$

$$\hat{P}^x_{k|k} = D_k P^z_{k|k} D'_k \tag{57}$$

where

$$z_{k|k-1} = A^{22}_{k-1} z_{k-1|k-1} + A^{21}_{k-1} y^1_{k-1} + B^2_{k-1} u_{k-1} \tag{58}$$

$$z_{k|k} = z_{k|k-1} + K_k^z \left(y_k^z - S_k z_{k|k-1}\right) \tag{59}$$

$$P_{k|k-1}^z = A_{k-1}^{22} P_{k-1|k-1}^z \left(A_{k-1}^{22}\right)' + Q_{k-1}^{22} \tag{60}$$

$$K_k^z = P_{k|k-1}^z S_k' \left\{S_k P_{k|k-1}^z S_k' + R_k^z\right\}^{-1} \tag{61}$$

$$P_{k|k}^z = (I - K_k^z S_k) P_{k|k-1}^z. \tag{62}$$

*Remark:* The reduced-order filter, i.e., (58)–(62), is a standard KF corresponding to the following reduced-order system:

$$z_{k+1} = A_k^{22} z_k + [\, A_k^{21} \quad B_k^2 \,]\, [\, (y_k^1)' \quad (u_k)' \,]' + w_k^z$$
$$y_k^z = S_k z_k + w_k^y$$

where $z_k \in R^{n-m_1}$ is the reduced-order system state, $[(y_k^1)' \; (u_k)']' \in R^{m_1+q}$ is the new control input, and $y_k^z \in R^m$ is the new measurement vector. The new process noise $w_k^z$ and the new measurement noise $w_k^y$ are zero-mean white Gaussian sequences with the following covariances: $E[w_k^z (w_l^z)'] = Q_k^{22} \delta_{kl}$, $E[w_k^y (w_l^y)'] = R_k^z \delta_{kl}$, and $E[w_k^z (w_l^y)'] = 0$.

## IV. Connection With the Yoshikawa–O'Reilly Estimator

To connect the OMOLSE filter with the Yoshikawa–O'Reilly estimator [11], we first introduce four auxiliary variables $z_{k|k}$, $z_{k|k-1}$, $\Sigma_{k|k-1}^z$, and $K_k^z$ as follows:

$$z_{k|k} = \hat{z}(k), \qquad z_{k|k-1} = D(k)\hat{x}\,(k|k-1)$$
$$\Sigma_{k|k-1}^z = D(k)\Sigma\,(k|k-1)\; D'(k), \qquad K_k^z = G(k) \tag{63}$$

where $\hat{z}(k)$, $\hat{x}(k|k-1)$, $\Sigma(k|k-1)$, and $D(k)$ are defined in [11]. Using (63), one can reformulate the Yoshikawa–O'Reilly estimator, i.e., [11, eqs. (11)–(13) and (15)–(17)], as follows:

$$z_{k|k} = z_{k|k-1} + K_k^z \left(y_k - C_k \hat{x}\,(k|k-1)\right) \tag{64}$$

$$z_{k|k-1} = A_{k-1}^{22} z_{k-1|k-1} + A_{k-1}^{21} y_{k-1}^1 + B_{k-1}^2 u_{k-1} \tag{65}$$

$$\hat{x}\,(k|k) = F_k y_k^1 + D_k z_{k|k} \tag{66}$$

$$K_k^z = D(k)\Sigma(k|k-1)\, C_k' \left\{C_k \Sigma(k|k-1)\, C_k' + R_k\right\}^{-1} \tag{67}$$

$$\Sigma_{k|k-1}^z = A_{k-1}^{22} \Sigma_{k-1|k-1}^z (A_{k-1}^{22})' + Q_{k-1}^{22} \tag{68}$$

$$\Sigma_{k|k}^z = \Sigma_{k|k-1}^z - K_k^z C_k \Sigma(k|k-1)\; D'(k). \tag{69}$$

Note that in deriving the above results, [11, (9), (33)] are assumed. Then, it is easily verifying the following identities:

$$y_k - C_k \hat{x}\,(k|k-1) = y_k^z - S_k z_{k|k-1} \tag{70}$$

$$C_k \Sigma\,(k|k-1)\; D'(k) = S_k P_{k|k-1}^z \tag{71}$$

$$C_k \Sigma\,(k|k-1)\, C_k' + R_k = S_k P_{k|k-1}^z S_k' + R_k^z \tag{72}$$

where $y_k^z$, $S_k$, $P_{k|k-1}^z$, and $R_k^z$ are given by (53), (49), (40), and (55), respectively. Using the above identities (70)–(72), the equivalence of the Yoshikawa–O'Reilly estimator [(64)–(69)] and the proposed OMOLSE [(56)–(62)] is established.

One advantage of expressing the Yoshikawa–O'Reilly estimator [11] in the form (56)–(62) is that it conforms to the standard KF form. Thus, the optimality of the reduced-order estimator is readily obtained. Even more important than the above is that it provides a simple way to establish the equivalence of the Yoshikawa–O'Reilly estimator [11] and the Fairman's hybrid estimator [13]. This gives an alternative proof other than [13] to evaluate the optimality, in the minimum-mean-square-error (MMSE) sense, of the latter; furthermore, we show that the latter is more efficient and stable than the former in view of computational complexity and numerical accuracy, respectively, since the smaller matrix inverse is involved.

## V. The Fairman's Hybrid Estimator

Although the Yoshikawa–O'Reilly estimator [11] is simple, it involves the inversion of an $m \times m$ matrix in the gain equation, where $m$ represents the measurement dimension. This may cause numerical problems if the measurement number is large; furthermore, the computational load may be excessive. In fact, these disadvantages can be avoided if the gain equation involves a smaller matrix inverse. In this section, we show that this aim can be achieved by applying the classical sequential measurement update technique [14], and the obtained simplified estimator is shown to be equivalent to the Fairman's hybrid estimator [13].

Owing to that the measurement noise covariance in (55) is in block diagonal form, the reduced-order filter (58)–(62) can be further simplified by applying the following two stages measurement update equations [14]:

Stage 1:

$$\tilde{z}_{k|k} = (A_{k-1}^{22} - \tilde{K}_k^z A_{k-1}^{12}) z_{k-1|k-1} + (A_{k-1}^{21} - \tilde{K}_k^z A_{k-1}^{11}) \bullet y_{k-1}^1 + (B_{k-1}^2 - \tilde{K}_k^z B_{k-1}^1) u_{k-1} + \tilde{K}_k^z y_k^1 \tag{73}$$

$$\tilde{K}_k^z = P_{k|k-1}^z U_k' \left\{A_{k-1}^{12} P_{k-1|k-1}^z (A_{k-1}^{12})' + Q_{k-1}^{11}\right\}^{-1} \tag{74}$$

$$\tilde{P}_{k|k}^z = (I - \tilde{K}_k^z U_k) P_{k|k-1}^z. \tag{75}$$

Stage 2:

$$\overline{z}_{k|k} = \tilde{z}_{k|k} + \overline{K}_k^z \left(y_k^2 - C_k^{22} \tilde{z}_{k|k}\right) \tag{76}$$

$$\overline{K}_k^z = \tilde{P}_{k|k}^z (C_k^{22})' \left\{C_k^{22} \tilde{P}_{k|k}^z (C_k^{22})' + R_k^{22}\right\}^{-1} \tag{77}$$

$$\overline{P}_{k|k}^z = (I - \overline{K}_k^z C_k^{22}) \tilde{P}_{k|k}^z. \tag{78}$$

At last, one has the optimal estimates $z_{k|k} = \overline{z}_{k|k}$ and $P_{k|k}^z = \overline{P}_{k|k}^z$ (for the proof, see [14, Th. 1]). Note that in deriving (73), the following equation is first obtained:

$$\tilde{z}_{k|k} = z_{k|k-1} + \tilde{K}_k^z \cdot \left(y_k^1 - A_{k-1}^{12} z_{k-1|k-1} - A_{k-1}^{11} y_{k-1}^1 - B_{k-1}^1 u_{k-1}\right)$$

and then applied (38). Furthermore, the matrix $U_k$ in (74) and (75) is redundant and can be eliminated as follows. Using (50) in (74) and (75) yields

$$\tilde{K}_k^z = \left(A_{k-1}^{22} P_{k-1|k-1}^z (A_{k-1}^{12})' + Q_{k-1}^{21}\right) \bullet \left\{A_{k-1}^{12} P_{k-1|k-1}^z (A_{k-1}^{12})' + Q_{k-1}^{11}\right\}^{-1} \tag{79}$$

$$\tilde{P}_{k|k}^z = (A_{k-1}^{22} - \tilde{K}_k^z A_{k-1}^{12}) P_{k-1|k-1}^z (A_{k-1}^{22})' + Q_{k-1}^{22} - \tilde{K}_k^z Q_{k-1}^{12}. \tag{80}$$

Next, using the following notations:

$$\Omega_k = \tilde{K}_k^z \quad \Sigma_{k|k-1}^z = \tilde{P}_{k|k}^z$$
$$\Sigma_{k|k}^z = \overline{P}_{k|k}^z \quad G_k = \overline{K}_k^z \tag{81}$$

and introducing auxiliary variables $\hat{z}_{k|k-1}$ and $\hat{z}_{k|k}$ such that

$$\tilde{z}_{k|k} = \hat{z}_{k|k-1} + \Omega_k y_k^1 \tag{82}$$

$$\overline{z}_{k|k} = \hat{z}_{k|k} + \Omega_k y_k^1 \tag{83}$$

one can reformulate the compact OMOLSE filter [(56)–(62)] by using (73)–(78) as

$$\hat{x}_{k|k} = \tilde{F}_k y_k^1 + D_k \hat{z}_{k|k} \tag{84}$$

$$\hat{P}_{k|k}^x = D_k \Sigma_{k|k}^z\, D_k' \tag{85}$$

where

$$\hat{z}_{k|k-1} = (A_{k-1}^{22} - \Omega_k A_{k-1}^{12})\hat{z}_{k-1|k-1} + (A_{k-1}^{21} - \Omega_k A_{k-1}^{11}) \bullet y_{k-1}^1 + (B_{k-1}^2 - \Omega_k B_{k-1}^1)u_{k-1} \quad (86)$$

$$\hat{z}_{k|k} = \hat{z}_{k|k-1} + G_k \left(\tilde{y}_k^z - C_k^{22}\hat{z}_{k|k-1}\right) \quad (87)$$

$$\Sigma_{k|k-1}^z = (A_{k-1}^{22} - \Omega_k A_{k-1}^{12})\Sigma_{k-1|k-1}^z (A_{k-1}^{22})' + Q_{k-1}^{22} - \Omega_k Q_{k-1}^{12} \quad (88)$$

$$G_k = \Sigma_{k|k-1}^z (C_k^{22})' \left\{C_k^{22}\Sigma_{k|k-1}^z (C_k^{22})' + R_k^{22}\right\}^{-1} \quad (89)$$

$$\Sigma_{k|k}^z = (I - G_k C_k^{22})\Sigma_{k|k-1}^z \quad (90)$$

$$\tilde{F}_k = [\, I_{m_1} \quad \Omega_k' \,]' \quad (91)$$

$$\tilde{y}_k^z = y_k^2 - C_k^{22}\Omega_k y_k^1 \quad (92)$$

$$\Omega_k = \left(A_{k-1}^{22}\Sigma_{k-1|k-1}^z (A_{k-1}^{12})' + Q_{k-1}^{21}\right) \bullet \left\{A_{k-1}^{12}\Sigma_{k-1|k-1}^z (A_{k-1}^{12})' + Q_{k-1}^{11}\right\}^{-1}. \quad (93)$$

Then, we show that (88) is equivalent to the following one:

$$\Sigma_{k|k-1}^z = (A_{k-1}^{22} - \Omega_k A_{k-1}^{12})\Sigma_{k-1|k-1}^z \cdot (A_{k-1}^{22} - \Omega_k A_{k-1}^{12})' + \Gamma_k Q_{k-1}\Gamma_k' \quad (94)$$

where $\Gamma_k = [-\Omega_k \quad I_{n-m_1}]$. Expanding (94) yields

$$\begin{aligned}\Sigma_{k|k-1}^z = &(A_{k-1}^{22} - \Omega_k A_{k-1}^{12})\Sigma_{k-1|k-1}^z (A_{k-1}^{22})' \\ &+ \Omega_k \left(A_{k-1}^{12}\Sigma_{k-1|k-1}^z (A_{k-1}^{12})' + Q_{k-1}^{11}\right)\Omega_k' \\ &- \left(A_{k-1}^{22}\Sigma_{k-1|k-1}^z (A_{k-1}^{12})' + Q_{k-1}^{21}\right)\Omega_k' \\ &+ Q_{k-1}^{22} - \Omega_k Q_{k-1}^{12}. \end{aligned} \quad (95)$$

Using (79) and (81), one obtains that the second and third lines of (95) equal to zero, and, hence, (94) is verified. Thus, the obtained simplified filter [(84)–(90)] is equivalent to that proposed by Fairman *et al.* [13].

In view of (61), (89), and (93), it is clear that the Fairman estimator [13] is numerically more efficient than the Yoshikawa–O'Reilly estimator [11] since the smaller matrix inverse is involved. Comparing (56) and (84), it is clear from the above derivation that the significance of the coordination used by Fairman, e.g., $P_{21}(k)$ in [11, (36)] and $\Omega_k$ in this paper, is that it transforms the original measurement equation (2) into a new one (also see [13, (14)]) which facilitates the further derivation for reducing the computational complexity of the reduced-order estimator.

## VI. Conclusion

This note derives the OMOLSE via the GTSKF. Using this new filter, the equivalence of the Yoshikawa–O'Reilly estimator [11] and the Fairman's hybrid estimator [13] is established. A practical implementation issue to consider these two estimators is addressed. It is shown that the former can be further simplified to the latter by using the classical sequential measurement update equations [14]. Furthermore, the significance of the coordination used by Fairman's hybrid estimator is also addressed.

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