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Method for solving quasi-concave and non-concave fuzzy multi-objective programming problems

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Abstract

This paper proposes a method based on linear programming techniques to treat quasi-concave and non-concave fuzzy multi-objective programming (FMOP) problems. The proposed method initially presents a piecewise linear expression to interpreting a quasi-concave membership function. Then we find the convex-type break points and transform all quasi-concave membership functions into concave functions. After that, the converted program is solved by linear programming techniques to obtain a global optimum. In addition to not containing any of the zero–one variables, the proposed method does not require dividing the quasi-concave FMOP problem into large sub-problems as in conventional methods. The extension of the proposed method can treat general non-concave FMOP problems by merely adding less number of zero–one variables. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Since Zimmermann first introduced conventional linear programming and multi-objective linear programming into fuzzy set theory [14,15], various methods using linear programming (LP) have been developed to solve fuzzy multi-objective programming (FMOP) problems. Many studies [1–3,5,8–11,13] indicate in practice that most applications in engineering, physical, business, social, and management fields are not pure linear, triangular, concave, or convex FMOP problems but rather are quasi-concave or more general non-concave FMOP problems. One of the most promising techniques of linearizing non-concave functions is the piecewise linear programming. Hence, FMOP problems with piecewise linear membership functions has been studied by Narasimham [10], Hannan [2], Nakamura [9], Inuiguchi et al. [4], and Yang et al. [13]. In general, an FMOP

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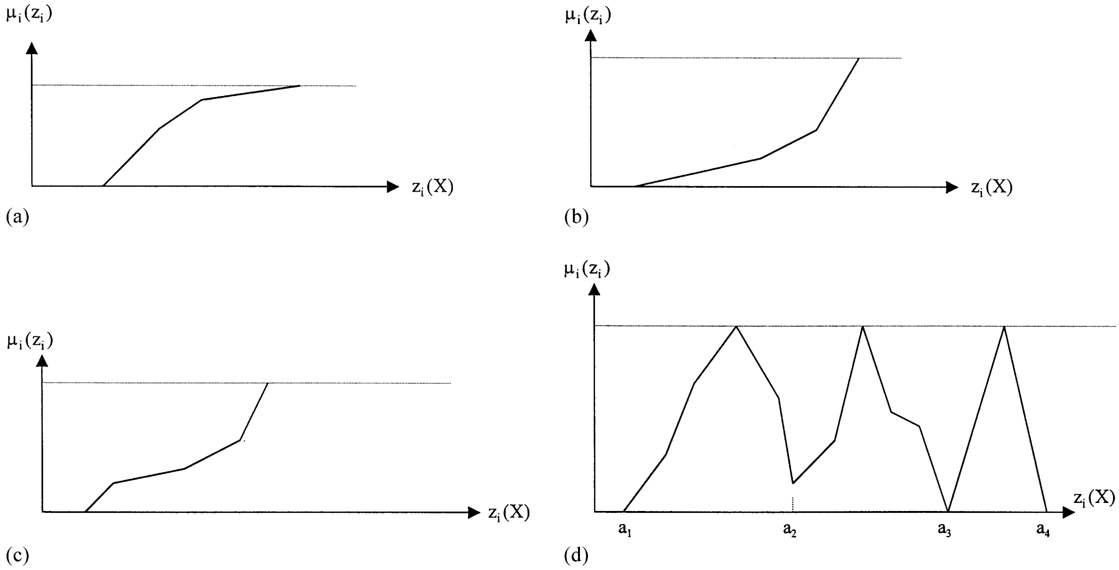

Fig. 1. (a) A concave membership function. (b) A convex membership function. (c) A concave–convex mixed membership function. (d) A more general non-concave membership function.

problem, in which the aggregated goal is the minimum operator of individual goals, is widely formulated as follows:

FMOP Problem

$$\begin{aligned} &\text{Maximize} \quad \lambda \\ &\text{Subject to} \quad \lambda \leqslant \mu_i(z_i), \quad i = 1, 2, \ldots, n, \\ &\qquad\qquad\quad \mu_i(z_i) = |z_i(X) - g_i|, \ z_i(X) \in F \ (\text{a feasible set}), \end{aligned} \tag{1.1}$$

where $\mu_i(z_i)$ is the membership function of ith objective function, g_i denotes the fuzzy goal of ith objective function, $z_i(X)$ is the ith objective function, and X is a vector of decision variables.

A membership function $\mu_i(z_i)$ may be concave-shaped or convex-shaped, as shown in Figs. 1(a) and (b) respectively. The marginal possibility with respect to a concave membership function is decreasing, whereas the marginal possibility with respect to a convex membership function is increasing. If the marginal possibility increases first and then decreases, or decreases first and then increases, then the membership function becomes a convex–concave or concave–convex mixed shape as shown in Fig. 1(c). Many empirical evidences [1–3,5,8–11,13] indicated that membership functions are not concave or convex but the mixed shapes composed by concave and convex curves or even more general non-concave curves as shown in Fig. 1(d).

Conventional FMOP methods [2,9,4,10,13] for solving Problem (1.1), however, have some disadvantages as discussed below:

(i) Conventional methods lack a clear and simple way to represent a general piecewise membership function $\mu_i(z_i)$. Most methods use complicated expressions to represent a quasi-concave membership function.

(ii) Narasimham's method [10] and Hannan's method [2] can only solve FMOP problems where all $\mu_i(z_i)$ are triangular or concave functions.

(iii) Nakamura's method [4] needs to divide the original quasi-concave FMOP problem into $2^{\sum_{i=1}^{n} m_i}$ sub-problems, where m_i is the number of intersections between concave and convex functions in $\mu_i(z_i)$, then uses LP to solve these sub-problems repeatedly.
(iv) Inuiguchi et al.'s method [4] involves tedious process of computing all break points for transforming entire original membership functions into new membership functions. If the number of break points in the membership functions is large, then it causes tiresome computational burden to convert these membership functions into concave functions.
(v) Yang et al.'s method [4] requires adding $\sum_{i=1}^{n} m_i$ zero–one variables in their model for solving a quasi-concave FMOP problem, where m_i represents the number of intersections between concave and convex functions in $\mu_i(z_i)$.
(vi) These methods are difficult to treat more general non-concave FMOP problems as shown in Fig. 1(d).

This paper proposes a method based on LP techniques to solve an FMOP problem in (1.1). The features of the proposed method are listed below:

(i) It uses a more convenient and clear way to express general piecewise membership functions such as quasi-concave shape.
(ii) It utilizes LP techniques to directly solve a piecewise quasi-concave FMOP problem without adding any zero–one variables or dividing the problem into several sub-problems.
(iii) It can also solve a more general piecewise non-concave FMOP problem by adding only one zero–one variable between two quasi-concave functions.

2. Review of conventional FMOP models

Several commonly used approaches for solving a FMOP problem in (1.1) are briefly reviewed in this section. In 1980, Narasimham [10] first proposed a LP approach to solving an FMOP problem with triangular membership functions. However, two primary drawbacks exist in Narasimham's method. First, an FMOP problem has to be divided into 2^n sub-problems where n is the number of fuzzy goals. Second, all membership functions are restricted to triangular or trapezoidal shapes.

Extending triangular or trapezoidal to general concave shaped membership functions, Hannan [2] presented a piecewise linear function $\sum_{j=1}^{N_i} \alpha_{ij}|z_i - g_{ij}| + \beta_i z_i + r_i$ to interpret a concave membership function $\mu_i(z_i)$. Where $|z_i - g_{ij}| = d_{ij}^- + d_{ij}^+$, g_{ij} are the change points of segments, d_{ij}^- and d_{ij}^+ are deviation variables, α_{ij}, β_i, and r_i are parameters. The serious limitation in Hannan's method is that all $\mu_i(z_i)$ should be concave functions.

For tackling a quasi-concave FMOP problem, Inuiguchi et al. [4] developed a approach of transforming all quasi-concave functions into concave functions. Consider the following example slightly modified from Inuiguchi et al. [4].

Example 1

$$
\begin{aligned}
&\text{Maximize} && \lambda \\
&\text{Subject to} && \lambda \leqslant \mu_1(z_1), \quad \lambda \leqslant \mu_2(z_2), \\
& && z_1 = -x_1 + 2x_2, \quad z_2 = 2x_1 + x_2, \\
& && -x_1 + 3x_2 \leqslant 21, \quad x_1 + 3x_2 \leqslant 27, \\
& && 4x_1 + 3x_2 \leqslant 45, \quad 3x_1 + x_2 \leqslant 30, \quad x_1, x_2 \geqslant 0,
\end{aligned}
$$

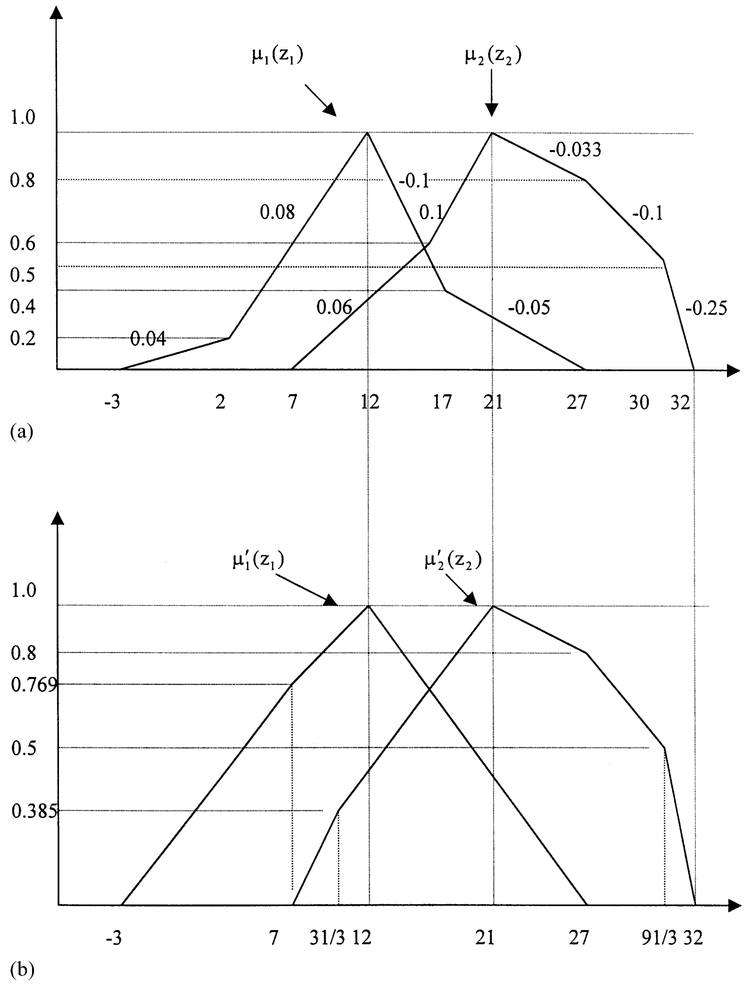

Fig. 2. (a) $\mu_1(z_1)$ and $\mu_2(z_2)$ in Example 1. (b) Two converted $\mu_1'(z_1)$ and $\mu_2'(z_2)$ in Inuiguchi et al. Model.

$$\mu_1(z_1) = \begin{cases} 0, & z_1 \leqslant -3, \\ 0.04z_1, & -3 \leqslant z_1 \leqslant 2, \\ 0.08z_1 + 0.2, & 2 \leqslant z_1 \leqslant 12, \\ 1, & z_1 = 12, \\ -0.1z_1 + 2.2, & 12 \leqslant z_1 \leqslant 17, \\ -0.05z_1 + 0.5, & 17 \leqslant z_1 \leqslant 27, \\ 0, & z_1 \geqslant 27, \end{cases} \qquad \mu_1(z_2) = \begin{cases} 0, & z_2 \leqslant 7, \\ 0.06z_2, & 7 \leqslant z_2 \leqslant 17, \\ 0.1z_2 + 0.6, & 17 \leqslant z_2 \leqslant 21, \\ 1, & z_2 = 21, \\ -0.033z_2 + 1.7, & 21 \leqslant z_2 \leqslant 27, \\ -0.1z_2 + 0.8, & 27 \leqslant z_2 \leqslant 30, \\ -0.25z_2 + 0.5, & 30 \leqslant z_2 \leqslant 32, \\ 0, & z_2 \geqslant 32, \end{cases}$$

where $\mu_1(z_1)$ and $\mu_2(z_2)$ are specified in Fig. 2(a).

Notably both $\mu_1(z_1)$ and $\mu_2(z_2)$ are quasi-concave functions, as depicted in Fig. 2(a). Inuiguchi et al. first convert $\mu_1(z_1)$ and $\mu_2(z_2)$ into two concave functions $\mu_1'(z_1)$ and $\mu_2'(z_2)$ respectively, as shown in Fig. 2(b). Example 1 then can be solved by the following LP model:

FMOP Model 1 (*Inuiguchi et al.'s* [4] *method for Example* 1)

$$\begin{aligned}&\text{Maximize} \quad \lambda'\\&\text{Subject to} \quad \lambda' \leqslant \mu_1'(z_1), \quad \lambda' \leqslant \mu_2'(z_2),\\&z_1 = -x_1 + 2x_2, \quad z_2 = 2x_1 + x_2,\\&-x_1 + 3x_2 \leqslant 21, \quad x_1 + 3x_2 \leqslant 27,\\&4x_1 + 3x_2 \leqslant 45, \quad 3x_1 + x_2 \leqslant 30, \quad x_1, x_2 \geqslant 0,\end{aligned}$$

$$\mu_1'(z_1) = \begin{cases} 0, & z_1 \leqslant -3,\\ \min(\frac{1}{13}z_1 + \frac{3}{13}, \frac{3}{65}z_1 + \frac{29}{65}), & -3 \leqslant z_1 \leqslant 12,\\ 1, & z_1 = 12,\\ -\frac{1}{15}z_1 + \frac{9}{5}, & 12 \leqslant z_1 \leqslant 27,\\ 0, & z_1 \geqslant 27,\end{cases}$$

$$\mu_2'(z_2) = \begin{cases} 0, & z_2 \leqslant 7,\\ \min(\frac{3}{26}z_2 - \frac{21}{26}, \frac{3}{52}z_2 - \frac{11}{52}), & 7 \leqslant z_2 \leqslant 17,\\ 1, & z_2 = 21,\\ \min(-\frac{1}{5}z_2 + \frac{32}{3}, -\frac{1}{15}z_2 + \frac{8}{3}, -\frac{1}{45}z_2 + \frac{53}{45}), & 21 \leqslant z_2 \leqslant 32,\\ 0, & z_2 \geqslant 32.\end{cases}$$

Although Inuiguchi et al.'s idea is very useful in formulating quasi-concave functions into concave functions, there are three shortcomings in Inuiguchi et al.'s method as described below:

(i) If number of break points is large, then it causes tedious computational burden to convert these membership functions into concave functions.
(ii) That transforming procedure is complicated and cannot effectively deal with an FMOP problem with more general non-concave functions.
(iii) That method still requires zero–one variables to treat converted concave functions (i.e., $\mu_1'(z_1)$ and $\mu_2'(z_2)$).

Take Example 1 for instance, five break points are required to do transforming computing. Suppose there are n objective functions and each of these functions have m_i break points then the number of transforming computing is $\sum_{i=1}^{n} m_i$. The situation would become more complicated for treating more general non-concave FMOP problems.

Yang et al. [13] presented another method for treating a quasi-concave FMOP problem. Take Example 1 for instance. Yang et al.'s method could formulate Example 1 as the following zero–one programming model (as depicted in Fig. 3(a) and 3(b)):

FMOP Model 2 (*Yang et al.'s* [13] *method for Example* 1)

$$\begin{aligned}&\text{Maximize} \quad \lambda\\&\text{Subject to} \quad \lambda \leqslant 1 - \frac{a_4 - z_1}{d_1} + M(1-\delta_1) + M\delta_2, \ \lambda \leqslant 1 - \frac{12 - z_1}{d_2} + M\delta_1 + M\delta_2,\\&\lambda \leqslant 1 - \frac{a_3 - z_1}{d_3} + M\delta_1 + M\delta_2, \ \lambda \leqslant 1 - \frac{27 - z_1}{d_4} + M(1-\delta_2) + M\delta_1,\\&\lambda \leqslant 1 - \frac{a_6 - z_2}{d_5} + M(1-\delta_3), \ \lambda \leqslant 1 - \frac{21 - z_2}{d_6} + M\delta_3,\\&\lambda \leqslant 1 - \frac{a_{10} - z_2}{d_7} + M\delta_3, \ \lambda \leqslant 1 - \frac{a_9 - z_2}{d_8} + M\delta_3,\\&\lambda \leqslant 1 - \frac{32 - z_2}{d_9} + M\delta_3, \ z_1 = -x_1 + 2x_2, \ z_2 = 2x_1 + x_2, \ -x_1 + 3x_2 \leqslant 21,\\&x_1 + 3x_2 \leqslant 27, \ 4x_1 + 3x_2 \leqslant 45, \ 3x_1 + x_2 \leqslant 30, \ x_1, x_2 \geqslant 0,\end{aligned}$$

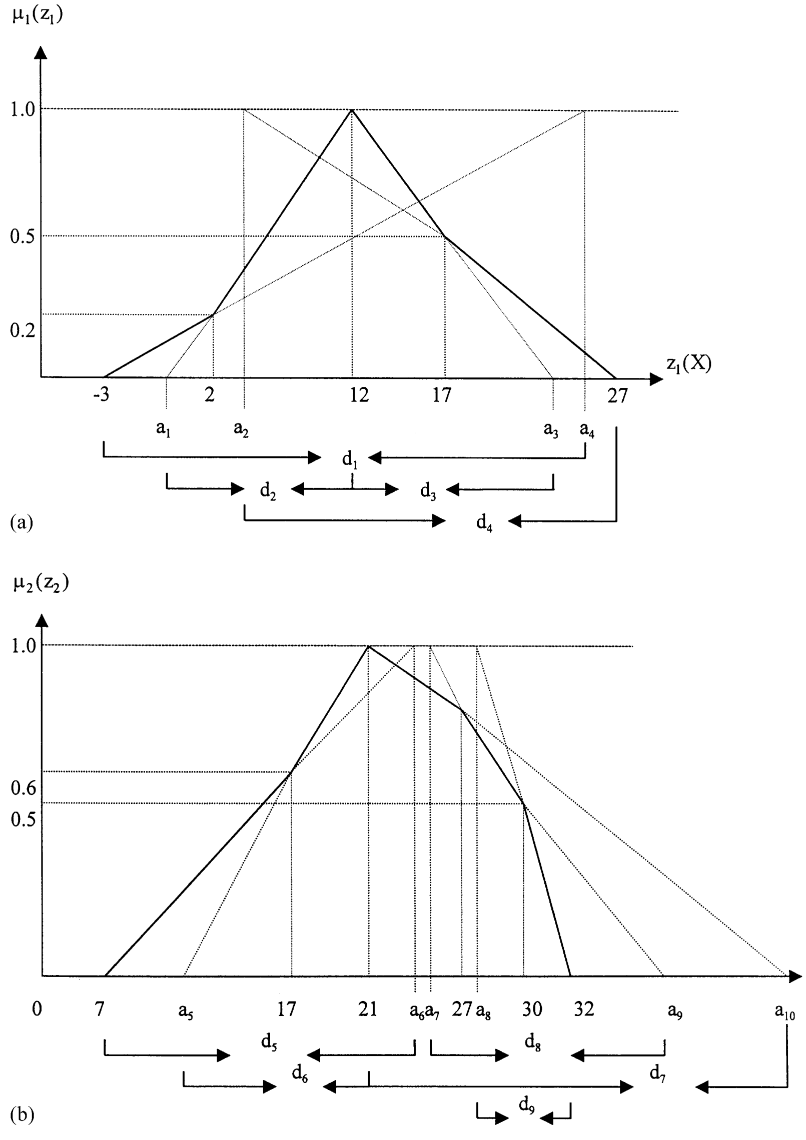

Fig. 3. (a) $\mu_1(z_1)$ in Yang et al. Model. (b) $\mu_2(z_2)$ in Yang et al. Model.

where δ_1, δ_2, and δ_3 are zero–one variables, M is a big number, and $a_1, a_2, \ldots, a_{10}$ are approximated end-point values as depicted in Figs. 3(a) and (b).

A major disadvantage in Yang et al.'s method is that it involves too many zero–one variables for treating quasi-concave FMOP problems. The number of zero–one variables equals the number of intersections between convex and concave functions. Besides, many end-point approximations are required before formulating a quasi-concave FMOP program. Take Example 1 for instance, $\mu_i(z_i)$ contains two convex–concave intersections and $\mu_2(z_2)$ contains one convex–concave intersection. Therefore, three zero–one variables (i.e., $\delta_1, \delta_2, \delta_3$) are

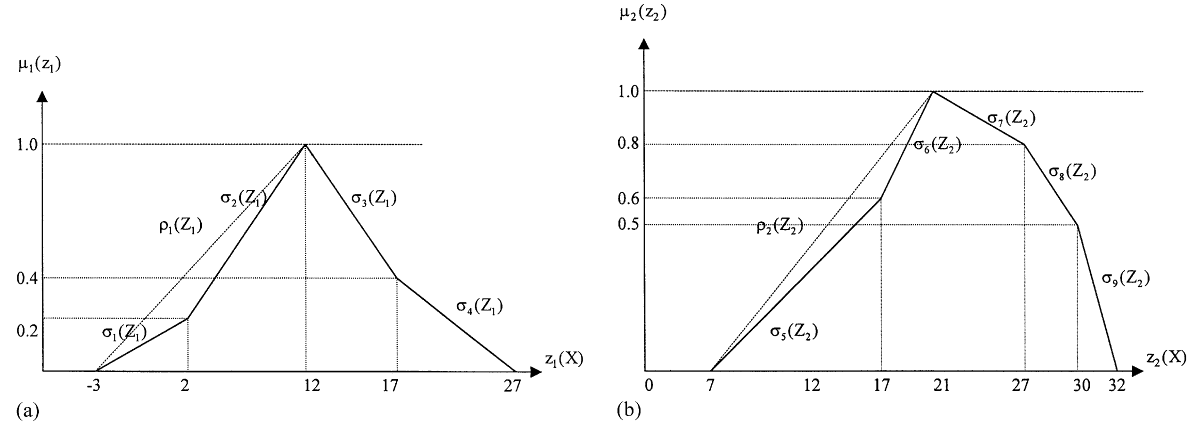

Fig. 4. (a) $\mu_1(z_1)$ in Nakamura's Model. (b) $\mu_2(z_2)$ in Nakamura's Model.

added in the solution process. In addition, ten times end-point approximations (i.e., $a_1, a_2, \ldots, a_{10}$) are required in formulating FMOP model 2. A detailed discussion is given in Li and Yu [7].

Considering $\mu_i(z_i)$ in Problem (1.1) could be concave, convex, or concave–convex mixed type functions, Nakamura developed a method to expressing a general piecewise membership function [9]. He reformulates Problem (1.1) as follows:

$$\begin{aligned}&\text{Maximize} \quad \lambda\\&\text{Subject to} \quad \lambda \leqslant \mu_{\tilde{D}}(z_i) \quad \text{for } i = 1, 2, \ldots, n,\end{aligned}$$

where

$$\mu_{\tilde{D}}(z_i) = \left[\left\{ \left(\bigvee_{j=1}^{i_{m'}} \sigma_j(z_i) \right) \wedge \bigvee_{j=1}^{i_{m''}} \rho_j(z_i) \right\} \wedge \left\{ \bigwedge_{j=1}^{i_{m'''}} \sigma_j(z_i) \right\} \wedge 1 \right] \vee 0, \text{ in which } j = 1, 2, \ldots, i_{m'}$$

are the change points over the convex part of each $\mu_i(z_i)$, $j = 1, 2, \ldots, i_{m''}$ are the number of linear functions for separating concave or convex parts over each $\mu_i(z_i)$, $j = 1, 2, \ldots, i_{m'''}$ are the change points over concave part of each $\mu_i(z_i)$, and $\sigma_i(z_i)$ are linear functions representing part of $\mu_i(z_i)$.

Nakamura's method encounters two major difficulties:

(i) Expression of piecewise membership functions is intricate, it requires repetitive use of LP computation for solving an FMOP problem.

(ii) That method divides an FMOP problem into $\prod_{i=1}^{n} 2k_i$ sub-problems and requires $2\sum_{i=1}^{n} k_i$ constraints, where k_i is the number of segments for each $\mu_i(z_i)$.

Take Example 1 for instance, Nakamura expresses the membership functions, depicted in Fig. 4(a) and (b), as follows:

$$\mu_1(z_1) = [\{\sigma_1(z_1) \vee \sigma_2(z_1)\} \wedge \{\rho_1(z_1)\} \wedge \{\sigma_3(z_1) \vee \sigma_4(z_1)\} \wedge 1] \vee 0,$$
$$\mu_2(z_2) = [\{\sigma_5(z_2) \vee \sigma_6(z_2)\} \wedge \rho_2(z_2) \wedge \sigma_7(z_2) \wedge \sigma_8(z_2) \wedge \sigma_9(z_2) \wedge 1] \vee 0,$$

where $\vee$ stands for maximum, $\wedge$ stands for minimum $\{\sigma_1(z_1) \vee \sigma_2(z_1)\}, \{\sigma_3(z_1) \vee \sigma_4(z_1)\}$, and $\{\sigma_5(z_2) \vee \sigma_6(z_2)\}$ are the sets of the convex parts.

Nakamura's method then divides Example 1 into eight sub-problems. Some of these sub-problems are expressed as follows:

FMOP Model 3 (*Nakamura's* [9] *method for Example* 1)

Subproblem 1
Maximize λ
Subject to $\lambda \leqslant \sigma_1(z_i) \wedge \rho_1(z_1) \wedge \sigma_3(z_1)$
$\lambda \leqslant \sigma_5(z_2) \wedge \rho_2(z_2) \wedge \sigma_7(z_2) \wedge \sigma_8(z_2) \wedge \sigma_9(z_2)$

Subproblem 2
Maximize λ
Subject to $\lambda \leqslant \sigma_2(z_i) \wedge \rho_1(z_1) \wedge \sigma_3(z_1)$
$\lambda \leqslant \sigma_5(z_2) \wedge \rho_2(z_2) \wedge \sigma_7(z_2) \wedge \sigma_8(z_2) \wedge \sigma_9(z_2)$
: :
: :
Subproblem 6
Maximize λ
Subject to $\lambda \leqslant \sigma_2(z_i) \wedge \rho_1(z_1) \wedge \sigma_3(z_1)$
$\lambda \leqslant \sigma_6(z_2) \wedge \rho_2(z_2) \wedge \sigma_7(z_2) \wedge \sigma_8(z_2) \wedge \sigma_9(z_2)$
: :
: :

After using LP computation repeatedly, Nakamura's method finds the optimal solution in Subproblem 6.

To improve conventional FMOP models, this paper first develops a convenient way to express a piecewise linear membership function. The proposed expression is simpler than Nakamura's method [9]. Then we propose Algorithm 1 for solving an FMOP problem with quasi-concave membership functions. We will demonstrate the way to apply Algorithm 1 to resolve Example 1 in a more efficient way. From the basis of Algorithm 1, we develop Algorithm 2 for solving an FMOP problem with more general non-concave membership functions.

3. Preliminary

The FMOP problem given in (1.1) with piecewise quasi-concave functions is termed as a quasi-concave FMOP problem. Some propositions of solving a quasi-concave FMOP problem are described in this section.

Proposition 1. *Let $\mu_i(z_i)$ be a piecewise linear membership function of $z_i(X)$, as depicted in Fig.* 5(a), *where a_k, $k = 1, 2, \ldots, m$, are the break points of $\mu_i(z_i), s_k$, $k = 1, 2, \ldots, m-1$, are the slopes of line segments between a_k and a_{k+1}, and*

$$s_k = \frac{\mu_i(a_{k+1}) - \mu_i(a_k)}{a_{k+1} - a_k}.$$

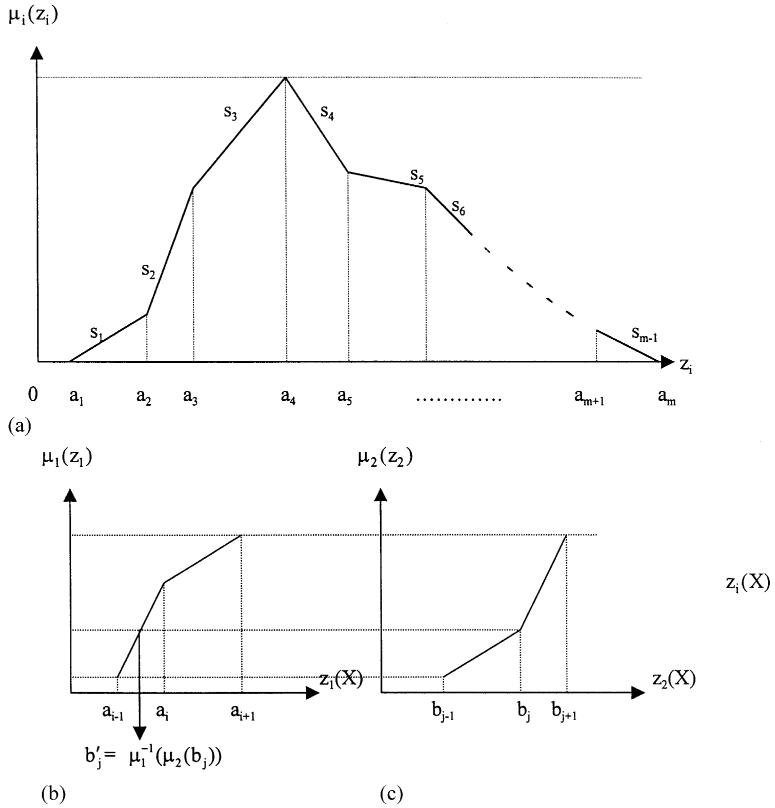

Fig. 5. (a) A general piecewise linear membership function. (b) A concave function $\mu_1(z_1)$. (c) A convex function $\mu_2(z_2)$.

$\mu_i(z_i)$ can then be expressed as follows:

$$\mu_i(z_i) = \mu_i(a_1) + s_1(z_i(X) - a_1) + \sum_{k=2}^{m-1} \frac{s_k - s_{k-1}}{2}(|z_i(X) - a_k| + z_i(X) - a_k), \tag{3.1}$$

where $|o|$ is the absolute value of o.

Proof. This proposition can be examined as follows:

(i) If $z_i(X) \leqslant a_2$ then

$$\mu_i(z_i) = \mu_i(a_1) + \frac{\mu_i(a_2) - \mu_i(a_1)}{a_2 - a_1}(z_i(X) - a_1) = a_1 + s_1(z_i(X) - a_1).$$

(ii) If $z_i(X) \leqslant a_3$ then

$$\begin{aligned} \mu_i(z_i) &= \mu_i(a_1) + s_1(a_2 - a_1) + s_2(z_i(X) - a_2) \\ &= \mu_i(a_1) + s_1(z_i(X) - a_1)\frac{s_2 - s_1}{2}(|z_i(X) - a_2| + z_i(X) - a_2). \end{aligned}$$

(iii) If $z_i(X)\leqslant a_{k'}$ then $\sum_{k\geqslant k'}^{m-1}(|z_i(X)-a_k|+z_i(X)-a_k)=0$ and $\mu_i(z_i)$ becomes

$$\mu_i(a_1)+s_1(z_i(X)-a_1)+\sum_{k=2}^{k'-1}\frac{s_k-s_{k-1}}{2}(|z_i(X)-a_k|+z_i(X)-a_k).$$

Take $\mu_1(z_1)$ and $\mu_2(z_2)$ in Example 1 (as depicted in Fig. 2(a)) for instances, $\mu_1(z_1)$ and $\mu_2(z_2)$ can be represented by Proposition 1 as

$$\begin{aligned}\mu_1(z_1) = {} & 0.04(z_1+3)+\frac{0.08-0.04}{2}(|z_1-2|+z_1-2)\frac{-0.1-0.08}{2}(|z_1-12|+z_1-12)\\ & +\frac{-0.05+0.1}{2}(|z_1-17|+z_1-17),\end{aligned} \tag{3.2}$$

$$\begin{aligned}\mu_2(z_2) = {} & 0.06(z_2-7)+\frac{0.1-0.06}{2}(|z_2-17|+z_2-17)\frac{-0.033-0.1}{2}(|z_2-21|+z_2-21)\\ & +\frac{-0.1+0.033}{2}(|z_2-27|+z_2-27)+\frac{-0.25+0.1}{2}(|z_2-30|+z_2-30).\end{aligned} \tag{3.3}$$

An advantage of expressing a quasi-concave membership function by (3.1) is the convenience of knowing the intervals of convexity and concavity for $\mu_i(z_i)$, as described below:

Remark 1 (Convex-type break point)**.** For a $\mu_i(z_i)$ expressed by Eq. (3.1), if $s_{k+1}>s_k$ then $=\mu_i(z_i)$ is a convex function for $a_{k-1}\leqslant z_i(X)\leqslant a_{k+1}$ and a_k is called a *convex-type break point* of z_i.

Take Expression (3.2) for instance, it is convenient to check that $\mu_1(z_1)$ is concave when $2\leqslant z_1(X)\leqslant 17$ and $\mu_1(z_1)$ is convex when $-3\leqslant z_1(X)\leqslant 12$ and $12\leqslant z_1(X)\leqslant 27$. Therefore, the point $z_1(X)=2$ and $z_1(X)=17$ are convex-type break points of z_i. Similarly for Expression (3.3), $\mu_2(z_2)$ is convex for $7\leqslant z_2(X)\leqslant 21$ and concave for $17\leqslant z_2(X)\leqslant 32$. $z_2(X)=17$ is a convex-type break point of z_2.

Remark 2 (Concave-type break point)**.** For a $\mu_i(z_i)$ expressed by Eq. (3.1) if $s_{k+1}<s_k$ then $\mu_i(z_i)$ is a concave function for $a_{k-1}\leqslant z_i(X)\leqslant a_{k+1}$ and a_k is called a *concave-type break point* of z_1.

Remark 3 (Mapping point)**.** For $\mu_1(z_1)$ and $\mu_2(z_2)$ shown in Fig. 5(b) and (c) respectively, we can find a convex-type break point b_j in z_2 by using Remark 1. Then a corresponding point of b_j can be found in z_1, and this has the same value of membership functions as b_j. Such a point is called a *mapping point* of b_j, denoted as b_j', which is mapped from z_2 to z_1 and calculated by $b_j'=\mu_1^{-1}(\mu_2(b_j))$.

Remark 4 (Converted concave function)**.** Now let us consider two piecewise linear functions μ_1 and μ_2 specified in Fig. 6(a).

$$\mu_1(f(X))=\mu_1(a_1)+s_1(f(X)-a_1)+\frac{s_2-s_1}{2}(|f(X)-a_2|+f(X)-a_2), \tag{3.4}$$

$$\mu_2(f(X))=\mu_2(b_1)+t_1(f(X)-b_1)+\frac{t_2-t_1}{2}(|f(X)-b_2|+f(X)-b_2), \tag{3.5}$$

where $s_1>s_2>0,\ t_2>t_1>0,\ a_1=b_1$, and $a_3=b_3$.

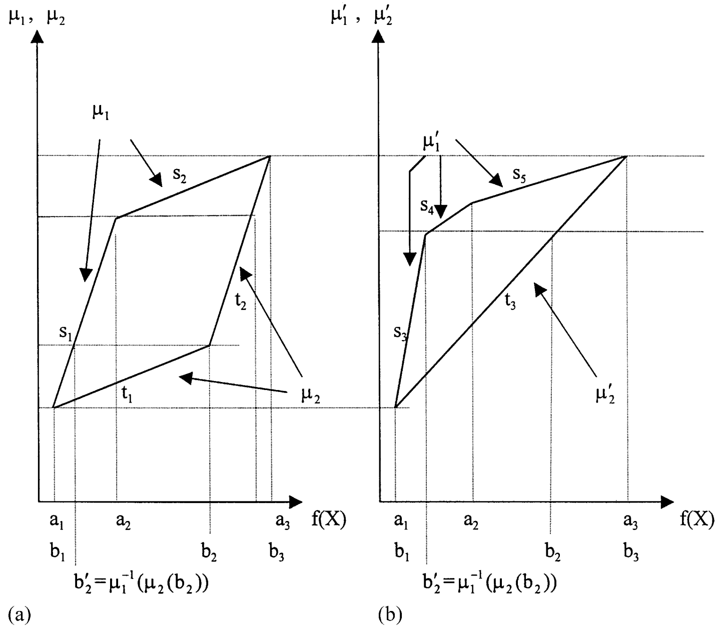

Fig. 6. (a) Two piecewise linear membership functions μ_1 and μ_2. (b) Two piecewise linear membership functions μ_1' and μ_2'.

Then two converted concave functions μ_1' and μ_2', shown in Fig. 6(b), can be specified as follows:

$$\mu_1'(f(X)) = \mu_1'(a_1) + s_3(f(X) - a_1) + \frac{s_4 - s_3}{2}(|f(X) - b_2'| + f(X) - b_2') + \frac{s_5 - s_4}{2}(|f(X) - a_2| + f(X) - a_2), \tag{3.6}$$

$$\mu_2'(f(X)) = \mu_2'(b_1) + t_3(f(X) - b_1), \tag{3.7}$$

where

$$a_1 = b_1,\ a_3 = b_3,\ t_3 > 0,\ s_3 > s_4 > s_5 > 0, \tag{3.8}$$

$$\begin{aligned} &\mu_1(a_1) = \mu_1'(a_1) = 0, \quad \mu_1(a_3) = \mu_1'(a_3) = 1, \\ &\mu_2(b_1) = \mu_2'(b_1) = 0, \quad \mu_2(b_3) = \mu_2'(b_3) = 1, \end{aligned} \tag{3.9}$$

$$\frac{\mu_1'(b_2')}{\mu_2'(b_2)} = \frac{\mu_1(b_2')}{\mu_2(b_2)} = 1, \tag{3.10}$$

$$b_2' = \mu_1^{-1}[\mu_2(b_2)]. \tag{3.11}$$

Next, Proposition 2 is presented below:

Proposition 2. *Function* μ_1' *specified in* (3.6)–(3.11) *is a concave function.*

Proof. Due to $s_3 > s_4 > s_5$, based on Remark 2, b_2' and a_2 become concave-type points on μ_1'. Consequently, μ_1' is a concave function.

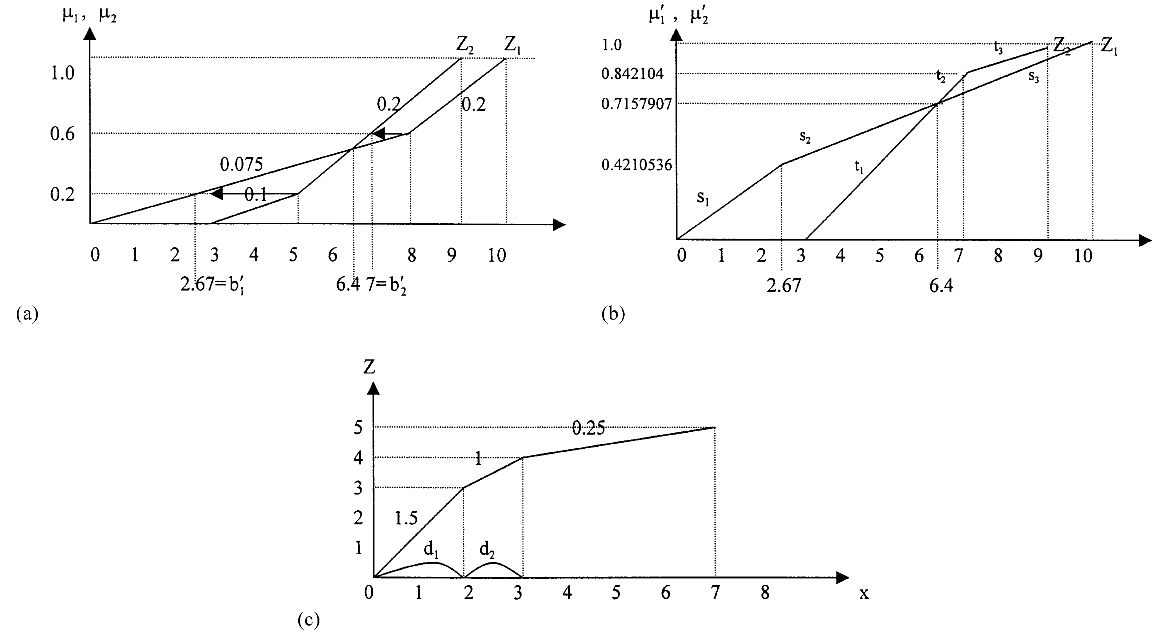

Fig. 7. (a) Two functions μ_1 and μ_2 in Example 2. (b) Two functions μ_1' and μ_2' in Example 2. (c) A function Z in Example 3.

Consider the following example:

Example 2

$$
\begin{aligned}
&\text{Maximize} && \lambda \\
&\text{Subject to} && \lambda \leqslant \mu_1(z_1),\ \lambda \leqslant \mu_2(z_2), \\
& && z_1 = -x_1 + 2x_2,\ z_2 = 2x_1 + x_2, -x_1 + 3x_2 \leqslant 6,\ x_1 + 3x_2 \leqslant 12, 4x_1 + 3x_2 \leqslant 30, \\
& && 3x_1 + x_2 \leqslant 15,\ x_1, x_2 \geqslant 0,
\end{aligned}
$$

where $\mu_1(z_1)$ and $\mu_2(z_2)$ are depicted in Fig. 7(a).

Referring to Remark 1, we know $\mu_1(z_1 = 8)$ is a convex-type point in z_1 and $\mu_2(z_2 = 5)$ is a convex-type point in z_2. Then, based on Remark 3, the mapping points can be computed by $b_1' = \mu_1^{-1}(\mu_2(z_2 = 5)) = 2.67$ and $b_2' = \mu_2^{-1}(\mu_1(z_1 = 8)) = 7$.

In reference to Remark 4, we have two converted functions below (as shown in Fig. 7(b)):

$$\mu_1'(z_1) = s_1(z_1 - 0) + \frac{s_2 - s_1}{2}(|z_1 - 2.67| + z_1 - 2.67) + \frac{s_2 - s_1}{2}(|z_1 - 6.4| + z_1 - 6.4), \tag{3.12}$$

$$\mu_2'(z_2) = t_1(z_2 - 3) + \frac{t_2 - t_1}{2}(|z_2 - 6.4| + z_2 - 6.4) + \frac{s_2 - s_1}{2}(|z_2 - 7| + z_2 - 7), \tag{3.13}$$

where

$$\mu_1(0) = \mu_1'(0) = 0, \quad \mu_1(10) = \mu_1'(10) = 2.67s_1 + 3.73s_2 + 3.6s_3 = 1,$$

$$\mu_2(3) = \mu_2'(3) = 0, \quad \mu_2(9) = \mu_2'(9) = 3.4t_1 + 0.6t_2 + 2t_3 = 1,$$

$$\frac{\mu_1(2.67)}{\mu_2(5)} = \frac{\mu_1'(2.67)}{\mu_2'(5)} = \frac{2.67s_1}{2t_1} = 1, \quad \frac{\mu_1(6.4)}{\mu_2(6.4)} = \frac{\mu_1'(6.4)}{\mu_2'(6.4)} = \frac{2.67s_1 + 3.73s_2}{3.4t_1} = 1,$$

$$\frac{\mu_1(8)}{\mu_2(7)} = \frac{\mu_1'(8)}{\mu_2'(7)} = \frac{2.67s_1 + 3.73s_2 + 1.6s_3}{3.4t_1 + 0.6t_2} = 1, \quad s_1 > s_2 > s_3 > 0, \text{ and } t_1 > t_2 > t_3 > 0.$$

After computation, the slopes of two converted concave functions are $s_1 = 0.157698$, $s_2 = 0.079018$, $s_3 = 0.078947$, $t_1 = 0.210526$, $t_2 = 0.210526$, and $t_3 = 0.078947$. Hence, Example 2 can be transformed into

Example 2′

$$\begin{aligned} &\text{Maximize} && \lambda' \\ &\text{Subject to} && \lambda' \leqslant \mu_1'(z_1),\ \lambda' \leqslant \mu_2'(z_2),\ z_1 = -x_1 + 2x_2,\ z_2 = 2x_1 + x_2, \\ &&& -x_1 + 3x_2 \leqslant 6,\ x_1 + 3x_2 \leqslant 12,\ 4x_1 + 3x_2 \leqslant 30,\ 3x_1 + x_2 \leqslant 15, \quad x_1, x_2 \geqslant 0, \end{aligned}$$

where $\mu_1'(z_1)$ and $\mu_2'(z_2)$ are expressed in (3.14) and (3.15), respectively.

$$\mu_1'(z_1) = 0.157698z_1 - \frac{0.07968}{2}(|z_1 - 2.67| + z_1 - 2.67) + \frac{0.000071}{2}(|z_1 - 6.4| + z_1 - 6.4), \tag{3.14}$$

$$\mu_2'(z_2) = 0.210526(z_2 - 3) + \frac{0}{2}(|z_2 - 6.4| + z_2 - 6.4) + \frac{0.131579}{2}(|z_2 - 7| + z_2 - 7). \tag{3.15}$$

Assume that R is the universal set of real numbers, D is an arbitrary domain, and R^n denotes n-dimensional Euclidean space. For any real-valued function $u : D \to R$, the image of D by u is denoted by $u(D)$, i.e. $u(D) = \{u(d) \mid d \in D\}$. Then Inuiguchi et al. [4] proved that there exists a strictly increasing and bijective function $g : u(D) \to u'(D)$ such that $u'(d) = g(u(d))$ for any d belonging to D, where $u : D \to R$ and $u' : D \to R$.

Define an r-level set of $u : D \to R$ by $[u]_r = \{d \in D \mid f(d) \geqslant r\}$ where $r \in R$. Inuiguchi et al. [4] proved that the solution maximizing a function u is equal to the solution maximizing a function u' in any restricted domain when $\{[u]_r \mid r \in u(D)\} = \{[u']_{r'} \mid r' \in u'(D)\}$ and $[u']_{r'}$ is a bijective function of $[u]_r$. Accordingly, we have the following proposition.

Proposition 3. *The optimal solution of* P1 *is the same as that of* P2; P1 *and* P2 *are given below in which* μ_1, μ_2, μ_1', *and* μ_2' *are specified in* (3.6)–(3.11):

P1		P2	
Maximize	λ	*Maximize*	λ'
Subject to	$\lambda \leqslant \mu_1(f(X))$,	*Subject to*	$\lambda' \leqslant \mu_1'(f(X))$,
	$\lambda \leqslant \mu_2(f(X))$,		$\lambda' \leqslant \mu_2'(f(X))$,
	$a_1 = b_1 \leqslant f(X) \leqslant a_3 = b_3$,		$a_1 = b_1 \leqslant f(X) \leqslant a_3 = b_3$,
	$f(X) \in F$ (F *is a feasible set*),		$f(X) \in F$ (F *is a feasible set*).

Proof. For an $f(X)$ in the restricted domain $[a_1, a_3]$ or $[b_1, b_3]$, we have

(i) $\mu_1(a_1) = \mu_1'(a_1)$, $\mu_1(a_3) = \mu_1'(a_3)$, $\mu_2(b_1) = \mu_2'(b_1)$, $\mu_2(b_3) = \mu_2'(b_3)$,

(ii) $\dfrac{\mu_1(b_2')}{\mu_2(b_2')} = \dfrac{\mu_1'(b_2')}{\mu_2'(b_2')}$, $\dfrac{\mu_1(a_2)}{\mu_2(a_2)} = \dfrac{\mu_1'(a_2)}{\mu_2'(a_2)}$, $\dfrac{\mu_1(b_2)}{\mu_2(b_2)} = \dfrac{\mu_1'(b_2)}{\mu_2'(b_2)}$,

(iii) $t_3 > 0$, $s_3 > s_4 > s_5 > 0$.

Since $\{\mu_1'(f(X)), \mu_2'(f(X))\}$ is the strictly increasing and bijective function of $\{\mu_1(f(X)), \mu_2(f(X))\}$ $\max_{f(X)} \min\{\mu_1(f(X)), \mu_2(f(X))\}$ is equivalent to

$$\max_{f(X)} \min\{\mu_1'(f(X)), \mu_2'(f(X))\}.$$

Therefore, the optimal solution of P1 is the same as the optimal solution of P2.

Take Example 2 for instance. Solve Example 2′ by utilizing Proposition 4, discussed next, the obtained solution $z_1 = 3.525553$, $z_2 = 5.321128$, $x_1 = 1.423341$, $x_2 = 2.474447$. The optimal solution of Example 2 is the same as the optimal solution of Example 2′.

Proposition 4. *By referring to Proposition* 1, *consider an FMOP problem as follows*:

$$\begin{aligned} &\textit{Maximize} \quad \lambda \\ &\textit{Subject to} \quad \lambda \leqslant \mu_i(z_i), \quad X \in F \ (\textit{a feasible set}), \end{aligned}$$

where

$$\mu_i(z_i) = \mu_i(a_1) + s_1(z_i(X) - a_1) + \sum_{k=2}^{m-1} \frac{s_k - s_{k-1}}{2}(|z_i(X) - a_k| + z_i(X) - a_k)$$

is a concave function (*i.e.*, $s_k - s_{k-1} < 0$ *for* $k = 2, 3, \ldots, m-1$).

This FMOP problem can then be reformulated as follows:

$$\begin{aligned} &\textit{Maximize} \quad \lambda \\ &\textit{Subject to} \quad \lambda \leqslant \mu_i(z_i) \\ &\qquad \mu_i(z_i) = \mu_i(a_1) + s_1(z_i(X) - a_1) + \sum_{k=2}^{m-1}(s_k - s_{k-1})\left(z_i(X) - a_k + \sum_{l=2}^{k-1} d_l\right), \\ &\qquad z_i(X) - a_{m-1} + \sum_{l=2}^{m-1} d_{l-1} \geqslant 0, \ 0 \leqslant d_{l-1} \leqslant a_l - a_{l-1} \ \text{for all } l, l = 2, 3, \ldots, m-1, \\ &\qquad X \in F \ (\textit{a feasible set}). \end{aligned} \tag{3.16}$$

Proof. By referring to Li [6], a GP problem

$$\left\{ \begin{aligned} &\text{Maximize } w = \sum_{k=2}^{m-1}(|z_i(X) - a_k| + z_i(X) - a_k) \\ &\text{subject to } z_i(X) \geqslant 0 \text{ and } 0 < a_2 < a_3 < \cdots < a_{m-1} \end{aligned} \right\}$$

is equivalent to

$$\left\{ \begin{aligned} &\text{Maximize } w = 2\sum_{k=2}^{m-1}(z_i(X) - a_k + r_{k-1}) \\ &\text{subject to } z_i(X) - a_k + r_{k-1} \geqslant 0 \text{ for } k = 2, 3, \ldots, m-1, \ r_{k-1} \geqslant 0, \ x_i \geqslant 0, \\ &\quad \text{where } r_{k-1} \text{ are deviation variables} \end{aligned} \right\}. \tag{3.17}$$

Expression (3.17) implies if $z_i(X) < a_k$ then at optimal solution $r_{k-1} = a_k - z_i(X)$; if $z_i(X) \geqslant a_k$ then at optimal solution $r_{k-1} = 0$. Substitute r_{k-1} by $\sum_{l=1}^{k-1} d_l$, where d_l is within the bounds as $0 \leqslant d_l \leqslant a_{l+1} - a_l$, Expression (3.17) then becomes

$$\begin{aligned} \text{Maximize} \quad & w = 2\sum_{k=2}^{m-1}\left(z_i(X) - a_k + \sum_{l=l}^{k-1} d_l\right) \\ \text{Subject to} \quad & z_i(X) + d_1 \geqslant a_2, \\ & z_i(X) + d_1 + d_2 \geqslant a_3, \\ & \quad \vdots \\ & z_i(X) + d_1 + d_2 + \cdots + d_{m-2} \geqslant a_{m-1}, \\ & 0 \leqslant d_l \leqslant a_{l+1} - a_l \quad \text{for } l = 1, 2, \ldots, m-2, \quad z_i(X) \geqslant 0. \end{aligned} \tag{3.18}$$

Since $a_{l+1} - a_l \geqslant d_l$ for all l, it is clear that

$$z_i(X) \geqslant a_{m-1} - \sum_{l=1}^{m-2} d_l \geqslant a_{m-2} - \sum_{l=1}^{m-3} d_l \geqslant \cdots \geqslant a_3 - d_1 - d_2 \geqslant a_2 - d_1 \geqslant 0.$$

The first $(m-3)$ constraints in Model (3.18) therefore are covered by the $(m-2)$th constraint in Model (3.18). Proposition 4 is then proven. □

Consider the following example as depicted in Fig. 7(c):

Example 3

$$\begin{aligned} \text{Maximize} \quad & z = 1.5x - \frac{0.5}{3}(|x-2| + x - 2) - \frac{0.75}{2}(|x-3| + x - 3) \\ \text{Subject to} \quad & x \leqslant 2.5. \end{aligned}$$

Referring to Proposition 4, Example 3 can be linearized as

Example 3′

$$\begin{aligned} \text{Maximize} \quad & z = 1.5x - 0.5(x - 2 + d_1) - 0.75(x - 3 + d_1 + d_2) \\ \text{Subject to} \quad & x + d_1 + d_2 \geqslant 3, \quad 0 \leqslant d_1 \leqslant 2, \; 0 \leqslant d_2 \leqslant 1, \text{ and } x \leqslant 2.5. \end{aligned}$$

After running on the LINDO [12], we obtained $z = 3.5$, $x = 2.5$, $d_1 = 0$, and $d_2 = 0.5$.

4. Solution algorithms

From the basis of Proposition 1 to Proposition 4, we propose Algorithm 1 for treating a quasi-concave FMOP problem.

Algorithm 1 (*Solve a quasi-concave FMOP problem*)

Step 1: Use Proposition 1 to express each piecewise membership function as $\mu_i(z_i) = \mu_i(a_{i1}) + s_{i1}(z_i(X) - a_{i1}) + \sum_{k=2}^{M(i)-1} s_{ik} - s_{ik-1}/2(|z_i(X) - a_{ik}| + z_i(X) - a_{ik})$ where a_{ik}, $k = 1, 2, \ldots, m$, are the break points of $\mu_i(z_i)$, s_{ik}, $k = 1,2,\ldots,m-1$, are the slopes of line segments between a_{ik} and $a_{i,k+1}$, and $i = 1,2,\ldots,n$.

Step 2: Use Remark 1 to find the convex-type break points and Remark 3 to obtain the corresponding mapping points.

Step 3: Use Remark 4 to specify the converted concave membership functions.

Step 4: Use Eqs. (3.8), (3.9), and (3.10) to compute the slopes of the Converted concave membership functions.

Step 5: Use Proposition 4 to linearize the converted functions and then solve it by LP techniques.

Based on the above discussion, for tackling more general non-concave FMOP problems we first have the following remark.

Remark 5 (*Model the union of some quasi-concave membership functions*). Any piecewise membership function can be regarded as the union of some quasi-concave membership functions. Take Fig. 1(d) for instance, $\mu_i(z_i)$ can be regarded as the union of three quasi-concave functions $\mu_{i1}(a_1 \leqslant z_i \leqslant a_2)$, $\mu_{i2}(a_2 \leqslant z_i \leqslant a_3)$, and $\mu_{i3}(a_3 \leqslant z_i \leqslant a_4)$.

The program of

$$\begin{aligned} &\text{Maximize} \quad \lambda \\ &\text{Subject to} \quad \lambda \leqslant \mu_i(z_i) \quad \text{for } i = 1,2,\ldots,n, \end{aligned}$$

can be rewritten as the following program by referring to Li and Yu [7].

$$\begin{aligned} &\text{Maximize} \quad \lambda \\ &\text{Subject to} \quad \lambda \leqslant \mu_{i1}(z_i) + M\delta_1, \ \lambda \leqslant \mu_{i2}(z_i) + M\delta_2, \\ &\qquad\qquad\quad \lambda \leqslant \mu_{i3}(z_i) + M\delta_3, \ \delta_1 + \delta_2 + \delta_3 = 1, \end{aligned}$$

where M is a big number and $\delta_1, \delta_2, \delta_3$ are zero-one variables.

From the basis of Remark 5, we propose Algorithm 2 for solving a general non-concave FMOP problem.

Algorithm 2 (*Solve a general non-concave FMOP problem*)

Step 0: Convert the piecewise membership functions into the union of some quasi-concave membership functions by adding some $0-1$ variables.

Step 1–*Step* 5: are the same as in Algorithm 1.

5. Numerical examples

Now we use Algorithm 1 to solve Example 1:

Step 1: Utilizing Proposition 1 to represent $\mu_1(z_1)$ and $\mu_2(z_2)$ as the following expressions (as depicted in Fig. 2(a)).

$$\mu_1(z_1) = 0.04(z_1+3) + 0.02(|z_1-2| + z_1 - 2) - 0.1(|z_1-12| + z_1 - 12) + 0.04(|z_1-17| + z_1 - 17),$$

$$\begin{aligned} \mu_2(z_2) = {} & 0.06(z_2-7) + 0.02(|z_2-17| + z_2 - 17) - 0.0665(|z_2-21| + z_2 - 21) \\ & - 0.03335(|z_2-27| + z_2 - 27) - 0.075(|z_2-30| + z_2 - 30). \end{aligned}$$

Step 2: Using Remark 1 to find convex-type points and Remark 3 to calculate their corresponding mapping points as follows: (as depicted in Fig. 8(a))

$$b_{11}=\mu_1^{-1}[\mu_2(17)]=7,\quad b_{21}=\mu_2^{-1}[\mu_1(2)]=\tfrac{31}{3}\quad\text{and}\quad b_{22}=\mu_2^{-1}[\mu_1(17)]=\tfrac{91}{3}.$$

Step 3: Use Remark 4 to specify the converted concave membership functions $\mu_1'(z_1)$ and $\mu_2'(z_2)$, as shown in Fig. 8(b).

$$\mu_1'(z_1)=s_1(z_1+3)+\frac{s_2-s_1}{2}(|z_1-7|+z_1-7)+\frac{s_3-s_2}{2}(|z_1-12|+z_1-12), \tag{5.1}$$

$$\begin{aligned}\mu_2'(z_2)=&\,t_1(z_2-7)+\frac{t_2-t_1}{2}\left(\left|z_2-\tfrac{31}{3}\right|+z_2-\tfrac{31}{3}\right)+\frac{t_3-t_2}{2}(|z_2-21|+z_2-21)\\&+\frac{t_4-t_3}{2}(|z_2-27|+z_2-27)+\frac{t_5-t_4}{2}(|z_2-30|+z_2-30)\\&+\frac{t_6-t_5}{2}\left(\left|z_2-\tfrac{91}{3}\right|+z_2-\tfrac{91}{3}\right).\end{aligned} \tag{5.2}$$

Step 4: Use Eqs. (3.8)–(3.10) to compute the slopes s_i and t_j, $i=1,2,3$ and $j=1,2,\ldots,6$ in (5.1) and (5.2). Then

$$\begin{aligned}&\mu_1(12)=\mu_1'(12)=10s_1+5s_2=1,\quad \mu_1(27)=\mu_1'(27)=10s_1+5s_2+15s_3=0,\\&\mu_2(21)=\mu_2'(21)=\tfrac{26}{3}t_1+\tfrac{16}{3}t_2=1,\quad \mu_2(32)=\mu_2'(32)=\tfrac{10}{3}t_1+\tfrac{32}{3}t_2+6t_3+3t_4+\tfrac{1}{3}t_5+\tfrac{5}{3}t_6=0,\\&\frac{\mu_1(2)}{\mu_2(\frac{31}{3})}=\frac{\mu_1'(2)}{\mu_2'(\frac{31}{3})}=\frac{5s_1}{\frac{10}{3}t_1}=1,\quad \frac{\mu_1(7)}{\mu_2(17)}=\frac{\mu_1'(7)}{\mu_2'(17)}=\frac{10s_1}{\frac{10}{3}t_1+\frac{20}{3}t_2}=1,\\&\frac{\mu_1(14)}{\mu_2(27)}=\frac{\mu_1'(14)}{\mu_2'(27)}=\frac{10s_1+5s_2+2s_3}{\frac{10}{3}t_1+\frac{32}{3}t_2+6t_3}=1,\quad \frac{\mu_1(16)}{\mu_2(30)}=\frac{\mu_1'(16)}{\mu_2'(30)}=\frac{10s_1+5s_2+4s_3}{\frac{10}{3}t_1+\frac{32}{3}t_2+6t_3+3t_4}=1,\\&\frac{\mu_1(17)}{\mu_2(\frac{91}{3})}=\frac{\mu_1'(17)}{\mu_2'(\frac{91}{3})}=\frac{10s_1+5s_2+5s_3}{\frac{10}{3}t_1+\frac{32}{3}t_2+6t_3+3t_4+\frac{1}{3}t_5}=1,\quad s_1>s_2>s_3,\ t_1>t_2>t_3>t_4>t_5>t_6.\end{aligned}$$

After running on the LINDO [12], the solutions found are $s_1=0.077$, $s_2=0.046$, $s_3=-0.067$, $t_1=0.11539$, $t_2=0.058$, $t_3=-0.022$, $t_4=-0.044$, $t_5=-0.2$, and $t_6=-0.4$. Therefore, we have

$$\begin{aligned}\mu_1'(z_1)=&\,0.077(z_1+3)-0.015(|z_1-7|+z_1-7)-0.056(|z_1-12|+z_1-12),\\\mu_2'(z_2)=&\,0.115(z_2-7)-0.029\left(\left|z_2-\tfrac{31}{3}\right|+z_2-\tfrac{31}{3}\right)\\&-0.0399(|z_2-21|+z_2-21)-0.011(|z_2-27|+z_2-27)-0.078(|z_2-30|+z_2-30)\\&-0.1\left(\left|z_2-\tfrac{91}{3}\right|+z_2-\tfrac{91}{3}\right).\end{aligned}$$

Step 5: Use Proposition 4 to linearize the converted functions and then solve it by linear programming techniques.

Based on Proposition 4, the linearized program is described below:

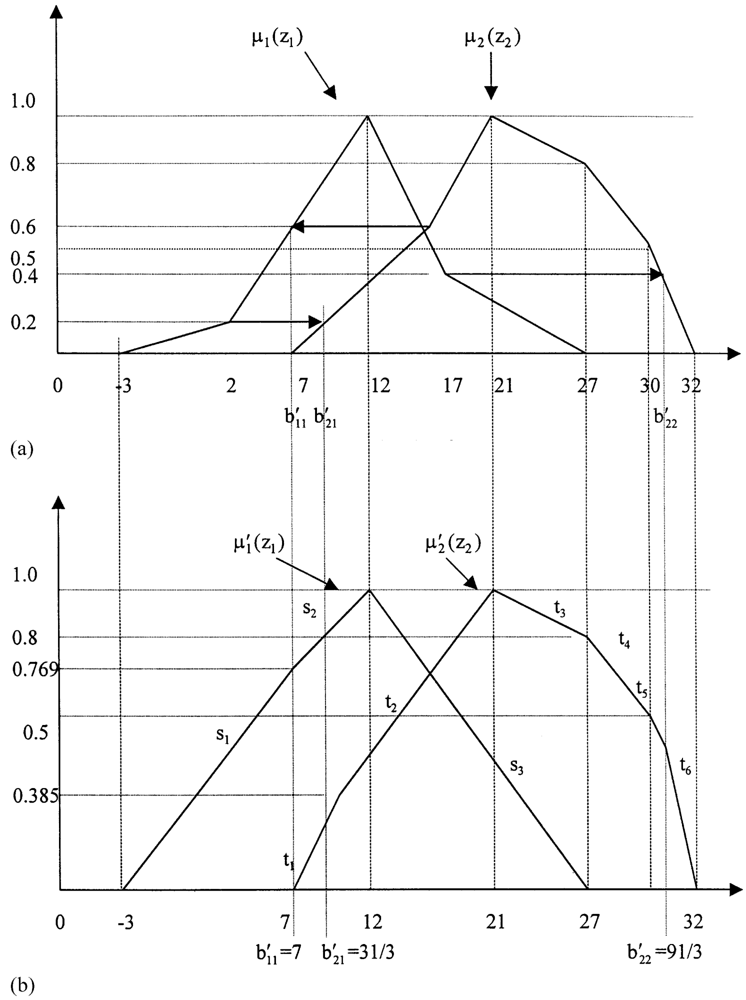

Fig. 8. (a) Two quasi-concave membership functions in Example 1. (b) Two converted concave membership functions in Algorithm 1.

FMOP Model 4 (*Proposed method for Example* 1)

$$
\begin{aligned}
\text{Maximize} \quad & \lambda' \\
\text{Subject to} \quad & \lambda' \leqslant \mu_1'(Z_1 = 0.067Z_1 - 0.031d_1 - 0.113d_2 + 1.804, \\
& \lambda' \leqslant \mu_2'(Z_2) = -0.4Z_2 - 0.058d_3 - 0.0798d_4 - 0.022d_5 - 0.156d_6 - 0.2d_7 + 12.808, \\
& z_1 - 7 + d_1 \geqslant 0,\ z_1 - 12 + d_2 \geqslant 0,\ z_2 - \tfrac{31}{3} + d_3 \geqslant 0, \\
& z_2 - 21 + d_4 \geqslant 0,\ z_2 - 27 + d_5 \geqslant 0,\ z_2 - 30 + d_6 \geqslant 0, \\
& z_2 - \tfrac{91}{3} + d_7 \geqslant 0,\ z_1 = -x_1 + 2x_2,\ z_2 = 2x_1 + x_2,\ -x_1 + 3x_2 \leqslant 21, \\
& x_1 + 3x_2 \leqslant 27,\ 4x_1 + 3x_2 \leqslant 45,\ 3x_1 + x_2 \leqslant 30,\ x_1, x_2 \geqslant 0.
\end{aligned}
$$

Table 1
Efficiency comparison for solving Example 1

FMOP models	Required zero–one variables	Required extra constraints	Required subproblems	Required LP computation	Required point calculation
Narasimhan's and Hannan's methods	Cannot treat Example 1.				
FMOP Model 1 (Inuiguchi et al. method)	3	2	0	1	5
FMOP Model 2 (Yang et al. method)	3	9	0	1	10
FMOP Model 3 (Nakamura's method)	0	9	8	8	0
FMOP Model 4 (Proposed method)	0	2	0	1	3

By solving on the LINDO [12], we obtained $x_1 = 5.62$, $x_2 = 7.13$, $z_1 = 8.64$ and $z_2 = 18.36$ which is exactly the optimal solution of Example 1. Table 1 summarizes the efficiency comparison between Algorithm 1 and conventional FMOP methods for solving Example 1.

Now let us consider the following piecewise non-concave FMOP problem.

Example 4

$$\begin{aligned}
&\text{Maximize} \quad \lambda \\
&\text{Subject to} \quad \lambda \leqslant \mu_1(z_1) = 0.04(z_1+3) + 0.02(|z_1-2| + z_1 - 2) - 0.1(|z_1-12| + z_1 - 12) \\
&\qquad + 0.04(|z_1-17| + z_1 - 17) + 0.04(|z_1-27| + z_1 - 27) \\
&\qquad + 0.02(|z_2-42| + z_2 - 42), \\
&\lambda \leqslant \mu_2(z_2) = 0.06(z_2-7) + 0.02(|z_2-17| + z_2 - 17) - 0.0665(|z_2-21| + z_2 - 21) \\
&\qquad - 0.03335(|z_2-27| + z_2 - 27) - 0.075(|z_2-30| + z_2 - 30) \\
&\qquad + 0.145(|z_2-32| + z_2 - 32) + 0.03(|z_2-37| + z_2 - 37),
\end{aligned}$$

where $\mu_1(z_1)$ and $\mu_2(z_2)$ are non-concave functions as depicted in Fig. 9(a).

By referring to Algorithm 2, we have the following steps:

Step 0: Here $\mu_1(z_1)$ can be regarded as the union of two quasiconcave functions $\mu_{11}(-3 \leqslant z_1 \leqslant 27)$ and $\mu_{12}(27 \leqslant z_1 \leqslant 47)$. $\mu_2(z_2)$ can be regarded as the union of two quasiconcave functions $\mu_{21}(7 \leqslant z_2 \leqslant 32)$ and $\mu_{22}(32 \leqslant z_2 \leqslant 45)$.

In reference to Remark 1, Example 2 can reformulated as

$$\begin{aligned}
&\text{Maximize} \quad \lambda \\
&\text{Subject to} \quad \lambda \leqslant \mu_{11}(-3 \leqslant z_1 \leqslant 27) + M\delta_1, \ \lambda \leqslant \mu_{12}(27 \leqslant z_1 \leqslant 47) + M(1-\delta_1), \\
&\qquad \lambda \leqslant \mu_{21}(7 \leqslant z_2 \leqslant 32) + M\delta_2, \ \lambda \leqslant \mu_{22}(32 \leqslant z_2 \leqslant 45) + M(1-\delta_2),
\end{aligned} \tag{5.3}$$

where M is a big number and δ_1, δ_2 are 0–1 variables.

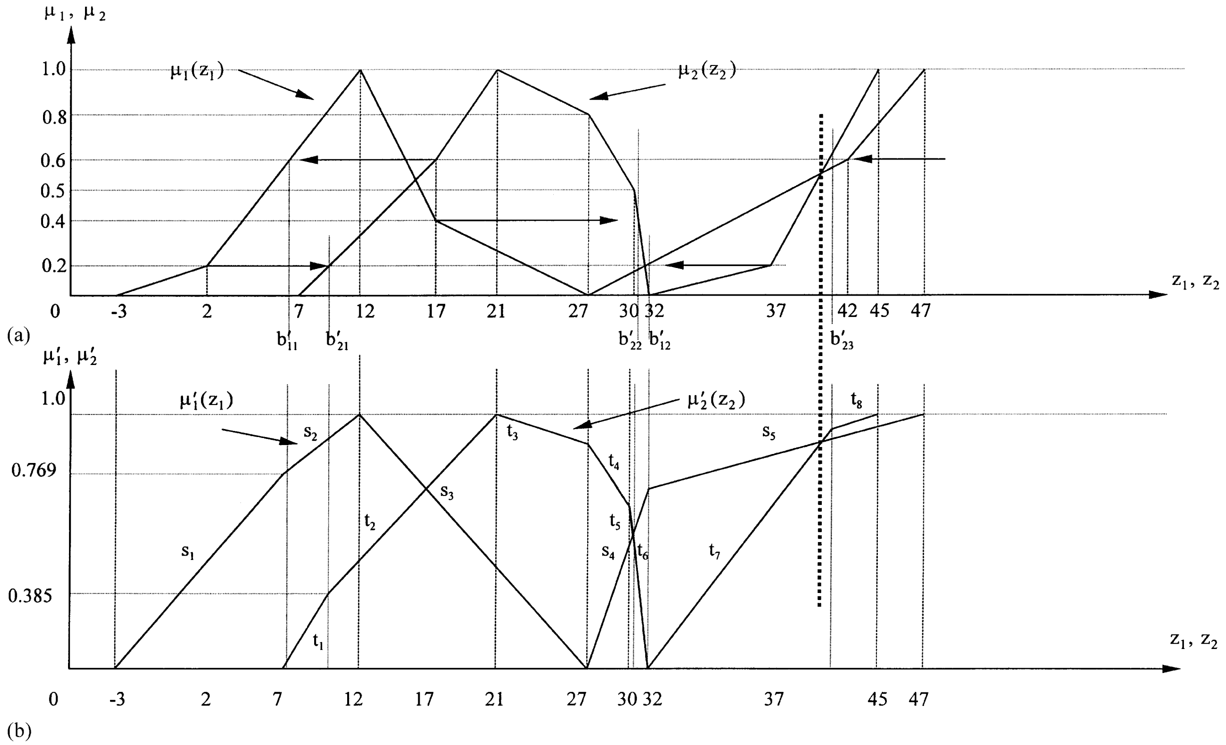

Fig. 9. (a) Two non-concave membership functions in Example 4. (b) Two converted concave membership functions in Algorithm 2.

Step 1: Employ Proposition 1 to represent $\mu_{11}(-3 \leqslant z_1 \leqslant 27)$, $\mu_{12}(27 \leqslant z_1 \leqslant 47)$, $\mu_{21}(7 \leqslant z_2 \leqslant 32)$ and $\mu_{22}(32 \leqslant z_2 \leqslant 45)$ as follows:

$$\mu_{11}(-3 \leqslant z_1 \leqslant 27) = 0.04(z_1 + 3) + 0.02(|z_1 - 2| + z_1 - 2) - 0.1(|z_1 - 12| + z_1 - 12)$$
$$+ 0.04(|z_1 - 17| + z_1 - 17),$$
$$\mu_{12}(27 \leqslant z_1 \leqslant 47) = 0.04(z_1 - 27) + 0.02(|z_1 - 42| + z_1 - 42),$$
$$\mu_{21}(7 \leqslant z_2 \leqslant 32) = 0.06(z_2 - 7) + 0.02(|z_2 - 17| + z_2 - 17) - 0.0665(|z_2 - 21| + z_2 - 21)$$
$$- 0.03335(|z_2 - 27| + z_2 - 27) - 0.075(|z_2 - 30| + z_2 - 30),$$
$$\mu_{22}(32 \leqslant z_2 \leqslant 45) = 0.04(z_2 - 32) + 0.03(|z_2 - 37| + z_2 - 37).$$

Step 2: Based on Remarks 1 and 3, after finding the convex-type point then the mapping points can be obtained by the following equations:

$$b'_{11} = \mu_1^{-1}[\mu_2(17)] = 7, \quad b'_{12} = \mu_1^{-1}[\mu_2(37)] = 32, \quad b'_{21} = \mu_2^{-1}[\mu_1(2)] = \tfrac{31}{3},$$
$$b'_{22} = \mu_2^{-1}[\mu_1(17)] = \tfrac{91}{3} \quad \text{and} \quad b'_{23} = \mu_2^{-1}[\mu_1(42)] = 41.$$

Step 3: Using Remark 4 to specify the converted functions $\mu'_{11}(z_1), \mu'_{12}(z_1), \mu'_{21}(z_2)$ and $\mu'_{22}(z_2)$, as shown in Fig. 9(b), respectively:

$$\mu'_{11}(z_1) = s_1(z_1+3) + \frac{s_2 - s_1}{2}(|z_1 - 7| + z_1 - 7) + \frac{s_3 - s_2}{2}(|z_1 - 12| + z_1 - 12), \tag{5.4}$$

$$\mu'_{12}(z_1) = s_4(z_1 - 27) + \frac{s_5 - s_4}{2}(|z_1 - 32| + z_1 - 32), \tag{5.5}$$

$$\begin{aligned}\mu'_{21}(z_2) = {} & t_1(z_2 - 7) + \frac{t_2 - t_1}{2}\left(\left|z_2 - \tfrac{31}{3}\right| + z_2 - \tfrac{31}{3}\right) + \frac{t_3 - t_2}{2}(|z_2 - 21| + z_2 - 21) \\ & + \frac{t_4 - t_3}{2}(|z_2 - 27| + z_2 - 27) + \frac{t_5 - t_4}{2}(|z_2 - 30| + z_2 - 30) \\ & + \frac{t_6 - t_5}{2}\left(\left|z_2 - \tfrac{91}{3}\right| + z_2 - \tfrac{91}{3}\right),\end{aligned} \tag{5.6}$$

$$\mu'_{22}(z_2) = t_7(z_2 - 32) + \frac{t_8 - t_7}{2}(|z_2 - 41| + z_2 - 41), \tag{5.7}$$

Step 4: In reference to Eqs. (3.8)–(3.10), the slopes s_i and t_j, $i = 1, 2, \ldots, 5$ and $j = 1, 2, \ldots, 8$, in (5.4)–(5.7) can be computed by solving the following equations:

$$\begin{aligned}
& s_1 > s_2 > s_3,\ s_4 > s_5,\ t_1 > t_2 > t_3 > t_4 > t_5 > t_6,\ t_7 > t_8, \\
& \mu_{11}(12) = \mu'_{11}(12) = 10s_1 + 5s_2 = 1, \quad \mu_{11}(27) = \mu'_{11}(27) = 10s_1 + 5s_2 + 15s_3 = 0, \\
& \mu_{12}(47) = \mu'_{12}(47) = 5s_4 + 15s_5 = 1, \quad \mu_{21}(21) = \mu'_{21}(21) = \tfrac{26}{3}t_1 + \tfrac{16}{3}t_2 = 1, \\
& \mu_{21}(32) = \mu'_{21}(32) = \tfrac{10}{3}t_1 + \tfrac{32}{3}t_2 + 6t_3 + 3t_4 + \tfrac{1}{3}t_5 + \tfrac{5}{3}t_6 = 0, \\
& \mu_{22}(45) = \mu'_{22}(45) = 9t_7 + 4t_8 = 1, \quad \frac{\mu_{11}(2)}{\mu_{21}(\frac{31}{3})} = \frac{\mu'_{11}(2)}{\mu'_{21}(\frac{31}{3})} = \frac{5s_1}{\frac{10}{3}t_1} = 1, \\
& \frac{\mu_{11}(7)}{\mu_{21}(17)} = \frac{\mu'_{11}(7)}{\mu'_{21}(17)} = \frac{10s_1}{\frac{10}{3}t_1 + \frac{20}{3}t_2} = 1, \quad \frac{\mu_{11}(14)}{\mu_{21}(27)} = \frac{\mu'_{11}(14)}{\mu'_{21}(27)} = \frac{10s_1 + 5s_2 + 2s_3}{\frac{10}{3}t_1 + \frac{32}{3}t_2 + 6t_3} = 1, \\
& \frac{\mu_{11}(16)}{\mu_{21}(30)} = \frac{\mu'_{11}(16)}{\mu'_{21}(30)} = \frac{10s_1 + 5s_2 + 4s_3}{\frac{10}{3}t_1 + \frac{32}{3}t_2 + 6t_3 + 3t_4} = 1, \\
& \frac{\mu_{11}(17)}{\mu_{21}(\frac{91}{3})} = \frac{\mu'_{11}(17)}{\mu'_{21}(\frac{91}{3})} = \frac{10s_1 + 5s_2 + 5s_3}{\frac{10}{3}t_1 + \frac{32}{3}t_2 + 6t_3 + 3t_4 + \frac{1}{3}t_5} = 1, \\
& \frac{\mu_{12}(32)}{\mu_{22}(37)} = \frac{\mu'_{12}(32)}{\mu'_{22}(37)} = \frac{10s_1 + 5s_2 + 15s_3 + 5s_4}{\frac{10}{3}t_1 + \frac{32}{3}t_2 + 6t_3 + 3t_4 + \frac{1}{3}t_5 + \frac{5}{3}t_6 + 5t_7} = 1, \\
& \frac{\mu_{12}(42)}{\mu_{22}(41)} = \frac{\mu'_{12}(42)}{\mu'_{22}(41)} = \frac{10s_1 + 5s_2 + 15s_3 + 5s_4 + 10s_5}{\frac{10}{3}t_1 + \frac{32}{3}t_2 + 6t_3 + 3t_4 + \frac{1}{3}t_5 + \frac{5}{3}t_6 + 9t_7} = 1.
\end{aligned}$$

After computing on the LINDO [12], the solutions found are $s_1 = 0.077$, $s_2 = 0.046$, $s_3 = -0.067$, $s_4 = 0.091$, $s_5 = 0.0364$, $t_1 = 0.11539$, $t_2 = 0.058$, $t_3 = -0.022$, $t_4 = -0.044$, $t_5 = -0.2$, $t_6 = -0.4$, $t_7 = 0.091$, and $t_8 = 0.0455$.

Table 2
Efficiency comparison for solving Example 2

FMOP Models	Required zero–one variables	Required extra constraints	Required subproblems	Required LP computation	Required point calculation
Narasimhan's, Hannan's and Inuiguchi et al. methods	Cannot treat Example 2				
Yang et al. method	9	26	0	1	14
Nakamura's method	0	26	26	26	0
Proposed method	2	4	0	1	5

Therefore, the program (5.3) becomes

$$\begin{aligned}
\text{Maximize}\quad & \lambda' \\
\text{Subject to}\quad & \lambda' \leqslant 0.077(Z_1+3) - 0.0154(|Z_1-7|+Z_1-7) - 0.056576(|Z_1-12|+Z_1-12) + M\delta_1, \\
& \lambda' \leqslant 0.158(Z_1-27) - 0.0273(|Z_1-42|+Z_1-42) + M(1-\delta_1), \\
& \lambda' \leqslant 0.11539(Z_2-7) - 0.029\left(\left|Z_2-\tfrac{31}{2}\right| + Z_2 - \tfrac{31}{3}\right) - 0.0399(|Z_2-21|+Z_2-21) \\
& \quad - 0.011(|Z_2-27|+Z_2-27) - 0.078(|Z_2-30|+Z_2-30) \\
& \quad - 0.1\left(\left|Z_2-\tfrac{91}{3}\right| + Z_2 - \tfrac{91}{3}\right) + M\delta_2, \\
& \lambda' \leqslant 0.49(Z_2-32) - 0.023(|Z_2-41|+Z_2-41) + M(1-\delta_2),
\end{aligned}$$

where M is a big number and δ_1, δ_2 are 0–1 variables.
Step 5: Employing Proposition 4, the above problem can then be linearized below:

$$\begin{aligned}
\text{Maximize}\quad & \lambda' \\
\text{Subject to}\quad & \lambda' \leqslant 0.067Z_1 - 0.031d_1 - 0.113d_2 + 1.804 + M\delta_1, \\
& \lambda' \leqslant 0.103Z_1 - 0.0556d_3 - 1.973 + M(1-\delta_1), \\
& \lambda' \leqslant 0.4Z_2 - 0.058d_4 - 0.0798d_5 - 0.022d_6 - 0.156d_7 - 0.2d_8 + 12.808 + M\delta_2, \\
& \lambda' \leqslant 0.444Z_2 - 0.046d_9 - 13.794 + M(1-\delta_2), \\
& z_1 - 7 + d_1 \geqslant 0, z_1 - 12 + d_2 \geqslant 0, z_1 - 42 + d_3 \geqslant 0, \\
& z_2 - (31/3) + d_4 \geqslant 0, z_2 - 21 + d_5 \geqslant 0, z_2 - 27 + d_6 \geqslant 0, \\
& z_2 - 30 + d_7 \geqslant 0, z_2 - (91/3) + d_8 \geqslant 0, z_2 - 41 + d_9 \geqslant 0, \\
& z_1 = -x_1 + 2x_2, z_2 = 2x_1 + x_2, -x_1 + 3x_2 \leqslant 21, x_1 + 3x_2 \leqslant 27, \\
& 4x_1 + 3x_2 \leqslant 45, 3x_1 + x_2 \leqslant 30, x_1, x_2 \geqslant 0,
\end{aligned}$$

After running on LINDO [12], the solutions obtained are $x_1 = 5.62$, $x_2 = 7.13$, $z_1 = 8.64$ and $z_2 = 18.36$. This is exactly the optimal solution of Example 2. Comparing the traditional FMOP methods with Algorithm 2, we have the following Table 2.

6. Concluding remarks

Real-world membership functions in engineering, physical, business, social, and management fields are not pure linear, triangular, concave, or convex shapes but rather are more general non-concave curves, this paper devotes its effort to solve a quasi-concave or more general non-concave FMOP problem. Comparing with conventional FMOP methods, the proposed method can directly solve a quasi-concave FMOP problem by using standard LP techniques. Besides, there is no requirement to add extra zero–one variables or to divide the original problem into several sub-problems for solving a quasi-concave FMOP problem. Without tiresome solution process, the proposed method can be extended to solve more general non-concave FMOP problems by adding less number of zero–one variables.

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