noting seems to be the formulation of initial values for the general differential-integro-algebraic representations. As a matter of a fact, transfer equivalence where initial conditions are not considered, has been investigated in [20] for rational equations in Rosenbrock's form.

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# Optimal Multistage Kalman Estimators

Fu-Chuang Chen and Chien-Shu Hsieh

***Abstract*—An optimal multistage Kalman estimator (OMSKE) is proposed as a generalization of the optimal two-stage Kalman estimator for the reduction of the computational burden of the Kalman estimator (KE) for discrete-time linear time-varying systems with triangular transition matrices. This new filter is obtained by applying a multistage $U-V$ transformation to decouple the covariances of the KE. It is shown analytically that the computational complexity of the OMSKE is less than that of the KE and is minimum when the system transition matrix has the maximum stage number.**

***Index Terms*—Augmented state Kalman estimator, multistage Kalman estimator, optimal filter, two-stage Kalman estimator.**

## I. INTRODUCTION

Consider the problem of estimating the state of a dynamic system in which the system transition matrix has an upper triangular form. A special case of this problem is illustrated by the estimation problem of a dynamical system subject to an unknown bias [1]–[4]. In this specific problem, the system state is augmented to include the bias state, and the system is reformulated as an augmented system in which the system transition matrix is in an upper triangular form. It is common to use the augmented state Kalman estimator (ASKE) to estimate the augmented state. However, to reduce the computational cost and numerical errors, a two-stage Kalman estimator (TSKE) proposed by Friedland [1] can be used to obtain the Kalman estimate. Unfortunately, Friedland's filter is optimal only for constant biases. Ignagni [2] considered the stochastic case for applying Friedland's filter. However, the result he obtained is suboptimal. Alouani *et al.* [3] have proposed a sufficient condition for Friedland's filter to be optimal when it is applied to stochastic systems. However, this sufficient condition is seldom satisfied in practical systems. Another illustration of the considering problem is the state estimation in a maneuvering target tracking application, where the state is composed of position, velocity, and acceleration.

Recently [4], we have generalized Friedland's filter by accounting for the bias noise effect to obtain the optimal two-stage Kalman estimator (OTSKE), which is optimal in the sense that it can generate the minimum-mean-square-error (MMSE) estimate of the system state without any constraint. The OTSKE is derived by applying a two-stage $U-V$ transformation to decouple the state and the covariance equations of the ASKE. The main reason that the OTSKE can reduce the computational complexity of the ASKE is due to order reduction. It is more expensive to implement an "$n1+n2$"-order Kalman filter than two Kalman filters with orders $n1$ and $n2$. Hence, it is expected that a multistage filter, when applicable, can do better than a two-stage filter in terms of computation.

The objective of this note is to propose a generalization of the OTSKE for the problem at hand to further simplify the computational complexity of the OTSKE, and the obtained filter will be denoted as the optimal multistage Kalman estimator (OMSKE). This new filter is derived by applying a multistage $U-V$ transformation, which is a generalization of the conventional two-stage $U-V$ transformation, to

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decouple the covariance matrices of the Kalman estimator (KE), and is composed of covariance-decoupled subfilters. It is shown analytically that the computational complexity of the OMSKE is less than that of the KE and is minimum when the system transition matrix has the maximum stage number.

This paper is organized as follows. In Section II, we state the problem of interest. In Section III, the OMSKE is derived, and an algorithm for implementing it is provided. In Section IV, the computational complexity of the OMSKE is analyzed analytically, and its computational advantage over the KE is also shown. Section V is the conclusion. Detailed proofs are provided in the Appendix.

## II. Statement of the Problem

Consider the following dynamical system:

$$X_{k+1} = A_k X_k + w_k \tag{1}$$
$$Y_k = H_k X_k + \eta_k \tag{2}$$

where $X_k \in R^n$ is the system state, $Y_k \in R^m$ is the measurement vector, $A_k$ is an $r$-by-$r$ block upper triangular matrix in which $3 \leq r \leq n$, and the matrix $H_k$ has an appropriate dimension. The vectors $w_k$ and $\eta_k$ are zero-mean white noise sequences governed by $E\{w_k w_l'\} = Q_k \delta_{kl}$, $E\{\eta_k \eta_l'\} = R_k \delta_{kl}$ and $E\{w_k \eta_l'\} = 0$, where $'$ denotes transpose and $\delta_{kl}$ denotes the Kronecker delta function. The initial state $X_0$ is assumed to be a random variable with $E\{X_0\} = \overline{X}_0$ and Cov$\{X_0\} = \overline{P}_0$ and is uncorrelated with the white noise sequences $w_k$ and $\eta_k$. Note that the assumption of the upper triangular block form in the system transition matrix is not a limitation since it can be obtained by using Householder transformations or using the modifying equations presented in Remark B of the next section. On the other hand, the system (1), (2) may stand for an augmented-state system and is best illustrated by the maneuvering target tracking application; furthermore, if $r = 2$, then the treated problem in this paper is reduced to the conventional two-stage problem in [1]–[4].

To obtain an optimal state estimate for the system (1), (2), the conventional KE may be used, with the initial estimates $X_{0|0} = \overline{X}_0$ and $P_{0|0} = \overline{P}_0$. However, the computational cost and the estimation error of the KE increases drastically with the state dimension. Hence, the KE model may be impractical to implement. One method for solving the above-mentioned problems of the KE is to apply covariance decoupled algorithms. To this end, the approach of the two-stage $U-V$ transformation in deriving the OTSKE [4] may be used. However, the OTSKE applies only for $r = 2$. In the next section, we propose a generalization of the two-stage approach to further reduce the complexity of the OTSKE for $r > 2$, and the obtained new estimator will be denoted as the optimal multistage Kalman estimator (OMSKE).

## III. Optimal Multistage Kalman Estimators

The derivation of a multistage Kalman estimator is through a block-diagonalizing of the covariance matrices of the KE, i.e., $P_{k|k-1}$ and $P_{k|k}$. This can effectively reduce the complexity of calculating these covariances, and hence the overall Kalman filter algorithm. The problem is to find out some $U_k$ and $V_k$ matrices that satisfy the following: $P_{k|k-1} = U_k \overline{P}_{k|k-1} U_k'$ and $P_{k|k} = V_k \overline{P}_{k|k} V_k'$, where $\overline{P}_{(\cdot)} = \text{diag}\{\overline{P^1}_{(\cdot)}, \ldots, \overline{P^r}_{(\cdot)}\}$. We define the structures of the $U_k$ and $V_k$ matrices as follows:

$$U_k = \begin{bmatrix} I_{n_1} & U_k^{12} & \cdots & U_k^{1r} \\ 0 & I_{n_2} & \cdots & U_k^{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n_r} \end{bmatrix} \tag{3}$$

$$V_k = \begin{bmatrix} I_{n_1} & V_k^{12} & \cdots & V_k^{1r} \\ 0 & I_{n_2} & \cdots & V_k^{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n_r} \end{bmatrix}. \tag{4}$$

These $U_k^{ij}$, $V_k^{ij}$ and $\overline{P^i}_{(\cdot)}$ terms are to be determined, where $1 \leq i < j \leq r$.

To facilitate the derivation, we have used the following notation: $M^{ij}$ to denote the $ij$ element of matrix $M$. As a first step, we define the following multistage $U-V$ transformation, which is to be applied to the KE expressed by $\{X, K, P\}$, as

$$X_{k|k-1} = U_k \overline{X}_{k|k-1}, \quad P_{k|k-1} = U_k \overline{P}_{k|k-1} U_k' \tag{5}$$
$$X_{k|k} = V_k \overline{X}_{k|k}, \quad K_k = V_k \overline{K}_k, \quad P_{k|k} = V_k \overline{P}_{k|k} V_k' \tag{6}$$

where $\overline{X}$, $\overline{K}$, and $\overline{P}$ denote the transformed state, gain, and covariance, respectively. Then, denote the inverse transformations of $U_k$ and $V_k$ as

$$U_k^{-1} = \tilde{U}_k = \begin{bmatrix} I_{n_1} & \tilde{U}_k^{12} & \cdots & \tilde{U}_k^{1r} \\ 0 & I_{n_2} & \cdots & \tilde{U}_k^{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n_r} \end{bmatrix} \tag{7}$$

$$V_k^{-1} = \tilde{V}_k = \begin{bmatrix} I_{n_1} & \tilde{V}_k^{12} & \cdots & \tilde{V}_k^{1r} \\ 0 & I_{n_2} & \cdots & \tilde{V}_k^{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n_r} \end{bmatrix}. \tag{8}$$

Thus, (5) and (6) become

$$\overline{X}_{k|k-1} = \tilde{U}_k X_{k|k-1}, \quad \overline{P}_{k|k-1} = \tilde{U}_k P_{k|k-1} \tilde{U}_k' \tag{9}$$
$$\overline{X}_{k|k} = \tilde{V}_k X_{k|k}, \quad \overline{K}_k = \tilde{V}_k K_k, \quad \overline{P}_{k|k} = \tilde{V}_k P_{k|k} \tilde{V}_k'. \tag{10}$$

Next, based on (5), (6), (9), and (10), the transformed filter can be calculated recursively via the *two-step recursive substitution* method [4] as

$$\overline{X}_{k|k-1} = \tilde{U}_k A_{k-1} V_{k-1} \overline{X}_{k-1|k-1} \tag{11}$$
$$\overline{X}_{k|k} = \tilde{V}_k U_k \overline{X}_{k|k-1} + \overline{K}_k (Y_k - H_k U_k \overline{X}_{k|k-1}) \tag{12}$$
$$\overline{P}_{k|k-1} = \tilde{U}_k (A_{k-1} V_{k-1} \overline{P}_{k-1|k-1} (A_{k-1} V_{k-1})' + Q_{k-1}) \tilde{U}_k' \tag{13}$$
$$\overline{K}_k = \tilde{V}_k U_k \overline{P}_{k|k-1} (H_k U_k)' \times \{H_k U_k \overline{P}_{k|k-1} (H_k U_k)' + R_k\}^{-1} \tag{14}$$
$$\overline{P}_{k|k} = (\tilde{V}_k U_k - \overline{K}_k H_k U_k) \overline{P}_{k|k-1} (\tilde{V}_k U_k)'. \tag{15}$$

To express (11)–(15) in subfilters form, we define the following notations:

$$A_{k-1} V_{k-1} = \begin{bmatrix} \overline{U}_k^{11} & \overline{U}_k^{12} & \cdots & \overline{U}_k^{1r} \\ 0 & \overline{U}_k^{22} & \cdots & \overline{U}_k^{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \overline{U}_k^{rr} \end{bmatrix} \tag{16}$$

$$H_k U_k = [\, S_k^1 \quad \cdots \quad S_k^r \,] \tag{17}$$

where

$$\overline{U}_k^{ij} = A_{k-1}^{ij} + u_s^{(j-i-1)} \sum_{l=i}^{j-1} A_{k-1}^{il} V_{k-1}^{lj} \tag{18}$$

$$S_k^i = H_k^i + u_s^{(i-2)} \sum_{l=1}^{i-1} H_k^l U_k^{li} \tag{19}$$

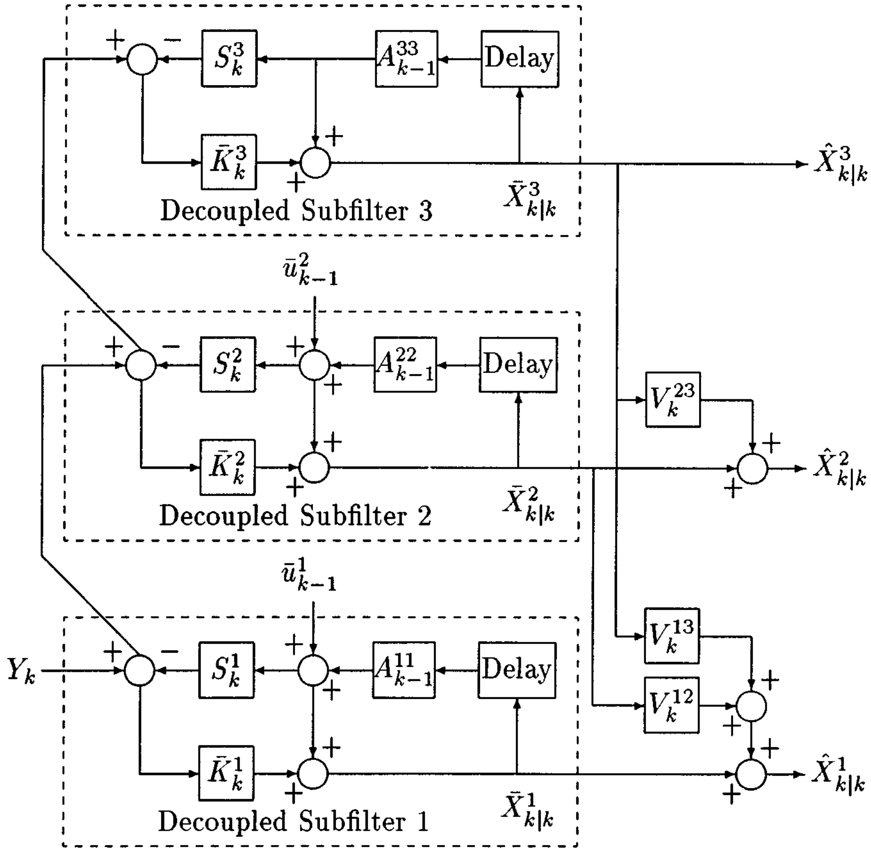


Fig. 1. Block diagram of the optimal multistage Kalman estimator ($r = 3$).

in which $u_s^{(\bullet)}$ denotes the unit-step function and $1 \leq i \leq j \leq r$. Using (8) and (16)–(19), and expanding (11)–(15), we obtain the following covariance-decoupled subfilter $i$ ($i = 1, \ldots, r$):

$$\overline{X}_{k|k-1}^i = A_{k-1}^{ii}\overline{X}_{k-1|k-1}^i + \overline{u}_{k-1}^i \tag{20}$$

$$\overline{X}_{k|k}^i = \overline{X}_{k|k-1}^i + \overline{K}_k^i(\overline{Y}_k^i - S_k^i\overline{X}_{k|k-1}^i) \tag{21}$$

$$\overline{P}_{k|k-1}^i = A_{k-1}^{ii}\overline{P}_{k-1|k-1}^i(A_{k-1}^{ii})' + \overline{Q}_{k-1}^i \tag{22}$$

$$\overline{K}_k^i = \overline{P}_{k|k-1}^i(S_k^i)'\{S_k^i\overline{P}_{k|k-1}^i(S_k^i)' + \overline{R}_k^i\}^{-1} \tag{23}$$

$$\overline{P}_{k|k}^i = (I - \overline{K}_k^i S_k^i)\overline{P}_{k|k-1}^i \tag{24}$$

where

$$\overline{u}_{k-1}^i = u_s^{(r-i-1)}\sum_{l=i+1}^{r}\left(\overline{U}_k^{il}\overline{X}_{k-1|k-1}^l - U_k^{il}\overline{X}_{k|k-1}^l\right) \tag{25}$$

$$\overline{Y}_k^i = \overline{Y}_k^{i-1} - u_s^{(i-2)}S_k^{i-1}\overline{X}_{k|k-1}^{i-1}, \quad \overline{Y}_k^0 = Y_k \tag{26}$$

$$\overline{Q}_{k-1}^i = Q_{k-1}^{ii} + u_s^{(r-i-1)} \times \sum_{l=i+1}^{r} \times \left(\overline{U}_k^{il}(\overline{U}_k^{il}\overline{P}_{k-1|k-1}^l)' - U_k^{il}(U_k^{il}\overline{P}_{k|k-1}^l)'\right) \tag{27}$$

$$\overline{R}_k^i = \overline{R}_k^{i-1} + u_s^{(i-2)}S_k^{i-1}\overline{P}_{k|k-1}^{i-1}(S_k^{i-1})', \quad \overline{R}_k^0 = R_k. \tag{28}$$

The blending matrices $U_k$ and $V_k$ are then calculated by

$$U_k^{ij} = \left(\overline{U}_k^{ij}\overline{P}_{k-1|k-1}^j(A_{k-1}^{jj})' + \overline{Q}_{k-1}^{ij}\right)(\overline{P}_{k|k-1}^j)^{-1} \tag{29}$$

$$V_k^{ij} = U_k^{ij} - \left(\overline{K}_k^i + u_s^{(j-i-2)}\sum_{l=i+1}^{j-1}V_k^{il}\overline{K}_k^l\right)S_k^j \tag{30}$$

where

$$\overline{Q}_{k-1}^{ij} = Q_{k-1}^{ij} + u_s^{(r-j-1)} \times \sum_{l=j+1}^{r} \times \left(\overline{U}_k^{il}(\overline{U}_k^{jl}\overline{P}_{k-1|k-1}^l)' - U_k^{il}(U_k^{jl}\overline{P}_{k|k-1}^l)'\right) \tag{31}$$

and $1 \leq i < j \leq r$. Equations (20)–(24), (29), and (30) are derived in the Appendix. Note that in the above algorithm, the measurement vector and the measurement noise covariance of subfilter $i$ are replaced by the innovation vector and the innovation covariance of subfilter $i-1$.

As a last step, the estimate of the KE can be obtained by the following OMSKE:

$$\hat{X}_{k|k} = V_k\overline{X}_{k|k} \tag{32}$$

$$\hat{P}_{k|k} = V_k\overline{P}_{k|k}V_k' \tag{33}$$

with the following initial conditions:

$$\overline{X}_{0|0}^i = \overline{X}_0^i - u_s^{(r-i-1)}\sum_{l=i+1}^{r}V_0^{il}\overline{X}_{0|0}^l \tag{34}$$

$$\overline{P}_{0|0}^i = \overline{P}_0^{ii} - u_s^{(r-i-1)}\sum_{l=i+1}^{r}V_0^{il}\overline{P}_{0|0}^l(V_0^{il})' \tag{35}$$

$$V_0^{ji} = \overline{P}_0^{ji}(\overline{P}_{0|0}^i)^{-1} \tag{36}$$

where $i = r, \ldots, 1$ and $j = 1, \ldots, i-1$. The structure of the proposed OMSKE with $r = 3$ is shown in Fig. 1.

*Remark A:* The OMSKE is mathematically equivalent to the KE. This is deduced from the inductive reasoning imbedded in the proposed two-step iterative substitution method [4], and can be verified by using inductive reasoning as in [3]. In fact, it can be checked that the KE and the OTSKE can be thought of as special cases of the OMSKE with

$r = 1$ and $r = 2$, respectively. In order for (29) to be applicable, it is assumed that $\overline{P}^j_{k|k-1}$ is nonsingular. This is the most often encountered situation in practical applications since the KE is applied mainly to stochastic systems where the process noise covariance $Q_k$ is positive. However, this restriction can also be removed and is described as follows. If $\overline{P}^j_{k|k-1} = 0$, then the matrices $U^{ij}_k$, where $1 \le i < j$, have no influence on the obtained filter. Thus, one can simply replace (29) by $U^{ij}_k = 0$. On the other hand, if $\overline{P}^i_{k|k-1}$ is singular ($\neq 0$), then one only need to replace (29) by

$$U^{ij}_k = \left(\overline{U}^{ij}_k \overline{P}^j_{k-1|k-1}(A^{jj}_{k-1})' + \overline{Q}^{ij}_{k-1}\right)\left(\overline{P}^j_{k|k-1}\right)^+ \tag{37}$$

where $M^+$ is the *Moore–Penrose* pseudoinverse of $M$ [11].

*Remark B:* As mentioned in the preceding section, the proposed multistage algorithm can be modified to be applied to general systems. This is done by modifying (18), (25), (27), and (31) by

$$\overline{U}^{ij}_k := \overline{U}^{ij}_k + u_s^{(i-2)} \sum_{l=1}^{i-1} A^{il}_{k-1} V^{lj}_{k-1} \tag{38}$$

$$\overline{u}^i_{k-1} := \overline{u}^i_{k-1} + u_s^{(i-2)} \sum_{l=1}^{i-1} \overline{U}^{il}_k \overline{X}^l_{k-1|k-1} \tag{39}$$

$$\overline{Q}^i_{k-1} := \overline{Q}^i_{k-1} + u_s^{(i-2)} \sum_{l=1}^{i-1} \overline{U}^{il}_k (\overline{U}^{il}_k \overline{P}^l_{k-1|k-1})' \tag{40}$$

$$\overline{Q}^{ij}_{k-1} := \overline{Q}^{ij}_{k-1} + \sum_{l=1}^{j-1} \overline{U}^{il}_k (\overline{U}^{jl}_k \overline{P}^l_{k-1|k-1})' \tag{41}$$

where the symbol ":=" means "is replaced by." Basically, in the above modification, only the time update equations need to be revised.

*Remark C:* If the system [(1) and (2)] represents a parallelly connected system, i.e., $A^{ij}_k = 0, Q^{ij}_k = 0$, and $H^j_k = 0$ for $1 \le i < j \le r$, then the blending matrices $U_k$ and $V_k$ become identity matrices, and hence the OMSKE will be equivalent to a decoupled Kalman filter.

*Remark D:* The OMSKE can be thought of as a general state-decentralized filtering algorithm in which the state space is decoupled while the output space remains coupled. This is different from conventional output-decentralized filtering algorithms, e.g., [5]–[8], where the complexity reduction is mainly due to the decoupling of the measurement space.

*Remark E:* The filtered error covariance given by (33) is not needed in the calculation of the recursive algorithm, e.g., (20)–(24). Hence, it may be calculated only when filtered covariance data are needed. On the other hand, the proposed algorithm can be thought of as an alternative to the well-known $U - D$ covariance factorization algorithm [9] under the assumption that the system transition matrix is triangularized. It is known [10] that numerical stability of the covariance factorization algorithm was due to their lack of sensitivity to the choice of *a priori* variances and process noise levels. Thus, it is expected that the proposed OMSKE has the advantage of both numerical accuracy and computational efficiency. In this paper, we focus our attention on the computational efficiency of the OMSKE.

*Remark F:* To implement the OMSKE, (22), (27), and (29) are calculated as follows:

$$\begin{bmatrix} U^{1i}_k \overline{P}^i_{k|k-1} \\ \vdots \\ U^{i-1,\,i}_k \overline{P}^i_{k|k-1} \\ \overline{P}^i_{k|k-1} \end{bmatrix} = \begin{bmatrix} Q^{1i}_{k-1} \\ \vdots \\ Q^{ii}_{k-1} \end{bmatrix} - [U_k]^{1r}_{i,\,i+1} \times \begin{bmatrix} (U^{i,\,i+1}_k \overline{P}^i_{k|k-1})' \\ \vdots \\ (U^{ir}_k \overline{P}^r_{k|k-1})' \end{bmatrix} + [\overline{U}_k]^{1r}_{ii} \begin{bmatrix} (\overline{U}^{ii}_k \overline{P}^i_{k-1|k-1})' \\ \vdots \\ (\overline{U}^{ir}_k \overline{P}^r_{k-1|k-1})' \end{bmatrix} \tag{42}$$

$$\begin{bmatrix} U^{1i}_k \\ \vdots \\ U^{i-1,\,i}_k \end{bmatrix} = \begin{bmatrix} U^{1i}_k \overline{P}^i_{k|k-1} \\ \vdots \\ U^{i-1,\,i}_k \overline{P}^i_{k|k-1} \end{bmatrix} (\overline{P}^i_{k|k-1})^{-1} \tag{43}$$

where

$$[M]^{ij}_{mn} = \begin{bmatrix} M^{in} & \cdots & M^{ij} \\ \vdots & \ddots & \vdots \\ M^{mn} & \cdots & M^{mj} \end{bmatrix}.$$

Finally, we give the procedure for implementing the OMSKE in each cycle as follows.

Step 1) Calculate $\overline{U}^{ij}_k$, $1 \le i < j \le r$, by (18).

Step 2) Calculate the time update equations for subfilter $j$ ($j = r, \ldots, 1$) through:

- 2.1) calculate $\overline{u}^j_{k-1}$ and $\overline{X}^j_{k|k-1}$ by (25) and (20), respectively.
- 2.2) calculate $\overline{P}^j_{k|k-1}$ and $U^{ij}_k$, $i = 1, \ldots, j-1$, by (42) and (43), respectively.

Step 3) Calculate the new measurement matrix $S^i_k$, $i = 1, \ldots, r$, by (19).

Step 4) Calculate the measurement update equations for subfilter $i$ ($i = 1, \ldots, r$), by (21), (23), and (24).

Step 5) Calculate the blending matrix $V^{ij}_k$, $1 \le i < j \le r$, by (30).

Step 6) Calculate the Kalman estimate by (32) and (33).

## IV. Computation Evaluations

To demonstrate the computational advantage of the OMSKE over the KE, we use floating-point operations or "flops" in Matlab as a measure of the computational complexity. Each multiplication and each addition contributes one flop count. To simplify the discussion of the computational load, we consider the special case that subfilters have the same dimensions ($n_i = \overline{n} = n/r$).

As a first step, we list the flops count of a general KE (KE$^g$), which has state dimension $n$ and measurement dimension $m$, as follows:

$$\text{flops}(\text{KE}^g) = 6n^3 + (4m+3)n^2 + (4m^2+4m+2)n + 2m^3 + m^2 + m. \tag{44}$$

Note that if the symmetric property of the covariance matrices has been used, then the above flops count is reduced to

$$\text{flops}(\text{KE}^s) = 4n^3 + (4m+4.5)n^2 + (3m^2+5m+2.5)n + 2m^3 + 0.5m^2 + 1.5m \tag{45}$$

where we used "KE$^s$" to represent the above simplified KE. Furthermore, if the upper triangular property of the system transition matrix, which has block number $r$, is used, the flops count in (45) is further reduced to

$$\text{flops}(\text{KE}^t) = \left(2 + \frac{1.5}{r} + \frac{0.5}{r^2}\right)n^3 + \left(4m + 3 + \frac{1.5}{r}\right) \times n^2 + (3m^2+5m+2.5)n + 2m^3 + 0.5m^2 + 1.5m \tag{46}$$

and the obtained algorithm is referenced to by the name "KE$^t$."

Then, the complexity of the OMSKE is evaluated as follows. From (45), it is clear that the flops count of the covariance-decoupled subfilters is

$$\text{flops(Subfilters)} = \frac{4}{r^2}n^3 + \frac{4m+4.5}{r}n^2 + (3m^2+5m+2.5)n + r(2m^3+0.5m^2+1.5m). \tag{47}$$

The flops count of the auxiliary calculations specifically needed by the OMSKE, e.g., the quantities calculated in Steps 1), 2.1), 2.3), 2.5), 3), 5), and 6) of the preceding section, is given as follows:

$$\begin{aligned}&\text{flops(Auxiliary)}\\&=\left(\frac{4}{3}+\frac{4}{r}-\frac{10}{3r^2}-\frac{2}{r^3}\right)n^3+(3m+6)\times n^2\\&\quad-\left(\frac{5m+6}{r}-\frac{2m}{r^2}\right)n^2+\left(0.5m(r-1)+3-\frac{2}{r}\right)n. \qquad (48)\end{aligned}$$

Using (47) and (48), the flops saving, denoting by $\triangle$flops, of the OMSKE as compared to the KE$^g$, the KE$^s$, and the KE$^t$ are given, respectively, by

$$\begin{aligned}\triangle\text{flops}_{\text{KE}^g}^{\text{OMSKE}} &= \left(\frac{14}{3}-\frac{4}{r}-\frac{2}{3r^2}+\frac{2}{r^3}\right)n^3\\&\quad+\left(m-3+\frac{m+1.5}{r}-\frac{2m}{r^2}\right)n^2\\&\quad+\left(m^2-0.5m(r+1)-3.5+\frac{2}{r}\right)n\\&\quad-(r-1)(2m^3+0.5m^2+1.5m)\\&\quad+0.5m(m-1) \qquad (49)\\ \triangle\text{flops}_{\text{KE}^s}^{\text{OMSKE}} &= \left(\frac{8}{3}-\frac{4}{r}-\frac{2}{3r^2}+\frac{2}{r^3}\right)n^3\\&\quad+\left(m-1.5+\frac{m+1.5}{r}-\frac{2m}{r^2}\right)n^2\\&\quad-\left(0.5m(r-1)+3-\frac{2}{r}\right)n\\&\quad-(r-1)(2m^3+0.5m^2+1.5m) \qquad (50)\\ \triangle\text{flops}_{\text{KE}^t}^{\text{OMSKE}} &= \left(\frac{2}{3}-\frac{2.5}{r}-\frac{1}{6r^2}+\frac{2}{r^3}\right)n^3\\&\quad+\left(m-3+\frac{m+3}{r}-\frac{2m}{r^2}\right)n^2\\&\quad-\left(0.5m(r-1)+3-\frac{2}{r}\right)n\\&\quad-(r-1)(2m^3+0.5m^2+1.5m). \qquad (51)\end{aligned}$$

Note that the above performance is satisfied for $r < n$. However, if $r = n$, then the performance in (51) will increase to

$$\begin{aligned}&\triangle\text{flops}_{\text{KE}^t}^{\text{OMSKE}}\\&=\tfrac{2}{3}n^3+(m-3.5)n^2+(0.5m^2+2.5m+\tfrac{35}{6})n\\&\quad-(n-1)(2m^3+0.5m^2+1.5m). \qquad (52)\end{aligned}$$

This additionally gained efficiency is mainly due to the fact that matrix multiplications are replaced by scalar multiplications.

To show the computational advantage of the OMSKE, we use the relative improvement ratio (RIR) of the OMSKE, which is defined by

$$\text{RIR}_{\text{KE}^i}^{\text{OMSKE}} = \lim_{n\to\infty}\frac{\triangle\text{flops}_{\text{KE}^i}^{\text{OMSKE}}}{\text{flops}(KE^i)} \qquad (53)$$

as the performance index. The RIR performance is intended for the situation where the number of the measurements is negligible as compared to the number of the system states. This is assumed for applying the OMSKE and is usually the case encountered in practical systems. However, if this is not the case, as can occur in large-scale or multisensor systems, then it is always possible in practice to process the measurements sequentially [8] so as to satisfy the above mentioned situation. Thus, subject to $n \gg m$, it is clear from (44)–(46), (49)–(51), and (53) that the RIRs of the OMSKE as compared to the KE$^g$, the KE$^s$, and the KE$^t$ are given, respectively, by

$$\text{RIR}_{\text{KE}^g}^{\text{OMSKE}} = \frac{7r^3-6r^2-r+3}{9r^3} \qquad (54)$$

$$\text{RIR}_{\text{KE}^s}^{\text{OMSKE}} = \frac{4r^3-6r^2-r+3}{6r^3} \qquad (55)$$

$$\text{RIR}_{\text{KE}^t}^{\text{OMSKE}} = \frac{4r^3-15r^2-r+12}{12r^3+9r^2+3r}. \qquad (56)$$

Note that if $r$ is sufficiently large, then the above RIR values will approach 78%, 67%, and 33%, respectively. Seeing from the above results, although the conventional KE, i.e., KE$^g$, can be simplified by applying the symmetric and triangular properties of the KE, the proposed OMSKE can further reduce the complexity of this simplified KE, i.e., KE$^t$, for the case that the state number is much larger than the measurement number.

Finally, from the RIR performances [(54)–(56)], we claim the following key concept about the complexity issue of the multistage algorithm: the computational complexity of the multistage Kalman estimator is less if the larger $r$ is chosen and has the minimum value when the system transition matrix has the maximum stage number.

## V. Conclusion

In this paper, the OMSKE is proposed. The OMSKE is a generalization of the OTSKE [4] and is used instead of the OTSKE when the upper triangular block number of the system transition matrix is greater than two. It is shown by analytical results that the computational complexity of the OMSKE is less than that of the simplified KE (KE$^t$), which can be obtained by applying the symmetric and triangular properties of the KE, and is minimum when the system transition matrix has the maximum stage number. Our result suggests that the OMSKE may serve as an alternative to the KE for estimating the system state of linear dynamical systems subject to upper triangular system transition matrices when the number of the system state is large and is much larger than the number of the measurement.

As inspired by the work of Bierman [10], the numerical reliability of the OMSKE may be better than that of the conventional KE. The problem of exploring the numerical stability of the OMSKE is under investigation.

## Appendix

1) Using (3) and (5), we have

$$X_{k|k-1}^i = \overline{X}_{k|k-1}^i + u_s^{(r-i-1)}\sum_{l=i+1}^{r} U_k^{il}\overline{X}_{k|k-1}^l \qquad (57)$$

where $1 \le i \le r$. Comparing (9) with (11) and using (16), we have

$$\begin{aligned}X_{k|k-1}^i &= A_{k-1}^{ii}\overline{X}_{k-1|k-1}^i + u_s^{(r-i-1)}\sum_{l=i+1}^{r}\\&\quad\times\overline{U}_k^{il}\overline{X}_{k-1|k-1}^l. \qquad (58)\end{aligned}$$

Substituting (58) into (57) and using (25), we obtain

$$\begin{aligned}\overline{u}_{k-1}^i &= \overline{X}_{k|k-1}^i - A_{k-1}^{ii}\overline{X}_{k-1|k-1}^i\\&= u_s^{(r-i-1)}\sum_{l=i+1}^{r}\left(\overline{U}_k^{il}\overline{X}_{k-1|k-1}^l - U_k^{il}\overline{X}_{k|k-1}^l\right). \qquad (59)\end{aligned}$$

2) Using (3), (5), and the diagonalizing structure of $\overline{P}_{k|k-1}$, we have

$$P^{ii}_{k|k-1} = \overline{P}^{i}_{k|k-1} + u_s^{(r-i-1)} \sum_{l=i+1}^{r} U^{il}_k \overline{P}^{l}_{k|k-1} (U^{il}_k)' \tag{60}$$

where $1 \le i \le r$. Comparing (9) with (13) and using (16), we have

$$P^{ii}_{k|k-1} = A^{ii}_{k-1} \overline{P}^{i}_{k-1|k-1} (A^{ii}_{k-1})' + Q^{ii}_{k-1} + u_s^{(r-i-1)} \sum_{l=i+1}^{r} \overline{U}^{il}_k \overline{P}^{l}_{k-1|k-1} (\overline{U}^{il}_k)'. \tag{61}$$

Substituting (61) into (60) and using (27), we obtain

$$\begin{aligned}\overline{Q}^{i}_{k-1} &= \overline{P}^{i}_{k|k-1} - A^{ii}_{k-1} \overline{P}^{i}_{k-1|k-1} (A^{ii}_{k-1})' \\ &= Q^{ii}_{k-1} + u_s^{(r-i-1)} \sum_{l=i+1}^{r} \\ &\quad \times \{\overline{U}^{il}_k (\overline{U}^{il}_k \overline{P}^{l}_{k-1|k-1})' - U^{il}_k (U^{il}_k \overline{P}^{l}_{k|k-1})'\}.\end{aligned} \tag{62}$$

3) Using (15), (17), and the diagonalizing structure of $\overline{P}_{k|k}$, we have

$$\overline{P}_{k|k} = \begin{bmatrix} \{\bullet\}^{11}_2 & \{\bullet\}^{12}_2 & \cdots & \{\bullet\}^{1r}_2 \\ \{\bullet\}^{21}_2 & \{\bullet\}^{22}_2 & \cdots & \{\bullet\}^{2r}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \{\bullet\}^{r1}_2 & \{\bullet\}^{r2}_2 & \cdots & \{\bullet\}^{rr}_2 \end{bmatrix} \times \begin{bmatrix} I_{n_1} & 0 & \cdots & 0 \\ ((\tilde{V}_k U_k)^{12})' & I_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ ((\tilde{V}_k U_k)^{1r})' & ((\tilde{V}_k U_k)^{2r})' & \cdots & I_{n_r} \end{bmatrix}$$

with

$$\{\bullet\}^{ii}_2 = (I - \overline{K}^{i}_k S^{i}_k) \overline{P}^{i}_{k|k-1} = \overline{P}^{i}_{k|k} \tag{63}$$
$$\{\bullet\}^{ij}_2 = ((\tilde{V}_k U_k)^{ij} - \overline{K}^{i}_k S^{j}_k) \overline{P}^{j}_{k|k-1} = 0 \tag{64}$$

where $1 \le i \le r$ and $i < j \le r$. From (64) and the fact that $\overline{P}^{j}_{k|k-1}$ is nonsingular, we have

$$(\tilde{V}_k U_k)^{ij} = \overline{K}^{i}_k S^{j}_k. \tag{65}$$

4) Substituting (65) into (12), we have

$$\begin{aligned}\overline{X}^{i}_{k|k} &= \overline{X}^{i}_{k|k-1} + u_s^{(r-i-1)} \sum_{j=i+1}^{r} \overline{K}^{i}_k S^{j}_k \overline{X}^{j}_{k|k-1} \\ &\quad + \overline{K}^{i}_k \left( Y_k - \sum_{j=1}^{r} S^{j}_k \overline{X}^{j}_{k|k-1} \right) \\ &= \overline{X}^{i}_{k|k-1} + \overline{K}^{i}_k \left( Y_k - \sum_{j=1}^{i} S^{j}_k \overline{X}^{j}_{k|k-1} \right).\end{aligned} \tag{66}$$

5) From (14) and (65), we obtain

$$\begin{aligned}&\overline{K}^{i}_k \left( \sum_{j=1}^{r} S^{j}_k \overline{P}^{j}_{k|k-1} (S^{j}_k)' + R_k \right) \\ &= \overline{P}^{i}_{k|k-1} (S^{i}_k)' + u_s^{(r-i-1)} \sum_{j=i+1}^{r} \overline{K}^{i}_k S^{j}_k \overline{P}^{j}_{k|k-1} (S^{j}_k)'.\end{aligned} \tag{67}$$

Canceling the same terms on both sides of (67) and solving for $\overline{K}^{i}_k$, we obtain

$$\overline{K}^{i}_k = \overline{P}^{i}_{k|k-1} (S^{i}_k)' \left\{ \sum_{j=1}^{i} S^{j}_k \overline{P}^{j}_{k|k-1} (S^{j}_k)' + R_k \right\}^{-1}. \tag{68}$$

6) Using (3), (5), (9), and (13), we have

$$\begin{aligned}P^{ij}_{k|k-1} &= U^{ij}_k \overline{P}^{j}_{k|k-1} + u_s^{(r-j-1)} \sum_{l=j+1}^{r} U^{il}_k \overline{P}^{l}_{k|k-1} (U^{jl}_k)' \\ &= \overline{U}^{ij}_k \overline{P}^{j}_{k-1|k-1} (A^{jj}_{k-1})' + Q^{ij}_{k-1} \\ &\quad + u_s^{(r-j-1)} \sum_{l=j+1}^{r} \overline{U}^{il}_k \overline{P}^{l}_{k-1|k-1} (\overline{U}^{jl}_k)'\end{aligned} \tag{69}$$

where $1 \le i < j \le r$. Assuming that $\overline{P}^{j}_{k|k-1} > 0$ and solving (69) for $U^{ij}_k$, we obtain

$$U^{ij}_k = \left( \overline{U}^{ij}_k \overline{P}^{j}_{k-1|k-1} (A^{jj}_{k-1})' + \overline{Q}^{ij}_{k-1} \right) (\overline{P}^{j}_{k|k-1})^{-1} \tag{70}$$

where $\overline{Q}^{ij}_{k-1}$ is defined in (31).

7) Solving (8) for $\tilde{V}^{ij}_k$, we obtain

$$\tilde{V}^{ij}_k = -V^{ij}_k - u_s^{(j-i-2)} \sum_{l=i+1}^{j-1} V^{il}_k \tilde{V}^{lj}_k. \tag{71}$$

Using (65), (8), (4), and (71), we have

$$\begin{aligned}\overline{K}^{i}_k S^{j}_k = (\tilde{V}_k U_k)^{ij} &= U^{ij}_k + u_s^{(j-i-2)} \sum_{l=i+1}^{j-1} \tilde{V}^{il}_k U^{lj}_k + \tilde{V}^{ij}_k \\ &= U^{ij}_k - V^{ij}_k + u_s^{(j-i-2)} \sum_{l=i+1}^{j-1} (\tilde{V}^{il}_k U^{lj}_k - V^{il}_k \tilde{V}^{lj}_k).\end{aligned} \tag{72}$$

Solving (72) for $V^{ij}_k$ and using (71) and (72), we obtain

$$\begin{aligned}V^{ij}_k &= U^{ij}_k - \overline{K}^{i}_k S^{j}_k + u_s^{(j-i-2)} \sum_{l=i+1}^{j-1} V^{il}_k \\ &\quad \times \left\{ V^{lj}_k - U^{lj}_k + u_s^{(j-l-2)} \sum_{t=l+1}^{j-1} (V^{lt}_k \tilde{V}^{tj}_k - \tilde{V}^{lt}_k U^{tj}_k) \right\} + \{\bullet\} \\ &= U^{ij}_k - \left( \overline{K}^{i}_k + u_s^{(j-i-2)} \sum_{l=i+1}^{j-1} V^{il}_k \overline{K}^{l}_k \right) S^{j}_k\end{aligned} \tag{73}$$

where

$$\begin{aligned}\{\bullet\} &= \sum_{l=i+1}^{j-1} \left( \sum_{t=l+1}^{j-1} V^{il}_k \tilde{V}^{lt}_k U^{tj}_k - \sum_{t=i+1}^{l-1} V^{it}_k \tilde{V}^{tl}_k U^{lj}_k \right) \\ &= \left( \sum_{l=i+1}^{j-2} \sum_{t=l+1}^{j-1} V^{il}_k \tilde{V}^{lt}_k U^{tj}_k - \sum_{l=i+2}^{j-1} \sum_{t=i+1}^{l-1} V^{it}_k \tilde{V}^{tl}_k U^{lj}_k \right) \\ &= \left( \sum_{l=i+1}^{j-2} \sum_{t=l+1}^{j-1} V^{il}_k \tilde{V}^{lt}_k U^{tj}_k - \sum_{t=i+1}^{j-2} \sum_{l=t+1}^{j-1} V^{it}_k \tilde{V}^{tl}_k U^{lj}_k \right) \\ &= 0.\end{aligned}$$

From (59), (66), (62), (68), (63), (70), and (73), we obtain (20)–(24), (29), and (30).

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# Reliable Guaranteed Cost Control for Uncertain Nonlinear Systems

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***Abstract*—This paper is concerned with the reliable control design problem for uncertain nonlinear systems. A more practical model of actuator failures than outage is adopted. Based on the Hamilton–Jacobi inequality (HJI) approach from nonlinear $H_\infty$ control theory, a method for designing reliable state feedback controllers is presented. The resulting control systems are robustly stable and with an $H_2$ performance bound against plant uncertainty and actuator failures.**

***Index Terms*—Actuator failures, $H_2$ control, Hamilton–Jacobi inequalities, nonlinear systems, reliable control, uncertain systems.**

## I. Introduction

In the area of reliable control system design, several design methods have been developed for the resulting closed-loop systems to tolerate the failures of control components and retain the desired control system properties, see; [3], [5], [7], [8], and [10]–[14]. In [10], Veillette *et al.* presented a methodology for the design of reliable linear control systems such that the resulting design guaranteed closed-loop stability and $H_\infty$ performance not only when all control components are operational, but also in the case of some admissible control component outages. Veillette [10] developed a procedure for the design of $H_2$ state feedback controllers, where the resulting design could tolerate the outages within a selected subset of actuators while retaining the stability and the known quadratic performance bound. In the nonlinear case, Yang *et al.* [13] presented a reliable $H_\infty$ controller design for nonlinear systems, which is a nonlinear version of the results given in [10].

However, the above reliable controller design methods are all based on a basic assumption that control component failures are modeled as outages, which neglects the case of partial degradation. In this paper, a more general failure model is adopted for actuator failures, which covers the cases of normal operation, partial degradation and outage. Specifically, this paper is concerned with the problem of reliable guaranteed cost control for uncertain nonlinear systems.

The paper is organized as follows. The model of actuator failures, problem, and some preliminaries are given in Section II. In Section III, a design method for reliable guaranteed cost control is presented, the resulting design guarantees the robust stability and an $H_2$ performance bound against plant uncertainty and actuator failures. A numerical example is given to illustrate the design method in Section IV. Section V concludes the paper.

## II. Problem Statement and Preliminaries

Consider a class of uncertain nonlinear systems described by equations of the form

$$\begin{aligned}\dot{x}(t) &= f(x) + f_{11}(x)\Delta(x,\theta)f_{22}(x)\\ &\quad + [g(x) + f_{11}(x)\Delta(x,\theta)g_{22}(x)]u(t), \qquad x(0) = x_0\end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, $f(x)$, $g(x)$, $f_{11}(x)$, $f_{22}(x)$, and $g_{22}(x)$ are known smooth mappings with $f(0) = 0$, $f_{22}(0) = 0$, and $\Delta(x,\theta)$ satisfies

$$\Delta^T(x,\theta)\Delta(x,\theta) \le I \tag{2}$$

with $\theta$ being an uncertain parameter vector. The cost function associated with this system is

$$J = \int_0^\infty [Q(x) + u^T(t)R_2u(t)]\,dt \tag{3}$$

where $Q(x)$ is a smooth positive definite function with $Q(0) = 0$, and $R_2 > 0$ is a symmetric constant matrix. For the control input $u_i$, $i = 1, 2, \cdots, m$, let $u_i^F$ denote the signal from the actuator that has failed. In this paper the following actuator failure model will be adopted:

$$u_i^F = \overline{\alpha}_i u_i + \phi_i(u_i), \qquad i = 1, 2, \cdots, m \tag{4}$$

where $\overline{\alpha}_i > 0$, and the uncertain function $\phi_i(u_i)$ satisfies

$$\phi_i^2(u_i) \le \underline{\alpha}_i^2 u_i^2, \qquad i = 1, 2, \cdots, m \tag{5}$$

with $\overline{\alpha}_i \ge \underline{\alpha}_i \ge 0$.

*Remark 2.1:* In the above model of actuator failure, if $\overline{\alpha}_i = 1$, $\underline{\alpha}_i = 0$, then it corresponds to the normal case $u_i^F = u_i$. When $\overline{\alpha}_i = \underline{\alpha}_i$, it covers the outage case. If $\overline{\alpha}_i > \underline{\alpha}_i$, it corresponds to the partial failure case and also can be regarded as partial degradation of the actuator.

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