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On the Isomorphism between Cyclic-Cubes and Wrapped Butterfly Networks

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Abstract—We show that the cyclic-cubes defined by Ada W.C. Fu and S.C. Chau [1] are isomorphic to k-ary wrapped butterfly networks.

Index Terms—Cyclic-cubes, wrapped butterfly networks.

Fu and Chau [1] proposed a new family of Cayley graphs, called cyclic-cubes, which have even fixed degrees. Let G_n^k denote k-ary n-dimensional cyclic-cubes. In [1], Fu and Chou also proposed optimal routing algorithms for G_n^k. Moreover, they showed that G_n^k has a Hamiltonian cycle, a diameter of $\lfloor \frac{3n}{2} \rfloor$, and connectivity of $2k$ if $n \geq 3$. In this short comment, we show that this family of graphs are indeed isomorphic to k-ary *wrapped butterfly networks* $WB(n,k)$ which are defined in [2, pp. 442-446].

For a graph G, we use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. To define G_n^k, let $t_1, t_2, \cdots, t_n$ be n distinct symbols with ordering $t_1 > t_2 \cdots > t_n$. Each symbol t_j is assigned a rank i for $1 \leq i \leq k$, and this ranked symbol is denoted by t_j^i. The graph G_n^k has $n \cdot k^n$ vertices, and each vertex of G_n^k is represented by an n-bit vector which is a circular permutation of $t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$ for $1 \leq i_1, i_2, \cdots, i_n \leq k$. For example, in G_4^2 $t_3^2 t_4^1 t_1^1 t_2^2$ is a vertex and $t_3^2 t_4^1 t_2^2 t_1^1$ is not. In other words,

$$V(G_n^k) = \{ t_j^{i_j} t_{j+1}^{i_{j+1}} \cdots t_n^{i_n} t_1^{i_1} \cdots t_{j-1}^{i_{j-1}} \mid \text{for } 1 \leq j \leq n \text{ and } 1 \leq i_1, i_2, \cdots, i_n \leq k \}.$$

To define edges in G_n^k, we first define function f_s, for every $1 \leq s \leq k$, mapping $V(G_n^k)$ onto itself as follows:

$$f_s(t_j^{i_j} t_{j+1}^{i_{j+1}} \cdots t_n^{i_n} t_1^{i_1} \cdots t_{j-1}^{i_{j-1}}) = t_{j+1}^{i_{j+1}} \cdots t_n^{i_n} t_1^{i_1} \cdots t_{j-1}^{i_{j-1}} t_j^s \quad \text{for any } 1 \leq s \leq k.$$

Note that all f_s are bijective functions. Each vertex $x \in V(G_n^k)$ is linked to exactly $2k$ vertices $f_s(x)$ and $f_s^{-1}(x)$ for all $1 \leq s \leq k$. For example, in G_4^2 the vertex $t_3^2 t_4^1 t_1^1 t_2^2$ is linked to $t_4^1 t_1^1 t_2^2 t_3^1$, $t_4^1 t_1^1 t_2^2 t_3^2$, $t_2^1 t_3^2 t_4^1 t_1^1$, and $t_2^2 t_3^2 t_4^1 t_1^1$.

Now we introduce the definition of wrapped butterfly networks $WB(n,k)$. The network $WB(n,k)$ has $n \cdot k^n$ vertices and each vertex is represented by an $(n+1)$-bit vector $a_0 a_1 \cdots a_{n-1} i$, where $0 \leq i \leq n-1$ and $1 \leq a_j \leq k$ for all $0 \leq j \leq n-1$. Two vertices $a_0 a_1 \cdots a_{n-1} i$ and $b_0 b_1 \cdots b_{n-1} j$ are adjacent in $WB(n,k)$ if and only if $j - i = 1 (\mathbf{mod}\ n)$ and $a_t = b_t$ for all $0 \leq t \neq j \leq n-1$.

In fact, G_n^k is isomorphic to $WB(n,k)$, as stated in the following theorem.

Theorem 1. *G_n^k is isomorphic to $WB(n,k)$.*

Proof. For each vertex $a_0 a_1 \cdots a_{n-1} i$ in $WB(n,k)$, we define a function π mapping $V(WB(n,k))$ to $V(G_n^k)$ as follows:

$$\pi(a_0 a_1 \cdots a_{n-1} i) = t_{i+2}^{a_{i+1}} t_{i+3}^{a_{i+2}} \cdots t_n^{a_{n-1}} t_1^{a_0} t_2^{a_1} \cdots t_{i+1}^{a_i}.$$

The function π is obviously bijective.

Let $u = a_0 a_1 \cdots a_{n-1} i$ and $v = b_0 b_1 \cdots b_{n-1} j$ be two distinct vertices in $WB(n,k)$. It follows that $\pi(u)$ and $\pi(v)$ are two distinct vertices in G_n^k given as follows:

$$\pi(u) = t_{i+2}^{a_{i+1}} t_{i+3}^{a_{i+2}} \cdots t_n^{a_{n-1}} t_1^{a_0} t_2^{a_1} \cdots t_{i+1}^{a_i},$$
$$\pi(v) = t_{j+2}^{b_{j+1}} t_{j+3}^{b_{j+2}} \cdots t_n^{b_{n-1}} t_1^{b_0} t_2^{b_1} \cdots t_{j+1}^{b_j}.$$

Suppose that u and v are adjacent in $WB(n,k)$. Without loss of generality, we may assume that $j = i + 1 (\mathbf{mod}\ n)$. It follows that $a_t = b_t$ for all $0 \leq t \neq j \leq n-1$, i.e.,

$$v = a_0 a_1 \cdots a_i b_{i+1} a_{i+2} \cdots a_{n-1} (i+1).$$

Therefore,

$$\pi(v) = t_{i+3}^{a_{i+2}} t_{i+4}^{a_{i+3}} \cdots t_n^{a_{n-1}} t_1^{a_0} t_2^{a_1} \cdots t_{i+1}^{a_i} t_{i+2}^{b_{i+1}} = f_{b_{i+1}}(\pi(u)).$$

Thus, $\pi(u)$ and $\pi(v)$ are adjacent in G_n^k. Hence, $(u,v) \in E(WB(n,k))$ implies $(\pi(u), \pi(v)) \in E(G_n^k)$.

On the other hand, let $\pi(u)$ and $\pi(v)$ be adjacent in G_n^k. It follows that $\pi(v)$ can be $f_s(\pi(u))$ or $f_s^{-1}(\pi(u))$ for some $1 \leq s \leq k$. Consider $\pi(v) = f_s(\pi(u))$ for some $1 \leq s \leq k$. It follows that

$$\pi(v) = t_{i+3}^{a_{i+2}} t_{i+4}^{a_{i+3}} \cdots t_n^{a_{n-1}} t_1^{a_0} t_2^{a_1} \cdots t_{i+1}^{a_i} t_{i+2}^s,$$

and $v = a_0 a_1 \cdots a_i s a_{i+2} \cdots a_{n-1} (i+1)$. Therefore, u, v are adjacent in $WB(n,k)$ and furthermore, $\pi(v) = f_s(\pi(u))$ implies $(u,v) \in E(WB(n,k))$. Since every f_s is a bijective function, it follows that $\pi(v) = f_s^{-1}(\pi(u))$ also implies $(u,v) \in E(WB(n,k))$. Hence, $(\pi(u), \pi(v)) \in E(G_n^k)$ implies $(u,v) \in E(WB(n,k))$.

Since $(u,v) \in E(WB(n,k))$ if and only if

$$(\pi(u), \pi(v)) \in E(G_n^k),$$

the two graphs $WB(n,k)$ and G_n^k are isomorphic. This theorem is proven. □

References

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