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Computing the noncentral beta distribution with S-system

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Abstract

Based on the recasting techniques of Rust and Voit (1990, J. Amer. Statist. Assoc. 85, 572–578), an S-system form of the noncentral beta distribution is extended from that of the noncentral F distribution and the other one is newly derived. The computing methods of this distribution have received much attention during the last decade. Its cumulative probabilities, densities, quantiles and related distributional values can be calculated in one S-system form. We demonstrate the new computational results using the S-system numerical solver ESSYNS. Consistent results are obtained from these two S-system forms under various situations. In addition, we compare the performance with an ad hoc computing method by evaluating the cumulative probabilities and densities jointly. The S-system formulation provides significant numerical advantages over its original form. Further properties are also discussed.

Keywords: Statistical computing; Differential equation; Recasting; Power law; Taylor series

1. Introduction

For most univariate statistical distributions, tabulations for frequently cited values of parameters can be made quite extensive; those for other parameterized values are usually interpolated or extrapolated. However, those tables for some sophisticated or new generated distributions may not be available or their computing procedures have not yet been implemented in any software package. The ad hoc calculations from scratch raise some practical problems since it involves integrals, implicit forms, or

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infinite series. Most of these properties usually make numerical computing difficult. On the other hand, many statistical distributions such as χ^2, F, and t-distributions can be recast in the S-system canonical form with a set of simultaneous nonlinear first-order differential equations. These formulations offer significant numerical advantages over representations of the distributions in their original forms (Rust and Voit, 1990; Savageau, 1982; Voit, 1991; Voit and Rust, 1987, 1988, 1990, 1992). However, the noncentral beta distribution with S-system is not available. This distribution has received much attention during the last decade (Chattamvelli, 1995; Ding, 1994; Frick, 1990; Lam, 1995; Lee, 1992; Lenth, 1987; Norton, 1983; Posten, 1993). In this article, we newly recast the noncentral beta distribution into the S-system formulation. Moreover, the S-system form of the noncentral F distribution (Rust and Voit, 1990) is extended for computing the noncentral beta since these two distributions are closely related.

The power-law formalism and associated S-system differential equations give us a systematic methodology for nonlinear analysis. The dynamics of a nonlinear system can be well represented by the *S-system* that indicates its ability to capture *S*aturable and *S*ynergistic characteristics of nonlinear systems. Its canonical form is represented as

$$\dot{X}_i = \alpha_i \prod_{j=1}^{n} X_j^{g_{ij}} - \beta_i \prod_{j=1}^{n} X_j^{h_{ij}}, \quad i = 1, 2, \ldots, n,$$

where $\dot{X}_i = \mathrm{d}X_i/\mathrm{d}t, \alpha_i$ and β_i are nonnegative, and g_{ij} and h_{ij} are any real numbers. Irvine and Savageau (1990) and Irvine (1988) propose a convenient numerical algorithm with Taylor's approximation in the logarithmic space. So the variables X_i are confined to be positive (Savageau and Voit, 1987). Based on their implemented codes ESSYNS (Voit et al., 1990), our recasting S-system formulation of noncentral beta distribution is demonstrated and compared with the extended one from the noncentral F distribution. The computing results are quite consistent. This soft computing is very general as well as powerful for calculating the statistical distributions in the S-system formulation. A fruitful application area that is just beginning to emerge is *computational statistics* (Kennedy and Gentle, 1980; Voit, 1991).

In addition to briefly describe the noncentral beta distribution in Section 2, we derive its recasting form and related theorem. Moreover, the S-system form of F distribution in Rust and Voit (1990) is extended for computing the noncentral beta distribution. Their equivalent representations are also discussed. In Section 3, we demonstrate the computational results of the two formulations using the S-system numerical solver ESSYNS. Moreover, the ad hoc computing method of Ding (1994) is chosen for comparison in calculating the densities and cumulative probabilities. Finally, concluding remarks are presented in Section 4.

2. S-system forms of noncentral beta

The standard form of noncentral beta distribution is in terms of the distribution of the chi-square variates, $X = \chi^2_{2a}(\lambda)/(\chi^2_{2a}(\lambda) + \chi^2_{2b})$ where $\chi^2_{2a}(\lambda)$ denotes a noncentral

χ^2 random variate with $2a$ degrees of freedom with noncentrality parameter λ and χ^2_{2b} is the central one. It was considered by many researchers (e.g., see, Chattamvelli, 1995). Other forms of noncentral beta distribution can be seen in Johnson et al. (1994). The distribution of X is usually represented as an infinite sum of Poisson weights of central beta distribution, that is,

$$F_B(s;a,b,\lambda)=\sum_{i=0}^{\infty}u_iF_B(s;a+i,b); \quad f_B(s;a,b,\lambda)=\sum_{i=0}^{\infty}u_if_B(s;a+i,b),$$

where $F_B(s;a,b,\lambda), f_B(s;a,b,\lambda)$ denote the respective noncentral beta distribution function and density with shape parameters a,b and noncentrality parameter $\lambda; F_B(s;a,b), f_B(s;a,b)$ are the corresponding central ones, with $0\le s\le 1$, $a>0$, $b>0$, $\lambda\ge 0$, $u_i=\mathrm{e}^{-\lambda/2}(\lambda/2)^i/i!$. Here we have

$$F_B(s;a,b)=F_B(s;a,b,0)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^s x^{a-1}(1-x)^{b-1}\,\mathrm{d}x,$$

$$f_B(s;a,b)=f_B(s;a,b,0)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}s^{a-1}(1-s)^{b-1}.$$

Computing the noncentral beta distribution has received much attention during the last decade. It can be used in the calculation of the power functions of tests of general linear hypotheses. These include standard tests used in the analysis of variance. This distribution is not only interesting in itself, but also related to other distributions, such as χ^2, F, student t, binomial, and negative binomial (Johnson et al., 1994; Lee, 1992). In particular, the noncentral F distribution function can be evaluated through the noncentral beta since the noncentral F variate is simply a monotone function of the noncentral beta one, that is, the noncentral F distribution $G_F(\cdot)$ in terms of noncentral beta distribution by

$$G_F(t;m,n,\lambda)=G_F(t;2a,2b,\lambda)=F_B(s;a,b,\lambda)$$

where $t=(b/a)s/(1-s)$ or $s=mt/(mt+n)$; hereafter this relation is called the *beta-F link condition*. Most of the evaluation methods of noncentral beta distribution except for Ding (1994) relies heavily on the central beta one (the incomplete beta integral). So the conventional computing methods are to develop an efficient and accurate algorithm for calculating the central beta distribution. With our proposed S-system formulation for this distribution, it can evaluate distributional values of interest jointly and more conveniently.

2.1. Recasting the noncentral beta distribution

Based on the recasting techniques proposed by Savageau and Voit (1987) as well as the applications to the statistical distributions illustrated by Rust and Voit (1990), we derive the noncentral beta distribution in terms of S-system form in the following.

The density function of noncentral beta distribution is represented as

$$
\begin{aligned}
&f_B(s;a,b,\lambda)\\
&=\sum_{i=0}^{\infty}\left(\frac{e^{-(\lambda/2)}(\lambda/2)^i}{i!}\right)\frac{\Gamma(a+b+i)}{\Gamma(a+i)\Gamma(b)}s^{a+i-1}(1-s)^{b-1}\\
&=e^{-(\lambda/2)}\sum_{i=0}^{\infty}\frac{1}{i!}\left(\frac{\lambda}{2}s\right)^i\frac{(a+i-1+b)\times\cdots\times(a+b)}{(a+i-1)\times\cdots\times(a)}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}s^{a-1}(1-s)^{b-1}\\
&=e^{-(\lambda/2)}\psi(s)f_B(s;a,b,0)
\end{aligned}
$$

where $0<s<1$, $a,b>0$, are the shape parameters, and λ is the noncentrality parameter, and where we put $\psi(s)=\sum_{j=0}^{\infty}(1/j!)((\lambda/2)s)^j\kappa(j)$ with $\kappa(j)=\prod_{i=1}^{j}((a+b+i-1)/(a+i-1))$. We show in Appendix A that $\psi(s)$ satisfies the following differential equation:

$$(2s)\psi''(s)=(\lambda s-2a)\psi'(s)+\lambda(a+b)\psi(s).$$

Based on these, its S-system formulation for all positive values of the shape parameters a and b can be recast as follows:

$$
\begin{aligned}
&X_1=s, && \dot{X}_1=1,\\
&X_2=1-s, && \dot{X}_2=-1,\\
&X_3=a^{-1}bX_1X_2^{-1}, && \dot{X}_3=a^{-1}bX_2^{-2},\\
&X_4=s(1-s)f_B(s;a,b,0), && \dot{X}_4=aX_1^{-1}X_4-bX_2^{-1}X_4,\\
&X_5=F_B(s;a,b,0), && \dot{X}_5=X_1^{-1}X_2^{-1}X_4,\\
&X_6=\psi(s), && \dot{X}_6=X_7X_8,\\
&X_7=\psi'(s)X_8^{-1}, && \dot{X}_7=\frac{\lambda}{2}X_7-aX_1^{-1}X_7,\\
&X_8=\psi'(s)X_7^{-1}, && \dot{X}_8=\frac{\lambda}{2}(a+b)X_1^{-1}X_6X_7^{-1},\\
&X_9=F_B(s;a,b,\lambda), && \dot{X}_9=e^{-\lambda/2}X_1^{-1}X_2^{-1}X_4X_6\ (=f_B(s;a,b,\lambda)),
\end{aligned}
\tag{BX1}
$$

where $\dot{X}_i=dX_i/ds$ and details for deriving X_7, X_8 can be seen in Appendix A. Note that $X_3=(b/a)s/(1-s)=t$ is for computing the quantities corresponding to the F distribution. The S-system formulation is usually not unique; another derivation for $a>1, b>1$ can be seen in Chou (1996). This system is for computing $X_i(X_1)$, in general. The central/noncentral densities and cumulative probabilities (in terms of X_1, $X_2(X_1)$, $X_4(X_1)$, $X_5(X_1)$, $X_6(X_1)$, $X_9(X_1)$) are of particular interest. Moreover, by using the chain rule for evaluating variables with the independent variable X_9 instead of $X_1 = s$, and putting $\tilde{X}_i = \mathrm{d}X_i/\mathrm{d}s/\mathrm{d}X_9/\mathrm{d}s$, we have the inverted form:

$$\begin{aligned}
\tilde{X}_1 &= \mathrm{e}^{\lambda/2} X_1 X_2 X_4^{-1} X_6^{-1}, \\
\tilde{X}_2 &= -\mathrm{e}^{\lambda/2} X_1 X_2 X_4^{-1} X_6^{-1}, \\
\tilde{X}_3 &= \mathrm{e}^{\lambda/2} a^{-1} b X_1 X_2^{-1} X_4^{-1} X_6^{-1}, \\
\tilde{X}_4 &= \mathrm{e}^{\lambda/2} a X_2 X_6^{-1} - \mathrm{e}^{\lambda/2} b X_1 X_6^{-1}, \\
\tilde{X}_5 &= \mathrm{e}^{\lambda/2} X_6^{-1}, \\
\tilde{X}_6 &= \mathrm{e}^{\lambda/2} X_1 X_2 X_4^{-1} X_6^{-1} X_7 X_8, \\
\tilde{X}_7 &= \mathrm{e}^{\lambda/2} \frac{\lambda}{2} X_1 X_2 X_4^{-1} X_6^{-1} X_7 - \mathrm{e}^{\lambda/2} a X_2 X_4^{-1} X_6^{-1} X_7, \\
\tilde{X}_8 &= \mathrm{e}^{\lambda/2} \frac{\lambda}{2} (a+b) X_2 X_4^{-1} X_7^{-1}.
\end{aligned} \tag{BX9}$$

This system is for computing $X_i(X_9)$, in general. The quantiles of the noncentral beta distribution, $X_1(X_9)$, as well as those of the noncentral F distribution, $X_3(X_9)$, and the type I error probabilities, $1 - X_5(X_9)$, for given levels of power for the noncentral beta distribution are of particular interest. We call the above formulation as beta-based derivation (BX). The other one is in the following.

2.2. F-based derivation

Based on the S-system form of noncentral F distribution (Rust and Voit, 1990), it can be extended for computing the noncentral beta distribution with the addition of $Y_3 = mt/(n+mt) = s$ to their original system:

$$\begin{aligned}
Y_1 &= t, & \dot{Y}_1 &= 1, \\
Y_2 &= n + mt, & \dot{Y}_2 &= m, \\
Y_3 &= m Y_1 Y_2^{-1}, & \dot{Y}_3 &= mn Y_2^{-2}, \\
Y_4 &= t g_F(t; m, n, 0), & \dot{Y}_4 &= \frac{m}{2} Y_1^{-1} Y_4 - \frac{m}{2}(m+n) Y_2^{-1} Y_4, \\
Y_5 &= G_F(t; m, n, 0), & \dot{Y}_5 &= Y_1^{-1} Y_4, \\
Y_6 &= \mathscr{M}(t), & \dot{Y}_6 &= Y_7 Y_8, \\
Y_7 &= \mathscr{M}'(t) Y_8^{-1}, & \dot{Y}_7 &= \frac{\lambda}{2} mn Y_2^{-2} Y_7 - \frac{1}{2} mn Y_1^{-1} Y_2^{-1} Y_7, \\
Y_8 &= \mathscr{M}'(t) Y_7^{-1}, & \dot{Y}_8 &= \frac{\lambda}{4} mn^2 (m+n) Y_1^{-1} Y_2^{-3} Y_6 Y_7^{-1} - 2m Y_2^{-1} Y_8, \\
Y_9 &= G_F(t; m, n, \lambda), & \dot{Y}_9 &= \mathrm{e}^{-\lambda/2} Y_1^{-1} Y_4 Y_6 \ (= g_F(t; m, n, \lambda)),
\end{aligned} \tag{FY1}$$

where $\dot{Y}_i = \mathrm{d}Y_i/\mathrm{d}t$. Consequently, we have the inverted form with the independent variable Y_9 instead of Y_1, that is, $\tilde{Y}_i = (\mathrm{d}Y_i/\mathrm{d}t)/(\mathrm{d}Y_9/\mathrm{d}t)$:

$$\begin{aligned}
\tilde{Y}_1 &= \mathrm{e}^{\lambda/2} Y_1 Y_4^{-1} Y_6^{-1},\\
\tilde{Y}_2 &= \mathrm{e}^{\lambda/2} m Y_1 Y_4^{-1} Y_6^{-1},\\
\tilde{Y}_3 &= \mathrm{e}^{\lambda/2} mn Y_1 Y_2^{-2} Y_4^{-1} Y_6^{-1},\\
\tilde{Y}_4 &= \mathrm{e}^{\lambda/2} \frac{m}{2} Y_6^{-1} - \mathrm{e}^{\lambda/2} \frac{m}{2} (m+n) Y_1 Y_2^{-1} Y_6^{-1},\\
\tilde{Y}_5 &= \mathrm{e}^{\lambda/2} Y_6^{-1},\\
\tilde{Y}_6 &= \mathrm{e}^{\lambda/2} Y_1 Y_4^{-1} Y_6^{-1} Y_7 Y_8,\\
\tilde{Y}_7 &= \mathrm{e}^{\lambda/2} \frac{\lambda}{2} mn Y_1 Y_4^{-1} Y_6^{-1} Y_7 - \mathrm{e}^{\lambda/2} \frac{mn}{2} Y_2^{-1} Y_4^{-1} Y_6^{-1} Y_7,\\
\tilde{Y}_8 &= \mathrm{e}^{\lambda/2} \frac{\lambda}{4} mn^2 (m+n) Y_2^{-3} Y_4^{-1} Y_7^{-1} - \mathrm{e}^{\lambda/2} 2m Y_1 Y_2^{-1} Y_4^{-1} Y_6^{-1} Y_8.
\end{aligned} \tag{FY9}$$

This system is for computing $Y_i(Y_9)$, in general. The quantiles of the noncentral F distribution, $Y_1(Y_9)$, as well as those of the noncentral beta distribution, $Y_3(Y_9)$, and the type I error probabilities, $1 - Y_5(Y_9)$, given the levels of power for the noncentral F distribution are of particular interest (Rust and Voit, 1990). On the other hand, in order to use this system for computing the central/noncentral densities and cumulative probabilities of the noncentral *beta* distribution, we can derive the inverted form with the independent variable Y_3 instead of Y_1, that is, $\bar{Y}_i = (\mathrm{d}Y_i/\mathrm{d}t)/(\mathrm{d}Y_3/\mathrm{d}t)$:

$$\begin{aligned}
\bar{Y}_1 &= \frac{1}{mn} Y_2^2,\\
\bar{Y}_2 &= \frac{1}{n} Y_2^2,\\
\bar{Y}_3 &= 1,\\
\bar{Y}_4 &= \frac{1}{2n} Y_1^{-1} Y_2^2 Y_4 - \frac{m+n}{2n} Y_2 Y_4,\\
\bar{Y}_5 &= \frac{1}{mn} Y_1^{-1} Y_2^2 Y_4,\\
\bar{Y}_6 &= \frac{1}{mn} Y_2^2 Y_7 Y_8,\\
\bar{Y}_7 &= \frac{\lambda}{2} Y_6 - \frac{1}{2} Y_1^{-1} Y_2 Y_7,\\
\bar{Y}_8 &= \frac{\lambda}{4} n (m+n) Y_1^{-1} Y_2^{-1} Y_6 Y_7^{-1} - \frac{2}{n} Y_2 Y_8,\\
\bar{Y}_9 &= \mathrm{e}^{-\lambda/2} \frac{1}{mn} Y_1^{-1} Y_2^2 Y_4 Y_6 \ (= g_F(t; m, n, \lambda)).
\end{aligned} \tag{FY3}$$

This system is for computing $Y_i(Y_3)$, in general. The central/noncentral densities and cumulative probabilities (in terms of $Y_1(Y_3)$, $Y_2(Y_3)$, $Y_4(Y_3)$, $Y_5(Y_3)$, $Y_6(Y_3)$, $Y_9(Y_3)$) of the beta distribution are of particular interest.

Note that the equations in (FY3) can be reduced to eight variables as those of (BX9) and (FY9) after removing the independent variable Y_3 from the system. To further examine the close relationship between these two S-system forms under the *beta-F link condition*, $X_1 = s = mt/(mt+n) = Y_3$ or $Y_1 = t = (b/a)s/(1-s) = X_3$, the density functions and the cumulative probabilities of the noncentral beta and F distributions are related by

$$F_B(s;a,b,0) = G_F(t;m,n,0) \quad \text{(i.e., } X_5(s) = Y_5(t)\text{)},$$

$$F_B(s;a,b,\lambda) = G_F(t;m,n,\lambda) \quad \text{(i.e., } X_9(s) = Y_9(t)\text{)},$$

$$f_B(s;a,b,0) = \frac{(n+mt)^2}{mn} g_F(t;m,n,0) \quad (cbf),$$

$$f_B(s;a,b,\lambda) = \frac{(n+mt)^2}{mn} g_F(t;m,n,\lambda) \quad (nbf),$$

$$g_F(t;m,n,0) = \frac{a}{b}(1-s)^2 f_B(s;a,b,0) \quad (cfb),$$

$$g_F(t;m,n,\lambda) = \frac{a}{b}(1-s)^2 f_B(s;a,b,\lambda).$$

So, we have $s(1-s)f_B(s;a,b,0) = tg_F(t;m,n,0)$ (i.e., $X_4(s) = Y_4(t)$) as well as $\psi(s) = \mathscr{M}(t)$ (i.e., $X_6(s) = Y_6(t)$) in the two S-system forms, which are derived by

$$\frac{a}{b}(1-s)^2 f_B(s;a,b,0) = g_F(t;m,n,0) \quad \text{from } (cfb),$$

$$\frac{bs}{a(1-s)}\frac{a}{b}(1-s)^2 f_B(s;a,b,0) = \frac{bs}{a(1-s)} g_F(t;m,n,0).$$

So, it is

$$s(1-s)f_B(s;a,b,0) = tg_F(t;m,n,0).$$

Similarly,

$$f_B(s;a,b,\lambda) = \frac{(n+mt)^2}{mn} g_F(t;m,n,\lambda) \quad \text{from } (nbf),$$

$$\mathrm{e}^{-\lambda/2} f_B(s;a,b,0)\,\psi(s) = \mathrm{e}^{-\lambda/2}\frac{(n+mt)^2}{mn} g_F(t;m,n,0)\mathscr{M}(t).$$

Therefore, we have

$$\psi(s) = \mathscr{M}(t) \quad \text{by } (cbf).$$

Note that $X_1 = Y_3$, $X_3 = Y_1$, $X_4 = Y_4$, $X_5 = Y_5$, $X_6 = Y_6$, and $X_9 = Y_9$ under the *beta-F link condition* can also be confirmed with the initial values reported in Tables 1 and 2 of Appendix A. Moreover, we can compute the central/noncentral densities from the outputs of either form as follows:

$$\begin{array}{llll} & \text{Beta-based system} & & \text{F-based system} \\ f_B(s;a,b,0) & = X_1^{-1}X_2^{-1}X_4 & = & \dfrac{1}{mn}Y_1^{-1}Y_2^2Y_4 \\ f_B(s;a,b,\lambda) & = \mathrm{e}^{-\lambda/2}X_1^{-1}X_2^{-1}X_4X_6 & = & \dfrac{1}{mn}\mathrm{e}^{-\lambda/2}Y_1^{-1}Y_2^2Y_4Y_6 \\ g_F(t;m,n,0) & = \dfrac{a}{b}X_1^{-1}X_2X_4 & = & Y_1^{-1}Y_4 \\ g_F(t;m,n,\lambda) & = \dfrac{a}{b}\mathrm{e}^{-\lambda/2}X_1^{-1}X_2X_4X_6 & = & \mathrm{e}^{-\lambda/2}Y_1^{-1}Y_4Y_6 \end{array}$$

These can be used to compute the distributional values of interest in one S-system form of F or beta. Note that after the S-system form of beta-based derivation is inverted with the $X_3=(b/a)s/(1-s)$ as the independent variable, it is an equivalent F-based system for computing densities and cumulative probabilities.

3. Demonstrations and comparisons

In order to evaluate the computing results of the noncentral beta by our new S-system derivation, we conduct the demonstration by the following steps:

1. Choose the alternative procedure (FY) in Section 2.2 as the comparison-based system and the same various situations in Rust and Voit (1990) as our pivotal cases (Tables 1 and 2).
2. Check the consistency of the computing results by the same procedure (FY) running on the two different PC systems (Tables 3 and 4).
3. Compare the testing results of the noncentral beta by our newly derived form (BX) with those by the (FY) S-system (Tables 3 and 4).
4. Perform the calculation of the noncentral beta densities and cumulative probabilities using the two S-system forms and an ad hoc method of Ding (1994), reported in Table 5 of Appendix A.

According to the pivotal cases, a convenient set of initial conditions is defined at $t=1$ or $s=a/(a+b)=m/(m+n)$, where central/noncentral F and beta distributions have nonzero densities and cumulative probabilities. Their initial values are computed by our implemented C codes primarily based on the algorithms of Ding (1994) and Singh and Relyea (1992). They are reported in Table 1 (BX) and Table 2 (FY) of Appendix A. Note that the initial values in the 1st and 6th rows of Table 2 coincide with those of Rust and Voit (1988, p. 276, their values shown in square brackets) that seem to be used in Rust and Voit (1990). These values are made on a SPARC 5 SUN workstation.

Subsequently, the S-system form evaluation using the numerical solver ESSYNS (implemented only for PC) is conducted on a DEC (Venturis FP 575) PC 586 system. All the conductions reported in Tables 3–5 are under the ESSYNS system error bound 10^{-12}. We first check the consistency of the results under the different PC systems by running the same outputs (Tables 1 and 2 in Rust and Voit, 1990) of the type I error probabilities (reported in Table 3) as well as quantiles (reported in Table 4) for given levels of power (0.8,0.9). Their original results are shown in the 1st rows (R&V) while ours are put in the 2nd rows (FY9) of Tables 3 and 4 by running the extended form (FY9) *for each case*. Table 3 shows that the pairs of type I error probabilities have the same values except for the case 3. Similarly, Table 4 shows the values of various quantiles in the 1st and 2nd rows of each case are almost the same except for the cases 6 and 8. Note that the values of case 3 in Table 3 and case 6 in Table 4, which are partially underlined, seem the typing errors from Rust and Voit (1990). So, the computational variation between these two PC systems is negligible.

Under the same situations, our derived formulation (BX9) is evaluated in the following. To see the type I error probabilities for given levels of power (0.8,0.9), $1-X_5(X_9)$ is the desired output values shown in the 3rd rows (BX9) while the comparion ones are $1-Y_5(Y_9)$ by the form (FY9) shown in 2nd rows (FY9) for *each case* of Table 3. Note that $X_9(s)=Y_9(t)$ whenever $t=bs/a(1-s)$. The reported values of two forms are all equal up to 10 significant digits for the cases under study. On the other hand, for computing the noncentral beta (or F) quantiles corresponding to the same two power points, X_1 (or X_3 for F) is the desired output values shown in the rows (BX9) while the comparion ones are Y_3 (or Y_1 for F) from the form (FY9) reported in the rows (FY9) of Table 4. To compare the F *quantiles*, the reported values between the 2nd and the 3rd rows are all equal upto 10 significant digits for cases 1–5 whose noncentrality parameter is 25 while those for cases 6–10, whose noncentrality parameter is 50, show at least 7 significant digits agreed. Similarly, the values in the 4th and 5th rows for comparing the *beta quantiles* are the same for cases 1–5 (the noncentrality being 25) while those for cases 6–10 (the noncentrality being 50) show at least 8 significant digits agreed. The difference may be due to the error propagation of the data representations and the approximation methods. It seems that the bigger the value of noncentrality is, the larger the error propagates during the proceeding of the numerical solver. Chattamvelli (1995) pointed out the considerations for numerical calculations with large noncentrality parameter values. It is also under our further study. However, our demonstrated results are quite consistent and satisfied.

For calculating the noncentral beta distributional values, most algorithms are for computing cumulative probabilities except for Ding's one (1994) allowing joint evaluation of the distribution function and the density. This method is chosen for computing the probabilitiy as well as cumulative density functions (p.d.f and c.d.f.) at the point 0.5 of the noncentral beta and for comparing with those by the two S-system forms (BX1) and (FY3); all the error bounds are set to be 10^{-12}. Table 5 shows that the computing results from these three methods are equal upto 12 significant digits under the specified precision for all cases.

4. Concluding remarks

With our newly derived S-system form, various demonstration results are obtained in Section 3. This system (BX) can be used for directly computing various distributional values of those statistics with the noncentral beta and F distribution. Alternatively, the (FY) S-system form extended from Rust and Voit (1990) can be used for doing the same things equally well. The computing results under both systems are quite consistent and satisfied. Further features of S-system formulation can be seen in Rust and Voit (1990). Some remarks are pointed as follows:

1. The most interesting parts for computing the noncentral distributions are on the tail part or extreme probabilities and associated quantities. The initial values for solving the canonical differential equations may be hard to pre-calculate for some canonical S-system forms. So it is crucial to calculate the initial conditions conveniently as well as precisely for the canonical S-system under study. Subsequently, the computing speed, the adaptive step length, the calculation accuracy, and the order of the approximation algorithm are correlated and influenced mutually in the proceeding of numerical calculation. Our implemented C codes can be easily modified to compute the various initial values of both S-system forms precisely except for the cases with very large noncentrality parameter values.
2. By performing the two equivalent S-system forms (BX1) and (FY3) to compute the quantiles of the beta distribution, theoretically they should produce the same values under study. However, the numerical solver ESSYNS results with larger diversity in precision for the cases with the large noncentrality; e.g., 50 being in various cases. The numerical considerations (e.g., stiffness, error propagation, stability and accuracy, etc.) implemented in the solver need to be further investigated. One numerical solver to take more characteristics of statistical applications is under our study.
3. The canonical S-system form for a tractable problem by recasting techniques is not unique. To recast the statistical distributions optimal in some sense is of interest. The insensitivity to the initial conditions and the neatness of the canonical form may be the criteria for consideration.

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Appendix A.

A.1. Theorem and proof

Theorem. *The infinite series relating to the noncentral beta distribution by*

$$\psi(s) = \sum_{j=0}^{\infty} \kappa(j) \frac{1}{j!} \left(\frac{\lambda}{2} s\right)^{j} \quad \textit{where } \kappa(j) = \prod_{i=1}^{j} \left(\frac{a+b+i-1}{a+i-1}\right)$$

solves the differential equation

$$(2s)\psi''(s) = (\lambda s - 2a)\psi'(s) + \lambda(a+b)\psi(s).$$

Proof.

$$\psi'(s) = \sum_{j=1}^{\infty} \kappa(j)\frac{j}{j!}\left(\frac{\lambda}{2}s\right)^{j-1}\left(\frac{\lambda}{2}\right)$$

$$= \frac{\lambda}{2}\sum_{j=0}^{\infty} \kappa(j+1)\frac{1}{j!}\left(\frac{\lambda}{2}s\right)^{j}$$

$$= \frac{\lambda}{2}\sum_{j=0}^{\infty} \left(\frac{a+b+j}{a+j}\right)\kappa(j)\frac{1}{j!}\left(\frac{\lambda}{2}s\right)^{j}$$

$$= \frac{\lambda}{2}\sum_{j=0}^{\infty} \left[1+\frac{b}{a}-\frac{b}{a}\frac{j}{a+j}\right]\kappa(j)\frac{1}{j!}\left(\frac{\lambda}{2}s\right)^{j}$$

$$= \frac{\lambda}{2}\left(1+\frac{b}{a}\right)\sum_{j=0}^{\infty} \kappa(j)\frac{1}{j!}\left(\frac{\lambda}{2}s\right)^{j} - \frac{\lambda}{2}\sum_{j=1}^{\infty}\frac{b}{a}\frac{1}{a+j}\kappa(j)\frac{j}{j!}\left(\frac{\lambda}{2}s\right)^{j-1}\left(\frac{\lambda}{2}s\right)$$

$$= \frac{\lambda}{2}\left(1+\frac{b}{a}\right)\psi(s) - \frac{\lambda}{2}\frac{b}{a}\left(\frac{\lambda}{2}s\right)\phi(s), \tag{A.1}$$

where

$$\phi(s) = \sum_{j=1}^{\infty}\frac{1}{a+j}\kappa(j)\frac{j}{j!}\left(\frac{\lambda}{2}s\right)^{j-1} = \sum_{j=0}^{\infty}\frac{1}{a+j+1}\kappa(j+1)\frac{1}{j!}\left(\frac{\lambda}{2}s\right)^{j}.$$

Then we have

$$\phi(s) = \frac{2(a+b)}{\lambda bs}\psi(s) - \frac{4a}{\lambda^2 bs}\psi'(s). \tag{A.2}$$

Therefore, the result can be further derived with (A.1):

$$\psi''(s) = \left(\frac{\lambda}{2}\right)^2\sum_{j=1}^{\infty}\kappa(j+1)\frac{j}{j!}\left(\frac{\lambda}{2}s\right)^{j-1}$$

$$= \left(\frac{\lambda}{2}\right)^2\sum_{j=0}^{\infty}\kappa(j+2)\frac{1}{j!}\left(\frac{\lambda}{2}s\right)^{j}$$

$$= \left(\frac{\lambda}{2}\right)^2\sum_{j=0}^{\infty}\left(1+\frac{b}{a+j+1}\right)\kappa(j+1)\frac{1}{j!}\left(\frac{\lambda}{2}s\right)^{j}$$

$$= \frac{\lambda}{2}\left[\left(\frac{\lambda}{2}\right)\sum_{j=0}^{\infty}\kappa(j+1)\frac{1}{j!}\left(\frac{\lambda}{2}s\right)^{j}\right]$$

$$+ \left(\frac{\lambda}{2}\right)^2\sum_{j=0}^{\infty}\left(\frac{b}{a+j+1}\right)\kappa(j+1)\frac{1}{j!}\left(\frac{\lambda}{2}s\right)^{j}$$

$$= \frac{\lambda}{2}\psi'(s) + \left(\frac{\lambda}{2}\right)^2 b \sum_{j=0}^{\infty} \left(\frac{1}{a+j+1}\right) \kappa(j+1)\frac{1}{j!}\left(\frac{\lambda}{2}s\right)^j$$

$$= \frac{\lambda}{2}\psi'(s) + \left(\frac{\lambda}{2}\right)^2 b\phi(s) \;\leftarrow\; \text{which is replaced by (A.2)}$$

$$= \frac{\lambda}{2}\psi'(s) + \frac{b\lambda^2}{4}\left[\frac{2(a+b)}{\lambda bs}\psi(s) - \frac{4a}{\lambda^2 bs}\psi'(s)\right]$$

$$= \frac{\lambda}{2}\psi'(s) + \frac{\lambda(a+b)}{2s}\psi(s) - \frac{a}{s}\psi'(s)$$

$$= \left(\frac{\lambda s - 2a}{2s}\right)\psi'(s) + \frac{\lambda(a+b)}{2s}\psi(s).$$

That is, $(2s)\psi''(s) = (\lambda s - 2a)\psi'(s) + \lambda(a+b)\psi(s)$.

Note that the $\dot{X}_7$ and $\dot{X}_8$ for recasting the noncentral beta into S-system form are derived by the above theorem as follows:

Put

$$\psi'(s) = X_7 X_8.$$

Then

$$\psi''(s) = \dot{X}_7 X_8 + \dot{X}_8 X_7.$$

So, we have

$$\psi''(s) = \frac{1}{2s}[(\lambda s - 2a)\psi'(s) + \lambda(a+b)\psi(s)]$$

$$= \frac{1}{2}X_1^{-1}[(\lambda X_1 - 2a)X_7X_8 + \lambda(a+b)X_6]$$

$$= \frac{\lambda}{2}X_7X_8 - aX_1^{-1}X_7X_8 + \frac{\lambda}{2}(a+b)X_1^{-1}X_6.$$

Therefore,

$$\dot{X}_7 = \frac{\lambda}{2}X_7 - aX_1^{-1}X_7$$

and

$$\dot{X}_8 = \frac{\lambda}{2}(a+b)X_1^{-1}X_6X_7^{-1}.$$

A.2. The tables for evaluation results

The evaluation results are given in Tables 1–5.

Table 1
The initial values for calculating the form (BX)

a (m)	b (n)	λ	X_1 (= s for beta) X_2 X_3 (= t for F)	X_4 X_5 (= cdf: central for both) X_6	X_7 X_8 X_9 (= cdf: noncentral for both)
5.5	30		1.54929577464788720e−01	8.46794328864283186e−01	2.39499166968949517e+05
(11)	(60)	25	8.45070422535211252e−01	5.42749109369440874e−01	1.0
			1.0	5.44613129301171557e+03	2.85822822437959128e−03
5.5	45		1.08910891089108910e−01	8.69743827810119829e−01	2.80808457314026426e+05
(11)	(90)	25	8.91089108910891103e−01	5.47131915885933573e−01	1.0
			1.0	4.63862215013610512e+03	2.46344614244440241e−03
5.5	60		8.39694656488549629e−02	8.81906822658790057e−01	3.28676156048240024e+05
(11)	(120)	25	9.16030534351145009e−01	5.49421653132873500e−01	1.0
			1.0	4.26071453188992928e+03	2.27780137905972532e−03
5.5	80		6.43274853801169555e−02	8.91356037870450701e−01	3.96115234716603358e+05
(11)	(160)	25	9.35672514619882989e−01	5.51185380665772717e−01	1.0
			1.0	3.98878812226587024e+03	2.14378753342197963e−03
5.5	100		5.21327014218009449e−02	8.97166973831606152e−01	4.65348291890082299e+05
(11)	(200)	25	9.47867298578199069e−01	5.52263619572691100e−01	1.0
			1.0	3.83040415258585199e+03	2.06554295877193692e−03
5.5	30		1.54929577464788720e−01	8.46794328864283186e−01	1.72061435073492944e+08
(11)	(60)	50	8.45070422535211252e−01	5.42749109369440874e−01	1.0
			1.0	2.40494326061564079e+06	3.01026903603926603e−06
5.5	45		1.08910891089108910e−01	8.69743827810119829e−01	1.56031809229001552e+08
(11)	(90)	50	8.91089108910891103e−01	5.47131915885933573e−01	1.0
			1.0	1.60820327936249366e+06	2.06282167974606275e−06
5.5	60		8.39694656488549629e−02	8.81906822658790057e−01	1.59632394959358096e+08
(11)	(120)	50	9.16030534351145009e−01	5.49421653132873500e−01	1.0
			1.0	1.30171143932178780e+06	1.69244706267223280e−06
5.5	80		6.43274853801169555e−02	8.91356037870450701e−01	1.73434024696825802e+08
(11)	(160)	50	9.35672514619882989e−01	5.51185380665772717e−01	1.0
			1.0	1.10563026075734245e+06	1.45308479877452971e−06
5.5	100		5.21327014218009449e−02	8.97166973831606152e−01	1.91233382993567914e+08
(11)	(200)	50	9.47867298578199069e−01	5.52263619572691100e−01	1.0
			1.0	1.00047812972492748e+06	1.32377481742240787e−06

Table 2
The initial values for calculating the form (FY)

m	n	λ	X_1 (= t for F)	X_4	X_7
(a)	(b)		X_2	X_5 (= cdf: central for both)	X_8
			X_3 (= s for beta)	X_6	X_9 (= cdf: noncentral for both)
11	60		1	8.46794328864283186e−01	3.13567645704238566e+04
				[8.46794328864278e−01]	[3.13567645704239e+04]
(5.5)	(30)	25	71	5.42749109369440874e−01	1.0
				[5.42749109369445e−01]	
			1.54929577464788720e−01	5.44613129301171557e+03	2.85822822437959128e−03
				[5.44613129301172e+03]	[2.858228224387e−03]
11	90		1	8.69743827810119829e−01	2.72522667131542148e+04
(5.5)	(45)	25	101	5.47131915885933573e−01	1.0
			1.08910891089108910e−01	4.63862215013610512e+03	2.46344614244440241e−03
11	120		1	8.81906822658790057e−01	2.52813079647850827e+04
(5.5)	(60)	25	131	5.49421653132873500e−01	1.0
			8.39694656488549629e−02	4.26071453188992928e+03	2.27780137905972532e−03
11	160		1	8.91356037870450701e−01	2.38419620772621274e+04
(5.5)	(80)	25	171	5.51185380665772717e−01	1.0
			6.43274853801169555e−02	3.98878812226587024e+03	2.14378753342197963e−03
11	200		1	8.97166973831606152e−01	2.29951313348348194e+04
(5.5)	(100)	25	211	5.52263619572691100e−01	1.0
			5.21327014218009449e−02	3.83040415258585199e+03	2.06554295877193692e−03
11	60		1	8.46794328864283186e−01	2.25273848737364337e+07
				[8.46794328864278e−01]	[2.25273848737365e+07]
(5.5)	(30)	50	71	5.42749109369440874e−01	1.0
				[5.42749109369445e−01]	
			1.54929577464788720e−01	2.40494326061564079e+06	3.01026903603926603e−06
				[2.40494326061564e+06]	[3.01026905e−06]
11	90		1	8.69743827810119829e−01	1.51427792507314496e+07
(5.5)	(45)	50	101	5.47131915885933573e−01	1.0
			1.08910891089108910e−01	1.60820327936249366e+06	2.06282167974606275e−06
11	120		1	8.81906822658790057e−01	1.22786994549474213e+07
(5.5)	(60)	50	131	5.49421653132873500e−01	1.0
			8.39694656488549629e−02	1.30171143932178780e+06	1.69244706267223280e−06
11	160		1	8.91356037870450701e−01	1.04389002929589767e+07
(5.5)	(80)	50	171	5.51185380665772717e−01	1.0
			6.43274853801169555e−02	1.10563026075734245e+06	1.45308479877452971e−06
11	200		1	8.97166973831606152e−01	9.44977522036453336e+06
(5.5)	(100)	50	211	5.52263619572691100e−01	1.0
			5.21327014218009449e−02	1.00047812972492748e+06	1.32377481742240787e−06

Table 3
Comparisons of type I error probabilities for given levels of power

Case	$a/(m)$	$b/(n)$	λ	$1-\beta=0.8$	$1-\beta=0.9$	S-system types
	(11)	(60)		0.0177269641	0.0483114486	R & V
1	(11)	(60)	25	0.0177269641	0.0483114486	(FY9): $1-Y_5$
	5.5	30		0.0177269641	0.0483114486	(BX9): $1-X_5$
	(11)	(90)		0.0129191238	0.0381974512	R & V
2	(11)	(90)	25	0.0129191238	0.0381974512	(FY9): $1-Y_5$
	5.5	45		0.0129191238	0.0381974512	(BX9): $1-X_5$
	(11)	(120)		0.0107659820	0.0333939053	R & V
3	(11)	(120)	25	0.0107659820	0.0333930953	(FY9): $1-Y_5$
	5.5	60		0.0107659820	0.0333930953	(BX9): $1-X_5$
	(11)	(160)		0.0092710697	0.0299241466	R & V
4	(11)	(160)	25	0.0092710697	0.0299241466	(FY9): $1-Y_5$
	5.5	80		0.0092710697	0.0299241466	(BX9): $1-X_5$
	(11)	(200)		0.0084257620	0.0279042347	R & V
5	(11)	(200)	25	0.0084257620	0.0279042347	(FY9): $1-Y_5$
	5.5	100		0.0084257620	0.0279042347	(BX9): $1-X_5$
	(11)	(60)		0.0001220607	0.0005292381	R & V
6	(11)	(60)	50	0.0001220607	0.0005292381	(FY9): $1-Y_5$
	5.5	30		0.0001220607	0.0005292381	(BX9): $1-X_5$
	(11)	(90)		0.0000412818	0.0002143243	R & V
7	(11)	(90)	50	0.0000412818	0.0002143243	(FY9): $1-Y_5$
	5.5	45		0.0000412818	0.0002143243	(BX9): $1-X_5$
	(11)	(120)		0.0000212401	0.0001236418	R & V
8	(11)	(120)	50	0.0000212401	0.0001236417	(FY9): $1-Y_5$
	5.5	60		0.0000212401	0.0001236417	(BX9): $1-X_5$
	(11)	(160)		0.0000120231	0.0000773930	R & V
9	(11)	(160)	50	0.0000120231	0.0000773930	(FY9): $1-Y_5$
	5.5	80		0.0000120231	0.0000773930	(BX9): $1-X_5$
	(11)	(200)		0.0000082531	0.0000568557	R & V
10	(11)	(200)	50	0.0000082531	0.0000568557	(FY9): $1-Y_5$
	5.5	100		0.0000082531	0.0000568557	(BX9): $1-X_5$

Table 4
Computing quantiles corresponding to two power points

Case	$a/(m)$	$b/(n)$	λ	0.1 quantile	0.2 quantile	S-system types
	(11)	(60)		1.9654266674	2.3449621010	R & V for computing F: t
	(11)	(60)		1.9654266674	2.3449621010	(FY9) for computing F: $t=Y_1$
1	5.5	30	25	1.9654266674	2.3449621010	(BX9) for computing F: $t=X_3$
	(11)	(60)		0.2648832954	0.3006551483	(FY9) for computing beta: $s=Y_3$
	5.5	30		0.2648832954	0.3006551483	(BX9) for computing beta: $s=X_1$
	(11)	(90)		1.9919675879	2.3648867276	R & V for computing F: t
	(11)	(90)		1.9919675879	2.3648867276	(FY9) for computing F: $t=Y_1$
2	5.5	45	25	1.9919675879	2.3648867276	(BX9) for computing F: $t=X_3$
	(11)	(90)		0.1957941353	0.2242299133	(FY9) for computing beta: $s=Y_3$
	5.5	45		0.1957941353	0.2242299133	(BX9) for computing beta: $s=X_1$

Table 4 (Continued.)

Case	$a/(m)$	$b/(n)$	λ	0.1 quantile	0.2 quantile	S-system types
	(11)	(120)		2.0059550346	2.3754201875	R & V for computing F: t
	(11)	(120)		2.0059550346	2.3754201875	(FY9) for computing F: t = Y_1
3	5.5	60	25	2.0059550346	2.3754201875	(BX9) for computing F: t = X_3
	(11)	(120)		0.1553192333	0.1788112615	(FY9) for computing beta: s = Y_3
	5.5	60		0.1553192333	0.1788112615	(BX9) for computing beta: s = X_1
	(11)	(160)		2.0167885285	2.3835948392	R & V for computing F: t
	(11)	(160)		2.0167885285	2.3835948392	(FY9) for computing F: t = Y_1
4	5.5	80	25	2.0167885285	2.3835948392	(BX9) for computing F: t = X_3
	(11)	(160)		0.1217702529	0.1407990954	(FY9) for computing beta: s = Y_3
	5.5	80		0.1217702529	0.1407990954	(BX9) for computing beta: s = X_1
	(11)	(200)		2.0234376418	2.3886193570	R & V for computing F: t
	(11)	(200)		2.0234376418	2.3886193570	(FY9) for computing F: t = Y_1
5	5.5	100	25	2.0234376418	2.3886193570	(BX9) for computing F: t = X_3
	(11)	(200)		0.1001441239	0.1161190350	(FY9) for computing beta: s = Y_3
	5.5	100		0.1001441239	0.1161190350	(BX9) for computing beta: s = X_1
	(11)	(60)		3.6600614410	4.2283318822	R & V for computing F: t
	(11)	(60)		3.6608614410	4.2283318822	(FY9) for computing F: t = Y_1
6	5.5	30	50	3.6608614347	4.2283318725	(BX9) for computing F: t = X_3
	(11)	(60)		0.4016125098	0.4366813433	(FY9) for computing beta: s = Y_3
	5.5	30		0.4016125094	0.4366813427	(BX9) for computing beta: s = X_1
	(11)	(90)		3.7246885683	4.2750566757	R & V for computing F: t
	(11)	(90)		3.7246885683	4.2750566757	(FY9) for computing F: t = Y_1
7	5.5	45	50	3.7246885614	4.2750566652	(BX9) for computing F: t = X_3
	(11)	(90)		0.3128279895	0.3431885384	(FY9) for computing beta: s = Y_3
	5.5	45		0.3128279891	0.3431885379	(BX9) for computing beta: s = X_1
	(11)	(120)		3.7590024162	4.3002402791	R & V for computing F: t
	(11)	(120)		3.7590024439	4.3002403172	(FY9) for computing F: t = Y_1
8	5.5	60	50	3.7590024366	4.3002403061	(BX9) for computing F: t = X_3
	(11)	(120)		0.2562706927	0.2827369760	(FY9) for computing beta: s = Y_3
	5.5	60		0.2562706923	0.2827369754	(BX9) for computing beta: s = X_1
	(11)	(160)		3.7859322067	4.3200399563	R & V for computing F: t
	(11)	(160)		3.7859322067	4.3200399563	(FY9) for computing F: t = Y_1
9	5.5	80	50	3.7859321991	4.3200399448	(BX9) for computing F: t = X_3
	(11)	(160)		0.2065273216	0.2289916098	(FY9) for computing beta: s = Y_3
	5.5	80		0.2065273212	0.2289916094	(BX9) for computing beta: s = X_1
	(11)	(200)		3.8026228336	4.3323283664	R & V for computing F: t
	(11)	(200)		3.8026228336	4.3323283664	(FY9) for computing F: t = Y_1
10	5.5	100	50	3.8026228258	4.3323283546	(BX9) for computing F: t = X_3
	(11)	(200)		0.1729688206	0.1924269417	(FY9) for computing beta: s = Y_3
	5.5	100		0.1729688203	0.1924269413	(BX9) for computing beta: s = X_1

Table 5
Computing the p.d.f. and c.d.f. of the noncentral beta distribution at the point s (X_1 or Y_3) = 0.5

Case	$a/(m)$	$b/(n)$	λ	p.d.f.	c.d.f.	Procedure
	5.5	30		1.492192250467	0.937698141355	(BX1)
1	(11)	(60)	25	1.492192250467	0.937698141355	(FY3)
	5.5	30		1.492192250466	0.937698141355	Ding
	5.5	45		0.056737126536	0.998790001677	(BX1)
2	(11)	(90)	25	0.056737126536	0.998790001677	(FY3)
	5.5	45		0.056737126536	0.998790001677	Ding
	5.5	60		0.000637517151	0.999991063720	(BX1)
3	(11)	(120)	25	0.000637517151	0.999991063720	(FY3)
	5.5	60		0.000637517151	0.999991063719	Ding
	5.5	80		0.000000510002	0.999999995149	(BX1)
4	(11)	(160)	25	0.000000510002	0.999999995149	(FY3)
	5.5	80		0.000000510002	0.999999995148	Ding
	5.5	100		0.000000000172	0.999999999999	(BX1)
5	(11)	(200)	25	0.000000000172	0.999999999998	(FY3)
	5.5	100		0.000000000172	0.999999999998	Ding
	5.5	30		5.176367428689	0.486833691139	(BX1)
6	(11)	(60)	50	5.176367428689	0.486833691139	(FY3)
	5.5	30		5.176367428689	0.486833691138	Ding
	5.5	45		2.120314308968	0.924837196375	(BX1)
7	(11)	(90)	50	2.120314308968	0.924837196375	(FY3)
	5.5	45		2.120314308967	0.924837196374	Ding
	5.5	60		0.183799195055	0.996300698618	(BX1)
8	(11)	(120)	50	0.183799195055	0.996300698618	(FY3)
	5.5	60		0.183799195055	0.996300698618	Ding
	5.5	80		0.001601446203	0.999980118429	(BX1)
9	(11)	(160)	50	0.001601446203	0.999980118429	(FY3)
	5.5	80		0.001601446203	0.999980118429	Ding
	5.5	100		0.000004493590	0.999999960158	(BX1)
10	(11)	(200)	50	0.000004493590	0.999999960158	(FY3)
	5.5	100		0.000004493590	0.999999960158	Ding

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