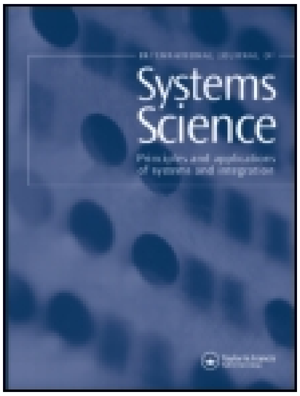

International Journal of Systems Science
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tsys20

Design of observers with unknown inputs using eigenstructure assignment

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Published online: 26 Nov 2010.

International Journal of Systems Science, 2000, volume 31, number 6, pages 705–711

Design of observers with unknown inputs using eigenstructure assignment

SHENG-FUU LIN† and AN-PING WANG†

Obsevers with unknown inputs using eigenstructure assignment are established in this paper. Orders of observers may vary from minimum order to full order. Complete and parametric solutions for observer matrices and generalized eigenvectors are obtained. Owing to the completeness and parametric forms of the solution, more properties of the observer may be obtained; hence, the solution is quite suitable for advanced applications.

1. Introduction

Over the past few decades, many researchers have developed the reduced and/or full order unknown input observers by different approaches (Meditch and Hostetter 1974, Wang *et al.* 1975, Kudva *et al.* 1980, Kobayashi and Nakamizo 1982, Miller and Mukundan 1982, Fairman *et al.* 1984, Hou and Muller 1992, 1994, Syrmos 1993, Yang and Wilde 1988, Darouach *et al.* 1994, Chang and Hsu 1994, Schreier *et al.* 1995), because some of the inputs to the system are inaccessible. It is well known that the observer design problem where all inputs are measurable is dual to the state feedback design problem. However, the way to apply the existent results of the state feedback design to the unknown input observer design is not trivial. Eigenstructure assignment is one of the existing methods for state feedback design. The method designs the feedback gain by finding eigenvectors with assigned eigenvalues of the resulting system (Moore 1976, Fahmy and O'Reilly 1982, Andry *et al.* 1984) and the solution is often represented in parametric forms. More control objectives such as robustness, sensitivity, minimum gain design etc can be achieved by choosing a set of suitable parameters from optimizing certain objective functions (Roppenecker 1983, Andry *et al.* 1984, Owens and O'Reilly 1987, Burrows and Patton 1991, Liu and Patton 1996).

Received 9 October 1998. Accepted 15 June 1999.

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In this paper, unknown input observers with orders varying from minimum order to full order using an eigenstructure assignment method are designed. Complete solutions of the observer matrices are represented in parametric forms. The solution is suitable for other optimization processes. In most eigenstructure assignment approaches based on state feedback design, the system is assumed to be controllable. Therefore, these approaches did not develop solutions of generalized eigenvectors with any uncontrollable eigenvalue. Transmission zeros in the unknown input observer design play the same role as uncontrollable eigenvalues in the feedback design. If a system contains a transmission zero, the transmission zero must be an eigenvalue of any possible unknown input observers. Here, solution for transmission zeros are found.

This paper is organized as follows. In section 2, an unknown input observer is introduced. Some preliminary results are presented in section 3. In section 4 the problem is formulated and in section 5 solutions for unknown input observers are established. Some illustrative examples are provided in section 6, and section 7 concludes the paper.

2. Observer design

Consider the following linear system

$$\left.\begin{aligned} \dot{x} &= Ax + Bu + Dv, \\ y &= Cx, \end{aligned}\right\} \qquad (1)$$

where $x \in R^n$ is the state variable, $u \in R^p$ is the input variable, $v \in R^m$ is the unknown disturbance, $y \in R^q$ is the measurable output and A, B, D, C are matrices with

International Journal of Systems Science ISSN 0020–7721 print/ISSN 1464–5319 online © 2000 Taylor & Francis Ltd
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appropriate dimensions. The main purpose of this paper is to design an rth order observer of the following type

$$\left.\begin{aligned}\dot{z} &= Nz + Ly + Gu,\\ \hat{x} &= Pz + Qy,\end{aligned}\right\} \tag{2}$$

without any knowledge of disturbance v, where r is an integer in the range $n - q \leq r \leq n$, $\hat{x}$ is the estimation of x, $z \in R^r$ is the state variable of the observer, and N, L, G, P, Q are matrices with appropriate dimensions. If there exists a matrix $T \in R^{r \times n}$ satisfying the following conditions:

$$NT - TA + LC = 0, \tag{3}$$

$$TD = 0, \tag{4}$$

$$G - TB = 0, \tag{5}$$

$$PT + QC = I, \tag{6}$$

then we have

$$(\dot{z} - T\dot{x}) = N(z - Tx),$$
$$(\hat{x} - x) = P(z - Tx).$$

If N is stable, then $\hat{x} \to x$. Hence, the problem of the rth order unknown input observer is to find matrices N, L, G, P, Q and T satisfying (3), (4), (5) and (6).

From right-multiplying both sides of (6), it follows that $PTD + QCD = QCD = D$; hence, if $\operatorname{rank} CD < \operatorname{rank} D$, the matrix Q does not exist. This means that the observer (2) cannot be found. From this fact, without loss of generality, assumptions that C is of full row rank, D is of full column rank and $q \geq m$ are made.

3. Some preliminary results

Before solving the problem, we will first discuss the possible eigenvalues of matrix N.

Definition 1: A complex number λ_i is a *transmission zero* of system (1) if and only if

$$\operatorname{rank}\begin{bmatrix}(A - \lambda_i I) & D\\ C & 0\end{bmatrix} < n + m \qquad \square$$

Definition 2: If there exists a set of linear independent vectors

$$\begin{bmatrix} v_{ij}^k \\ w_{ij}^k \end{bmatrix},\ j = 1, \ldots, \bar{\theta}_i;\ k = 1, \ldots, \bar{\rho}_{ij},$$

satisfying that

$$\begin{bmatrix}(A - \lambda_i I) & D\\ C & 0\end{bmatrix}\begin{bmatrix} v_{ij}^k \\ w_{ij}^k \end{bmatrix} = \begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} v_{ij}^{k-1} \\ w_{ij}^{k-1} \end{bmatrix};\quad v_{ij}^0 = 0, \tag{7}$$

then we say that v_{ij}^k is the *right generalized transmission vector* of order k with transmission zero λ_i. Furthermore, if

$$\operatorname{rank}\begin{bmatrix}(A - \lambda_i I) & D\\ C & 0\end{bmatrix} = n + m - \bar{\theta}_i$$

and any non-zero linear combination of

$$\begin{bmatrix} v_{i1}^{\bar{\rho}_{i1}} \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} v_{i\bar{\theta}_i}^{\bar{\rho}_{i\bar{\theta}_i}} \\ 0 \end{bmatrix}$$

is not in the column space of

$$\begin{bmatrix}(A - \lambda_i I) & D\\ C & 0\end{bmatrix},$$

then we say that v_{ij}^k, $j = 1, \ldots, \bar{\theta}_i$; $k = 1, \ldots, \bar{\rho}_{ij}$, form a *complete set* of right transmission vectors with transmission zero λ_i. $\square$

The following lemma discusses the role of transmission zeo in an observer with unknown inputs.

Lemma 1: *If observer* (2) *exists, the eigenvalues of* N *contains the transmission zeros of* (1) *counting multiplicity and the rest of eigenvalues can be assigned arbitrarily.*

Proof: Let Λ_{TZ} be a Jordan form matrix with the transmission zeros as its eigenvalues counting multiplicity and $F \in C^{n \times s}$ be a matrix whose columns are all generalized transmission vectors arranged in the corresponding order. Since $CF = 0$, TF is of full column rank; otherwise, from (6), the observer does not exist. We can, therefore, define a matrix $K \in C^{n \times (r-s)}$ such that $[F \quad K]$ and $T[F \quad K]$ are of full column rank. Three matrices C_2, L_1 and L_2 can then be found such that

$$C[F \quad K] = [0 \quad C_2] \text{ and } L = [TF \quad TK]\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}. \tag{8}$$

By definition, a matrix W exists such that $AF + DW = F\Lambda_{\mathrm{TZ}}$; hence, $TAF = TF\Lambda_{\mathrm{TZ}}$ and A_1 and A_2 can be found such that

$$TA[F \quad K] = [TF \quad TK]\begin{bmatrix}\Lambda_{\mathrm{TZ}} & A_1\\ 0 & A_2\end{bmatrix}. \tag{9}$$

From (3), we have $NTF = TAF = TF\Lambda_{\mathrm{TZ}}$; hence, N_1 and N_2 can be found such that

$$N[TF \quad TK] = [TF \quad TK]\begin{bmatrix}\Lambda_{\mathrm{TZ}} & N_1\\ 0 & N_2\end{bmatrix}. \tag{10}$$

It follows that the eigenvalues of N are those of Λ_{TZ} and N_2. From right-multiplying by $[F \quad K]$ and left-multiplying by $T[F \quad K]^{-1}$ to (3), we have

$$\begin{bmatrix}\Lambda_{\mathrm{TZ}} - \Lambda_{\mathrm{TZ}} & N_1 - A_1 + L_1 C_2\\ 0 & N_2 - A_2 + L_2 C_2\end{bmatrix} = 0. \tag{11}$$

Noticing that if the matrices pair (A_2, C_2) is unobservable, then its unobservable eigenvalue is a transmission zero of the original system. Since Λ_{TZ} contains all transmission zeros, (A_2, C_2) is an observable pair and the eigenvalues of N_2 can be assigned arbitrarily. □

For a transmission zero λ_i, assume that elements of set $\{\bar{\rho}_{i1}, \ldots, \bar{\rho}_{i\bar{\theta}_i}\}$ are arranged as follows: $\bar{\rho}_{i1} \le \bar{\rho}_{i2} \le \cdots \le \bar{\rho}_{i\bar{\theta}_i}$. Denote ϕ_i as the number of distinct elements in the set $\{\bar{\rho}_{i1}, \ldots, \bar{\rho}_{i\bar{\theta}_i}\}$. In addition, the notations $\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{i\phi_i}$, satisfying $\sigma_{il} < \sigma_{i2} < \cdots < \sigma_{i\phi_i}$, represent all distinct elements of the set $\{\bar{\rho}_{i1}, \ldots, \bar{\rho}_{i\bar{\theta}_i}\}$. Assume that there are η_{il} elements with value σ_{il}, $l = 1, \ldots, \phi_i$, within the set $\{\bar{\rho}_{i1}, \ldots, \bar{\rho}_{i\bar{\theta}_i}\}$. Let

$$\begin{aligned} V_{il}^k &= [v^k_{i(\eta_{i0}+\eta_{i1}+\cdots+\eta_{i(l-1)}+1)} \quad v^k_{i(\eta_{i0}+\eta_{i1}+\cdots+\eta_{i(l-1)}+2)} \\ &\qquad \cdots v^k_{i(\eta_{i0}+\eta_{i1}+\cdots+\eta_{i(l-1)}+\eta_{il})}], \\ W_{il}^k &= [w^k_{i(\eta_{i0}+\eta_{i1}+\cdots+\eta_{i(l-1)}+1)} \quad w^k_{i(\eta_{i0}+\eta_{i1}+\cdots+\eta_{i(l-1)}+2)} \\ &\qquad \cdots w^k_{i(\eta_{i0}+\eta_{i1}+\cdots+\eta_{i(l-1}+\eta_{il})}], \\ k &= 1, \ldots, \sigma_{il};\ l = 1, \ldots, \phi_i \text{ and } \eta_{i0} = 0, \end{aligned}$$

where the column vectors of V_{il}^k are the right generalized transmission vectors of grade k, in all chains with length σ_{il}. Then by (7), we have

$$\begin{bmatrix} (A - \lambda_i I) & D \\ C & 0 \end{bmatrix} \begin{bmatrix} V_{il}^k \\ W_{il}^k \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{il}^{k-1} \\ W_{il}^{k-1} \end{bmatrix}, \quad k = 1, \ldots, \sigma_{il};\ V_{il}^0 = 0. \tag{12}$$

Let $U_{il} = [V_{i1}^{\sigma_{i1}} \cdots V_{il}^{\sigma_{il}}]$, which gathers all the last generalized transmission vectors in those chains with length less than or equal to σ_{il}.

4. Problem formulation

The key problem in this paper is now stated as follows. *Design an observer with unknown inputs using eigenstructure assignment.*

Given a symmetric set of complex numbers $\{\lambda_i, \ldots, \lambda_\pi\}$, which contain transmission zeros, and a set of positive integers $\rho_{i1}, \ldots, \rho_{i\theta_i}$, $i = 1, \ldots, \pi$, representing the multiplicities and satisfying $\sum_{i=1}^{\pi} \sum_{j=1}^{\theta_i} \rho_{ij} = r$, we want to find the parametric solutions of matrices N, L, G, P, Q and T over the field of real number satisfying (3), (4), (5) and (6) where the matrix N has eigenvalues $\{\lambda_i, \ldots, \lambda_\pi\}$ and left generalized eigenvectors $h_{ij}^1, \ldots, h_{ij}^{\rho_{ij}}$; $j = 1, \ldots, \theta_i$; $i = 1, \ldots, \pi$ satisfying the relation:

$$h_{ij}^k N = \lambda_i h_{ij}^k + h_{ij}^{k-1}, \text{ for } i = 1, \ldots, \pi;\ j = 1, \ldots, \theta_i; \quad k = 1, \ldots, \rho_{ij}, \text{ and } h_{ij}^0 = 0. \tag{13}$$

Since h_{ij}^k, $i = 1, \ldots, \pi$; $j = 1, \ldots, \theta_i$; $k = 1, \ldots, \rho_{ij}$ are linearly independent, (3) and (4) are equivalent to the following conditions:

$$[\tilde{t}_{ij}^k \quad \tilde{l}_{ij}^k] \begin{bmatrix} (A - \lambda_i I) & D \\ C & 0 \end{bmatrix} = [\tilde{t}_{ij}^{k-1} \quad 0]$$

$$i = 1, \ldots, \pi;\ j = 1, \ldots, \theta_i;\ k = 1, \ldots, \rho_{ij}, \tag{14}$$

where $\tilde{t}_{ij}^k = -h_{ij}^k T$ and $\tilde{l}_{ij}^k = h_{ij}^k L$. Define H as

$$H = \begin{bmatrix} H_1 \\ \vdots \\ H_\pi \end{bmatrix}, \quad \text{where } H_i = \begin{bmatrix} H_{i1} \\ \vdots \\ H_{i\theta_i} \end{bmatrix} \quad \text{and } H_{ij} = \begin{bmatrix} h_{ij}^1 \\ \vdots \\ h_{ij}^{\rho_{ij}} \end{bmatrix}.$$

The matrices $\tilde{T}$ and $\tilde{L}$ are defined in the same way. If solutions of $\tilde{T}$ and $\tilde{L}$ are found from (14), then by choosing a non-singular H, the solutions of T, L, N and G are:

$$T = -H^{-1}\tilde{T},\ L = H^{-1}\tilde{L},\ N = H^{-1}\Lambda H \text{ and } G = TB,$$

where Λ is a Jordan form matrix in lower case with eigenvalues $\{\lambda_1, \ldots, \lambda_\mu\}$ and multiplicity $\{\rho_{i1}, \ldots, \rho_{i\theta_i};\ i = 1, \ldots, \mu\}$. Let X^+ represent the pseudo-inverse of the matrix X satisfying $X^+ = (X^H X)^{-1} X^H$. If the matrix $\tilde{T}$ or T satisfies the following relation:

$$\text{rank} \begin{bmatrix} T \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} \tilde{T} \\ C \end{bmatrix} = n,$$

then solutions of P and Q are as follows:

$$[P \quad Q] = \begin{bmatrix} T \\ C \end{bmatrix}^+ + K\left(I_{r=q} - \begin{bmatrix} T \\ C \end{bmatrix} \begin{bmatrix} T \\ C \end{bmatrix}^+ \right),$$

where $K \in R^{n \times (r+q)}$ represents free parameters.

From the above discussion, the following lemma similar to that given by Moore (1976) and Klein and Moore (1977), characterizing all possible solutions of the observer, is given.

Lemma 2: *Let $\{\lambda_1, \ldots, \lambda_\mu\}$ be a symmetric set of complex numbers and let $\{\rho_{i1}, \ldots, \rho_{i\theta_i};\ i = 1, \ldots, \mu\}$ be a set of positive integers satisfying $\sum_{i=1}^{\mu} \sum_{j=1}^{\theta_i} \rho_{ij} = r$. There exist N, L, G, P, Q, T and $h_{ij}^1, \ldots, h_{ij}^{\rho_{ij}}$; $j = 1, \ldots, \theta_i$; $i = 1, \ldots, \pi$, satisfying* (3), (4), (5), (6) *and* (13) *if and only if*

(1) *matrix H is non-singular,*

(2) *for each $i \in \{1, \ldots, \pi\}$ there exists $i' \in \{1, \ldots, \pi\}$ such that $\lambda_i = (\lambda_{i'})^*$, $\theta_i = \theta_{i'}$, $\rho_{ij} = \rho_{i'j}$, $j = 1, \ldots, \theta_i$, and $v_{ij}^k = (v_{ij}^k)^*$, $i = 1, \ldots, \pi$; $j = 1, \ldots, \theta_i$; $k = 1, \ldots, \rho_{ij}$, where $(v_{ij}^k)^*$ means the component-wise conjugate of the vector v_{ij}^k.*

(3) *for each $i \in \{1, \ldots, \mu\}$, there exists a set of vectors $\{\tilde{t}_{ij}^k, \tilde{l}_{ij}^k, i = 1, \ldots, \pi;\ j = 1, \ldots, \theta_i;\ k = 1, \ldots, \rho_{ij}\}$ satisfying* (14) *and*

$$\text{rank} \begin{bmatrix} \tilde{T} \\ C \end{bmatrix} = n. \tag{15}$$

Proof: The sufficient part has been stated above. Assume that the solution exists. Since matrix H represents the left generalized eigenvectors, it is non-singular. If (15) is not satisfied, P and Q do not exist. Hence, the necessity part follows. □

5. Main results

Complete parameteric solutions of $\tilde{t}^{1}_{ij},\ldots,\tilde{t}^{\rho_{ij}}_{ij}$ and $\tilde{l}^{1}_{ij},\ldots,\tilde{l}^{\rho_{ij}}_{ij}$ with an eigenvalue λ_i from (14) according to whether the eigenvalue λ_i is a transmission zero or not will be shown in this section.

5.1. *Solutions for eigenvalues which are not transmission zeros*

If λ_i is not a transmission zero, then

$$\operatorname{rank}\begin{bmatrix}(A-\lambda_i I) & D\\ C & 0\end{bmatrix} = n+m.$$

There exists a non-singular transformation such that

$$\begin{bmatrix}P_i^{11} & P_i^{12}\\ P_i^{21} & P_i^{22}\end{bmatrix}\begin{bmatrix}(A-\lambda_i I) & D\\ C & 0\end{bmatrix}\begin{bmatrix}Q_i^{1}\\ Q_i^{2}\end{bmatrix} = \begin{bmatrix}I_{(n+m)\times(n+m)}\\ 0\end{bmatrix}, \tag{16}$$

where $P_i^{11}\in C^{(n+m)\times n}$, $P_i^{12}\in C^{(n+m)\times q}$, $P_i^{21}\in C^{(q-m)\times n}$, $P_i^{22}\in C^{(q-m)\times q}$, $Q_i^{1}\in C^{n\times(n+m)}$, $Q_i^{2}\in C^{m\times(n+m)}$ and $I_{(n+m)\times(n+m)}$ is an $(n+m)\times(n+m)$ identity matrix.

Theorem 1: *If λ_i is not a transmission zero, complete solutions of* (14) *can be represented as follows:*

$$[\tilde{t}^{k}_{ij}\quad \tilde{l}^{k}_{ij}] = [\tilde{t}^{k-1}_{ij}Q_i^{1}\quad f^{k-1}_{ij}]\begin{bmatrix}P_i^{11} & P_i^{12}\\ P_i^{21} & P_i^{22}\end{bmatrix},$$
$$k=1,\ldots,\rho_{ij};\ \tilde{t}^{0}_{ij}=0, \tag{17}$$

where f^{k-1}_{ij}, $k=1,\ldots,\rho_{ij}$, are free vectors and if $\lambda_i=(\lambda_{i'})^{*}$ then the free parameters should be chosen as $f^{k}_{ij}=(f^{k}_{i'j})^{*}$ for consideration of realness.

Proof: The constraint for consideration of realness can be seen easily. Here, we will show that (17) is equivalent to (14).

(Necessity) If the following variable transformation is introduced:

$$[\tilde{t}^{k}_{ij}\quad \tilde{l}^{k}_{ij}] = [a^{k}_{ij}\quad f^{k-1}_{ij}]\begin{bmatrix}P_i^{11} & P_i^{12}\\ P_i^{31} & P_i^{32}\end{bmatrix},$$
$$k=1,\ldots,\rho_{ij}, \tag{18}$$

then from (14) and (16), we have $a^{k}_{ij}=\tilde{t}^{k-1}_{ij}Q_i^{1}$, $k=1,\ldots,\rho_{ij}$. By applying this relation into (18), (17) is obtained.

(Sufficiency) From (17) and (16), we have

$$\begin{aligned}[\tilde{t}^{k}_{ij}\quad \tilde{l}^{k}_{ij}]\begin{bmatrix}(A-\lambda_i I) & D\\ C & 0\end{bmatrix} &= [\tilde{t}^{k-1}_{ij}Q_i^{1}\quad \tilde{z}^{k}_{ij}]\begin{bmatrix}I_{(n+m)\times(n+m)}\\ 0\end{bmatrix}\begin{bmatrix}Q_i^{1}\\ Q_i^{2}\end{bmatrix}^{-1}\\ &= [\tilde{t}^{k-1}_{ij}\quad 0].\end{aligned}$$

Therefore the vectors given by (17) satisfy (14). □

5.2. *Solutions for eigenvalues which are transmission zeros*

If λ_i is a transmission zero, then

$$\operatorname{rank}\begin{bmatrix}(A-\lambda_i I) & D\\ C & 0\end{bmatrix} = r_i < n+m.$$

The following non-singular transformation can be obtained:

$$\begin{bmatrix}P_i^{11} & P_i^{12}\\ P_i^{21} & P_i^{22}\end{bmatrix}\begin{bmatrix}(A-\lambda_i I) & D\\ C & 0\end{bmatrix}\begin{bmatrix}Q_i^{11} & Q_i^{12}\\ Q_i^{21} & Q_i^{22}\end{bmatrix} = \begin{bmatrix}I_{r_i} & 0\\ 0 & 0\end{bmatrix}, \tag{19}$$

where $P_i^{11}\in C^{r_i\times n}$, $P_i^{12}\in C^{r_i\times q}$, $P_i^{21}\in C^{(n+q-r_i)\times n}$, $P_i^{22}\in C^{(n+q-r_i)\times q}$, $Q_i^{11}\in C^{n\times r_i}$, $Q_i^{12}\in C^{n\times(n+m-r_i)}$, $Q_i^{21}\in C^{m\times r_i}$, $Q_i^{22}\in C^{m\times(n+m-r_i)}$; and I_{r_i} is an $r_i\times r_i$ identity matrix.

Lemma 3: *$P_i^{21}U_{il}$ is of full column rank for $l=1,\ldots,\phi_i$.*

Proof: Assume that a non-zero vector $\boldsymbol{f}$ cn be found such that $P_i^{21}U_{il}f=0$. Since

$$[I_{r_i}\quad 0]\begin{bmatrix}Q_i^{11} & Q_i^{12}\\ Q_i^{21} & Q_i^{22}\end{bmatrix}^{-1}$$

is of full row rank, a vector

$$\begin{bmatrix}\bar{v}\\ \bar{w}\end{bmatrix}$$

can be found such that

$$[I_{r_i}\quad 0]\begin{bmatrix}Q_i^{11} & Q_i^{12}\\ Q_i^{21} & Q_i^{22}\end{bmatrix}^{-1}\begin{bmatrix}\bar{v}\\ \bar{w}\end{bmatrix} = [P_i^{11}\quad P_i^{12}]\begin{bmatrix}U_{il}f\\ 0\end{bmatrix}.$$

Then it follows that

$$\begin{bmatrix}A-\lambda_i I & D\\ C & 0\end{bmatrix}\begin{bmatrix}\bar{v}\\ \bar{w}\end{bmatrix} = \begin{bmatrix}U_{il}f\\ 0\end{bmatrix}.$$

It is a contradiction to the definition of U_{il}. □

From lemma 3, for each U_{il}, $l=1,\ldots,\phi_i$, we can obtain the following non-singular transformation:

$$\begin{bmatrix}S^{1}_{il}\\ S^{2}_{il}\end{bmatrix}P_i^{21}U_{il} = \begin{bmatrix}I_{\eta_{i1}+\cdots+\eta_{il}}\\ 0\end{bmatrix}, \tag{20}$$

where

$$S_{il}^1 \in R^{(\eta_{i1}+\cdots+\eta_{il})\times(n+q-r_i)},$$
$$S_{il}^2 \in R^{(n+q-r_i-\eta_{i1}-\cdots-\eta_{il})\times(n+q-r_i)}.$$

Theorem 2: *Assume that λ_i is a transmission zero.*

(*a*) *If $\rho_{ij} \le \sigma_{i1}$, complete solutions of* (14) *can be represented as follows:*

$$[\tilde{t}_{ij}^k \quad \tilde{l}_{ij}^k] = [\tilde{t}_{ij}^{k-1} Q_i^{11} \quad f_{ij}^{k-1}] \begin{bmatrix} P^{11} & P^{12} \\ P^{21} & P^{22} \end{bmatrix} \tilde{t}_{ij}^0 = 0,$$
$$k = 1, \ldots, \rho_{ij}, \qquad (21)$$

where f_{ij}^{k-1}, $k = 1, \ldots, \rho_{ij}$ are free vectors.

(*b*) *If $\sigma_{i1} < \rho_{ij} \le \sigma_{i\phi_i}$, there exists a number b such that $\sigma_{ib} < \rho_{ij} < \sigma_{i(b+1)}$, and if $\rho_{ij} > \sigma_{i\phi_i}$, then $b = \phi_i$. Under the above conditions, the complete solutions of* (14) *can be represented as follows:*

$$[\tilde{t}_{ij}^k \quad \tilde{l}_{ij}^k] = [\tilde{t}_{ij}^{k-1} Q_i^{k-1} \quad f_{ij}^{k-1}] \begin{bmatrix} I & -P_i^{11} U_{ib} S_{ib}^1 \\ 0 & S_{ib}^2 \end{bmatrix}$$
$$\times \begin{bmatrix} P_i^{11} & P_i^{12} \\ P_i^{21} & P_i^{22} \end{bmatrix}, \quad \tilde{t}_{ij}^0 = 0, \ k = 1, \ldots, (\rho_{ij} - \sigma_{ib}),$$
$$[\tilde{t}_{ij}^k \quad \tilde{l}_{ij}^k] = [\tilde{t}_{ij}^{k-1} Q_i^{11} \quad f_{ij}^{k-1}] \begin{bmatrix} I & -P_i^{11} U_{il} S_{il}^1 \\ 0 & S_{il}^2 \end{bmatrix}$$
$$\times \begin{bmatrix} P_i^{11} & P_i^{12} \\ P_i^{21} & P_i^{22} \end{bmatrix},$$
$$k = (\rho_{ij} - \sigma_{i(l+1)}) + 1, \ldots, (\rho_{ij} - \sigma_{il}),$$
$$l = (b-1), \ldots, 1,$$
$$[\tilde{t}_{ij}^k \quad \tilde{l}_{ij}^k] = [\tilde{t}_{ij}^{k-1} Q_i^{11} \quad f_{ij}^{k-1}] \begin{bmatrix} P^{11} & P^{12} \\ P^{21} & P^{22} \end{bmatrix},$$
$$k = (\rho_{ij} - \sigma_{i1}) + 1, \ldots, \rho_{ij}, \qquad (22)$$

where f_{ij}^{k-1}, $k = 1, \ldots, \rho_{ij}$ are free vectors.

(*c*) *In* (a) *and* (b) *, if $\lambda_i = (\lambda_{i'})^*$ then the free parameters should be chosen as $f_{ij}^k = (f_{i'j}^k)^*$ for consideration of realness.*

Proof: The constraint for consideration of realness can be easily seen. Here, we will show that (21) is equivalent to (14), or (22) is equivalent to (14).

(Necessity) A variable transformation is adopted as follows:

$$[\tilde{t}_{ij}^k \quad \tilde{l}_{ij}^k] = [a_{ij}^k \quad c_{ij}^k] \begin{bmatrix} P^{11} & P^{12} \\ P^{21} & P^{22} \end{bmatrix}, \quad k = 1, \ldots, \rho_{ij}, \quad (23)$$

where $a_{ij}^k \in C^{1\times r}$ and $c_{ij}^k \in C^{1\times(m+n-r)}$; hence, (14) is equivalent to the following relations:

$$a_{ij}^k = \tilde{t}_{ij}^{k-1} Q_i^{11}, \qquad (24)$$
$$0 = \tilde{t}_{ij}^{k-1} Q_i^{12}, \quad k = 1, \ldots, \rho_{ij}. \qquad (25)$$

Since the column spaces of Q_i^{12} and V_{il}^1, $l = 1, \ldots, \phi_i$ are the same, (25) is equivalent to $\tilde{t}_{ij}^{k-1} V_{il}^1 = 0$, $l = 1, \ldots, \phi_i$; $k = 1, \ldots, \rho_{ij}$. From (12) and (14), it can be further shown that

$$\tilde{t}_{ij}^{k-1} V_{il}^1 = \begin{cases} \tilde{t}_{ij}^{(k-\sigma_{il})} V_{il}^{\sigma_{il}}, & \text{if } k > \sigma_{il}, \\ \tilde{t}_{ij}^0 V_{il}^k \equiv 0, & \text{if } k \le \sigma_{il}. \end{cases} \qquad (26)$$

We discuss the problem in the following two cases:

(*a*) From (26), if $\rho_{ij} \le \sigma_{i1}$, (14) is equivalent to (24). Let $f_{ij}^{k-1} = c_{ij}^k$. By applying (24) to (23), (21) is obtained.

(*b*) If $\sigma_{i1} < \rho_{ij} \le \sigma_{i\phi_i}$, find a number b such that $\sigma_{ib} < \rho_{ij} \le \sigma_{i(b+1)}$, and if $\rho_{ij} > \sigma_{i\phi_i}$, let $b = \phi_i$. From (26), (25) is equivalent to the following relations:

$$\tilde{t}_{ij}^{k-1} U_{ib} = 0, \quad \text{where } k = 2, \ldots, (\rho_{ij} - \sigma_i^b) + 1,$$
$$\tilde{t}_{ij}^{k-1} U_{i(b-1)} = 0, \quad \text{where } k = (\rho_{ij} - \sigma_{ib}) + 2, \ldots,$$
$$(\rho_{ij} - \sigma_{i(b-1)}) + 1,$$
$$\vdots$$
$$\tilde{t}_{ij}^{k-1} U_{i1} = 0, \quad \text{where } k = (\rho_{ij} - \sigma_{i2}) + 2, \ldots,$$
$$(\rho_{ij} - \sigma_{i1}) + 1.$$

From the above relations and (23), we have

$$(a_{ij}^{k-1} P_i^{11} + c_{ij}^{k-1} P_i^{21}) U_{ib} = 0,$$
$$k = 2, \ldots, (\rho_{ij} - \sigma_{ib}) = 1,$$
$$(a_{ij}^{k-1} P_i^{11} + c_{ij}^{k-1} P_i^{21}) U_{il} = 0,$$
$$k = (\rho_{ij} - \sigma_{i(l+1)}) + 2, \ldots, (\rho_{ij} - \sigma_{il}) + 1,$$
$$l = 1, \ldots, (b-1).$$

From (20) and (24), the above relations are equivalent to

$$c_{ij}^k = -\tilde{t}_{ij}^{k-1} Q_i^{11} P_i^{11} U_{1b} S_{ib}^1 + f_{ij}^{k-1} S_{ib}^2,$$
$$k = 1, \ldots, (\rho_{ij} - \sigma_{ib}),$$
$$c_{ij}^k = -\tilde{t}_{ij}^{k-1} Q_i^{11} P_i^{11} U_{1l} S_{il}^1 + f_{ij}^{k-1} S_{il}^2,$$
$$k = (\rho_{ij} - \sigma_{i(i-1)}) + 1, \ldots, (\rho_{ij} - \sigma_{il}),$$
$$l = 1, \ldots, (b-1), \qquad (27)$$

where f_{ij}^k are free vectors. Note that c_{ij}^k is free for $k = (\rho_{ij} - \sigma_{il}), \ldots, \rho_{ij}$. Let

$$c_{ij}^k = f_{ij}^{k-1}, \quad k = (\rho_{ij} - \sigma_{i1}) + 1, \ldots, \rho_{ij}. \qquad (28)$$

By applying (24), (27) and (28) to (23), (22) is obtained.

(Sufficiency) Since (23) is a non-singular transformation and (26) is an equivalent relation, (14) can be derived from (21) or (22) by reversing the procedure. □

6. Examples

Example 1: Consider system (1) with the following matrices

$$A = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

This example has been studied by Miller and Mukundan (1982), Yang and Wilde (1988), Hou and Muller (1992). This system has one transmission zero -4 with $\bar{\theta}_1 = 1$ and $\rho_{11} = 1$. If a first-order observer is designed, the eigenvalue must be -4. We have

$$T = h^{-1} f_{11}^{0} [0 \quad 0 \quad -1], \quad L = h^{-1} f_{11}^{0} [0 \quad 3],$$

$$[P \quad Q] = \begin{bmatrix} \dfrac{h}{f_{11}^{0}} & 1 & 0 \\ 0 & 0 & 1 \\ -\dfrac{h}{f_{11}^{0}} & 0 & 0 \end{bmatrix}.$$

Let $a = h^{-1} f_{11}^{0}$, then the parametric solutions of the observer are

$$\dot{z} = -4z + [0 \quad 3a]y, \quad \hat{x} = \begin{bmatrix} \dfrac{1}{a} \\ 0 \\ -\dfrac{1}{a} \end{bmatrix} z + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} y. \quad (29)$$

Putting the existing results in the previous research studies (Miller and Mukundan 1982. Yang and Wilde 1988, Hou and Muller 1992), only solutions when $a = -1$ are obtained. However, in (29), there is one more free parameter a, and a is not a trivial parameter. It provides a greater degree of freedom to assign a suitable observer according to the control requirement. □

Example 2: Consider system (1) with the following matrices (Hou and Muller 1992)

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

This system has one transmission zero -1 with $\theta_1 = 1$ and $\rho_{11} = 1$.

(*a*) First (minimum) order observer: the eigenvalue of the observer must be -1. Let $f_{11}^{0} = [g_1 \quad g_2]$. It follows that

$$T = \begin{bmatrix} 0 & \dfrac{1}{h} g_1 & -\dfrac{1}{h} g_2 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & \dfrac{1}{h} g_1 \end{bmatrix},$$

$$[P \quad Q] = \begin{bmatrix} 0 & 1 & -1 \\ \dfrac{1}{g_1} h & 0 & \dfrac{1}{g_1} f_2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $a_1 = h^{-1} g_1$ and $a_2 = h^{-1} g_2$, then the solutions are

$$\dot{z} = -z + [0 \quad a_1]y, \quad \hat{x} = \begin{bmatrix} 0 \\ \dfrac{1}{a_1} \\ 0 \end{bmatrix} z + \begin{bmatrix} 1 & -1 \\ 0 & \dfrac{a_2}{a_1} \\ 0 & 1 \end{bmatrix} y. \quad (30)$$

In (30), there are two more free parameters a_1 and a_2 then the first-order results given by Hou and Muller (1992) whose solutions are those with $a_1 = 1$ and $a_2 = 0$ in (30).

(*b*) Third (full) order observer: here, we assigned the eigenvalues to be the transmission zero $\lambda_1 = -1$ with $\theta_1 = 1$ and $\rho_{11} = 3$. By a simple computation, it follows that

$$[\tilde{t}_{1j}^{k} \quad \tilde{l}_{1j}^{k}] = [\tilde{t}_{1j}^{k-1} \quad f_{1j}^{k-1}] \begin{bmatrix} 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix};$$

$$\tilde{t}_{ij}^{0} = 0, \; k = 1, 2,$$

$$[\tilde{t}_{1j}^{k} \quad \tilde{l}_{1j}^{k-1}] = [\tilde{t}_{1j}^{k-1} \quad f_{1j}^{k-1}] \begin{bmatrix} 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$k = 3.$$

Choose $f_{11}^{0} = -1$, $f_{11}^{1} = 2$, $f_{11}^{2} = [1 \quad 1]$,

$$H = V^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and

$$K = \begin{bmatrix} 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 & 0 \end{bmatrix},$$

then the resulting system is

$$\dot{z} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & -2 \end{bmatrix} z + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -2 \end{bmatrix} y,$$

$$\hat{x} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} z + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} y.$$

7. Conclusions

In this paper, the eigenstructure assignment method for state feedback design has been further applied to the unknown input observer design with orders varied from minimum order to full order. Complete and parametric solutions of the observer matrices and the generalized eigenvector are obtained. In the illustrative examples, we can see that results obtained by the proposed method have more meaningful free parameters than the previous results. It is because the proposed solutions are complete, as shown in the main theorems. The completeness and parametric form of the solutions makes then more suitable for advanced applications.

Acknowledgements

This research was supported in part by the National Science Council of Taiwan, R.O.C., under contract NSC-87-2213-E-009-135.

References

Andry, A. N., Chung, J. C., and Shapiro, E. Y., 1984, Modalized observers. *IEEE Translations on Automatic Control,* **29,** 669–672.

Burrows, S. P., and Patton, R. J., 1991, Design of a low-sensitivity, minimum norm and structurally constrained control law using eigenstructure assignment. *Optim. Control Appl. Methods,* **12,** 131–140.

Chang, S. K., and Hsu, P. L., 1994, On the application of the {1}-inverse to the design of general structured unknown input observers. *International Journal of Systems Science,* **25,** 2167—2186.

Darouach, M., Zasadzinski, M., and Xu, S. J., 1994, Full-order observer for linear systems with unknown inputs. *IEEE Transactions on Automatic Control,* **39,** 606–609.

Fahmy, M. M., and O'Reilly, J., 1982, On eigenstructure assignment in linear multivariable systems. *IEEE Transactions on Automatic Control,* **27,** 690–693.

Fairman, F. W., Mahil, S. S., and Luk, L., 1984, Disturbance decouple observer design via singular value decomposition. *IEEE Transactions on Automatic Control,* **29,** 84–86.

Hou, M., and Muller, P. C., 1992, Design of observers for linear systems with unknown inputs. *IEEE Transactions on Automatic Control,* **37,** 871–875; 1994, Disturbance decoupled observer design: a unified viewpoint. *IEEE Transactions on Automatic Control,* **39,** 1338–1341.

Klein, G., and Moore, B. C., 1977, Eigenvalue-generalized eigenvector assignment with state feedback. *IEEE Transactions on Automatic Control,* **22,** 140–141.

Kobayashi, N., and Nakamizo, T., 1982, An Observer design for linear systems with unknown inputs. *International Journal of Control,* **35,** 605–619.

Kudva, P., Viswanadham, N., and Ramakrishna, A., 1980, Observers for linear systems with unknown inputs. *IEEE Transactions on Automatic Control,* **25,** 113–115.

Liu, G. P., and Patton, R. J., 1996, Robust control design using eigenstructure assignment and multi-objective optimization. *International Journal of Systems Science,* **27,** 871–879.

Meditch, J. S., and Hostetter, G. H., 1974, Observer for systems with unknown and inaccessible inputs. *International Journal of Control,* **19,** 473–480.

Miller, R. J., and Mukundan, R., 1982, On designing reduced order observers for linear time invariant systems subject to unknown inputs. *International Journal of Control,* **35,** 183–188.

Moore, B. C., 1976, On the flexibility offered by state feedback in multivariable systems beyond closed-loop eigenvalue assignment. *IEEE Transactions on Automatic Control,* **21,** 689–692.

Owens, T. J., and O'Reilly, J., 1987, Parametric state feedback control with response insensitivity. *International Journal of Control,* **45,** 791–809.

Roppenecker, G., 1983, Minimum norm output feedback design under specified eigenvalue areas. *Systems Control Letter,* **3,** 101–103.

Schreier, G., Gaddouna, B., and Ragot, J., 1995, Matrices computation of an unknown-input observer. *International Journal of Systems Science,* **26,** 991–998.

Yang, F., and Wilde, R. W., 1988, Observers for linear systems with unknown inputs. *IEEE Transactions on Automatic Control,* **33,** 677–681.

Wang, S. H., Davison, E. J., and Dorato, P., 1975. Observing the states of systems with unmeasurable disturbances. *IEEE Transactions on Automatic Control,* **20,** 716–717.