V. Conclusion

We have proposed the algorithm which enables us to obtain reduced-order Riccati equations for PRness, WSPRness and SPRness. WSPRness can be characterized by the existence of a symmetric positive definite stabilizing solution for the reduced-order Riccati equation and, in addition to the existence of the stabilizing solution, SPRness by zero or two inherent integration of $\Phi(s)$. While WSPRness and SPRness have been used with confusion in some literature, the necessary and sufficient condition in the time domain for a WSPR transfer function to be SPR has been demonstrated. This condition is very important in the adaptive control, Popov criterion, and absolute stability theory. We have also shown that the feedback system consisting of a nonlinear time-invariant passive system and a WSPR transfer function is asymptotically stable, which has been an open problem.

Acknowledgment

The authors acknowledge Prof. K. Sugimoto of Nagoya University and Prof. K. Hamada of Gifu University for helpful discussions.

References

[1] B. D. O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis*. Englewood Cliffs, NJ: Prentice Hall, 1973.
[2] C. I. Byrnes, A. Isidori, and J. C. Willems, "Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 1228–1240, Nov. 1991.
[3] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*. New York: Academic, 1975.
[4] W. M. Haddad and D. S. Bernstein, "Explicit construction of quadratic Lyapunov functions for the small gain, positivity, circle, and Popov theorems and their application to robust stability. Part I: Continuous-time theory," *Int. J. Robust and Nonlinear Contr.*, vol. 3, pp. 313–339, 1993.
[5] S. M. Joshi and S. Gupta, "On a class of marginally stable positive-real systems," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 152–155, Jan. 1996.
[6] A. Kelka and S. Joshi, *Control of Nonlinear Multibody Flexible Space Structures*: Springer, 1996, Lecture Notes in Control and Information Sciences 221.
[7] H. K. Khalil, *Nonlinear Systems*, 2nd ed: Macmillan, 1996.
[8] R. Lozano-Leal and S. M. Joshi, "Strictly positive real transfer functions revisited," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 1243–1245, Nov. 1990.
[9] R. Lozano-Leal and S. M. Joshi, "Reply to "Comments on 'Strictly positive real transfer functions revisited"'," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 1799–1800, Oct. 1995.
[10] H. J. Marquez and C. J. Damaren, "Comments on 'Strictly positive real transfer functions revisited'," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 478–479, Mar. 1995.
[11] T. Mita, Y. Chida, and J. W. Wang, "Strictly positive real condition and pseudo-strictly positive real condition," *Trans. SICE*, vol. 25, pp. 751–757, 1989. in Japanese.
[12] M. K. Sain and J. L. Massey, "Invertibility of linear time-invariant dynamical systems," *IEEE Trans. Automat. Contr.*, vol. 14, pp. 141–149, Apr. 1969.
[13] W. Sun, P. P. Khargonekar, and D. Shim, "Solution to the positive real control problem for linear time-invariant systems," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 2034–2046, Oct. 1994.
[14] G. Tao and P. A. Ioannou, "Strictly positive real matrices and the Lefschetz-Kalman-Yakubovich lemma," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 1183–1185, Dec. 1988.
[15] H. Weiss, Q. Wang, and J. L. Speyer, "System characterization of positive real conditions," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 540–544, Mar. 1994.
[16] J. T. Wen, "Time domain and frequency domain conditions for strict positive realness," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 988–992, Oct. 1988.
[17] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*: Prentice Hall, 1996.

General Two-Stage Kalman Filters

Chien-Shu Hsieh and Fu-Chuang Chen

***Abstract*—A general two-stage Kalman filter (GTSKF) that is equivalent to, but numerically more efficient than, the standard single-stage Kalman filter is developed for general, time-varying, linear discrete-time systems. Analytical results defining the reduction in computational burdens are presented. Simulation results that validate the predicted efficiency improvements are shown as well.**

***Index Terms*—Reduced-order estimator, reduced-order observer, single-stage Kalman filter, two-stage Kalman estimator.**

I. Introduction

The general state estimation problem in a stochastic linear system is solved by the well-known Kalman filter (KF). It is known that the KF may suffer from computational burden and numerical problems when state dimensions are large. To reduce the computational burden of the KF, researchers have tried the basis changes technique [1] or used the uncorrelated assumption of the measurement noises [2] to simplify the computation. However, their methods are either restricted to time-invariant systems with some off-line preprocessing requirement, or nonpractical for general systems. On the other hand, some researchers have tried to apply reduced-order observers, e.g., [3]–[6]. However, their results are optimal when all or some of the measurements are perfect.

The goal of this correspondence is to propose a general decoupled structure of the KF. With this new structure, a more efficient KF, suitable for parallel-computing, can be obtained. It is shown that the conventional two-stage decoupling technique, originally proposed by Friedland [7] to decouple the bias augmented state filter, can be generalized to achieve this. This correspondence presents the derivation of a general two-stage Kalman filter (GTSKF) which provides the optimal estimate of the system state and can be applied to general, time-varying, linear dynamic systems without a constraint on their structure. This new filter is more efficient than the single-stage KF. Analytical and numerical results are both presented to verify this computational advantage. It is also shown that this computational superiority will gain the most benefit when the proposed GTSKF is applied to implement the interacting multiple model (IMM) algorithm [10].

This correspondence is organized as follows. The problem is stated in Section II. In Section III, we apply the two-stage transformation to the KF to obtain the GTSKF which is optimal in the minimum-mean-square-error (MMSE) sense. In Section IV, the computational savings of the GTSKF are analyzed to demonstrate the superiority of the proposed filter. Section V gives a simulated example to verify the results of Section IV. Section VI gives a potential application of the GTSKF as an alternative to implement the IMM algorithm. Section VII has the conclusions.

II. Statement of the Problem

Consider the following discrete-time system:

$$X_{k+1} = A_k X_k + B_k u_k + w_k \tag{1}$$
$$Y_k = C_k X_k + \eta_k \tag{2}$$

Manuscript received December 22, 1997; revised June 11, 1998. Recommended by Associate Editor, J. C. Spall.

The authors are with the Department of Electrical and Control Engineering, National Chiao Tung University, Hsinchu, Taiwan, R.O.C. (e-mail: fcchen@cc.nctu.edu.tw).

Publisher Item Identifier S 0018-9286(00)02294-7.

where

$X_k \in R^n$ system state;

$u_k \in R^q$ control vector;

$Y_k \in R^m$ measurement vector.

Matrices A_k, B_k, and C_k have the appropriate dimensions (rank of C_k is $m < n$). The process noises w_k and the measurement noise η_k are zero-mean white Gaussian sequences with the following variances: $E[w_k(w_l)'] = Q_k\delta_{kl}$, $E[\eta_k(\eta_l)'] = R_k\delta_{kl}$, and $E[w_k(\eta_l)'] = 0$, where $'$ denotes transpose and δ_{kl} denotes the Kronecker delta function. The initial state X_0 is assumed to be uncorrelated with the white noise sequences w_k and η_k, and is assumed to be Gaussian random variables with $E[X_0] = \overline{X}_0$ and $\mathrm{Var}[X_0] = \overline{P}_0$.

It is well known that the KF may be used to produce the optimal state estimate. However, the computational cost and the numerical errors of the KF increase drastically with the state dimension. Hence, the KF may be impractical to implement. In such cases, reduced-order filters are preferable. The computational load of the KF mainly comes from the calculation of the error covariance updating equations. Thus, if one can simplify the calculation of these equations, a more efficient filter may be obtained. At the same time, the numerical errors can also be reduced.

Recently, the two-stage decoupling technique of [9] has been used to reduce the computational complexity of an augmented state Kalman filter (ASKF) by decoupling the ASKF into two quasiparallel, reduced-order Kalman filters. These two subfilters are simpler than the ASKF, and are suitable for parallel computing. The objectives here are: 1) to apply the two-stage approach to the standard single-stage Kalman filter to derive the optimal two-stage Kalman filter for general systems, which can exactly reconstruct the Kalman estimate, and 2) to evaluate its computing performance.

III. General Two-Stage Kalman Filters

The previously proposed optimal two-stage Kalman estimator (OTSKE) [9] sought to reduce the computational complexity of the ASKF in which the augmented state transition matrix was in a two-block upper triangular form. The efficiency, in terms of fewer computations required, of the two-stage decoupling technique is due to order reduction, i.e., implementing an "$n_1 + n_2$" order filter costs more than two lesser order "n_1" and "n_2" filters. In this section, the authors generalize the OTSKE to apply to the more general two-block partitioned state transition matrix, and the obtained filter will be called the general two-stage Kalman filter (GTSKF) which is equivalent to, but more efficient than, the single-stage KF.

Before proposing the GTSKF, some necessary assumptions must be made about the measurement matrix C_k. Without loss of generality, it will be assumed that the last "m" columns of C_k, with a possible renumbering of the states, are linearly independent [4]. Therefore, we assume C_k may be partitioned as follows:

$$C_k = \begin{bmatrix} C_k^1 & C_k^2 & C_k^3 \end{bmatrix}$$

where $C_k^1 \in R^{m\times(n-p)}$, $C_k^2 \in R^{m\times(p-m)}$, $C_k^3 \in R^{m\times m}$, and $m \le p < n$. Then, the following state transformation:

$$X_k = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -\left(C_k^3\right)^{-1} C_k^1 & 0 & I \end{bmatrix} X_k^t \tag{3}$$

will transform the original measurement equation (2) into the following desired form:

$$Y_k = [0 \quad \overline{C}_k]X_k + \eta_k \tag{4}$$

where $\overline{C}_k = [C_k^2 \quad C_k^3]$. To facilitate the derivation of the GTSKF, it will be assumed that the measurement equation is already of the form (4). However, the extra computational load for this state transformation will be considered in Section IV.

The key idea for developing the two-stage filter is based on state transformations that make the covariance matrices block diagonal. This can be achieved by applying the following two-stage transformation [9]:

$$T(M) = \begin{bmatrix} I_{n-p} & M \\ 0 & I_p \end{bmatrix}$$

to the KF. The transformed filter then becomes

$$\overline{X}_{k|k-1} = T(-U_k)X_{k|k-1} \tag{5}$$

$$\overline{X}_{k|k} = T(-V_k)X_{k|k} \tag{6}$$

$$\overline{P}_{k|k-1} = T(-U_k)P_{k|k-1}T'(-U_k) \tag{7}$$

$$\overline{K}_k = T(-V_k)K_k \tag{8}$$

$$\overline{P}_{k|k} = T(-V_k)P_{k|k}T'(-V_k) \tag{9}$$

where $\overline{P} = \mathrm{diag}\{\overline{P}^1, \overline{P}^2\}$. The blending matrices U_k and V_k are left to be determined to make the predicted covariance and the filtered covariance block diagonal, respectively.

Next, based on the *two-steps iterative substitution* method of [9], the transformed filter expressed by (5)–(9) can be recursively calculated as follows:

$$\overline{X}_{k|k-1} = T(-U_k)(A_{k-1}T(V_{k-1})\overline{X}_{k-1|k-1} + B_{k-1}u_{k-1}) \tag{10}$$

$$\overline{X}_{k|k} = T(U_k - V_k)\overline{X}_{k|k-1} + \overline{K}_k(Y_k - C_kT(U_k)\overline{X}_{k|k-1}) \tag{11}$$

$$\begin{aligned}\overline{P}_{k|k-1} =& T(-U_k)(A_{k-1}T(V_{k-1})\overline{P}_{k-1|k-1} \\ &\times T'(V_{k-1})A'_{k-1} + Q_{k-1})T'(-U_k)\end{aligned} \tag{12}$$

$$\begin{aligned}\overline{K}_k =& T(U_k - V_k)\overline{P}_{k|k-1}T'(U_k) \\ &\times C'_k\{C_kT(U_k)\overline{P}_{k|k-1}T'(U_k)C'_k + R_k\}^{-1}\end{aligned} \tag{13}$$

$$\overline{P}_{k|k} = (T(U_k - V_k) - \overline{K}_kC_kT(U_k))\overline{P}_{k|k-1}T'(U_k - V_k). \tag{14}$$

Define the following notations:

$$T(U_k) = [T_k \quad E_k], \qquad A_kT(V_k) = \begin{bmatrix} H_k & S_k \\ L_k & M_k \end{bmatrix}. \tag{15}$$

Then, using the fact that $C_kT_k = 0$ and $C_kE_k = \overline{C}_k$, (10)–(14) can be expanded into the following *two-stage decoupled subfilter one*:

$$\overline{X}^1_{k|k} = H_{k-1}\overline{X}^1_{k-1|k-1} + S_{k-1}\overline{X}^2_{k-1|k-1} + B^1_{k-1}u_{k-1} - U_k\overline{X}^2_{k|k-1} \tag{16}$$

$$\overline{P}^1_{k|k} = H_{k-1}\overline{P}^1_{k-1|k-1}H'_{k-1} + S_{k-1}\overline{P}^2_{k-1|k-1}S'_{k-1} + Q^{11}_{k-1} - U_k\left(P^{12}_{k-1}\right)' \tag{17}$$

where

$$P^{12}_k = H_k\overline{P}^1_{k|k}L'_k + S_k\overline{P}^2_{k|k}M'_k + Q^{12}_k \tag{18}$$

and the following *two-stage decoupled subfilter two*:

$$\overline{X}^2_{k|k-1} = M_{k-1}\overline{X}^2_{k-1|k-1} + L_{k-1}\overline{X}^1_{k-1|k-1} + B^2_{k-1}u_{k-1} \tag{19}$$

$$\overline{X}^2_{k|k} = \overline{X}^2_{k|k-1} + \overline{K}^2_k\left(Y_k - \overline{C}_k\overline{X}^2_{k|k-1}\right) \tag{20}$$

$$\overline{P}^2_{k|k-1} = M_{k-1}\overline{P}^2_{k-1|k-1}M'_{k-1} + L_{k-1}\overline{P}^1_{k-1|k-1}L'_{k-1} + Q^{22}_{k-1} \tag{21}$$

$$\overline{K}^2_k = \overline{P}^2_{k|k-1}\overline{C}'_k\left\{\overline{C}_k\overline{P}^2_{k|k-1}\overline{C}'_k + R_k\right\}^{-1} \tag{22}$$

$$\overline{P}^2_{k|k} = \left(I - \overline{K}^2_k\overline{C}_k\right)\overline{P}^2_{k|k-1}. \tag{23}$$

The blending matrices are given by

$$U_k = V_k = P^{12}_{k-1}\left(\overline{P}^2_{k|k-1}\right)^{-1}. \tag{24}$$

Equations (16)–(24) are derived in the Appendix. To guarantee that the matrix inverse in (24) exists, the covariance $\overline{P}^2_{k|k-1}$ must be positive definite. This is guaranteed by assuming that $Q^{22}_k > 0$. The blending matrix U_k can be viewed as the gain of subfilter one, and acts as the correcting factor to cancel the coupling effect from subfilter two. This can be seen by reformulating (16) and (17) into the following modified Kalman filter form:

$$\tilde{X}^1_{k|k-1} = H_{k-1}\overline{X}^1_{k-1|k-1} + S_{k-1}\overline{X}^2_{k-1|k-1} + B^1_{k-1}u_{k-1} \tag{25}$$

$$\overline{X}^1_{k|k} = \tilde{X}^1_{k|k-1} + U_k\left(0 - \overline{X}^2_{k|k-1}\right) \tag{26}$$

$$\tilde{P}^1_{k|k-1} = H_{k-1}\overline{P}^1_{k-1|k-1}H'_{k-1} + S_{k-1}\overline{P}^2_{k-1|k-1}S'_{k-1} + Q^{11}_{k-1} \tag{27}$$

$$U_k = P^{12}_{k-1}\left\{\overline{P}^2_{k|k-1} + 0\right\}^{-1} \tag{28}$$

$$\overline{P}^1_{k|k} = \tilde{P}^1_{k|k-1} - U_k\left(P^{12}_{k-1}\right)'. \tag{29}$$

It can be proven that this correcting gain vanishes when the system transition matrix is in diagonal form, i.e., $A^{12}_k = 0$, $A^{21}_k = 0$, and $Q^{12}_k = 0$. Then, from (18), one obtains $P^{12}_k = 0$. In this case, the two subfilters are completely decoupled.

Based on the above two-stage decoupled subfilters (16)(23) and (6) and (9), the Kalman estimate can be reconstructed as stated in the following theorem.

Theorem 1: If the predicted error covariance $\overline{P}^2_{k|k-1}$ of the decoupled subfilter two given by (21) is positive definite, and the following initial conditions

$$\begin{aligned}\overline{X}_0 &= T_0\overline{X}^1_{0|0} + E_0\overline{X}^2_{0|0}\\ \overline{P}_0 &= T_0\overline{P}^1_{0|0}T'_0 + E_0\overline{P}^2_{0|0}E'_0\end{aligned} \tag{30}$$

are satisfied, the following general two-stage Kalman filter (GTSKF):

$$\hat{X}_{k|k} = T_k\overline{X}^1_{k|k} + E_k\overline{X}^2_{k|k} \tag{31}$$

$$\hat{P}_{k|k} = T_k\overline{P}^1_{k|k}T'_k + E_k\overline{P}^2_{k|k}E'_k \tag{32}$$

gives the MMSE estimate of the system state.

Remarks:

1) The above GTSKF is optimal in the MMSE sense since it is equivalent to the KF, and this is deduced from the reasoning imbedded in the proposed two-steps recursive substitution method (see [9]), and can be verified by using the inductive reasoning as in [8].
2) From (16), (17), and (24), it is clear that subfilter one is uncorrelated with the matrix R_k, and hence it is insensitive to the measurement noise covariance uncertainty. Thus, only subfilter two involves this measurement uncertainty. This is different from the KF in which all estimates are affected by the measurement noise covariance.
3) The GTSKF is computationally attractive since only reduced-order filters, which are amenable for parallel computing, are involved in the computation. This computational advantage stems from the fact that the redundant computations and the coupling effects present in the KF have been reduced. Thus, the GTSKF can be thought of as a simplified version of the partitioned KF.
4) Since subfilter two is a standard KF, it is clear that subfilter two can further be simplified by using the same two-stage decoupling technique presented here.
5) Owing to the specific decoupled structure, the GTSKF is amenable for serving as a unified framework to derive reduced-order filters. One known result is the derivation of the optimal minimum-order observer [4] which can be obtained by substituting the following values:

$$\overline{C}_k = I_m, \qquad R_k = 0, \qquad p = m$$

into the GTSKF.

IV. Performance Evaluations

To demonstrate the computational advantage of the GTSKF over the KF, the authors used floating point operations, or "flops," in Matlab as a measure of the computational complexity. Each multiplication and each addition contributed one to the flops counts. First, the authors listed the flops count of the KF as follows:

$$\text{flops(KF)} = 6n^3 + 4(m+1)n^2 + (4m^2 + 4m + 2q + 2)n + 2m^3 + m^2 + m. \tag{33}$$

Second, the authors listed the flops count of the GTSKF via the flops of the two-stage decoupled subfilter one (TSDSO), the two-stage decoupled subfilter two (TSDST), and the output estimate $(\hat{X}_{k|k})$ as

$$\text{flops}(\text{TSDSO}) = 4n^3 + 5n^2 + (2q+3)n - 2p^3 + (4n-1)p^2 - (4n^2+3n+2q+3)p \quad (34)$$

$$\text{flops}(\text{TSDST}) = 6p^3 + (4m+3-2n)p^2 + (2n^2+4m^2+2n + 4m + 2q + 1)p + 2m^3 + m^2 + m \quad (35)$$

$$\text{flops}(\hat{X}_{k|k}) = 2np + n - 2p^2 - p. \quad (36)$$

Note that (32) is not included in the above evaluation since it is not involved in the recursive algorithm of the GTSKF.

Using (33)–(36), the flops savings, denoted by $\triangle$ flops, of the GTSKF as compared to the KF is given as

$$\triangle\text{flops}_{\text{KF}}(\text{GTSKF}) = 2n^3 + (4m-1)n^2 + (4m^2+4m-2)n - 4p^3 - (2n+4m)p^2 + (2n^2+3-4m^2-4m-n)p. \quad (37)$$

For $n \gg m$ and (37) having the maximum value, the authors set $p = 0.25n$ which was obtained by minimizing the following: $4p^3 + 2np^2 - 2n^2p$. Then (37) become

$$\triangle\text{flops}_{\text{KF}}(\text{GTSKF}) \approx 2.31n^3 + (3.75m - 1.25)n^2 + (3m^2 + 3m - 1.25)n. \quad (38)$$

From (38), the relative improvement ratio (RIR) of the GTSKF, which is defined by

$$\text{RIR}(\text{GTSKF}) = 1 - \lim_{n\to\infty} \frac{\text{flops(GTSKF)}}{\text{flops(KF)}} \quad (39)$$

was about 40%.

The above analytical results are for sequential processing. Since subfilter two is a reduced-order KF with external input $L_{k-1}\overline{X}^1_{k-1|k-1} + B^2_{k-1}u_{k-1}$, it can be run without waiting for the message from subfilter one. On the other hand, the only messages subfilter one needs from subfilter two are the prediction information, i.e., $\overline{X}^2_{k|k-1}$ and $\overline{P}^2_{k|k-1}$. Thus, they can be easily programmed to run concurrently. To maximize this parallel effect, the authors used $p = 0.7n$ to minimize the term "$\text{flops(TSDSO)} - \text{flops(TSDST)}$." In this case, the flops savings of the GTSKF as compared to the KF is modified from (38) to

$$\triangle\text{flops}_{\text{KF}}(\text{GTSKF}) = 3.52n^3 + (2.04m + 0.71)n^2 + (1.2m^2 + 1.2m + 0.6q + 1)n. \quad (40)$$

The RIR of this GTSKF is about 59%.

As noted in the preceding section, one may need to carry out a state transformation in order to get the desired measurement equation (4). In the following, the authors show that this computational overhead is not excessive. This can be seen by firstly listing the flops count needed by the state transformation (ST) (3) as follows:

$$\text{flops(ST)} = (6m+1)n^2 + (2m^2 + qm + 4m - 4mp - p)n - 2mp^2 - (2m+q+1)mp + 2m^3 + qm + m. \quad (41)$$

Then performance (37) become

$$\triangle\text{flops}_{\text{KF}}(\text{GTSKF}) = 2n^3 - 2(m+1)n^2 + (2m^2 - qm - 2)n - 4p^3 - 2(n+m)p^2 + (2n^2 + 4mn + qm + 3 - 2m^2 - 3m)p - 2m^3 - qm - m. \quad (42)$$

Comparing (42) with (37), it is clear that the flops counts in (41) will not affect the RIR value of the GTSKF when $n \gg m$. Similarly, it can be checked that the above claim is also true for the parallel result.

In summary, the GTSKF is computationally superior to the KF. Its computational savings are most significant when the computational load of the single-stage Kalman filter is heavy because of limited computer power.

V. Simulation Example

To verify the previous analytical results, the following target tracking simulation was conducted. Consider a target maneuvers a slow 90° turn with an acceleration of $\ddot{x} = \ddot{y} = 0.075$ m/s^2. The initial position and velocity of the target were $x(0) = 2000$ m, $\dot{x}(0) = 0$ m/s, $y(0) = 10\,000$ m, and $\dot{y}(0) = -15$ m/s. The sampling interval was $T = 10$ s; the simulation time was 500 s. The target position was measured. The system matrices were given by

$$A_k = \begin{bmatrix} 1 & 10 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 10 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 10 & 50 & 0 & 0 & 1 & 0 \\ 0 & 0 & 10 & 50 & 0 & 1 \end{bmatrix}, \qquad B_k = 0$$

$$C_k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}', \qquad R_k = \begin{bmatrix} 10\,000 & 0 \\ 0 & 10\,000 \end{bmatrix}$$

$$Q_k = \begin{bmatrix} 20 & 2 & 0 & 0 & 100 & 0 \\ 2 & 0.2 & 0 & 0 & 10 & 0 \\ 0 & 0 & 20 & 2 & 0 & 100 \\ 0 & 0 & 2 & 0.2 & 0 & 10 \\ 100 & 10 & 0 & 0 & 500 & 0 \\ 0 & 0 & 100 & 10 & 0 & 500 \end{bmatrix}$$

and the state vector was

$$X_k = [\dot{x}_k \quad \ddot{x}_k \quad \dot{y}_k \quad \ddot{y}_k \quad x_k \quad y_k]'.$$

The KF and the GTSKF with $p = 2$ and 3 are considered. All filters were initialized by taking the initial state estimate $\overline{X}_0$ and the corresponding covariance matrix $\overline{P}_0$ as $\overline{X}_0 \sim N(X_0, \overline{P}_0)$ and $\overline{P}_0 = Q_0$, where X_0 was the initial target state. The tracking error is defined by the root-mean-square (RMS) of the state estimating error.

A Monte Carlo simulation of 50 runs (using Matlab) was performed. The simulation results in Table I show the average RMS tracking error and the corresponding flops generated by Matlab. Table I shows that the tracking errors of the KF and the GTSKF are the same, but the flops of the GTSKF is fewer than that of the KF. Note that if one substitutes $n = 6$, $m = 2$, and $p = 2(3)$ into (37), the flops of the GTSKF with $p = 2$ and 3 as compared to the KF are reduced by 794 and 663, respectively. These results comply with the simulation results of 790 and 653, respectively.

VI. An Application of the GTSKF

To demonstrate the computational advantage of the GTSKF, the authors give a potential application of the GTSKF as an alternative to implement the IMM algorithm [10], an attractive way to reduce model mismatch. The IMM algorithm is characterized by estimating the system state with two or more system models operating in parallel. Because of its decoupling structure, the two-stage Kalman estimator has been proved to reduce the complexity of the IMM algorithm via the interacting acceleration compensation (IAC) algorithm when applied to the tracking of maneuvering targets, e.g., [11] and [12]. The IAC algorithm incorporates the concept of the IMM algorithm for two motion models into the framework of the two-stage estimator. Simulation results indicate that the tracking performance of the IAC algorithm approaches that of a comparative IMM algorithm while requiring only about one-half of the computations.

This section extends the multiple model approach to reduce the effect of uncertainty in the measurement noise. This uncertainty may be caused by time-varying parameter variation in the output matrix. For illustration, consider two models: one stands for small measurement noise, and the other stands for large measurement noise. Thus, two Kalman filters with different R_k are needed in the IMM algorithm. The flops of these two Kalman filters from (33) is of the order $12n^3$. Alternatively, since subfilter one is constant with this measurement noise uncertainty, it should be calculated only once in order to implement the IMM algorithm. On the other hand, subfilter two with different R_k is then needed to implement the IMM algorithm. In this case, the GTSKF is composed of three subfilters. If the state dimension is larger than the measurement dimension, one can choose $p = m$ to reduce the complexity of subfilter two. Substituting this p into (34) and (35), one finds that the complexity of subfilter two can be ignored. Hence, the flops of the GTSKF will be equal to that of subfilter one which is of the order $4n^3$. Thus, the RIR of the GTSKF becomes

$$\text{RIR(GTSKF)} = 1 - \frac{2}{3n_m} \tag{43}$$

where n_m is the model number used in the IMM algorithm. In this two-model case, the RIR is about 70%. It is clear that, from (43), this superiority will increase even more if more models are involved in the IMM algorithm.

VII. Conclusion

This correspondence presents the general two-stage Kalman filter (GTSKF). The proposed GTSKF provides the optimal state estimate which is equivalent to that of the KF. It is shown that the GTSKF is more efficient than the single-stage KF. This computational superiority gains the most benefit when the GTSKF is applied to implement complex KF-based filtering algorithms, such as the IMM algorithm. Simulation results agree with those predicted by the complexity analysis. Our results suggest that the proposed GTSKF can be used as a simplified model to replace the KF for obtaining state estimates in time-varying, linear discrete-time stochastic systems.

Owing to the specific decoupled structure, the proposed GTSKF may also serve as a unified framework to derive reduced-order filter algorithms. One known result is the derivation of the optimal minimum-order observer of Leondes and Novak [4]. The problem of applying the GTSKF to derive other reduced-order Kalman filters is under investigation.

TABLE I
Performances of the KF and GTSKF Filters

Performances	KF	GTSKF(p=2)	GTSKF(p=3)
$Tracking Error$	125.85	125.85	125.85
$flops$	1900	1110	1247
RIR	—	42%	34%

Appendix

Using (15), one can expand (10)–(14) into the following two-stage decoupled subfilter one:

$$\begin{aligned}\overline{X}^1_{k|k-1} = {} & H_{k-1}\overline{X}^1_{k-1|k-1} + S_{k-1}\overline{X}^2_{k-1|k-1} \\ & + B^1_{k-1}u_{k-1} - U_k\overline{X}^2_{k|k-1}\end{aligned} \tag{44}$$

$$\overline{X}^1_{k|k} = \overline{X}^1_{k|k-1} + \overline{K}^1_k\left(Y_k - C_kT_k\overline{X}^1_{k|k-1}\right) \tag{45}$$

$$\begin{aligned}\overline{P}^1_{k|k-1} = {} & H_{k-1}\overline{P}^1_{k-1|k-1}H'_{k-1} + S_{k-1}\overline{P}^2_{k-1|k-1}S'_{k-1} \\ & + Q^{11}_{k-1} - U_k\left(P^{12}_{k-1}\right)'\end{aligned} \tag{46}$$

$$\overline{K}^1_k = \overline{P}^1_{k|k-1}(C_kT_k)'\left\{C_kT_k\overline{P}^1_{k|k-1}(C_kT_k)' + R_k\right\}^{-1} \tag{47}$$

$$\overline{P}^1_{k|k} = \left(I - \overline{K}^1_kC_kT_k\right)\overline{P}^1_{k|k-1} \tag{48}$$

and the following two-stage decoupled subfilter two:

$$\overline{X}^2_{k|k-1} = L_{k-1}\overline{X}^1_{k-1|k-1} + M_{k-1}\overline{X}^2_{k-1|k-1} + B^2_{k-1}u_{k-1} \tag{49}$$

$$\overline{X}^2_{k|k} = \overline{X}^2_{k|k-1} + \overline{K}^2_k\left(Y_k - C_kT_k\overline{X}^1_{k|k-1} - C_kE_k\overline{X}^2_{k|k-1}\right) \tag{50}$$

$$\overline{P}^2_{k|k-1} = L_{k-1}\overline{P}^1_{k-1|k-1}L'_{k-1} + M_{k-1}\overline{P}^2_{k-1|k-1}M'_{k-1} + Q^{22}_{k-1} \tag{51}$$

$$\begin{aligned}\overline{K}^2_k = {} & \overline{P}^2_{k|k-1}(C_kE_k)'\left\{C_kT_k\overline{P}^1_{k|k-1}(C_kT_k)'\right. \\ & \left. + C_kE_k\overline{P}^2_{k|k-1}(C_kE_k)' + R_k\right\}^{-1}\end{aligned} \tag{52}$$

$$\overline{P}^2_{k|k} = \left(I - \overline{K}^2_kC_kE_k\right)\overline{P}^2_{k|k-1}. \tag{53}$$

The blending matrices U_k and V_k are given by

$$U_k = P^{12}_{k-1}\left(\overline{P}^2_{k|k-1}\right)^{-1} \tag{54}$$

$$V_k = U_k - \overline{K}^1_kC_kE_k. \tag{55}$$

Equation (54) is obtained by solving the following constraint:

$$\begin{aligned}0 = {} & H_{k-1}\overline{P}^1_{k-1|k-1}L'_{k-1} + S_{k-1}\overline{P}^2_{k-1|k-1}M'_{k-1} \\ & + Q^{12}_{k-1} - U_k\overline{P}^2_{k|k-1}.\end{aligned} \tag{56}$$

Substituting $C_kT_k = 0$ into (45), (47), and (48), one obtains

$$\overline{X}^1_{k|k} = \overline{X}^1_{k|k-1}, \quad \overline{K}^1_k = 0, \quad \overline{P}^1_{k|k} = \overline{P}^1_{k|k-1}. \tag{57}$$

Using (57) and the notation: $C_kE_k = \overline{C}_k$, (44)–(55) become (16)–(24).

Acknowledgment

The authors are grateful to the reviewers for their helpful comments and suggestions.

References

[1] I. B. Rhodes, "A parallel decomposition for Kalman filters," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 322–324, 1990.

[2] R. A. Singer and R. G. Sea, "Increasing the computational efficiency of discrete Kalman filters," *IEEE Trans. Automat. Contr.*, vol. 16, pp. 254–257, 1971.

[3] E. Tse and M. Athans, "Optimal minimal-order observer-estimators for discrete linear time-varying systems," *IEEE Trans. Automat. Contr.*, vol. 15, pp. 416–426, 1970.

[4] C. T. Leondes and L. M. Novak, "Optimal minimal-order observers for discrete-time systems—A unified theory," *Automatica*, vol. 8, pp. 379–387, 1972.

[5] E. Fogel and Y. F. Huang, "Reduced-order optimal state estimator for linear systems with partially noise corrupted measurement," *IEEE Trans. Automat. Contr.*, vol. 25, pp. 994–996, 1980.

[6] Y. Halevi, "The optimal reduced-order estimator for systems with singular measurement noise," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 777–781, 1989.

[7] B. Friedland, "Treatment of bias in recursive filtering," *IEEE Trans. Automat. Contr.*, vol. 14, pp. 359–367, 1969.

[8] A. T. Alouani, P. Xia, T. R. Rice, and W. D. Blair, "On the optimality of two-stage state estimation in the presence of random bias," *IEEE Trans. Automat. Contr.*, vol. 38, pp. 1279–1282, 1993.

[9] C.-S. Hsieh and F.-C. Chen, "Optimal solution of the two-stage Kalman estimator," *IEEE Trans. Automat. Contr.*, vol. 44, pp. 194–199, 1999.

[10] H. A. P. Blom and Y. Bar-Shalom, "The interacting multiple model algorithm for systems with Markovian switching coefficients," *IEEE Trans. Automat. Contr.*, vol. 23, pp. 780–783, 1988.

[11] G. A. Watson and W. D. Blair, "Interacting acceleration compensation algorithm for tracking maneuvering targets," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 31, pp. 1152–1159, 1995.

[12] W. D. Blair and G. A. Watson, "Interacting multiple bias model algorithm with application to tracking maneuvering targets," in *Proc. IEEE 31th Conf. Decision Contr.*, Tucson, AZ, Dec. 1992, pp. 3790–3795.

Adaptive Control of a Weakly Nonminimum Phase Linear System

Zongli Lin and Gang Tao

***Abstract*—For a weakly nonminimum phase linear system, we design an adaptive state feedback control law that causes the system output to track a desired trajectory to an arbitrarily high degree of precision. The key to this is the use of a low gain feedback design technique.**

***Index Terms*—Adaptive control, low gain feedback, nonminimum phase, tracking.**

I. Introduction

It is well known in both classical and modern control that the system minimum phase property facilitates control designs [1], [3]–[5], [7], [9], [10]. However, many practical systems are nonminimum phase [3]. The nonminimum phase property may prevent some desired control objectives from being achieved. One of such objectives is that the system output tracks a desired trajectory. Since the unstable system zeros cannot be cancelled by standard state or output feedback controllers, special designs are usually needed for output tracking purpose. Controllers based on the internal model principal [1], [4], [5] are commonly used for nonminimum phase systems in order to achieve output tracking of reference signals at the internal model frequencies.

In this paper we explore a different way of treating nonminimum phase systems and identify conditions under which certain design objective can be met for nonminimum phase systems. In particular, by utilizing the recent development of low gain feedback design techniques [8], we show how adaptive state feedback control laws can be constructed to cause the output of a weakly nonminimum phase linear system (a system whose invariant zeros are in the closed left-half of the complex plane) to track a given reference output trajectory to an arbitrarily high degree of precision. The reference output trajectory can be any smooth signals that do not contain the frequency components of the jw axis invariant zeros.

The rest of the paper is organized as follows. In Section II, we formulate the problem of designing adaptive state feedback controllers for weakly nonminimum phase linear systems to ensure desired tracking properties. In Section III, we will show that, with the use of a low gain feedback design, our adaptive controller, when applied to a weakly nonminimum phase linear system, ensures the closed-loop signal boundedness and an output tracking error whose steady-state trajectory can be made arbitrarily small. We will also present an example with simulation results to illustrate our low gain adaptive controller and its desired tracking performance. A brief concluding remark is made in Section 4.

II. Problem Statement

Consider the following linear system:

$$\begin{cases} \dot{x}_0 = A_0 x_0 + B_0 x_1, \quad x_0 \in \mathbb{R}^{n_0} \\ \dot{x}_1 = x_2, \quad x_1 \in \mathbb{R} \\ \quad \vdots \\ \dot{x}_{r-1} = x_r \\ \dot{x}_r = E_0 x_0 + a_1 x_1 + \cdots + a_r x_r + bu, \quad b > 0 \\ y = x_1 \end{cases} \tag{1}$$

where $x = [x_0', x_1, x_2, \ldots, x_r]' \in \mathbb{R}^{n_0+r}$ is the state vector, $u \in \mathbb{R}$ is the control input, $y \in \mathbb{R}$ is the system output, and b and a_i's are unknown system parameters. We have assumed that $b \neq 0$, and without loss of generality, further assumed that $b \geq b_0 > 0$ for some known b_0.

We note here that the dynamics of x_0 is the zero dynamics of the system and the eigenvalues of A_0 are the invariant zeros of the system. Also note that this system has a relative degree of r. We further make the following assumptions on the system.

Assumption 1: The system (1) is of weakly nonminimum phase, i.e., all the eigenvalues of A_0 lie in the closed left-half plane.

Assumption 2: The system (1) is stabilizable, i.e., the pair (A_0, B_0) is stabilizable.

Our objective is to construct an adaptive state feedback control law that causes the system output $y(t)$ to track a desired output trajectory y_d to an arbitrarily high degree of precision without knowing the values of the system parameters b and a_i's. Unlike in the adaptive control of minimum phase systems in which A_0 is stable, here A_0 is unstable and needs to be stabilized through B_0. For this reason, we require the knowledge of A_0 and B_0. We also make the following assumption on the desired trajectory y_d.

Manuscript received February 2, 1999; revised May 20, 1999. Recommended by Associate Editor, B. M. Chen. This work was supported in part by the U.S. Office of Naval Research Young Investigator Program under Grant N00014-99-1-0670 and the National Science Foundation under Grant ECS-9619363.

The authors are with the Department of Electrical Engineering, University of Virginia, Charlottesville, VA 22903-2442 USA.

Publisher Item Identifier S 0018-9286(00)01087-4.