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Fuzzy Sets and Systems 109 (2000) 59–81

FUZZY sets and systems

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A fuzzy multiobjective program with quasiconcave membership functions and fuzzy coefficients

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Received June 1997; received in revised form May 1998

Abstract

A method is proposed for solving a fuzzy multi-objective linear programming problem (FMP) with quasiconcave membership functions and fuzzy coefficients. The proposed method first expresses a piecewise function as the summation of absolute terms. Then we search for the interval where the optimal solution is allocated by finding the corresponding points with same value of membership functions. After that, the problem is solved by goal programming techniques. Comparing with other FMP methods, the proposed method does not need to add extra zero–one variables, to divide the original problem into several sub-problems, or transforming all original quasiconcave functions into concave functions. In addition, the proposed method could solve a FMP problem with fuzzy coefficients to obtain a solution closing to a global optimum. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Fuzzy multi-objective linear programming; Goal programming; Piecewise linear membership function; Quasiconcave function

1. Introduction

A fuzzy multi-objective linear programming problem (FMP) with crisp constraints discussed in this study is expressed as follows:

FMP problem:

$$\begin{aligned} \text{Maximize}\quad & \{\mu_1(z_1(X)), \mu_2(z_2(X)), \ldots, \mu_n(z_n(X))\} \\ \text{subject to}\quad & z_i(X) = \sum_{j=1}^{m} b_{ij}x_j + c_i, \quad i = 1, 2, \ldots, n, \\ & \sum_{j=1}^{m} b_{ij}x_j + c_i \geqslant 0, \quad i = n+1, n+2, \ldots, I, \\ & X = (x_1, x_2, \ldots, x_m), \end{aligned} \tag{1.1}$$

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PII: S0165-0114(98)00195-X

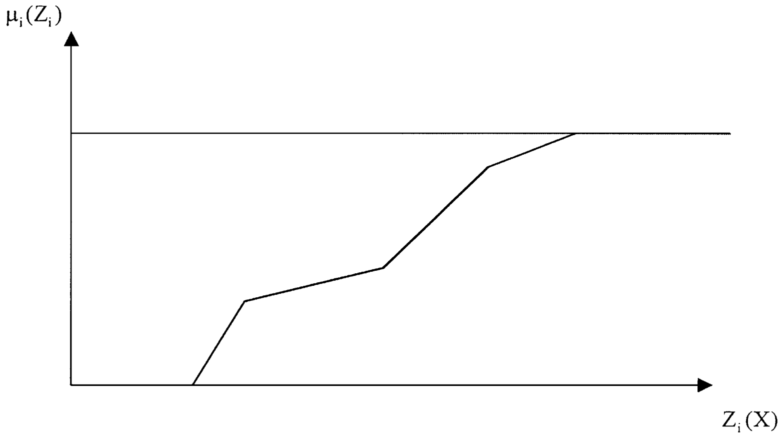

Fig. 1. A quasi-concave membership function.

where $\mu_i(z_i(X))$ are piecewise linear membership functions, $i=1,2,\ldots,n$, $0 \leqslant \mu_i(z_i(X)) \leqslant 1$, $z_i(X)$ are the objective functions, b_{ij} and c_i are constants, and X is a vector of decision variables.

In general, a piecewise linear membership function $\mu_i(z_i)$ may be concave shaped or convex shaped. The marginal possibility with respect to a concave membership function is decreasing, whereas the marginal possibility with respect to a convex membership function is increasing. If the marginal possibility increases first then decreases, or decreases first then increases, then the membership function becomes a quasiconcave shape as shown in Fig. 1. Since many empirical evidences [10, 11] have revealed that membership functions in real-life situations are usually not concave or convex but quasiconcave, this study emphasizes on solving a FMP problem in (1.1) where $\mu_i(z_i(X))$ are quasiconcave membership functions.

In many practical multi-objective decision models, decision makers (DM) are often difficult to specify the coefficients of variables and/or resources. Suppose the imprecise or uncertain nature of input data is modeled, then we have possibilistic multi-objective problems in which some constraints have fuzzy values for coefficients of variables or resources. Notable that the meaning of a membership function is different from the possibility distribution. The membership function is based on a DM's preference but the possibility distribution is based on the degree of a precise data set.

Suppose some constraints in (1.1) have fuzzy coefficients in variables and resources, then (1.1) can be expressed as the following fuzzy program:

FMP′ problem:

$$\begin{aligned}
&\text{Maximize} \quad \{\mu_1(z_1(X)), \mu_2(z_2(X)), \ldots, \mu_n(z_n(X))\} \\
&\text{subject to} \quad z_i(X) = \sum_{j=1}^{m} \tilde{b}_{ij} x_j + \tilde{c}_i, \quad i = 1,2,\ldots,n, \\
&\qquad \sum_{j=1}^{m} \tilde{b}_{ij} x_j + \tilde{c}_i \geqslant 0, \quad i = n+1, n+2, \ldots, I, \\
&\qquad X = (x_1, x_2, \ldots, x_m),
\end{aligned} \tag{1.2}$$

where $\tilde{b}_{ij}$ are imprecise variable coefficients of x_j and $\tilde{c}_i$ is uncertain resource of ith constraint.

Current FMP methods [2, 4, 11, 12, 16], however, are not computationally efficient for solving a FMP problem with quasiconcave membership functions. In addition, most of these methods can only treat the problem with crisp coefficients instead of imprecise coefficients. Some disadvantages of current FMP methods are discussed below:

(i) These methods lack a clear and simple way to present a general piecewise membership function $\mu_i(z_i)$. Most methods use complicated expressions to present a quasiconcave membership function.

(ii) These methods can only solve a FMP problem in (1.1) where b_{ij} and c_i are certain constants. They cannot treat a FMP′ problem in (1.2) where $\tilde{b}_{ij}$ and $\tilde{c}_i$ are fuzzy numbers.

(iii) Narasimham's method [12] and Hannan's method [2] can only solve FMP problems where all $\mu_i(z_i)$ are concave functions.

(iv) Nakamura's method [11] needs to divide the original quasiconcave FMP problem into $2^{\sum_{i=1}^{n} m_i}$ sub-problems, where m_i is the number of intersections between concave functions and convex functions in $\mu_i(z_i)$, then uses linear programming to solve these sub-problems repeatedly.

(v) Yang et al.'s method [16] requires to add $\sum_{i=1}^{n} m_i$ zero–one variables in their model for solving the quasiconcave FMP problem, where m_i represents the number of intersections between concave and convex functions in $\mu_i(z_i)$.

(vi) Inuiguchi et al.'s method [4] involves tedious process of transforming all original membership functions into new concave membership functions.

This study proposes two algorithms to solve fuzzy programs. Algorithm 1 is applied to solve the program with quasiconcave membership functions and crisp constraints in (1.1). Algorithm 2, which is the extension of Algorithm 1, is used to solve the program with quasiconcave membership functions and imprecise constraints in (1.2).

The features of the proposed algorithms are:

(i) It uses a more convenient and clear way to express general piecewise membership functions such as quasiconcave type.

(ii) It can directly solve a quasiconcave FMP problem without adding any zero–one variables, dividing the problem into several sub-problems, or transforming all original membership functions into new functions.

(iii) It could be extended to treat a fuzzy program with fuzzy coefficients.

Section 2 reviewed current FMP models of treating fuzzy multi-objective linear problems. Some propositions regarding the proposed algorithms are introduced in Section 3. The proposed algorithm of solving (1.1) is formulated in Section 4. Section 5 describes the algorithm of solving (1.2).

2. Review of current FMP models

This section briefly reviews Yang et al. method [16], Nakamura method [11], and Inuiguchi et al. method [4]; which are three commonly used approaches for solving a FMP problem in (1.1). First, consider the following example:

Example 1 (*Slightly modified from Inuiguchi et al.* [4]).

$$\text{Maximize} \quad \lambda$$

$$\text{subject to} \quad \lambda \leqslant \mu_1(z_1), \quad \lambda \leqslant \mu_2(z_2),$$

$$z_1 = -x_1 + 2x_2, \quad z_2 = 2x_1 + x_2,$$

$$-x_1+3x_2\leqslant 21,\quad x_1+3x_2\leqslant 27,$$
$$4x_1+3x_2\leqslant 45,\quad 3x_1+x_2\leqslant 30,$$
$$x_1,x_2\geqslant 0,$$

$$\mu_1(z_1)=\begin{cases}0, & z_1\leqslant -3,\\ 0.04z_1, & -3\leqslant z_1\leqslant 2,\\ 0.08z_1+0.2, & 2\leqslant z_1\leqslant 12,\\ 1, & z_1=12,\\ -0.1z_1+2.2, & 12\leqslant z_1\leqslant 17,\\ -0.05z_1+0.5, & 17\leqslant z_1\leqslant 27,\\ 0, & z_1\geqslant 27,\end{cases}\qquad \mu_2(z_2)=\begin{cases}0, & z_2\leqslant 7,\\ 0.06z_2, & 7\leqslant z_2\leqslant 17,\\ 0.1z_2+0.6, & 17\leqslant z_2\leqslant 21,\\ 1, & z_2=21,\\ -0.033z_2+1.7, & 21\leqslant z_2\leqslant 27,\\ -0.1z_2+0.8, & 27\leqslant z_2\leqslant 30,\\ -0.25z_2+0.5, & 30\leqslant z_2\leqslant 32,\\ 0, & z_2\geqslant 32,\end{cases}$$

where $\mu_1(z_1)$ and $\mu_2(z_2)$ are specified in Fig. 2(a) and (b).

Yang et al.'s method could formulate Example 1 as following zero–one programming model (as depicted in Fig. 3(a) and (b)):

FMP Model 1 (*Yang et al.'s method for Example* 1)

$$\begin{aligned}
&\text{Maximize}\quad \lambda\\
&\text{subject to}\quad \lambda\leqslant 1-\frac{b_5-z_1}{d_1}+M(1-\delta_1)+M\delta_2,\\
&\lambda\leqslant 1-\frac{b_4-z_1}{d_2}+M\delta_1+M\delta_2,\\
&\lambda\leqslant 1-\frac{b_6-z_1}{d_3}+M\delta_1+M\delta_2,\\
&\lambda\leqslant 1-\frac{b_7-z_1}{d_4}+M(1-\delta_2)+M\delta_1,\\
&\lambda\leqslant 1-\frac{b_{11}-z_2}{d_5}+M(1-\delta_3),\\
&\lambda\leqslant 1-\frac{b_{10}-z_2}{d_5}+M\delta_3,\quad \lambda\leqslant 1-\frac{b_{16}-z_2}{d_7}+M\delta_3,\\
&\lambda\leqslant 1-\frac{b_{15}-z_2}{d_8}+M\delta_3,\quad \lambda\leqslant 1-\frac{b_{14}-z_2}{d_9}+M\delta_3,\\
&z_1=-x_1+2x_2,\quad z_2=2x_1+x_2,\\
&-x_1+3x_2\leqslant 21,\quad x_1+3x_2\leqslant 27,\\
&4x_1+3x_2\leqslant 45,\quad 3x_1+x_2\leqslant 30,\\
&x_1,x_2\geqslant 0,
\end{aligned}\tag{2.1}$$

where M is a big number, and $\delta_1,\delta_2,\delta_3$ are 0–1 variables.

A major disadvantage in Yang et al.'s method is that it involves too many zero–one variables for treating non-concave functions. The number of zero–one variables equals the number of intersections between convex functions and concave functions. Take Example 1, for instance, $\mu_1(z_1)$ contains two convex–concave

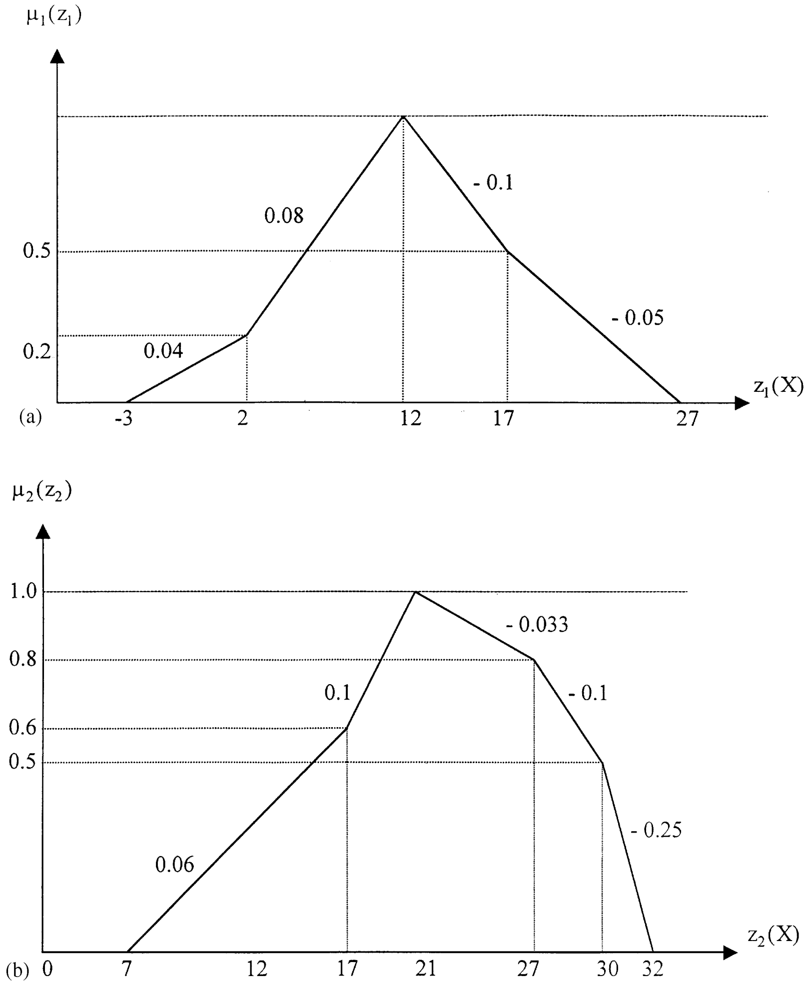

Fig. 2. (a) Membership function $\mu_1(z_1)$ in Example 1. (b) Membership function $\mu_2(z_2)$ in Example 1.

intersections and $\mu_2(z_2)$ contains one convex–concave intersection. Therefore, three zero–one variables (i.e., $\delta_1, \delta_2, \delta_3$) are required in the solution process. A detailed discussion is given in Li and Yu [9].

Nakamura [11] develops a method to expressing a general piecewise membership function $\mu_i(z_i)$ in (1.1). Take Example 1, for instance, Nakamura expresses $\mu_1(z_1)$ and $\mu_2(z_2)$ as follows:

$$\mu_1(z_1) = [\{\sigma_1(z_1) \vee \sigma_2(z_1)\} \wedge \{\rho_1(z_1)\} \wedge \{\sigma_3(z_1) \vee \sigma_4(z_1)\} \wedge 1] \vee 0,$$

$$\mu_2(z_2) = [\{\sigma_5(z_2) \vee \sigma_6(z_2)\} \wedge \rho_2(z_2) \wedge \sigma_7(z_2) \wedge \sigma_8(z_2) \wedge \sigma_9(z_2) \wedge 1] \vee 0,$$

where $\vee$ stands for maximum or disjunction operator, $\wedge$ stands for minimum or conjunction operator, $\{\sigma_1(z_1) \vee \sigma_2(z_1)\}, \{\sigma_3(z_1) \vee \sigma_4(z_1)\}$ and $\{\sigma_5(z_2) \vee \sigma_6(z_2)\}$ are the sets of the convex parts, as graphed in Fig. 4(a) and (b).

Nakamura's method then divides Example 1 into eight subproblems. Some of these subproblems are expressed as follows:

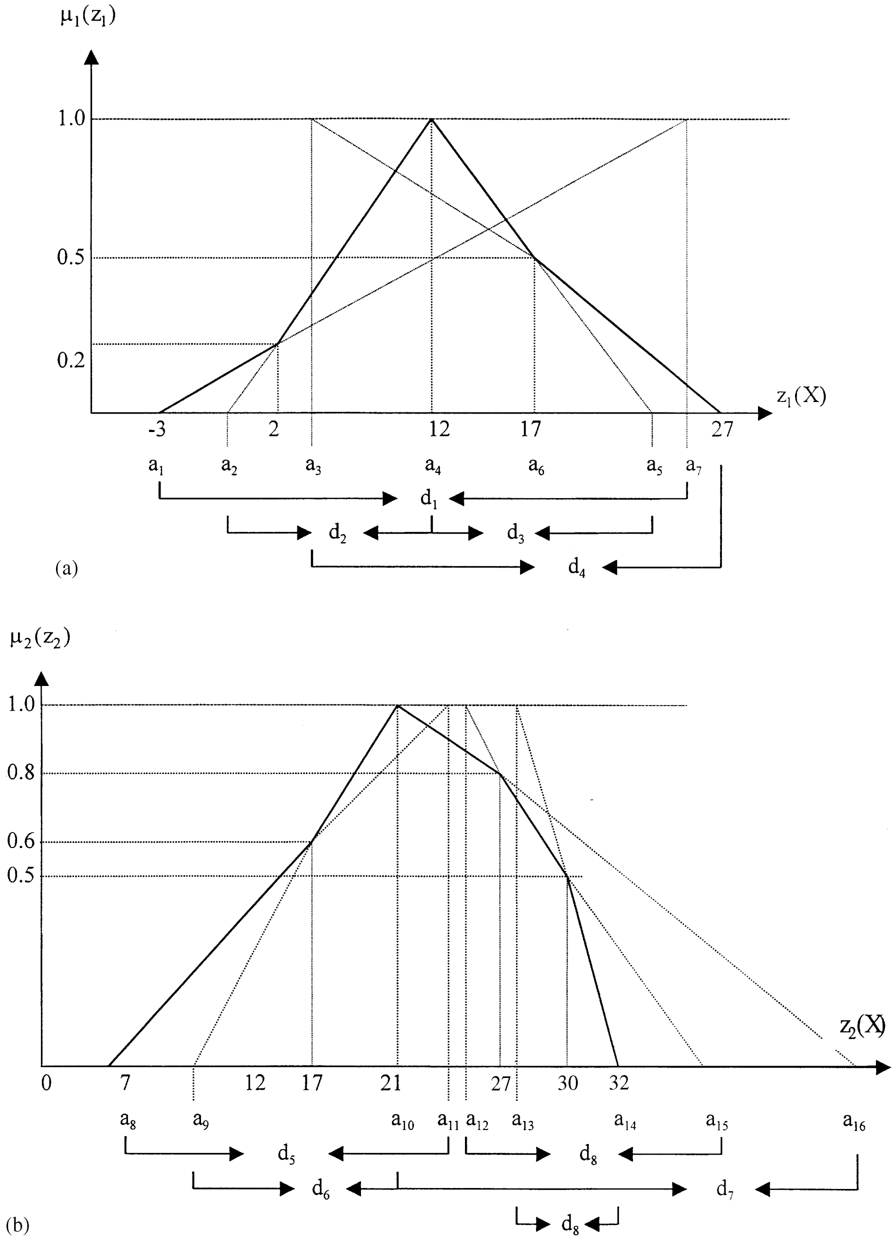

Fig. 3. (a) $\mu_1(z_1)$ in Yang et al.'s method. (b) $\mu_2(z_2)$ in Yang et al.'s method.

FMP Model 2 (*Nakamura Method for Example* 1)
Subproblem 1

$$\begin{aligned} &\text{Maximize} \quad \lambda \\ &\text{subject to} \quad \lambda \leqslant \sigma_1(z_i) \wedge \rho_1(z_1) \wedge \sigma_3(z_1), \\ &\qquad\qquad\quad \lambda \leqslant \sigma_5(z_2) \wedge \rho_2(z_2) \wedge \sigma_7(z_2) \wedge \sigma_8(z_2) \wedge \sigma_9(z_2). \end{aligned}$$

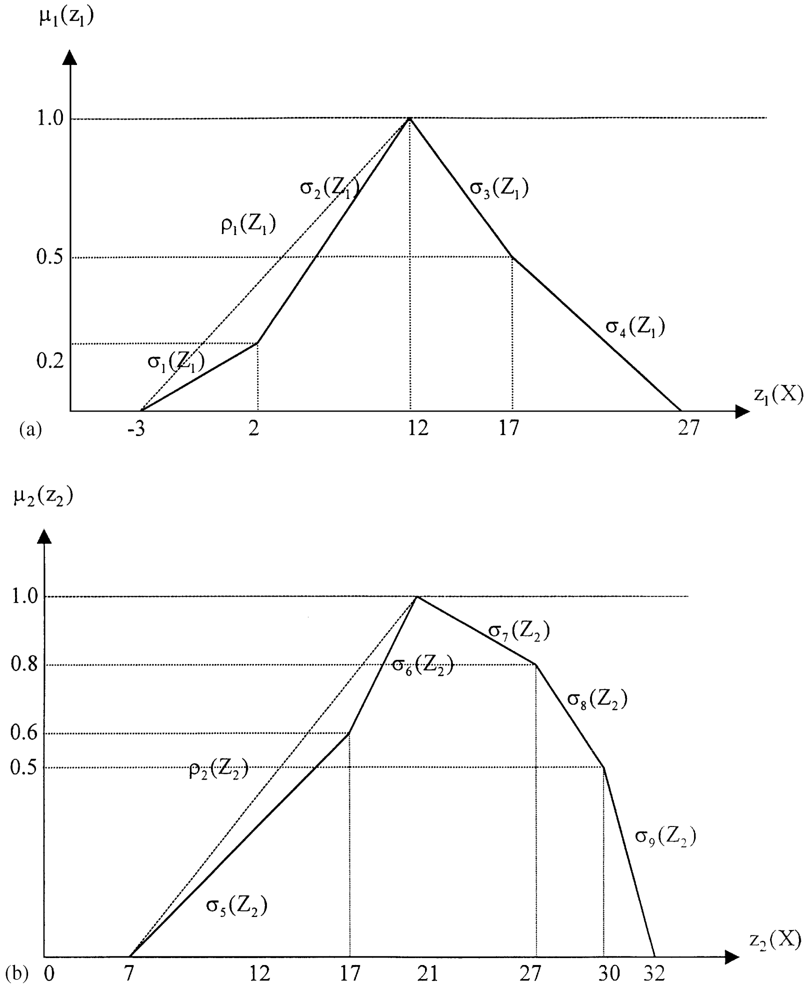

Fig. 4. (a) $\mu_1(z_1)$ in Nakamura's method. (b) $\mu_2(z_2)$ in Nakamura's method.

Subproblem 2

Maximize λ

subject to $\lambda \leqslant \sigma_2(z_i) \wedge \rho_1(z_1) \wedge \sigma_3(z_1)$,

$\lambda \leqslant \sigma_5(z_2) \wedge \rho_2(z_2) \wedge \sigma_7(z_2) \wedge \sigma_8(z_2) \wedge \sigma_9(z_2)$.

:

:

Subproblem 6

Maximize λ

subject to $\lambda \leqslant \sigma_2(z_i) \wedge \rho_1(z_1) \wedge \sigma_3(z_1)$,

$$\lambda \leqslant \sigma_5(z_2) \wedge \rho_2(z_2) \wedge \sigma_7(z_2) \wedge \sigma_8(z_2) \wedge \sigma_9(z_2).$$
$$\vdots \qquad (2.2)$$

Nakamura's method encounters two difficulties:

(i) Expression of piecewise membership functions is intricate, it requires repetitive use of LP computation for solving a FMP problem.

(ii) Nakamura's method has to divide a FMP problem into large subproblems, then solve it by LP computation repeatedly. Take Example 1, for instance, Nakamura's method involves eight subproblems and finds the optimal solution in Subproblem 6 after using LP computation repeatedly.

For tackling a FMP problem in (1.1), Inuiguchi et al. [4] developed an approach of transforming quasiconcave functions into concave functions. For instance, Inuiguchi et al. approach could transform Example 1 into the following linear program:

FMP Model 3 (*Inuiguchi et al.'s method for Example* 1)

$$\begin{aligned} &\text{Maximize} \quad \lambda' \\ &\text{subject to} \quad \lambda' \leqslant \mu_1'(z_1), \quad \lambda' \leqslant \mu_2'(z_2), \\ &z_1 = -x_1 + 2x_2, \quad z_2 = 2x_1 + x_2, \\ &-x_1 + 3x_2 \leqslant 21, \quad x_1 + 3x_2 \leqslant 27, \\ &4x_1 + 3x_2 \leqslant 45, \quad 3x_1 + x_2 \leqslant 30, \\ &x_1, x_2 \geqslant 0, \end{aligned} \qquad (2.3)$$

where

$$\mu_1'(z_1) = \begin{cases} 0, & z_1 \leqslant -3, \\ \min(\frac{1}{13}z_1 + \frac{3}{13}, \frac{3}{65}z_1 + \frac{29}{65}), & -3 \leqslant z_1 \leqslant 12, \\ 1, & z_1 = 12, \\ -\frac{1}{15}z_1 + \frac{9}{5}, & 12 \leqslant z_1 \leqslant 27, \\ 0 & z_1 \geqslant 27, \end{cases}$$

and

$$\mu_2'(z_2) = \begin{cases} 0, & z_2 \leqslant 7, \\ \min(\frac{3}{26}z_2 - \frac{21}{26}, \frac{3}{52}z_2 - \frac{11}{52}), & 7 \leqslant z_2 \leqslant 21, \\ 1, & z_2 = 21, \\ \min(-\frac{1}{5}z_2 + \frac{32}{3}, -\frac{1}{15}z_2 + \frac{8}{3}, -\frac{1}{45}z_2 + \frac{53}{45}), & 21 \leqslant z_2 \leqslant 32, \\ 0 & z_2 \geqslant 32, \end{cases}$$

where original quasiconcave functions $\mu_1(z_1)$ and $\mu_2(z_2)$ are transformed into concave functions $\mu_1'(z_1)$ and $\mu_2'(z_2)$, respectively, as shown in Fig. 5(a) and (b).

Although Inuiguchi et al.'s idea is very useful in formulating quasiconcave functions into concave functions, there are two major shortcomings in Inuiguchi et al.'s model as described below:

(i) Since the converted membership functions are quite different from the original membership functions, a decision maker has difficulty of imaging the relationship between these functions. For instance, a decision maker is hard to realize the mapping from $\mu_1(z_1)$ and $\mu_2(z_2)$ (Fig. 2(a) and (b)) to $\mu_1'(z_1)$ and $\mu_2'(z_2)$ (Fig. 5(a) and (b)).

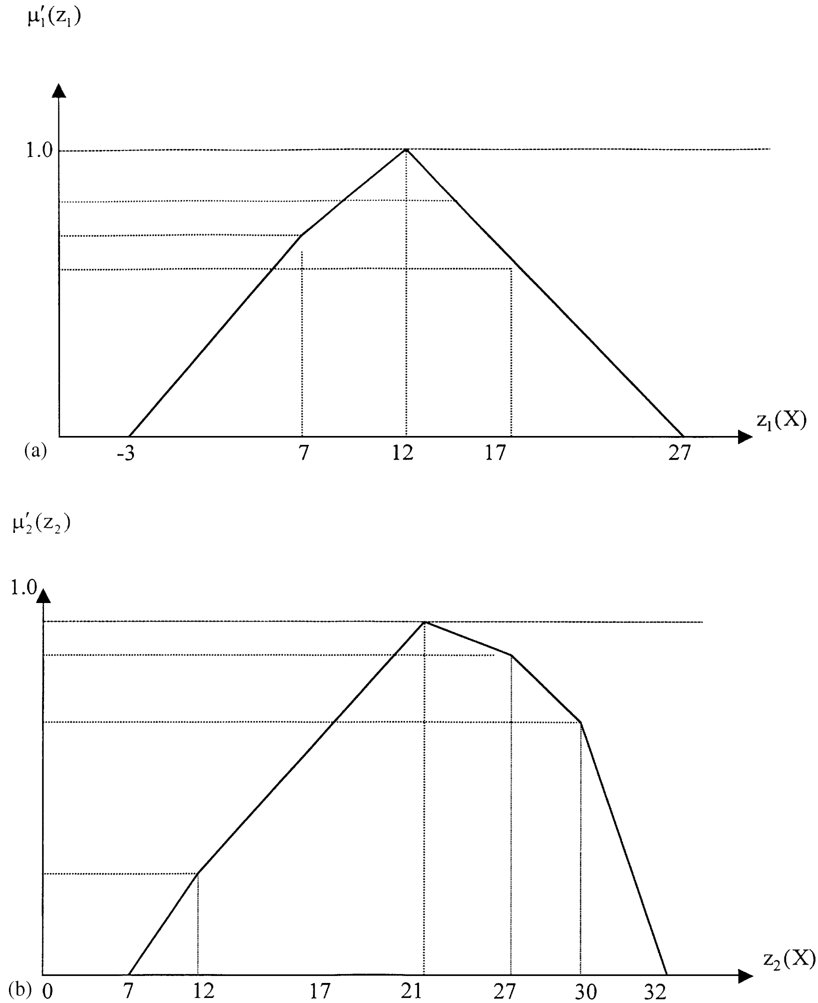

Fig. 5. (a) $\mu_1'(z_1)$ in Inuiguchi et al.'s method. (b) $\mu_2'(z_2)$ in Inuiguchi et al.'s method.

(ii) If the number of break points in the membership functions is large, then it causes tedious computational burden to convert these membership functions into concave functions.

Take Example 1 for instance, five break points are required to do transforming computing. Suppose there are n objective functions and each of these functions have m_i break points then the number of transforming computing is $\sum_{i=1}^{n} m_i$. The situation would become more complicated for treating problems with fuzzy coefficients.

To improve the above FMP methods, this paper first proposes a convenient way of expressing a piecewise linear function. The proposed expression is simpler than Nakamura's method [11]. Then we develop Algorithm 1 that can solve the FMP problem in (1.1) where $\mu_i(z_i)$ could be quasiconcave membership functions. The proposed Algorithm 1 could solve the problem effectively without adding any zero–one variables, dividing the problem into several subproblems, or calculating every break point to transforming original functions into new functions. The extension of Algorithm 1 named Algorithm 2 is then studied to solve the quasiconcave FMP program with fuzzy coefficients in (1.2).

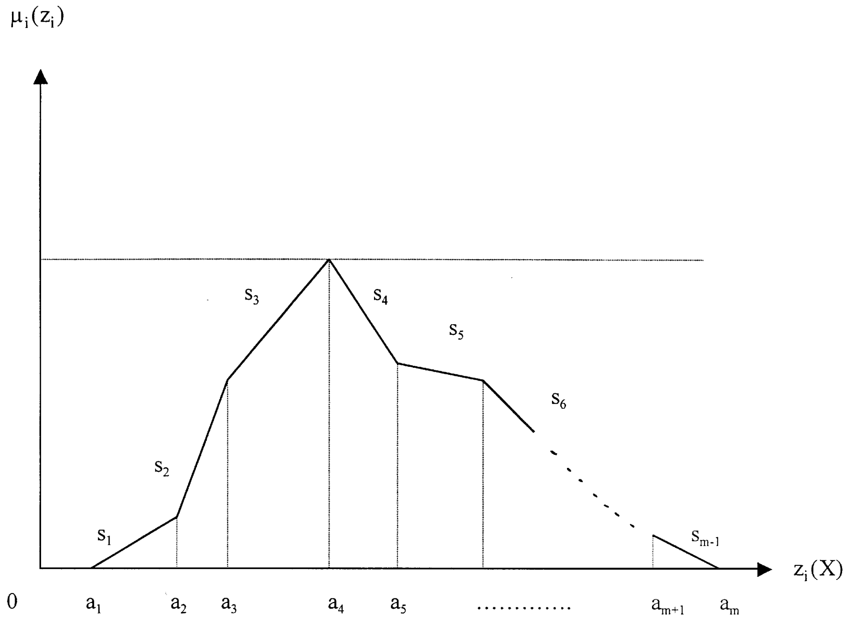

Fig. 6. A general piecewise linear membership function.

3. Preliminaries

Some propositions of linearizing a quasiconcave membership function $\mu_i(z_i)$ are described as follows:

Proposition 1. *Let $\mu_i(z_i)$ be a piecewise linear membership function of $z_i(X)$, as depicted in Fig.* 6, *where a_k, $k=1,2,\dots,m$ are the break points of $\mu_i(z_i)$, s_k, $k=1,2,\dots,m-1$, are the slopes of line segments between a_k and a_{k+1}, and*

$$s_k=\frac{\mu_i(a_{k+1})-\mu_i(a_k)}{a_{k+1}-a_k}.$$

$\mu_i(z_i)$ can then be expressed as:

$$\mu_i(z_i)=\mu_i(a_1)+s_1(z_i(X)-a_1)+\sum_{k=2}^{m-1}\frac{s_k-s_{k-1}}{2}(|z_i(X)-a_k|+z_i(X)-a_k), \tag{3.1}$$

where |o| is the absolute value of o.

This proposition can be examined as follows:

(i) If $z_i(X)\leqslant a_2$ then

$$\mu_i(z_i)=\mu_i(a_1)+\frac{z_i(a_2)-z_i(a_1)}{a_2-a_1}(z_i(X)-a_1)=a_1+s_1(z_i(X)-a_1).$$

(ii) If $z_i(X)\leqslant a_3$ then

$$\mu_i(z_i)=\mu_i(a_1)+s_1(a_2-a_1)+s_2(z_i(X)-a_2)$$
$$=\mu_i(a_1)+s_1(z_i(X)-a_1)+\frac{s_2-s_1}{2}(|z_i(X)-a_2|+z_i(X)-a_2).$$

(iii) If $z_i(X)\leqslant a_{k'}$ then

$$\sum_{k\geqslant k'}^{m-1}(|z_i(X)-a_k|+z_i(X)-a_k)=0$$

and $\mu_i(z_i)$ becomes

$$\mu_i(a_1)+s_1(z_i(X)-a_1)+\sum_{k=2}^{k'-1}\frac{s_k-s_{k-1}}{2}(|z_i(X)-a_k|+z(X)-a_k).$$

Take $\mu_1(z_1)$ and $\mu_2(z_2)$ in Example 1 (Fig. 2(a) and (b)) for instances, $\mu_1(z_1)$ and $\mu_2(z_2)$ can be represented by Proposition 1 as

$$\mu_1(z_1)=0.04(z_1+3)+\frac{0.08-0.04}{2}(|z_1-2|+z_1-2)+\frac{-0.1-0.08}{2}(|z_1-12|+z_1-12)$$
$$+\frac{-0.05+0.1}{2}(|z_1-17|+z_1-17). \tag{3.2}$$

$$\mu_2(z_2)=0.06(z_2-7)+\frac{0.1-0.06}{2}(|z_2-17|+z_2-17)+\frac{-0.033-0.1}{2}(|z_2-21|+z_2-21)$$
$$+\frac{-0.1+0.033}{2}(|z_2-27|+z_2-27)+\frac{-0.25+0.1}{2}(|z_2-30|+z_2-30). \tag{3.3}$$

An advantage of expressing a quasiconcave membership function by (3.1) is the convenience of knowing the intervals of convexity and concavity for $\mu_i(z_i)$, as described below:

Remark 1. For a $\mu_i(z_i)$ expressed by (3.1) if $s_{k+1}>s_k$ then $\mu_i(z_i)$ is a convex function for $a_{k-1}\leqslant z_i(X)\leqslant a_{k+1}$, and a_k is called a *convex-type break point* of z_i; if $s_{k+1}<s_k$ then $\mu_i(z_i)$ is a concave function for $a_{k-1}\leqslant z_i(X)\leqslant a_{k+1}$.

Take Expression (3.2), for instance, it is convenient to check that $\mu_1(z_1)$ is concave when $2\leqslant z_1(X)\leqslant 17$ and $\mu_1(z_1)$ is convex when $-3\leqslant z_1(X)\leqslant 12$ and $12\leqslant z_1(X)\leqslant 27$. The point $z_1(X)=2$ and 17 are convex-type break points of z_i. Similarly for Expression (3.2), $\mu_2(z_2)$ is convex for $7\leqslant z_2(X)\leqslant 21$ and concave for $17\leqslant z_2(X)\leqslant 32$. $z_2(X)=17$ is a convex-type break point of z_2.

Proposition 2. *Consider the following fuzzy program* (P1):

$$\text{P1:}\quad \mathit{Max}\quad \lambda$$
$$\text{s.t.}\quad \lambda\leqslant\mu_i(z_i(X)),\quad i=1,2,\ldots,m,$$
$$X\in F\ (F \text{ is a feasible set}).$$

Suppose the decrease of any $\mu_i(z_i(X))$ will cause the increase of $\mu_j(z_j(X))$ for all $j\neq i$, then at the optimal solution X^*, $\lambda=\mu_i(z_i(X^*))$ for $i=1,2,\ldots,m$.

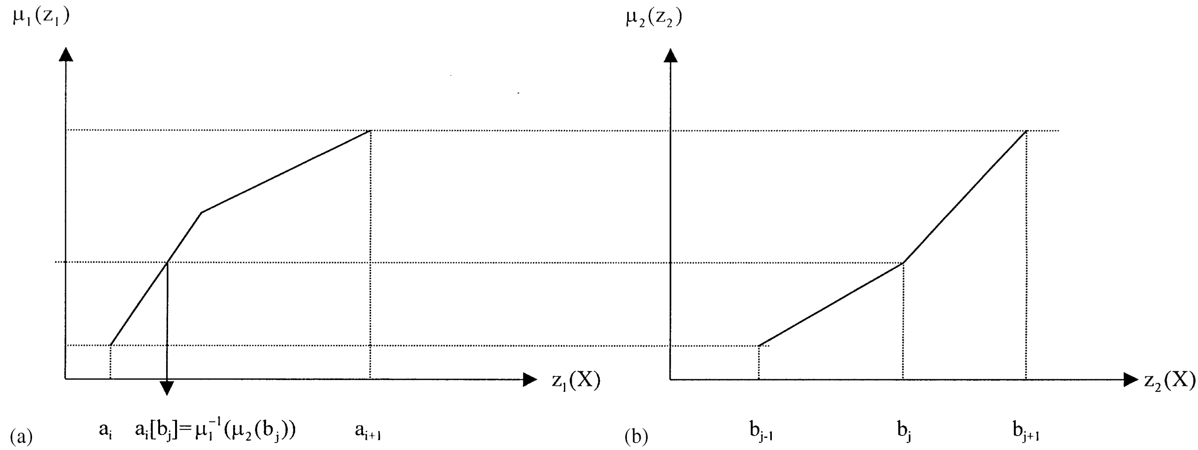

Fig. 7. (a) A concave function $\mu_1(z_1)$. (b) A convex function $\mu_2(z_2)$.

Proof. Problem (P1) is to find X^* such that $\lambda = \text{Max Min}(\mu_i(z_i(X^*)))$ for $i = 1,2,\dots,m$. Suppose there is a solution X^0 satisfying the following conditions:

$\lambda^0 = \text{Max Min}(\mu_i(z_i(X^0)))$,

$\lambda^0 = \mu_i(z_i(X^0))$ for all $i = 1,2,\dots,k$, $i \neq k$,

$\lambda^0 < \mu_k(z_k(X^0))$.

Since the decrease of $\mu_k(z_k(X^0))$ will cause the increase of $\mu_i(z_i(X^0))$ which finally improves the objective value, there should exist another solution X^Δ in which $\lambda < \lambda^\Delta = \mu_k(z_k(X^\Delta)) < \mu_k(z_k(X^0))$, $\lambda^\Delta = \mu_i(z_i(X^\Delta))$, $i \in (1,2,\dots,m)$, and $i \neq k$.

Since $\lambda^\Delta > \lambda^0$, λ^0 is not the optimal solution. Therefore the optimal solution X^* should satisfy $\lambda = \mu_i(z_i(X^*))$ for $i = 1,2,\dots,m$. □

Consider two piecewise linear membership functions $\mu_1(z_1)$ and $\mu_2(z_2)$ as depicted in Fig. 7(a) and (b), where

(i) μ_1 is a concave function within the interval $a_i \leqslant z_1 \leqslant a_{i+1}$, a_i and a_{i+1} are starting and ending points of that interval,

(ii) μ_2 is a convex function within the interval $b_{j-1} \leqslant z_2 \leqslant b_{j+1}$ and b_j is a *convex-type break point.*

(iii) $\mu_1(a_i) = \mu_2(b_{j-1})$ and $\mu_1(a_{i+1}) = \mu_2(b_{j+1})$.

Remark 2. For $\mu_1(z_1)$ and $\mu_2(z_2)$ in Fig. 7(a) and (b), we can find a corresponding point of b_j in z_1 which has the same value of membership functions as b_j. Such a point is called a *mapping point* of b_j, denoted as $a_i[b_j]$, which is mapped from z_2 to z_1, as follows:

$$a_i[b_j] = \mu_1^{-1}(\mu_2(b_j)).$$

By using b_j and $a_i[b_j]$, we can divide z_1 and z_2 into two pairs of segments as follows:

1st pair segments: $(a_i, a_i[b_j])$, (b_{j-1}, b_j);

2nd pair segments: $(a_i[b_j], a_{i+1})$, (b_j, b_{j+1}).

By utilizing the above concept, we can divide $\mu_1(z_1)$ and $\mu_2(z_2)$ in Fig. 2(a) and (b) of Example 1 into five pairs of segments below:

Table 1
Five pairs of segments

$\mu_1(z_1)=\mu_2(z_2)$	0	0.2	0.6	1	0.5	0
Z_1	−3	2	$\mu_1^{-1}(\mu_2(17))=7$	12	17	27
Z_2	7	$\mu_2^{-1}(\mu_1(2))=10.333$	17	21	$\mu_2^{-1}(\mu_1(17))=30$	32
Segment number	→1←	→2←	→3←	→4←	→5←	

Where the mapping points of $z_1=2$, $z_1=17$ and $z_2=17$ are computed as $\mu_2^{-1}(\mu_1(2))=\mu_2^{-1}(0.2)=10.333$, $\mu_2^{-1}(\mu_1(17))=\mu_2^{-1}(0.5)=30$, and $\mu_1^{-1}(\mu_2(17))=\mu_1^{-1}(0.6)=7$.

By referring to Proposition 2, we can find the optimal solution of Example 1. At first, we check the feasibility of each segment.

Segment 1: (z_1,z_2) between $(-3,7)$ and $(2,10.333)$.

Segment 2: (z_1,z_2) between $(2,10.333)$ and $(7,17)$.

Segment 3: (z_1,z_2) between $(7,17)$ and $(12,21)$.

Segment 4: (z_1,z_2) between $(12,21)$ and $(17,30)$.

Segment 5: (z_1,z_2) between $(17,30)$ and $(27,32)$.

Since $(7,17)$ are feasible and $(12,21)$ is infeasible, the optimal solution should fall into segment 3. That means $7\leqslant z_1(X^*)\leqslant 12$ and $17\leqslant z_2(X^*)\leqslant 21$. $\mu_1(z_1)$ and $\mu_2(z_2)$ in (3.2) and (3.3) can then be rewritten as $\mu_1(z_1)=0.08z_1+0.04$ and $\mu_2(z_2)=0.1z_2-1.1$. Example 1 is reformulated as the following LP model:

$$\begin{aligned}
&\text{Maximize} && \lambda \\
&\text{subject to} && \lambda\leqslant\mu_1(z_1)=0.08z_1+0.04,\\
&&& \lambda\leqslant\mu_2(z_2)=0.1z_2-1.1,\\
&&& z_1=-x_1+2x_2,\quad z_2=2x_1+x_2,\\
&&& -x_1+3x_2\leqslant 21,\quad x_1+3x_2\leqslant 27,\\
&&& 4x_1+3x_2\leqslant 45,\quad 3x_1+x_2\leqslant 30,\ x_1,x_2\geqslant 0.
\end{aligned}\tag{3.4}$$

After computing on the LINDO [13], the obtained solution is $(x_1=5.6,\ x_2=7.133,\ \lambda=0.733)$ which is the same as found by the Nakamura's and Yang et al.'s models.

Suppose a membership function within a given segment is not linear but concave, then we need to solve it by goal programming techniques [7]. Consider the following proposition:

Proposition 5. *By referring to Proposition* 1, *consider a FMP problem below*:

$$\begin{aligned}
&\textit{Maximize} && \lambda\\
&\textit{subject to} && \lambda\leqslant\mu_i(z_i),\quad X\in F\ (\textit{a feasible set}),
\end{aligned}$$

where

$$\mu_i(z_i)=\mu_i(a_1)+s_1(z_i(X)-a_1)+\sum_{k=2}^{m-1}\frac{s_k-s_{k-1}}{2}(|z_i(X)-a_k|+z_i(X)-a_k)$$

is a concave function (*i.e.*, $s_k-s_{k-1}<0$ *for* $k=2,3,\ldots,m-1$).

This FMP problem can then be reformulated as follows:

$$
\begin{aligned}
&\textit{Maximize} \quad \lambda \\
&\textit{subject to} \quad \lambda \leqslant \mu_i(z_i) \\
&\qquad \mu_i(z_i) = \mu_i(a_1) + s_1(z_i(X) - a_1) + \sum_{k=2}^{m-1} (s_k - s_{k-1}) \left(z_i(X) - a_k + \sum_{l=1}^{k-1} d_l \right), \\
&\qquad z_i(X) - a_{m-1} + \sum_{l=2}^{m-1} d_{l-1} \geqslant 0, \\
&\qquad 0 \leqslant d_{l-1} \leqslant a_l - a_{l-1} \quad \textit{for all } l, \ l = 2, 3, \ldots, m-1, \\
&\qquad X \in F \ (\textit{a feasible set}).
\end{aligned} \tag{3.5}
$$

Proof. By referring to Li [7], a goal programming problem {Maximize $w = \sum_{k=2}^{m-1} (|z_i(X) - a_k| + z_i(X) - a_k)$, subject to: $z_i(X) \geqslant 0$ and $0 < a_2 < a_3 < \cdots < a_{m-1}$} is equivalent to

$$
\left\{ \text{Maximize } w = 2 \sum_{k=2}^{m-1} (z_i(X) - a_k + r_{k-1}), \text{ subject to: } z_i(X) - a_k + r_{k-1} \geqslant 0 \text{ for } k = 2, 3, \ldots, m-1, \right.
$$

$$
\left. r_{k-1} \geqslant 0, \ x_i \geqslant 0, \ \text{where } r_{k-1} \text{ are deviation variables.} \right\} \tag{3.6}
$$

Expression (3.6) implies if $z_i(X) < a_k$ then at optimal solution $r_{k-1} = a_k - z_i(X)$; if $z_i(X) \geqslant a_k$ then at optimal solution $r_{k-1} = 0$. Substitute r_{k-1} by $\sum_{l=1}^{k-1} d_l$, where d_l is within the bounds as $0 \leqslant d_l \leqslant a_{l+1} - a_l$, (3.6) then becomes

$$
\begin{aligned}
&\text{Maximize} \quad w = 2 \sum_{k=2}^{m-1} \left(z_i(X) - a_k + \sum_{l=1}^{k-1} d_l \right) \\
&\text{Subject to} \quad z_i(X) + d_1 \geqslant a_2, \\
&\qquad z_i(X) + d_1 + d_2 \geqslant a_3, \\
&\qquad \vdots \\
&\qquad z_i(X) + d_1 + d_2 + \cdots + d_{m-2} \geqslant a_{m-1}, \\
&\qquad 0 \leqslant d_l \leqslant a_{l+1} - a_l \quad \text{for } l = 1, 2, \ldots, m-2, \\
&\qquad z_i(X) \geqslant 0.
\end{aligned} \tag{3.7}
$$

Since $a_{l+1} - a_l \geqslant d_l$ for all l, it is clear that

$$
z_i(X) \geqslant a_{m-1} - \sum_{l=1}^{m-2} d_l \geqslant a_{m-2} - \sum_{l=1}^{m-3} d_l \geqslant \cdots \geqslant a_3 - d_1 - d_2 \geqslant a_2 - d_1 \geqslant 0.
$$

The first $(m-3)$ constraints in Model (3.7) therefore are covered by the $(m-2)$th constraint in (3.7). Proposition 5 is then proven. □

4. Algorithm for quasiconcave FMP problems

From above discussion, an algorithm of solving a FMP problem in (1.1) where $\mu_i(z_i)$ are quasiconcave functions is formulated below:

Algorithm 1

Step 1: Express $\mu_i(z_i)$ as

$$\mu_i(z_i) = \mu_i(a_{i1}) + s_{i1}(z_i(X) - a_{i1}) + \sum_{k=2}^{M(i)-1} \frac{s_{ik} - s_{ik-1}}{2}(|z_i(X) - a_{ik}| + z_i(X) - a_{ik}) \quad \text{for } i = 1, 2, \ldots, n.$$

Step 2: Put $\{\mu_i(a_{ik}) \mid i = 1, 2, \ldots, m$ and $s_{i,k+1} > s_{i,k}$ for $k = 1, 2, \ldots, m(i) - 1\}$, which is the set of convex-type break points, in an ascending order. Let n be its cardinality. Name the elements r_g $(g = 1, 2, \ldots, n)$ in order of value. As shown in Table 2. Let $g = 1$.

Step 3: Compute the corresponding mapping points of r_g. Find t_{ig} for each i where $\mu_i(t_{ig}) = \mu(r_g)$.

Step 4: Check the feasibility of r_g.
Suppose there is an X satisfying

$$Z_i(X) = t_{ig}, \tag{4.1}$$

$$\sum_{j=1}^{m} b_{ij}x_j + c_i \geqslant 0 \quad \text{for all } i, \tag{4.2}$$

then r_g is feasible to the problem.
If r_g is infeasible, then let $g = g + 1$ and reiterate Step 3.
Otherwise, r_g is feasible, go to Step 5.

Step 5: The optimal solution is located between r_{g-1} and r_g, which is obtained by solving the following linear program:

$$\text{Maximize} \quad \lambda$$

$$\text{subject to} \quad \lambda \leqslant u_i(z_i(X)) = u_i(a_{ik}) + s_{ik}(z_i - a_{ik}) + \sum_{k \geqslant 2}^{N(i)-1} (s_{ik} - s_{i,k-1})(z_i - a_{ik} + d_k)$$

$$X \in F,$$

where $a_{ik} = t_{ip}$ and $a_{i,N(i)} = t_{i,\,p+1}$.

Table 2
Mapping point table

Element	r_1	r_2	r_g	....	r_n
Value order $0 \leqslant u \leqslant 1$	Highest $u(r_1) = 1$	....	$\mu(r_g)$	....	Lowest $u(r_n) = 0$
$Z_1(X)$	t_{11}	t_{12}	t_{1g}	....	t_{1n}
$Z_2(X)$	t_{21}	t_{22}	t_{2g}	....	t_{2n}
.	:	:	:		:
.	:	:	:		:
.	:	:	:		:
$Z_n(X)$	t_{n1}	t_{n2}	t_{ng}	....	t_{nn}

Consider the following example:

Example 2.

$$\begin{aligned}&\text{Maximize} \quad \lambda\\ &\text{Subject to} \quad \lambda \leqslant \mu_i(z_i), \quad i=1,2,3,\\ &\qquad z_1 = -2.2x_1 + x_2 + 2x_3,\ z_2 = 3.2x_1 + 3.2x_2 - x_3,\ z_3 = 3x_1 - 2x_2 + 3.5x_3,\\ &\qquad -x_1 + 3x_2 + x_3 \leqslant 50,\ x_1 + 3x_2 - x_3 \leqslant 40,\ 3x_1 - x_2 + 2x_3 \leqslant 80,\\ &\qquad x_1 + x_2 \leqslant 20,\ x_1 + x_3 \leqslant 20,\ x_1, x_2, x_3 \geqslant 0,\end{aligned}$$

where $\mu_1(z_1)$, $\mu_2(z_2)$, and $\mu_3(z_3)$ are depicted in Fig. 8(a)–(c), respectively.

Now, we take Example 2 for instance to demonstrate the solution process of Algorithm 1.

Step 1: By referring to Proposition 1, the $\mu_1(z_1)$, $\mu_2(z_2)$, and $\mu_3(z_3)$ can be expressed below:

$$\mu_1(z_1) = 0.05z_1 - \frac{0.03}{2}(|z_1 - 10| + z_1 - 10) - \frac{0.01}{2}(|z_1 - 20| + z_1 - 20) - \frac{0.0067}{2}(|z_1 - 40| + z_1 - 40), \tag{4.3}$$

$$\mu_2(z_2) = 0.03z_2 - \frac{0.01}{2}(|z_2 - 10| + z_2 - 10) - \frac{0.01}{2}(|z_2 - 20| + z_2 - 20) - \frac{0.005}{2}(|z_2 - 40| + z_2 - 40) + \frac{0.015}{2}(|z_2 - 60| + z_2 - 60), \tag{4.4}$$

$$\mu_3(z_3) = 0.01z_3 + \frac{0.05}{2}(|z_3 - 30| + z_3 - 30) - \frac{0.05}{2}(|z_2 - 40| + z_2 - 40). \tag{4.5}$$

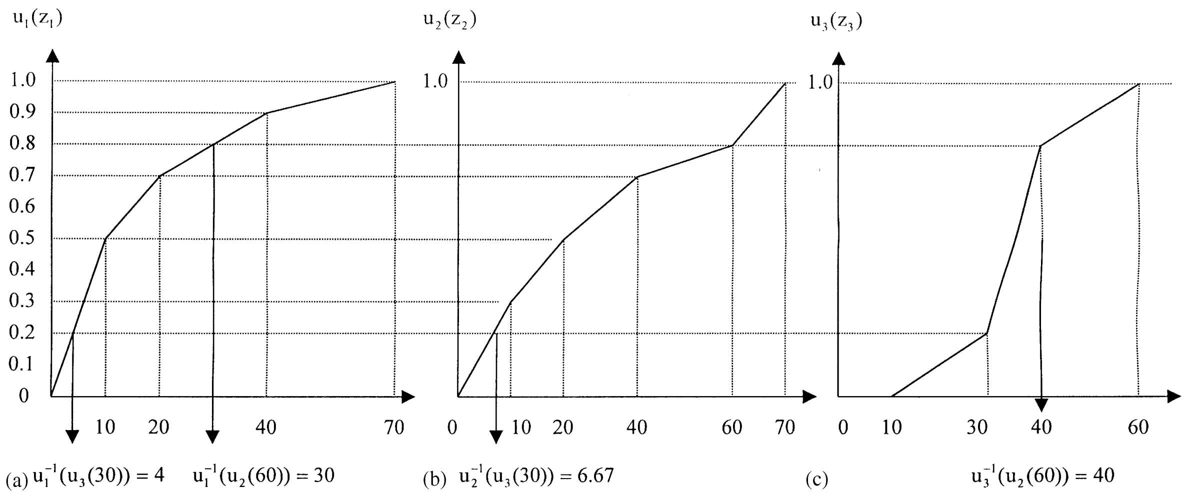

Fig. 8. (a) $u_1(z_1)$ in Example 2. (b) $u_2(z_2)$ in Example 2. (c) $u_3(z_3)$ in Example 2.

Table 3
Corresponding elements

Element	r_1	r_2	r_3	r_4
μ value	1	0.8	0.2	0
z_1	70	t_{12}	t_{13}	0
z_2	70	60	t_{23}	0
z_3	60	t_{32}	30	10

Step 2: The convex-type break points with "positive" slope in this example are $z_2=60$ and $z_3=30$ where $\mu_2(z_2=60)=0.8$ and $\mu_3(z_3=30)=0.2$. The corresponding r_g elements are listed in Table 3.

Step 3: Since $(z_1=70,\ z_2=70,\ z_3=60)$ is infeasible to Example 2, we compute t_{12} and t_{32} as $t_{12}=\mu_1^{-1}(\mu_2(60))=\mu_1^{-1}(0.8)=30$ and $t_{32}=\mu_3^{-1}(\mu_2(60))=\mu_3^{-1}(0.8)=40$.

Step 4: Since $(z_1=30,\ z_2=60,\ z_3=40)$ is infeasible to Example 3, we reiterate Step 3.

[Step 3] Compute t_{13} and t_{23} as $t_{13}=\mu_1^{-1}(\mu_3(30))=\mu_1^{-1}(0.2)=4$ and $t_{23}=\mu_2^{-1}(\mu_3(30))=\mu_2^{-1}(0.2)$ $=6.67$.

[Step 4] Since $(z_1=4,\ z_2=6.67,\ z_3=10)$ is feasible to Example 2, we go to Step 5.

Step 5: Since $(z_1=4,\ z_2=6.67,\ z_3=10)$ is feasible and $(z_1=30,\ z_2=60,\ z_3=40)$ is infeasible, the optimal solution should fall into the segment between r_2 and r_3. Therefore, referring to Proposition 5, $\mu_1(z_1)$, $\mu_2(z_2)$ and $\mu_3(z_3)$ can be expressed as $\mu_1(z_1)=0.01z_1-0.04d_1-0.01d_2+0.5$ where $z_1+d_1+d_2\geqslant 20$, $\mu_2(z_2)=0.005z_2-0.025d_3-0.015d_4-0.005d_5+0.5$ where $z_2+d_3+d_4+d_5\geqslant 40$, and $\mu_3(z_3)=0.06z_3-1.5$.

Example 2 is then reformulated as the LP model below:

$$
\begin{aligned}
&\text{Maximize} && \lambda\\
&\text{subject to} && \lambda\leqslant\mu_1(z_1)=0.01z_1-0.04d_1-0.01d_2+0.5,\quad z_1+d_1+d_2\geqslant 20,\\
&&& \lambda\leqslant\mu_2(z_2)=0.005z_2-0.025d_3-0.015d_4-0.005d_5+0.5,\ z_2+d_3+d_4+d_5\geqslant 40,\\
&&& \lambda\leqslant\mu_3(z_3)=0.06z_3-1.5,\\
&&& z_1=-2.2x_1+x_2+2x_3,\quad z_2=3.2x_1+3.2x_2-x_3,\quad z_3=3x_1-2x_2+3.5x_3,\\
&&& -x_1+3x_2+x_3\leqslant 50,\quad x_1+3x_2-x_3\leqslant 40,\quad 3x_1-x_2+2x_3\leqslant 80,\\
&&& x_1+x_2\leqslant 20,\quad x_1+x_3\leqslant 20,\quad x_1,x_2,x_3\geqslant 0.
\end{aligned}
$$

Solving by LINDO [13], the obtained solution is $\lambda=0.755$, $z_1=25.497$, $z_2=50.993$, $z_3=37.583$, $x_1=6.412$, $x_2=13.588$, and $x_3=13.007$.

The comparison of Algorithm 1 with current FMP methods for solving Examples 1 and 2 are summarized in Tables 4 and 5.

Tables 4 and 5 reveal that

(i) Nakamura's method needs to divide Example 1 into eight subproblems and to find the optimal solution by linear programming computation repeatedly, while the proposed method only uses linear programming computation one time for solving Example 1.

Table 4
Comparison of different methods for solving Example 1

	Number of LP subproblems	Number of divided extra constraints	Number of extra zero–one variables
Nakamura's model	8	9	0
Yang et al.'s model	1	9	3
The proposed model	1	2	0

Table 5
Comparison of different methods for solving Example 2

	Number of computed break points
Inuiguchi et al.'s model	12
The proposed model	4

(ii) Yang et al.'s method requires adding three extra zero–one variables in their method while none of zero–one variables is needed in the proposed method.
(iii) The proposed method uses less number of constraints than other methods. (i.e., proposed method uses two constraints only, whereas both Yang et al.'s method and Nakamura's method uses nine extra constraints in treating fuzzy objective functions in Example 1).
(iv) Inuiguchi et al.'s method requires to compute the mapping points of all break points, while the proposed method only needs to compute the mapping points of the convex-type break points.

5. Extension to quasiconcave FMP problems with fuzzy values in coefficients

Algorithm 1 can be extended to solve a FMP$'$ program in (1.2) where the constraints may have fuzzy coefficients. Suppose the fuzzy coefficients $\tilde{b}_{ij}$ and uncertain resources $\tilde{c}_i$ in (1.2) are symmetric triangular distribution where $(b_{i1}, b_{i2}, \ldots, b_{im}, c_i)$ are the most possible values (central values) and $(p_{i1}, p_{i2}, \ldots, p_{im}, q_i)$ are the most possible deviations from the central values as shown in Fig. 9.

Let

$$y_i = \sum_{j=1}^{m} \tilde{b}_{ij} x_j + \tilde{c}_i.$$

By referring to [14, 15], the triangular membership function could be expressed as

$$\prod_i (y_i) = 1 - \frac{|y_i - \sum_{j=1}^m b_{ij} x_j - c_i|}{\sum_{j=1}^m p_{ij} x_j + q_i}.$$

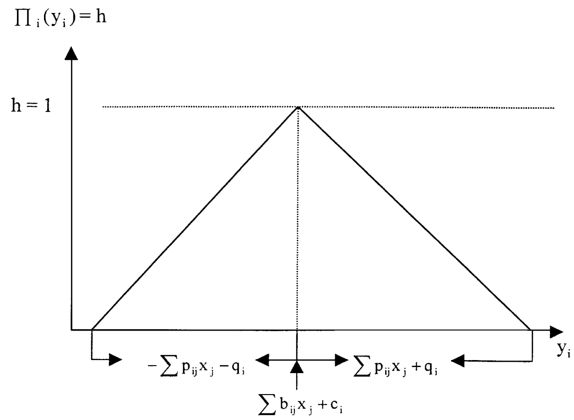

Fig. 9. Possibility distribution.

Denote h as $h=\prod_i(y_i)$. Since $0\leqslant\prod_i(y_i)\leqslant 1$, we have

$$h=1-\frac{\delta_i^+ + \delta_i^-}{\sum_{j=1}^m p_{ij}x_j + q_i}$$

where $y_i-\sum_{j=1}^m b_{ij}x_j - c_i=\delta_i^+ - \delta_i^-$, $\delta_i^+\geqslant 0$, $\delta_i^-\geqslant 0$, $0\leqslant h\leqslant 1$.

Utilizing the Max–Min operator [6, 17] FMP′ problem in (1.2) can then be converted into the following program:

$$\text{Maximize} \quad \lambda \tag{5.1}$$

$$\text{subject to} \quad \lambda\leqslant\mu_i(z_i(X)), \tag{5.2}$$

$$\lambda\leqslant wh, \tag{5.3}$$

$$h\left(\sum_{j=1}^m p_{ij}x_j+q_j\right)=\left(\sum_{j=1}^m p_{ij}x_j+q_i\right)-\delta_i^+ - \delta_i^- \quad \text{for } i=1,2,\ldots,I, \tag{5.4}$$

$$z_i(X)-\sum_{j=1}^m b_{ij}x_j-c_i=\delta_i^+ - \delta_i^- \quad \text{for } i=1,2,\ldots,n, \tag{5.5}$$

$$y_i-\sum_{j=1}^m b_{ij}x_j-c_i=\delta_i^+ - \delta_i^- \quad \text{for } i=n+1,n+2,\ldots,I, \tag{5.6}$$

$$\delta_i^+\geqslant 0,\quad \delta_i^-\geqslant 0,\quad 0\leqslant h\leqslant 1, \tag{5.7}$$

where w is the weight between the degree of preference for membership functions and the degree of possibility for possibility distribution.

The upper and lower bounds of w value could be computed by referring to the scaling algorithm recently developed by Biswal [1]. His algorithm is very helpful for a decision maker to specify w based on the information of the upper values of $\mu_i(z_i(X))$ and h. Suppose w is given, then Tanaka and Asia's method [14, 15] and Leung's method [6] could be applied to solve Problem (5.1)–(5.7). Their methods, however, could only find locally optimal solution of this problem. Hereby, we propose a method that could find a solution which is as close as possible to a global optimum.

A major difficulty of solving this program is that the constraint (5.4) contains product terms hx_j. In order to approximately linearize the product term hx_j, the following proposition is introduced, referring to Li and Chang [8].

Proposition 6. *A product term hx_j where $0 \leqslant h \leqslant 1$ and $x_j \geqslant 0$ can be approximately linearized as $v_j \doteq hx_j$. The relationships among h, x_j, and v_j are expressed as*

$$v_j \doteq \frac{1}{\sum_{k=1}^{K} 2^{k-1}} \left[\sum_{k=1}^{K} (2^{k-1} \alpha_{jk}) \right] \quad (K \text{ is an integer specified by the user}), \tag{5.8}$$

$$h \doteq \frac{1}{\sum_{k=1}^{K} 2^{k-1}} \left[\sum_{k=1}^{K} (2^{k-1} \theta_{jk}) \right], \tag{5.9}$$

$$M(\theta_{jk} - 1) + x_j \leqslant \alpha_{jk} \leqslant 1 - \theta_{jk} + x_j, \quad k = 1, 2, \ldots, K, \tag{5.10}$$

$$0 \leqslant \alpha_{jk} \leqslant M\theta_k, \quad \alpha_{jk} \leqslant x_j, \quad k = 1, 2, \ldots, K, \tag{5.11}$$

where θ_{jk} are 0–1 variables, α_{jk} are continuous variables and M is the upper bound of x_j. That is $M = \text{Max}\{x_j \; \forall j\}$.

Proof. Since $0 \leqslant h \leqslant 1$, there exists a set of 0–1 variables θ_{jk} such that (5.9) is true. We then have

$$v_j = hx_j \doteq \frac{1}{\sum_{k=1}^{K} 2^{k-1}} \left[\sum_{k=1}^{K} (2^{k-1} \theta_{jk} x_j) \right].$$

Replace the product term $\theta_{jk} x_j$ by new variables α_{jk}, we obtain (5.8). Expressions (5.10) and (5.11) ensure that if $\theta_{jk} = 1$ then $\alpha_{jk} = x_j$, and if $\theta_{jk} = 0$ then $\alpha_{jk} = 0$. □

Notably that K in the above expressions means the number of 0–1 variables used to express a product term. The larger the K is, the less the linearizing error becomes. For instance, for linearizing a product term hx where $0 \leqslant h \leqslant 1$ and $0 \leqslant x \leqslant 10$. Suppose the maximal tolerable error is $\frac{1}{15}$, then $v = hx$ can be approximately linearized as follows:

$$v = \tfrac{1}{15}(\delta_1 + 2\delta_2 + 4\delta_3 + 8\delta_4), \qquad h = \tfrac{1}{15}(\theta_1 + 2\theta_2 + 4\theta_3 + 8\theta_4),$$

$$M(\theta_i - 1) + x \leqslant \alpha_i \leqslant 1 - \theta_i + x, \quad i = 1, 2, 3, 4,$$

$$0 \leqslant \alpha_i \leqslant M\theta_i, \quad \alpha_i \leqslant x, \quad i = 1, 2, 3, 4, \text{ and } M \text{ is the upper bound of } x.$$

Algorithm 2. The algorithm of solving Quasiconcave FMP problems with fuzzy coefficients is formulated below:
Step 0: Ask the decision maker to specify w in (5.1), $0 \leqslant w \leqslant 1$.
Step 1 through *Step* 3 are the same as in Algorithm 1.
Step 4: Check whether r_g satisfies the following linear equalities

$$z_i(X) = t_{ig} \quad \text{for } i = 1, 2, \ldots, n,$$

$$h = 1 - \frac{\delta_i^+ + \delta_i^-}{\sum_{j=1}^{m} p_{ij}x_j + q_i} \quad \text{for } i = n+1, n+2, \ldots, I,$$

$$y_i - \sum_{j=1}^{m} b_{ij}x_j - c_i = \delta_i^+ - \delta_i^- \quad \text{for } i = n+1, n+2, \ldots, I,$$

where $h = (1/w)\mu(r_g)$.

If r_g is infeasible then let $g = g + 1$ and reiterate Step 3. Otherwise, go to Step 5.

Step 5: Solve the following linear mixed 0–1 program:

$$\begin{aligned} &\text{Maximize} \quad \lambda \\ &\text{subject to} \quad \lambda \leqslant \mu_i(z_i(X)),\ (5.3)\text{–}(5.11). \end{aligned}$$

Consider the following example which is slightly modified from Example 1 by adding fuzzy coefficients.

Example 3.

$$\begin{aligned} &\text{Maximize} \quad \lambda \\ &\text{subject to} \quad \lambda \leqslant \mu_1(z_1), \quad \lambda \leqslant \mu_2(z_2), \\ &\qquad z_1 = -x_1 + \tilde{2}x_2, \quad z_2 = 2x_1 + x_2, \\ &\qquad -x_1 + 3x_2 \leqslant 21, \quad x_1 + 3x_2 \leqslant 27, \\ &\qquad \tilde{4}x_1 + 3x_2 \leqslant 4\tilde{5}, \quad 3x_1 + x_2 \leqslant 30, \\ &\qquad x_1, x_2 \geqslant 0, \end{aligned}$$

where $0 \leqslant x_1 \leqslant 11.25$, $0 \leqslant x_2 \leqslant 15$, $\tilde{2} = (1,2,3)$, $\tilde{4} = (3,4,5)$, $4\tilde{5} = (43,45,47)$, and μ_1 and μ_2 are the same as in Example 1.

Step 0: Ask the decision maker to specify w. (Suppose $w = 0.3, 0.5, 0.7, 0.75, 0.8$ and 1.0 for illustration)

Steps 1–3: From the basis of Algorithm 1, Example 3 can be reformulated below (referring to (3.4)):

$$\begin{aligned} &\text{Maximize} \quad \lambda \\ &\text{subject to} \quad \lambda \leqslant \mu_1(\tilde{z}_1) = 0.08\tilde{z}_1 + 0.04, \quad \lambda \leqslant \mu_2(z_2) = 0.1z_2 - 1.1, \\ &\qquad \tilde{z}_1 = -x_1 + \tilde{2}x_2, \quad z_2 = 2x_1 + x_2, \\ &\qquad -x_1 + 3x_2 \leqslant 21, \quad x_1 + 3x_2 \leqslant 27, \\ &\qquad \tilde{4}x_1 + 3x_2 \leqslant 4\tilde{5}, \quad 3x_1 + x_2 \leqslant 30, \quad x_1, x_2 \geqslant 0, \end{aligned}$$

Step 4: Fuzzy inequality $\tilde{4}x_1 + 3x_2 \leqslant 4\tilde{5}$ can be expressed as

$$h = 1 - \frac{\delta_1^+ + \delta_1^-}{x_1 + 2}, \tag{5.12}$$

$$y_1 - (45 - 4x_1 - 3x_2) = \delta_1^+ - \delta_1^+, \tag{5.13}$$

$$\delta_1^+ \geqslant 0, \quad \delta_1^- \geqslant 0, \quad 0 \leqslant h \leqslant 1. \tag{5.14}$$

Suppose the maximal tolerable error is $\frac{1}{15}$ and from the basis of Proposition 6, (5.12) can be replaced by the following:

$$v_1 + 2h - x_1 - 2 = -\delta_1^+ - \delta_1^-, \tag{5.15}$$

Table 6
Solution table subject to w value

w	λ	h	μ_1	μ_2	z_1	z_2	x_1	x_2
0.3	0.3	1	0.3	0.886	3.25	19.864	7.295	5.273
0.5	0.5	1	0.5	0.841	5.75	19.409	6.614	6.182
0.7	0.7	1	0.7	0.775	8.25	18.75	5.85	7.05
0.75	0.733	1	0.733	0.733	8.667	18.333	5.6	7.13
0.8	0.746	0.933	0.746	0.746	8.83	18.644	5.786	7.071
1.0	0.755	0.933	0.755	0.755	8.931	18.544	5.727	7.091

$$v_1 = \tfrac{1}{15}(\alpha_{11} + 2\alpha_{12} + 4\alpha_{13} + 8\alpha_{14}), \tag{5.16}$$

$$h = \tfrac{1}{15}(\theta_{11} + 2\theta_{12} + 4\theta_{13} + 8\theta_{14}), \tag{5.17}$$

$$M(\theta_{1j} - 1) + x_1 \leqslant \alpha_{1j} \leqslant 1 - \theta_{1j} + x_1,\ \ 0 \leqslant \alpha_{1j} \leqslant M\theta_{1j}, \quad \alpha_{1j} \leqslant x_1, \quad j = 1,2,3,4, \tag{5.18}$$

where θ_{1j} are 0–1 variables, α_{1j} are continuous variables and M is the upper bound of $\max\{x_1, x_2\}$. Similarly, $z_1 = -x_1 + \tilde{2}x_2$ is solved by

$$h = 1 - \frac{\delta_2^+ + \delta_2^-}{x_2}, \tag{5.19}$$

$$z_1 - (-x_1 + 2x_2) = \delta_2^+ - \delta_2^-, \tag{5.20}$$

$$\delta_2^+ \geqslant 0, \quad \delta_2^- \geqslant 0,\ \ 0 \leqslant h \leqslant 1, \tag{5.21}$$

Eq. (5.15) can be replaced by the following:

$$v_2 - x_2 = -\delta_2^+ - \delta_2^-, \tag{5.22}$$

$$v_2 = \tfrac{1}{15}(\alpha_{21} + 2\alpha_{22} + 4\alpha_{23} + 8\alpha_{24}), \tag{5.23}$$

$$h = \tfrac{1}{15}(\theta_{21} + 2\theta_{22} + 4\theta_{23} + 8\theta_{24}), \tag{5.24}$$

$$M(\theta_{2j} - 1) + x_2 \leqslant \alpha_{2j} \leqslant 1 - \theta_{2j} + x_2,\ \ 0 \leqslant \alpha_{2j} \leqslant M\theta_{2j}, \quad \alpha_{2j} \leqslant x_2, \quad j = 1,2,3,4, \tag{5.25}$$

where θ_{2j} are 0–1 variables, α_{2j} are continuous variables, and M is the upper bound of $\max\{x_1, x_2\}$.

Step 5: Reformulate Example 3 as following linear mixed 0–1 program:

$$\begin{array}{ll} \text{Maximize} & \lambda \\ \text{subject to} & \lambda \leqslant \mu_1(z_1) = 0.08z_1 + 0.04, \quad \lambda \leqslant \mu_2(z_2) = 0.1z_2 - 1.1, \quad \lambda \leqslant wh, \\ & z_2 = 2x_1 + x_2, \quad -x_1 + 3x_2 \leqslant 21, \quad x_1 + 3x_2 \leqslant 27, \quad 3x_1 + x_2 \leqslant 30, \\ & (5.12)\text{–}(5.18),\ (5.19)\text{–}(5.25), \quad x_1, x_2 \geqslant 0 \end{array}$$

Solve the program by LINDO [13], the obtained solution is listed in Table 6. From Table 6, the decision maker could choose a suitable w which is a compromise value between the degree of preference for membership functions and the degree of possibility for possibility distribution.

6. Concluding remarks

Two algorithms are proposed for treating FMP problems with quasiconcave functions. Algorithm 1 is applied to treat the programs with crisp coefficients and Algorithm 2 could solve the programs with fuzzy coefficients. By comparing with current FMP models [4, 11, 16] three advantages of Algorithm 1 are: first, it uses a more convenient way to express quasiconcave functions. Second, it could directly solve the quasiconcave FMP problem without adding zero–one variables, dividing the problem into several subproblems, or converting the original membership functions into new functions. Third, it could be extended to treat a quasiconcave FMP problem with fuzzy coefficients. By comparing with current models [6, 14, 15] of treating fuzzy coefficients, the advantage of Algorithm 2 is that it could find an approximate solution closing to a global optimum.

References

[1] M.P. Biswal, Use of projective and scaling algorithm to solve multi-objective fuzzy linear programming problems, J. Fuzzy Math. 5 (1997) 439–448.
[2] E.L. Hannan, Linear programming with multiple fuzzy goals, Fuzzy Sets and Systems 6 (1981) 235–248.
[3] E.L. Hannan, On fuzzy goal programming, Decision Sci. 12 (1981) 522–531.
[4] M. Inuiguchi, H. Ichihashi, Y. Kume, A solution algorithm for fuzzy linear programming with piecewise linear membership functions, Fuzzy Sets and Systems 34 (1990) 15–31.
[5] Y.J. Lai, C.L. Hwang, Fuzzy Multiple Objective Decision Making, Springer, New York, 1994.
[6] Y. Leung, Compromise programming under fuzziness, Control Cybernet. 13 (1984) 203–215.
[7] H.L. Li, Technical note: an efficient method for solving linear goal programming problems, J. Optim. Theory Appl. 90 (1996) 465–469.
[8] H.L. Li, C.T. Chang, An approximately global optimization method for assortment problems, European J. Oper. Res. 105 (1998) 604–612.
[9] H.L. Li, C.S. Yu, Comments on "Fuzzy programming with nonlinear membership functions …", Fuzzy Sets and Systems 101 (1999) 109–113.
[10] K.M. Mjelde, Fractional resource allocation with S-shaped return functions, J. Oper. Res. Soc. 34 (1983) 627–632.
[11] K. Nakamura, Some extensions of fuzzy linear programming, Fuzzy Sets and Systems 14 (1984) 211–229.
[12] R. Narasimhan, Goal programming in a fuzzy environment, Decision Sci. 11 (1980) 325–336.
[13] L. Schrage, LINDO Release 6.0, LINDO System, INC., 1997.
[14] H. Tanaka, K. Asai, Fuzzy solution in fuzzy linear programming problems, IEEE Trans. Systems Man Cybernet. 14 (1984) 325–328.
[15] H. Tanaka, K. Asai, Fuzzy linear programming problems with fuzzy numbers, Fuzzy Sets and Systems 13 (1984) 1–10.
[16] T. Yang, J.P. Ignizio, H.J. Kim, Fuzzy programming with nonlinear membership functions: Piecewise linear approximation, Fuzzy Sets and Systems 41 (1991) 39–53.
[17] H.-J. Zimmermann, Fuzzy programming and linear programming with several objective functions, Fuzzy Sets and Systems 1 (1991) 45–55.