# Existence of Invariant Series Consecutive-$k$-out-of-$n$:G Systems

**Hsun-Wen Chang**
National Chiao Tung University, Hsinchu

**F. K. Hwang**
National Chiao Tung University, Hsinchu

**Key Words** — Consecutive-$k$-out-of-$n$:G system, Series system, Invariant assignment.

**Summary & Conclusions** — A consecutive-$k$-out-of-$n$:G line has an invariant optimal assignment iff $k \le n \le 2k$. This paper —

- studies a system consisting of consecutive-$k_i$-out-of-$n_i$:G lines with $k_i \le n_i \le 2k_i$,
- completely characterizes the existence of an invariant optimal assignment.

## 1. INTRODUCTION

*Notation*

- $n$ number of components/positions in a system
- $p_i$ reliability of component $i$
- $P$ $\{p_1, p_2, \cdots, p_n\}$
- $p_{[i]}$ the $i^{\text{th}}$ smallest reliability in $P$
- $R$ reliability function
- $\bar{\phi}$ $1-\phi$, $\phi$ is any item
- $\text{con}(k/n:\text{G})$ consecutive-$k$-out-of-$n$:G line
- $\text{con}(k/n:\text{F})$ consecutive-$k$-out-of-$n$:F system
- $S(k_1^{\alpha_1}, k_2^{\alpha_2}, \ldots, k_m^{\alpha_m})$, $k_1 < k_2 < \ldots < k_m$, $\alpha_i \ge 1$ for $1 \le i \le m$: a series system with $\Sigma_{i=1}^{m}\alpha_i$ subsystems consisting of $\alpha_i$ $\text{con}(k_i/2k_i:\text{G})$, $i = 1, \ldots, m$
- $S(k_1/n_1, \ldots, k_m/n_m)$: a series system with $m$ subsystems; $k_i/n_i \equiv \text{con}(k_i/n_i:\text{G})$

*Definitions*

- $\text{con}(k/n:\text{G})$: a system of $n$ components arranged into a line such that the system works iff some consecutive $k$ components of its $n$ positions all work.
- $\text{con}(k/n:\text{F})$: the system fails iff some consecutive $k$ components of its $n$ positions all fail.

The $\text{con}(k/n:\text{G})$ is the counterpart of the more well-known $\text{con}(k/n:\text{F})$, but has its own applications. The $\text{con}(k/n:\text{G})$ was first mentioned in [4], but has attracted more attention since the publication of [2]. ◁

Consider a set of $n$ components with reliabilities:

$$p_{[1]} \le p_{[2]} \le \cdots \le p_{[n]}.$$

The problem is to assign the $n$ components to the $n$ positions in a $\text{con}(k/n:\text{G})$ to maximize its reliability. An optimal assignment is *invariant* if it depends only on the ranks of the $p_i$ but not on their values. Thus an invariant assignment can be applied to a wide range of component reliabilities, even when the available information on these values is inexact. A system is invariant if there exists an invariant assignment.

Unfortunately, [5] proved the nonexistence of an invariant assignment for $2 \le k < n/2$. Therefore our attention is focused on the range $n/2 \le k \le n$. Ref [2] proved that for $n = 2k$, the invariant assignment is

$$p_{[1]}, p_{[3]}, \cdots, p_{[2k-1]}, p_{[2k]}, p_{[2k-2]}, \cdots, p_{[2]}.$$

For $n < 2k$, simply delete those $p_{[i]}$ with $i > n$.
However, [1] observed that the proof in [5] was not rigorous, and then gave a completely different proof.

Let,
$S(k_1^{\alpha_1}, k_2^{\alpha_2}, \ldots, k_m^{\alpha_m})$, $k_1 < k_2 < \ldots < k_m$,
$\alpha_i \ge 1$ for $1 \le i \le m$,

denote a series system with $\sum_{i=1}^{m} \alpha_i$ subsystems consisting of $\alpha_i$ $\text{con}(k_i/2k_i:\text{G})$, $i = 1, \ldots, m$. It is a series system since $S$ works iff all its subsystems work. Ref [3] proved that $S(k^\alpha, (k+1)^{\alpha'})$ is invariant for any $k \ge 1$, $\alpha \ge 0$, and $\alpha' \ge 0$. A reasonable question arises about the existence of other invariant $S(k_1^{\alpha_1}, \ldots, k_m^{\alpha_m})$; this paper shows that there is none. $S$ is allowed to contain $\text{con}(k/n:\text{G})$ with $k \le n \le 2k$; then the existence of invariant assignment is completely characterized.

## 2. MAIN RESULTS

### 2.1 Two Lines

*Lemma* 1. If $\text{con}(k/2k:\text{G})$ contains a component with reliability 1, then it is reduced to $\text{con}((k-1)/2(k-1):\text{G})$.

*Proof:* Without loss of generality, let $p_k = 1$. Then the system works iff the sequence
$(p_1, p_2, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{2k-1})$
contains $k-1$ consecutive working components. ◁

*Lemma* 2. Consider $S(k, k')$ with $k < k'$. In an invariant assignment, both the best and the worst components go to the $2k'$-line.

*Proof:* Let $P$ be the set of reliabilities
$\{p_{[1]} = 0, p_{[2]} = p_{[3]} = \ldots = p_{[2k+2k'-1]} = \epsilon, p_{[2k+2k']} = 1\}$, where $\epsilon \to 0$. Compare two assignments:

$A_1$: $p_{[2k+2k']}$ goes to the $2k$-line,

$A_2$: $p_{[2k+2k']}$ goes to the $2k'$-line.

Since $\epsilon \gg \epsilon^2$, it suffices to compare $A_1$ and $A_2$, conditional on the event that exactly $k + k' - 1$ $\epsilon$-components work. By lemma 1, $S(k, k')$ reduces to

$S(k-1, k')$ under $A_1$,

$S(k, k'-1)$ under $A_2$.

The conditional probabilities that the system works under $A_1$ and $A_2$ are, respectively,

$k(k'+1)\epsilon^{k+k'-1}$;

$(k+1)k'\epsilon^{k+k'-1}$.

The latter is larger since $k < k'$.

The reason that the worst element also goes to the $2k'$-line is that the worst component gets lost during the reduction, *ie*, the system reliability does not depend on that component. So it must be optimal to put the worst component in that position. ◀

*Corollary* 1. Consider $S(k, k')$ with $k < k'$. In an invariant assignment the $k' - k$ best and worst components all go to the $2k'$-line.

*Proof:* The proof is obvious. ◀

*Theorem* 1. $S(k, k+2)$ is not invariant.

*Proof:* By corollary 1, it suffices to identify a set $P$ of $4k+4$ reliabilities such that assigning the 3 worst components to the $(k+2)$-line is not optimal. Let

· $P = \{0 < p_{[1]} = p_{[2]} = p_{[3]} = p < p_{[4]} = \ldots = p_{[4k+4]} = r < 1\}$.

· $A$ be the assignment that $p_{[1]}$, $p_{[2]}$, $p_{[3]}$ all go to the $(k+2)$-line;

· $A'$ be the assignment that $p_{[1]}$ and $p_{[2]}$, but not $p_{[3]}$, go to the $(k+2)$-line.

By adding the probabilities that $p_i, p_{i+1}, \ldots, p_{i+k-1}$ is the first set of $k$ consecutive working components over $i = 1, 2, \ldots, k+1$; then —

Under $A$:

$$R(2k\text{–line}) = r^k + k \cdot \bar{r} \cdot r^k,$$

$$R((2k+4)\text{–line}) = p^2 \cdot r^k + \bar{p} \cdot p \cdot r^{k+1} + \bar{p} \cdot r^{k+2} + (k-1) \cdot \bar{r} \cdot r^{k+2} + \bar{r} \cdot r^{k+1} \cdot p;$$

Under $A'$:

$$R(2k\text{–line}) = p \cdot r^{k-1} + \bar{p} \cdot r^k + (k-1) \cdot \bar{r} \cdot r^k,$$

$$R((2k+4)\text{–line}) = p \cdot r^{k+1} + \bar{p} \cdot r^{k+2} + k \cdot \bar{r} \cdot r^{k+2} + \bar{r} \cdot r^{k+1} \cdot p.$$

Thus

$$R(A') - R(A) = r^{2k} \cdot \Big[ [p + \bar{p} \cdot r + (k-1) \cdot \bar{r} \cdot r] \cdot [p + \bar{p} \cdot r + k \cdot \bar{r} \cdot r + \bar{r} \cdot p] - [1 + k \cdot \bar{r}] \cdot [p^2 + \bar{p} \cdot p \cdot r + \bar{p} \cdot r^2 + (k-1) \cdot \bar{r} \cdot r^2 + \bar{r} \cdot r \cdot p] \Big]$$
$$= r^{2k} \cdot p \cdot \bar{r} \cdot (r - p) \cdot [(k-1) \cdot \bar{r} + r] > 0. \quad \triangleleft$$

*Corollary* 2. $S(k, k')$ is not invariant for $k' \geq k+2$.

*Proof:* True for $k' = k+2$. Prove the general $k'$ case by induction. By the inductive hypothesis, there exist two sets $P$ and $P'$ of $2(k + k' - 1)$ reliabilities with different optimal assignments. Consider the two sets of $2(k + k')$ reliabilities $P \cup \{0, 1\}$ and $P' \cup \{0, 1\}$. By lemma 2, the two components with reliabilities 0 and 1 go to the $2k'$-line. By lemma 1, the $S(k, k')$ problem is reduced to the $S(k, k'-1)$ problem on $P$ and $P'$, which does not have an invariant assignment. ◁

2.2 General Case

*Theorem* 2. $S(k_1^{\alpha_1}, \ldots, k_m^{\alpha_m})$ is not invariant if $k_m \geq k_1 + 2$.

*Proof:* Suppose to the contrary that there exists an invariant assignment $A$ for $S(k_1^{\alpha_1}, \ldots, k_m^{\alpha_m})$. Let

$L_1$ denote a $k_1$-line,

$L_m$ a $k_m$-line in $S$. Define:

$n \equiv 2\Sigma_{i=1}^{m} \alpha_i \cdot k_i$

$n' \equiv 2(k_1 + k_m)$.

Then $S(k_1^{\alpha_1}, \ldots, k_m^{\alpha_m})$ has $n$ components and $L_1 \cup L_m$ has $n'$ components. Let

$r \equiv \{r_1 < r_2 < \ldots < r_{n'}\}$ denote the ranks of the components of $L_1 \cup L_m$ among the $n$ components. Then $r$ induces a set of ranks $r' = \{1, \ldots, n'\}$ of components of $L_1 \cup L_m$ among themselves. Consequently, $A$ induces an assignment $A'$ of $r'$ to $L_1 \cup L_m$. By corollary 2, there exists a set $P'$ of $n'$ reliabilities for which $A'$ is not optimal, say, $B'$ is optimal. Extend $P'$ to $P$ with $n$ reliabilities by setting

$p_{r_i}$ of $P = p_{[i]}$ of $P'$

and filling in $p_{[j]}$ of $R$, $r_i < j < r_{i+1}$, with arbitrary numbers as long as they are consistent with the ranking. Let $B$ be an assignment of $R$ to $S(k_1^{\alpha_1}, \ldots, k_m^{\alpha_m})$ which differs from $A$ only by replacing $A'$ with $B'$ on the assignment of $L_1 \cup L_m$. Then clearly,

$R_B(S) > R_A(S)$,

contradicting the assumption that $A$ is invariant. ◁

Finally, generalize $S$ to the case that it can contain con($k/n$ : G) for $k \leq n \leq 2k$. Let $S(k_1/n_1, \ldots, k_m/n_m)$ denote such a system, where $k_i/n_i$ is con($k_i/n_i$ : G). Ref [3] showed that for $k \leq n \leq 2k$, con($k/n$ : G) works iff the middle $2k - n$ components all work and the remaining $2(n-k)$ components constitute a working

con($(n-k)/2(n-k)$ : G).

The $\sum_{i=1}^{m}(2k_i - n_i)$ largest reliabilities should be assigned to the middle positions, $2k_i - n_i$ of them to line $i$. The exact mapping of these reliabilities to the middle positions is irrelevant since the system fails if any one of them fails. Consider two assignments as equivalent if they differ only in positions which must work. Delete these middle positions; then $S(k_1/n_1, \ldots, k_m/n_m)$ is reduced to $S(n_1 - k_1, \ldots, n_m - k_m)$. Thus:

*Theorem* 3. $S(k_1/n_1, \ldots, k_m/n_m)$ is invariant (up to equivalence) iff $S(n_1 - k_1, \ldots, n_m - k_m)$ is invariant.

*Proof:* The proof is obvious.

2.3 Example

Assign 15 reliabilities, $p_1 < p_2 < \ldots < p_{15}$, to $S(1/1, 1/2, 2/3, 2/4, 3/5)$.

Then two equivalent invariant assignments are:

$$p_{13} \quad p_5 - p_8 \quad p_6 - p_{15} - p_7 \quad p_1 - p_9 - p_{12} - p_4$$
$$p_2 - p_{10} - p_{14} - p_{11} - p_3,$$
$$p_{15} \quad p_6 - p_7 \quad p_5 - p_{14} - p_8 \quad p_2 - p_{10} - p_{11} - p_3$$
$$p_1 - p_9 - p_{13} - p_{12} - p_4.$$

## ACKNOWLEDGMENT

This research was partially supported by NSC grant 88-2115-M-009-016.

## REFERENCES

[1] A. Jalali, A.G. Hawkes, L. Cui, F.K. Hwang, "The optimal consecutive-$k$-out-of-$n$:G line for $n \leq 2k$", submitted to *Naval Research Quarterly.*

[2] W. Kuo, W. Zhang, M. Zuo, "A consecutive-$k$-out-of-$n$:G system: The mirror image of a consecutive-$k$-out-of-$n$:F system", *IEEE Trans. Reliability,* vol 39, 1990 Jun, pp 244 – 253.

[3] J. Shen, M.J. Zuo, "Optimal design of series consecutive-$k$-out-of-$n$:G systems", *Reliability Engineering and System Safety,* vol 45, 1990, pp 277 – 283.

[4] Y.L. Tong, "A rearrangement inequality for the longest run with an application to network reliability", *J. Applied Probability,* vol 22, 1986, pp 386 – 393.

[5] M. Zuo, W. Kuo, "Design and performance analysis of consecutive-$k$-out-of-$n$ structure", *Naval Research Logistics,* vol 37, 1990, pp 203 – 230.

## AUTHORS

Hsun-Wen Chang; Dep't of Computer Science & Information Eng'g; National Chiao Tung Univ; Hsinchu 300-10 TAIWAN - R.O.C.

*Internet (e-mail):* hwchang@csie.nctu.edu.tw

**Hsun-Wen Chang** received her BS (1988) in Industrial Design from Tatung Institute of Technology (Taipei) and MS (1992) in Mathematical Sciences from National Chengchi University (Taipei). She is a PhD candidate in the Department of Computer Science and Information Engineering, National Chiao Tung University. Her research interests include reliability, group testing, interconnection networks, and parallel processing.

F.K. Hwang; Dep't of Applied Mathematics; National Chiao Tung Univ; Hsinchu 300-10 TAIWAN - R.O.C.

*Internet (e-mail):* fhwang@math.nctu.edu.tw

**F. K. Hwang** worked at the Mathematics and Statistics Center, Bell Laboratories from 1967 – 1996. He is now a University Chair–Professor at the Department of Applied Mathematics, National Chiao Tung University, Hsinchu. He has published books on Steiner Trees, Group Testing, and Switching Networks. He is working on books on the consecutive-$k$ systems and on optimal partitions.

Manuscript TR1999-069 received: 1999 June 5.
Responsible editor: W. Kuo
Publisher Item Identifier S 0018-9529(99)08987-3