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Bifurcation of fundamental Gaussian modes in Kerr-lens mode-locked lasers

Ming-Dar Wei, Wen-Feng Hsieh *

Institute of Electro-Optical Engineering, National Chiao-Tung University, Ta-Hsueh Rd. 1001, Hsinchu 30050, Taiwan

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Abstract

We propose that more than one spatial fundamental Gaussian mode may occur in concentric and confocal resonators under persisting nonlinear effect. Extensively studying the influence of the resonator's parameters in a Kerr-lens mode-locked resonator, pitchfork bifurcation results in more symmetrical configurations, and saddle-node bifurcation appears as the symmetry being broken. From the properties of bifurcation, we suggest that the equal-arm and near-confocal resonator is suitable for the emergence of bistability in Kerr-lens mode-locked lasers. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Bifurcation; Kerr effect and resonator

1. Introduction

Recently, bistability in Kerr-lens mode-locking (KLM) resonators has been studied by many researchers since the KLM laser was built [1–3]. Bistability was first predicted in a near-concentric unstable KLM resonator by considering saturable Gaussian gain and Kerr nonlinearity when the pumping rate is modulated about the threshold for laser operation [1]. When both spatial and temporal effects are simultaneously taken into account, the S-shaped bistable behavior of spot size, pulse width and pulse energy with varying pump power was found at a specific near-confocal configuration [2]. In these two papers, bistability results mainly from the mixing effect of absorptive (gain saturation) and dispersive (optical Kerr effect) nonlinearity. Excluding gain effect, bistable behavior was also studied in Ref. [3]. By applying the reduced self-focusing ABCD matrix for a Kerr medium and considering the nonlinear coupling of two transversal directions in the elliptical beam, they found that more than one TEM_{00} resonator mode exists around concentric configuration. Summarizing the previous results, multiple Gaussian modes are numerically obtained in some specific configurations with the different source of nonlinearity. However, why these configurations are sensitive to nonlinear effects and whether this bistable character is configuration-dependent is not known. In this paper, we show that the bistability depends upon resonator configuration, and we propose a reasonable illumination from studying the dynamics of Gaussian beam propagation.

In previous research, the iterative map was used to study Gaussian beam propagation in a general

* Corresponding author. Tel.: +886-3-5712121 ext 56316; fax: +886-3-5716631; e-mail: wfhsieh@cc.nctu.edu.tw

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resonator [4]. Analyzing the map with the help of Greene's residue theorem [5], we propose that bifurcation can easily occur around the confocal and concentric configurations whilst keeping a nonlinear perturbation. This result will be numerically verified in KLM resonators. For the sake of discussing the bifurcation of configuration dependence, we will only focus on the Gaussian beam propagation in a cold KLM resonator having pure self-focusing. The multiple fundamental Gaussian modes indeed exist in the above-mentioned configurations, even just considering the self-focusing. Moreover, we extensively explored the bifurcation behaviors under the influence of the resonator's parameters, such as crystal position and astigmatism angle of curved mirrors in equal- and unequal-arm resonators. These resonators have the different classifications of bifurcation that mainly result from the degree of symmetry. These results offer a useful analysis for bistability resonator design in KLM experiment, because self-focusing governs the character of the KLM resonator.

2. Theoretical prediction

In the paraxial approximation, the fundamental propagation of Gaussian beam follows the ABCD law for the q parameter. The q parameter of Gaussian beam is defined as $1/q = 1/R - \mathrm{i}\lambda/(\pi w_0^2)$, where R is the radius of curvature, w_0 the spot size and λ the wavelength of the beam. After using the q parameter to construct the iterative map, the character of fundamental resonator mode can be determined from the behavior of the map at the period-1 fixed point [4]. Analyzing the stability of the fixed point with the Greene's residue theorem, we obtained the residue in a linear system as

$$\mathrm{Re}s = 1 - (2G_1G_2 - 1)^2. \tag{1}$$

Here we have defined $G_1 = a - b/\rho_1$ and $G_2 = d - b/\rho_2$ as the G-parameters for general optical resonators, where ρ_1 and ρ_2 are their radii of curvature for the two end mirrors and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the transfer matrix of one-way pass between these two end mirrors. If a fixed point of a map without multiplier $+1$ is isolated, there are no other fixed points within its proximity [5]. In contrast, another periodic orbit could be created or destroyed when the fixed point has a multiplier $+1$. Such saddle-node or pitchfork bifurcation may occur under persisting nonlinear perturbation. Since the system with residue equal to zero has multiplier $+1$ [5], the confocal $(G_1G_2 = 0)$ or concentric $(G_1G_2 = 1)$ configuration could have the above-mentioned bifurcation with the help of Eq. (1). Thus, a general resonator at these configurations has the intrinsic character that it may have multiple Gaussian resonator modes under the nonlinear effect.

Based on the previous discussion, we will analyze the properties of bifurcation in a KLM resonator. The nonlinear self-focusing of Gaussian beam in the Kerr medium can be described by using the renormalized q parameter [6]. After renormalizing the q parameter to be

$$\frac{1}{q'} = \mathrm{Re}\left[\frac{1}{q}\right] - j\,\mathrm{Im}\left[\frac{1}{q}\right]\sqrt{1-K}, \tag{2}$$

q' will follow the free-space propagation in the Kerr medium. Here the Kerr parameter K is the cavity beam power over the critical power of self-trapping and the Re and Im represent the real and imaginary parts of a complex number. From self-consistency of the q parameters in a resonator, a simple analytical approach was proposed to design the four-mirror folded KLM laser [7]. The four-mirror folded KLM resonator, shown in Fig. 1, consists of two flat end mirrors (M_1 and M_4) and a pair of curved mirrors (M_2 and M_3) of focal length f. The distance between M_1 and M_2 is d_1, between M_3 and M_4 is d_2, between M_2 and the end face I of the Kerr medium

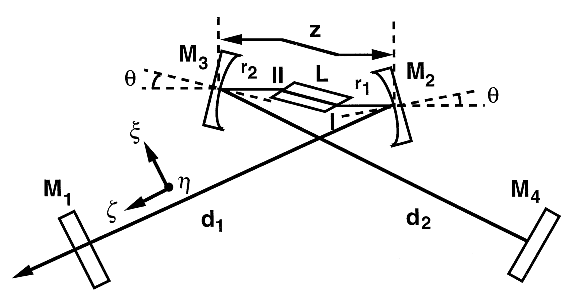

Fig. 1. The KLM resonator. The four-mirror folded KLM resonator consists of two flat end mirrors (M_1 and M_4) and a pair of curved mirrors (M_2 and M_3) with the same focal length f. The distance between M_1 and M_2 is d_1, M_3 and M_4 is d_2, M_2 and the end face I of Kerr medium is r_1 and two curved mirrors is z, respectively.

is r_1 and between two curved mirrors is z, respectively. Since the curvature of Gaussian resonator mode must match those of the end mirrors in lossless system, the curvature of parameter q at output flat mirror M_1 is zero. Thus, we can assume that the q parameter at M_1 is $q_1 = jy = j\pi w^2/\lambda$. Using the renormalized q parameter concept to transform q_1 through the Kerr medium and the self-consistence at end face II, one can obtain the spot size of fundamental mode at M_1 to satisfy a quartic equation of y^2 as [7]

$$h(w, K, \boldsymbol{\Gamma}) = a_4(y^2)^4 + a_3(y^2)^3 + a_2(y^2)^2 + a_1(y^2)a_0 = 0. \quad (3)$$

Here $\boldsymbol{\Gamma}$ represents a configuration variable that depends on d_1, d_2, r_1, r_2, z, f and the Kerr medium length L. The coefficients a_4, a_3, a_2, a_1, and a_0 are functions of K and $\boldsymbol{\Gamma}$. Owing to a quartic equation having analytic solutions, one can extensively study the bifurcation behavior of spatial fundamental Gaussian mode in the KLM resonator with various configurations $\boldsymbol{\Gamma}$s and Ks. This approach is used in the following numerical calculation.

3. Numerical results

We will concentrate initially on the symmetric confocal configuration in the KLM laser. This resonator has equal arms with $d_1 = d_2 = 850$ mm and the crystal is placed at the center of two curved mirrors with $r_1 = r_2$. The parameters of the optical elements referred to the experimental ones [8] in which the radii of curvature of the curved mirrors M_2 and M_3 are both 100 mm and the length of Brewster-cut Ti:sapphire rod is $L = 20$ mm. The curved mirrors have been tilted by an angle θ for the astigmatism compensation about Brewster-cut laser rod. Thus, the resonator could be divided into two astigmatic optical systems associated with the sagittal and tangential planes. These two planes are considered to be orthogonal. Because similar behavior can also be found in both planes, we focused the following numerical simulation on the sagittal plane. Since the beam's q-parameter contains only the spot size with zero curvature at the output mirror M_1, the spot size is chosen as the scalar measure in the numerical simulation. The Kerr parameter offers the nonlinear effect to be the bifurcation parameter.

Fig. 2(a) shows spot size at M_1 versus Kerr parameter for $z = 115.3$ mm, $r_1 = r_2 = 47.65$ mm and $\theta = 14.5°$ in the symmetrical stable resonator. The solid line in this figure indicates the stable solutions of self-consistent Gaussian beam and the dashed line represents the unstable one. The configuration is near confocal, since the confocal configuration for linear (cw) resonator corresponds to a separation of the curved mirrors $z_c = 115.267$ mm; therefore, as predicted in Section 2, the bifurcation appears as a pitchfork bifurcation with the bifurcation point at $(K_b, w_b) = (0.04278, 0.4513$ mm$)$, where K_b and w_b represent the critical bifurcation Kerr parameter and spot size, respectively. Furthermore, we numerically obtain $h = h_w = h_{ww} = h_K = 0$ at the bifurcation point, where the subscripts of

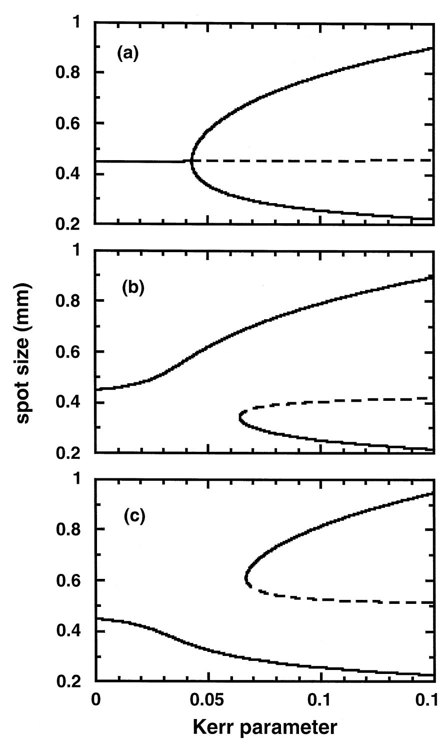

Fig. 2. Pitchfork bifurcation. The spot size at M_1 versus Kerr parameter in equal-arm resonator with $d_1 = d_2 = 85$ cm is shown for $z = 115.3$ mm and (a) $r_1 = 47.65$ mm, (b) $r_1 = 46.5$ mm and (c) $r_1 = 49$ mm. The solid line corresponds to the stable solutions of self-consistent Gaussian beam and the dashed line represents the unstable one.

function $h(w,K,\boldsymbol{\Gamma})$ represent the various order partial derivatives with respect to the variables indicated as subscripts. According to the singularity theory [9], this result also verifies that the classification belongs to the pitchfork bifurcation.

When the crystal is located away from the center of two curved mirrors, the characteristics of bifurcation will be changed. By constraining $z = 115.3$ mm, the bifurcation diagram with $r_1 = 46.5$ mm is shown in Fig. 2(b), note that $r_1 = 47.65$ mm for symmetry resonator. It is no longer a standard pitchfork bifurcation but a perturbed one. As pitchfork bifurcation is not generic, it usually results from some peculiar symmetry or the inadequacy of the idealization, in which some small effects are neglected [9]. Therefore, moving the crystal away from the center of two curved mirrors had broken the symmetry and induced the perturbed variation of bifurcation. In Fig. 2(b), the upper branch of this typical perturbed bifurcation corresponds to the continuous evolution branch from before to after bifurcation. On the contrary, the other typical perturbed pitchfork bifurcation takes place at r_1 greater than 47.65 mm. For example, $r_1 = 49$ mm in Fig. 2(c), the lower branch in this type is a continuous evolution one.

When a slit is inserted at M_1 in a hard aperturing KLM resonator, the KLM strength δ is determined by the rate of change of the spot size as increasing the laser power [7,10]. Experimentally, for achieving the larger self-amplitude modulation, in general the slit is inserted vertical to the tangential plane, which has the larger KLM strength. If the configuration has the pitchfork bifurcation in the sagittal plane with $\delta < 0$ in the tangential plane, it is suitable for observing the bistability in hard-aperturing KLM laser. Under well-matching astigmatism, δ has the same sign in both planes, thus the configuration with the type $r_1 = 49$ mm in Fig. 2(c) prefers to operate at KLM due to $\delta < 0$ in both planes. However, the configuration with the type $r_1 = 46.5$ mm in Fig. 2(b) will not be a preferred KLM one.

Further calculating the bifurcation points by varying the configuration-parameters z and r_1, we obtain the contour of critical bifurcation parameter K_b, shown in Fig. 3 with progressively increasing K_b from curves (a) to (e). The dashed line represents the symmetrical configuration that the crystal is located at the center of two curved mirrors. Clearly, the K_b increases as the crystal is located away from the center of two curved mirrors or as the configuration far from the confocal structure. Similar behavior can also be found in the tangential plane, but the bifurcation region will reduce to less than half of that in the sagittal plane. The smaller region results from shorter effective rod length and effective focal length of the curved mirrors in the tangential plane. The bifurcation region on the right hand side of dashed line belongs to the perturbed pitchfork bifurcation having the type of Fig. 2(c). Compared with the preferable KLM region in the experiments in Ref. [10] and the theory in Ref. [11], we find that the bifurcation region is located within the preferred KLM region though much smaller than the preferred one. Thus, we believe that bistability may be observed in such a bifurcation region.

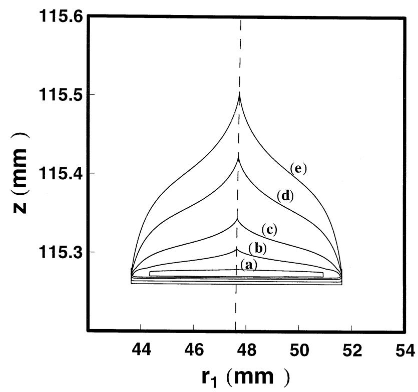

Fig. 3. Contour of the critical bifurcation parameter in equal-arm resonator. The contour values are (a) $K_b = 0.01$, (b) $K_b = 0.05$, (c) $K_b = 0.1$, (d) $K_b = 0.2$, and (e) $K_b = 0.3$, respectively. The dashed line represents the crystal locating at the center of two curved mirrors in which corresponds to the symmetric configuration.

If we consider the asymmetric resonator with $d_1 \neq d_2$ but $d_1 + d_2 = 170$ cm, a different classification of bifurcation will take place. In Fig. 4 with $d_1 = 70$ cm, $d_2 = 100$ cm, $z = 114.74$ mm and $r_1 = 41$ mm, saddle-node bifurcation occurs at critical bifurcation parameter $K_b \approx 0.08993$. There is no real solution of spot size as $K < K_b$, and two self-consistent spot sizes exist as $K > K_b$. The lower branch with solid line indicates stable solutions and the upper branch with dashed line represents unstable solutions. When the configuration varies close to the confocal one, the K_b decreases. In general, pitchfork bifurcation will occur in more symmetrical configurations and saddle-node bifurcation exists as this symmetry being broken [5]. Therefore, we think the emergence of different classification mainly results from the configurations having the unequal arms, which have broken the symmetry of resonator.

However, the symmetry broken from the slightly unequal arms can be compensated by the tilted angle of the curved mirrors. The critical bifurcation parameter against the tilted angle is shown in Fig. 5 with $d_1 = 83$ cm, $d_2 = 87$ cm, $z = 115$ mm and $r_1 = 40$ mm which corresponds to near-confocal configuration. We constrain our discussions on $K_b < 0.4$ that the associated power can be obtained from general experiments. As $\theta < \theta_a = 13.7709°$, the configuration has only one real solution of spot size and no bifurcation. Increasing the tilted angle to $\theta > \theta_a$, we find the saddle-node bifurcation takes place. The upper branch corresponds to the stable solution and the lower branch corresponds to the unstable solution. Moreover, K_b increases as the tilted angle is increased, but the stability reverses and the K_b decreases as increasing the tilted angle to $\theta > \theta_b = 13.9155°$. When θ is greater than $\theta_c = 14.1207°$, the bifurcation transits to the perturbed pitchfork bifurcation and K_b still decreases as increasing θ. The minimum K_b is 0.04955 at $\theta_d = 14.1361°$ corresponding to the standard pitchfork bifurcation. The other type of perturbed pitchfork bifurcation exists as $\theta_d < \theta < \theta_e = 14.2186°$, in which K_b increases as increasing θ. The configuration returns to having one reasonable spot size as $\theta > \theta_e$ if one constrains $K_b < 0.4$. In fact, dynamical characteristics in this region still belongs to the pitchfork bifurcation for $K_b > 0.4$. Because such a K value is not easy to reach in general KLM lasers, the phenomenon of bifurcation may not be observed in experiments. It is worth noting that the above regions all have $\delta < 0$ in the tangential plane, i.e., these regions prefer KLM operation with hard aperture. Although the tilted angle can compensate the broken symmetry resulting from unequal arms, the latter one governs the emergence of bifurcation. If we constrain $d_1 + d_2 = 170$ cm and $d_1 \leq d_2$, the region of pitchfork bifurcation is about hundreds of μm for z translation in equal-arm resonator. The region quickly shrinks as increasing $|d_1 - d_2|$ and becomes less than 10 μm as $|d_1 - d_2| \geq 4$ cm, even having the tilted angle compensation.

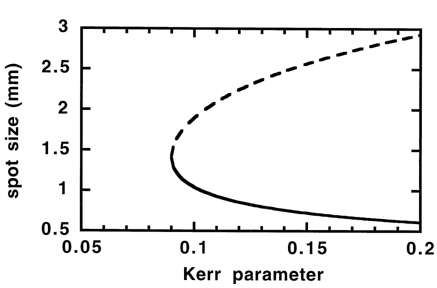

Fig. 4. Saddle-node bifurcation. The bifurcation diagram is shown as $d_1 = 70$ cm, $d_2 = 100$ cm, $z = 114.74$ mm and $r_1 = 41$ mm.

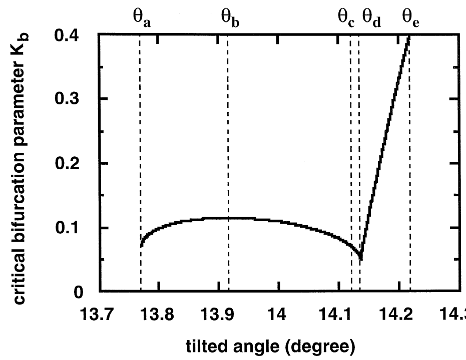

Fig. 5. The critical bifurcation parameter versus the tilted angle. It is shown the critical bifurcation parameter against the tilted angle with $d_1 = 83$ cm, $d_2 = 87$ cm, $z = 115$ mm and $r_1 = 40$ mm. The value of specific tilted angle are $\theta_a = 13.7709°$, $\theta_b = 13.9155°$, $\theta_c = 14.1207°$, $\theta_d = 14.1361°$ and $\theta_e = 14.2186°$. These tilted angles are transition point for the changing the character of bifurcation diagram.

4. Discussion

The bifurcation phenomenon can also be found in the near-concentric resonator. However, the region having bifurcation is smaller than that of near-confocal resonator, and its classification belongs to the saddle-node bifurcation. Since saddle-node bifurcation has only one stable solution, bistability will not be found in such configuration if we just consider the self-focusing effect. Moreover, no steady spot size in the range with $K < K_b$ represents that the resonator lacks a steady pulse generating mechanism from cw with $K = 0$ transiting to KLM with higher K. This system may not be spontaneous in KLM laser. On the contrary, bistability could be observed in the configuration with unperturbed or perturbed pitchfork bifurcation having two stable spot sizes in $K > K_b$ and a steady one in $K < K_b$; especially, the configurations having $\delta < 0$ in tangential plane is achievable to mode-locking operation with hard aperture. Thus, we suggest that the equal-arm and near-confocal resonator is suitable for the emergence of bistability in experiment due to the region with pitchfork bifurcation being large.

When we further consider the spatial and temporal effects in a KLM resonator, a simple quadratic equation is obtained to determine the pulse width from space–time analogy [12]. Owing to the spot size variation in Kerr medium couples to temporal self-phase modulation matrix, the bifurcation of spot size will result in the bifurcation of pulse width. We indeed found the same classification of bifurcation in pulse width versus Kerr parameter. It is also to be found in Ref. [2] that both spot size and pulse width appear to have the same bistability behavior. However, the S-shaped behavior in Ref. [2] is not found and instead of pitchfork bifurcation in this research, this difference may be attributed to the gain saturation effect.

5. Conclusion

Studying the iterative map constructed from the propagation of Gaussian q parameter in a resonator, we found that the confocal and concentric configuration corresponding to the map having multiplier $+1$. Such a map will induce bifurcation under nonlinear perturbation in general. This result gives a reasonable interpretation for the existence of multiple fundamental Gaussian modes, which often occur at the limit of stable region. When the numerical simulations contain only the self-focusing and ignore the gain saturation effect in KLM resonator, bistability takes place in the above-mentioned configurations. Under extensively studying the influence of different arm-lengths, crystal position and tilted angle, there are two main classification bifurcation corresponding to pitchfork and saddle-node ones. Pitchfork bifurcation occurs in configurations with higher symmetry and saddle-node bifurcation exists as this symmetry being broken. In addition, we suggest that the equal-arm and near-confocal resonator is suitable for the emergence of bistability in experiments.

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