A posteriori boundary element error estimation ☆

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Abstract

An a posteriori error estimator is presented for the boundary element method in a general framework. It is obtained by solving local residual problems for which a local concept is introduced to accommodate the fact that integral operators are nonlocal operators. The estimator is shown to have an upper and a lower bound by the constant multiples of the exact error in the energy norm for Symm's and hypersingular integral equations. Numerical results are also given to demonstrate the effectiveness of the estimator for these equations. It can be used for adaptive h, p, and hp methods. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Effective and efficient a posteriori error estimators play a key role in adaptive numerical methods for boundary value problems. We introduce an error estimator for the boundary element method (BEM) applied to boundary integral equations (BIEs).

The estimator is motivated by the weak residual a posteriori error estimation developed mainly for partial differential equations (PDEs) in connection with the finite element method (FEM). We refer to [19] for a general framework of the estimation and to [1–4,18–22] for further references on the application of such estimation in adaptive FEM and finite volume method (FVM). We find that the approach is even more natural for BIEs since the residual is inherently in integral form rather than differential form which entails specific treatments of the jumps in the normal derivatives of the finite element solutions on the interfaces between elements. In fact, the various error estimators for FEM differ essentially in the way the jumps are handled. This does not appear to be an issue in

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adaptive BEM; see [6–8,13,26,27]. Nevertheless, there is an intrinsic difference between boundary integral operators and differential operators; namely, differential operators are local operators whereas boundary integral operators are nonlocal [26]. Certain localization concepts such as the influence index of [26] and the augmented BEM of [13] have been introduced to accomodate this nonlocal property for the needed local and computable a posteriori error estimators.

Our approach is to construct local shape functions for the solution of local residual problems and then compute estimated errors in a localized energy norm which is induced by the diagonal (say, A) of the bilinear form (say, B) defined by the variational BIEs.

The error estimator is first presented in a general setting in Section 2 and then applied to Symm's and hypersingular integral equations in Section 3. We briefly describe the main results in this article. The estimator for a computed solution, say u_h, in some boundary element (BE) space $S_h \subset H$ is obtained by solving element-by-element local problems in a complementary BE space $S_h^c \subset H$. Here H denotes some Sobolev function space to which the exact solution u belongs. The local problems use the same Galerkin formulation of approximation except that the right side of these problems is a residual of the approximate solution. It is shown here that, for both Symm's and hypersingular integral equations in two space dimensions, the estimated error $\tilde{e} \in S_h^c$ satisfies the estimate

$$C_1 \leqslant \theta := \frac{\|\tilde{e}\|_A}{\|e\|_B} \leqslant C_2, \tag{1.1}$$

where C_1 and C_2 are positive constants independent of the mesh size h, $\|\cdot\|_A$ and $\|\cdot\|_B$ are the norms associated respectively with the bilinear forms A and B, $e = u - u_h$ is the exact error, and θ is called the effectivity index of the estimator.

A posteriori error estimates of the form as (1.1) are very important in practice since they are used to justify the effectiveness of the resulting adaptive scheme. While this is not comprehensive, we compare our estimator (denoted by JL-estimator) to the estimators of [6–8] (CES-estimator), of [13] (FHK-estimator), and of [26,27] (WY-estimator) which are all based on various local postprocessing schemes on the residual error instead of solving local problems. For a better view in comparison, we summarize the main results of all estimators in Table 1 in which, for simplicity, we restrict to the following conditions: Symm's and hypersingular BIEs in two space dimensions, unstructured mesh, and under the norms specified in the respective references.

The FHK-estimator is developed for the augmented Galerkin BEM described in [13] not for the standard Galerkin BEM. The augmented technique takes into account the behavior of the exact solution near points of singularity. Hence, one has to have an a priori information about the singularities. Moreover, all other estimators are obtained by postprocessing the residual error (e.g., differentiating

Table 1
Comparison of various estimators

Estimator	Symm's BIE	Hypersingular BIE
WY-estimator [26]		$C_1 \leqslant \theta$
CES-estimator [6]	$C_1 \leqslant \theta$	
FHK-estimator [13]		$C_1 \leqslant \theta \leqslant C_2$
JL-estimator	$C_1 \leqslant \theta \leqslant C_2$	$C_1 \leqslant \theta \leqslant C_2$

the residual) in some computable norm which often requires the data function (or equivalently the exact solution) to be smoother. Our approach, in contrast, does not incur the exact solution to be more regular than that is required by the standard a priori estimates. In other words, the estimator holds for the minimal regularity of the exact solution in the sense of Céa's lemma [10]. The only unproven assumption that we make for our error analysis is the saturation assumption. This assumption is very moderate and natural since it essentially says that the approximate solution in the larger BE space $S_h \oplus S_h^c$ is a better approximation to the solution u than $u_h \in S_h$. This is generally true in practice. If this assumption is replaced by other assumptions associated with higher regularity on the exact solution, one may be able to analyze the asymptotic exactness of the estimator such as that of [2] for FEM. We shall not consider this topic here.

Compared to the cost of computing the approximate solution u_h, the cost of computing the estimated error $\tilde{e}$ is fractional since the complementary BE space S_h^c can be constructed using, for instance, only one or two shape functions on each element. Consequently, we only have one or two equations in the solution of a local problem. In particular, if only one shape function is used for S_h^c, our estimator is then equivalent to the previous residual error estimators; see the numerical example for a hypersingular integral equation presented in Section 4. Furthermore, since the estimated error is explicitly calculated there is no restriction on the choice of the norm used to measure the errors. In other words, whichever the norm appropriate for the approximate solution u_h can also be used for the estimated error $\tilde{e}$. This can be very useful in practice when a more flexible norm is needed for assessing the computed solution. We however only prove estimate (1.1) in the energy norm.

2. General framework of the estimator

Let Ω be a bounded Lipschitz domain in R^n, $n = 2, 3$, with boundary $\hat{\Gamma}$. Let Γ be a closed or open connected subset of $\hat{\Gamma}$. Let H be a Sobolev space equipped with the norm $\|\cdot\|_H$ and be defined on Γ. Consider a general variational problem of the form: Find $u \in H$ such that

$$B(u, v) = F(v) \quad \forall v \in H. \tag{2.1}$$

Assumption 1. Let $F(\cdot)$ be a continuous linear functional on H and $B(\cdot, \cdot)$ be a symmetric bilinear form on $H \times H$ such that there exist two positive constants α_1 and β_1 for which

$$B(u, v) \leqslant \alpha_1 \|u\|_H \|v\|_H \quad \forall u, v \in H, \tag{2.2}$$

$$B(u, u) \geqslant \beta_1 \|u\|_H^2 \quad \forall u \in H \tag{2.3}$$

hold.

By Assumption 1, the energy norm $\|\cdot\|_B$ induced by the bilinear form $B(\cdot, \cdot)$ is equivalent to the norm $\|\cdot\|_H$ on H. Let S_h be a finite-dimensional subspace of H. The Galerkin approximation of (2.1) is to find $u_h \in S_h$ such that

$$B(u_h, v) = F(v) \quad \forall v \in S_h. \tag{2.4}$$

The Lax–Milgram theorem guarantees the existence and uniqueness of the solutions u and u_h of (2.1) and (2.4). Our main concern is to estimate the exact error $e = u - u_h$. Let $\mathcal{T}_h$ denote a finite partition of Γ associated with S_h. The partition $\mathcal{T}_h$ is expressed as

$$\mathcal{T}_h = \left\{ \Gamma_j : \ j = 1, 2, \ldots, N; \bigcup_{j=1}^{N} \bar{\Gamma}_j = \Gamma \right\}, \tag{2.5}$$

where the elements Γ_j are open disjoint intervals for $n = 2$ or triangles (or quadrangles or both) for $n = 3$ and $\bar{\Gamma}_j$ is the closure of Γ_j. The partition $\mathcal{T}_h$ is not necessarily quasi-uniform. More specifically, let σ_i denote the diameter of the inscribed circle for $\Gamma_i \in \mathcal{T}_h$ and let h_i denote the diameter of the element Γ_i. We assume there exists a positive constant δ independent of the mesh size h such that

$$\delta \leqslant \frac{\sigma_i}{h_i} \tag{2.6}$$

for all $\Gamma_i \in \mathcal{T}_h$.

Based on the current partition, we shall construct another finite-dimensional subspace S_h^c of H. We call S_h^c a complementary space of S_h. In order to obtain a practical and efficient error estimator, its construction is essential. For this, we require S_h^c to meet the following conditions.

Assumption 2. Assume the following conditions hold for S_h^c:

$$S_h^c := S_h^c(\Gamma_1) \oplus S_h^c(\Gamma_2) \oplus \cdots \oplus S_h^c(\Gamma_N), \tag{2.7}$$

$$S_{\bar{h}} := S_h \oplus S_h^c \subset H, \quad S_h \neq \emptyset, \ \ S_h^c \neq \emptyset, \tag{2.8}$$

$$\|u - u_{\bar{h}}\|_B \leqslant \rho \|u - u_h\|_B, \quad \rho \in [0, 1), \tag{2.9}$$

where $S_h^c(\Gamma_i)$, $i = 1, \ldots, N$, denote subspaces whose basis functions have supports only in their respective domain Γ_i, $u_{\bar{h}}$, is the approximate solution of (2.1) in the larger subspace $S_{\bar{h}}$, and ρ is a constant independent of the mesh size h.

We do not explicitly compute $u_{\bar{h}}$. It is merely for the analysis of the estimator. We now observe one of the most important properties that distinguishes boundary integral operators from differential operators; namely, differential operators are local operators whereas boundary integral operators are nonlocal. Translated into our setting, the difference is that

$$B(w_j, v_i) \neq 0 \quad \forall w_j \in S_h^c(\Gamma_j), \ \ v_i \in S_h^c(\Gamma_i), \tag{2.10}$$

for BIEs while it always vanishes for PDEs. To treat this nonlocal property which conflicts with the localization of a posteriori error estimation, we are led to consider an equivalent bilinear operator for $B(\cdot, \cdot)$. We define a new bilinear form $A(\cdot, \cdot)$ on $S_{\bar{h}} \times S_{\bar{h}}$ such that

$$A(w, v) = \sum_{i=1}^{N} B(w_i, v_i) \quad \forall w, \ v \in S_{\bar{h}}, \tag{2.11}$$

where

$$w_i(x) = \begin{cases} w(x) & \text{if } x \in \Gamma_i, \\ 0 & \text{otherwise} \end{cases} \tag{2.12}$$

and v_i are similarly defined for all $i = 1, \ldots, N$. Note that we can write $w = \sum_i w_i$ and $v = \sum_i v_i$ for all $w, v \in S_{\bar{h}}$. The bilinear form $A(\cdot, \cdot)$ is a block-diagonal form of $B(\cdot, \cdot)$ in $S_{\bar{h}}$.

Assumption 3. Let $A(\cdot, \cdot)$ be an inner product on $S_{\bar{h}} \times S_{\bar{h}}$ and let there exist two positive constants C_1 and C_2 (possibly depending on h) such that

$$C_1 \|w\|_A \leqslant \|w\|_B \quad \forall w \in S_{\bar{h}}, \tag{2.13}$$

$$\|w\|_B \leqslant C_2 \|w\|_A \quad \forall w \in S_h^c, \tag{2.14}$$

$$|A(w, v)| \leqslant \gamma \|w\|_A \|v\|_A, \quad \gamma \in [0, 1), \ \forall w \in S_h, \ \forall v \in S_h^c, \tag{2.15}$$

where the constant γ is independent of h and the norm $\|\cdot\|_A$ is induced by $A(\cdot, \cdot)$.

Theorem 1. *Let Assumptions 1–3 hold. Let $u \in H$ and $u_h \in S_h$ be the solutions of* (2.1) *and* (2.4), *respectively. Then there exist unique solutions $\tilde{e}_i \in S_h^c(\Gamma_i)$, for all $i = 1, \ldots, N$, such that*

$$B(\tilde{e}_i, v_i) = F(v_i) - B(u_h, v_i) \quad \forall v_i \in S_h^c(\Gamma_i). \tag{2.16}$$

Moreover, we have the estimate

$$C_1(1 - \rho)\sqrt{1 - \gamma^2} \|e\|_B \leqslant \|\tilde{e}\|_A \leqslant C_2 \|e\|_B, \tag{2.17}$$

where $e = u - u_h$ is the exact error, $\tilde{e} = \sum_i \tilde{e}_i$, and $C_1, C_2, \gamma \in [0, 1)$ and $\rho \in [0, 1)$ are constants given in Assumptions 2 *and* 3.

Proof. Assumption 1 ensures the uniqueness and existence of the solutions $\tilde{e}_i \in S_h^c(\Gamma_i)$, $i = 1, \ldots, N$, of (2.16) as well as the solutions $u_{\bar{h}}, \bar{e}$, and e satisfying, respectively,

$$B(u_{\bar{h}}, v) = F(v) \quad \forall v \in S_{\bar{h}}, \tag{2.18}$$

$$B(\bar{e}, v) = F(v) - B(u_h, v) \quad \forall v \in S_{\bar{h}}, \tag{2.19}$$

$$B(e, v) = F(v) - B(u_h, v) \quad \forall v \in H. \tag{2.20}$$

Eqs. (2.18) and (2.19) imply $\bar{e} = u_{\bar{h}} - u_h \in S_{\bar{h}}$. By (2.11), (2.14), (2.16), and (2.20) we have, for $\tilde{e} = \sum_i \tilde{e}_i \in S_h^c$,

$$\begin{aligned}
\|\tilde{e}\|_A^2 &= \sum_i B(\tilde{e}_i, \tilde{e}_i) \\
&= \sum_i B(e, \tilde{e}_i) \\
&= B(e, \tilde{e}) \\
&= \|e\|_B \|\tilde{e}\|_B \\
&\leqslant C_2 \|e\|_B \|\tilde{e}\|_A
\end{aligned}$$

and hence the right-hand side of (2.17) holds. Since

$$e - \bar{e} = (u - u_h) - (u_{\bar{h}} - u_h) = u - u_{\bar{h}},$$

we have, by (2.9) and a triangle inequality,

$$(1 - \rho)\|e\|_B \leqslant \|\bar{e}\|_B.$$

On the other hand, by (2.19) and (2.20), we have

$$\|\bar{e}\|_B^2 = B(\bar{e}, \bar{e}) = B(e, \bar{e}) \leqslant \|e\|_B \|\bar{e}\|_B.$$

Hence,

$$(1 - \rho)\|e\|_B \leqslant \|\bar{e}\|_B \leqslant \|e\|_B. \tag{2.21}$$

Let $\bar{e} = e_1 + e_2$ such that $e_1 \in S_h$ and $e_2 \in S_h^c$ where e_2 can also be written as $e_2 = \sum_i e_{2i}, e_{2i} \in S_h^c(\Gamma_i)$. Following (2.18),(2.4), (2.16) and (2.11), we have

$$\begin{aligned}
\|\bar{e}\|_B^2 &= B(\bar{e}, \bar{e}) \\
&= B(\bar{e}, e_1) + B(\bar{e}, e_2) \\
&= B(\bar{e}, e_2) \\
&= \sum_i B(\bar{e}, e_{2i}) \\
&= \sum_i B(\bar{e}_i, e_{2i}) \\
&= A(\tilde{e}, e_2) \\
&\leqslant \|\tilde{e}\|_A \|e_2\|_A.
\end{aligned}$$

On the other hand, by (2.15) and (2.13),

$$\begin{aligned}
\|\bar{e}\|_A^2 &= A(\bar{e}, \bar{e}) \\
&= A(e_1, e_1) + 2A(e_1, e_2) + A(e_2, e_2) \\
&\geqslant \|e_1\|_A^2 + \|e_2\|_A^2 - 2\gamma\|e_1\|_A \|e_2\|_A \\
&\geqslant (1 - \gamma^2)\|e_2\|_A^2
\end{aligned}$$

shows that

$$C_1 \sqrt{1 - \gamma^2} \|\bar{e}\|_B \leqslant \|\tilde{e}\|_A$$

which, together with (2.21), implies the left-hand side of (2.17). □

Note that (2.16) is a local problem since both trial and test functions all have the supports only in Γ_i. We therefore use

$$\|\tilde{e}_i\|_A = \|\tilde{e}_i\|_B = \sqrt{B(\tilde{e}_i, \tilde{e}_i)}$$

as an error indicator for each element Γ_i, $i = 1, \ldots, N$. Consequently, the error estimator for the approximate solution is defined by

$$\|\tilde{e}\|_A = \left(\sum_i \|\tilde{e}_i\|_A^2 \right)^{1/2}$$

and the effectivity index is defined by

$$\theta := \frac{\|\tilde{e}\|_A}{\|e\|_B}.$$

Although the error estimator can also be defined in the B norm, it is inefficient since we then have to have a global calculation for the norm due to (2.10).

Remark 1. The first paper using a formula similar to (2.16) for elliptic PDEs to develop an error estimator that we know of is by Adjerid and Flaherty in [1]. In [19], a general framework of this kind of error estimators is given for various types of variational problems in connection with FEM and FVM, while theoretical results are given in, e.g., [2,4,18,21,22]. Estimate (2.17) as well as its proof are slightly different from the previous works due to the fact of the nonlocal property (2.10). Inequality (2.9) is commonly used in these papers for error analysis. This saturation assumption is a very natural assumption since one expects that the approximate solution $u_{\bar{h}}$ is in general a better approximation to u than u_h. The assumption consequently yields a minimal regularity for the exact solution in H that is required to satisfy the optimal approximation for u_h in the sense of Céa's lemma [10]. If, in particular, $S_{\bar{h}}$ consists of polynomials with degrees higher than that of S_h, one can anticipate $\rho = \rho(h^r)$, $r > 0$, which then asymptotically results in a better estimator according to (2.17). If this assumption is replaced by other assumptions associated with higher regularity on the exact solution, asymptotic exactness of the estimator may be analyzed; see [2].

Remark 2. Let $\gamma = \sup\{B(w,v)|w \in S_h,\ \|w\|_B = 1,\ v \in S_h^c,\ \|v\|_B = 1\}$. Then $\gamma \leqslant 1$ and γ equals one exactly if w and v are linearly dependent. Thus $\gamma = 1$ would contradict the complementarity of S_h and S_h^c and the fact that both spaces are assumed to be nonempty. However, it is not clear that γ is independent of the mesh size h. It should be noted that our use of (2.15) is closely related to that of the strengthened Cauchy–Schwarz inequality widely used in the analysis of iterative methods based on hierarchical bases [12].

3. Model problems

While the weak residual error estimators have been extensively studied for PDEs, it lacks evidence that they have been investigated for BIEs. We now show that Symm's and hypersingular integral equations [6–8,13,14,16,23,25] indeed fit into the framework by verifying inequalities (2.13)–(2.15) for these model problems.

Let the connect subset $\Gamma \subseteq \hat{\Gamma}$ be such that $\Gamma = \hat{\Gamma}$ if Γ is closed and $\Gamma \neq \hat{\Gamma}$ if Γ is open. As in [17], we define the Sobolev spaces

$$H^t(\hat{\Gamma}) = \{u|_{\hat{\Gamma}}: u \in H^{t+1/2}(R^n)\}, \quad t > 0,$$
$$H^0(\hat{\Gamma}) = L^2(\hat{\Gamma})$$

and $H^{-t}(\hat{\Gamma})$, $t \geqslant 0$, is the dual space of $H^t(\hat{\Gamma})$ with respect to the duality $\langle \cdot, \cdot \rangle$ defined by

$$\langle w, v \rangle := \int_{\hat{\Gamma}} wv \, \mathrm{d}s \quad \forall w \in H^t(\hat{\Gamma}),\ \forall v \in H^{-t}(\hat{\Gamma}).$$

We further define, for all $t \in R$,

$$H^t(\Gamma) = \{u|_\Gamma : u \in H^t(\hat{\Gamma})\},$$

$$\hat{H}^t(\Gamma) = \{u \in H^t(\hat{\Gamma}) : \operatorname{supp} u \subseteq \bar{\Gamma}\}.$$

The duality properties are as follows:

$$(H^t(\Gamma))' = \hat{H}^{-t}(\Gamma) \quad \text{and} \quad (\hat{H}^t(\Gamma))' = H^{-t}(\Gamma).$$

For $t > 0$, the norms in $H^t(\hat{\Gamma})$, $H^t(\Gamma)$ and $\hat{H}^t(\Gamma)$ are defined by

$$\|u\|_{H^t(\hat{\Gamma})} = \inf\{\|v\|_{H^{t+1/2}(R^n)} : v|_{\hat{\Gamma}} = u\},$$

$$\|u\|_{H^t(\Gamma)} = \inf\{\|v\|_{H^t(\hat{\Gamma})} : v|_\Gamma = u\},$$

$$\|u\|_{\hat{H}^t(\Gamma)} = \|u\|_{H^t(\hat{\Gamma})}.$$

For $t < 0$, the norms are defined by duality. Associated with $\mathscr{T}_h$, let $S_h^p \subset L^2(\Gamma)$ denote the finite-dimensional vector space of piecewise polynomials with degree p.

3.1. Symm's integral equation

The Dirichlet problem for the Laplacian is related to the Symm's integral equation

$$Vu(x) := \int_\Gamma G(x, y)u(y)\,\mathrm{d}s_y = f(x), \quad x \in \Gamma, \tag{3.1}$$

where u is the unknown density, $G(x, y) = -(1/2\pi)\ln|x - y|$ for $n = 2$ and $G(x, y) = 1/(4\pi|x - y|)$ for $n = 3$, and f is determined by some given Dirichlet data. We assume that $f \in H^{1/2}(\Gamma)$.

The operator

$$V : u \in \hat{H}^{-1/2}(\Gamma) \to f \in H^{1/2}(\Gamma)$$

is a Fredholm operator of index zero and is an isomorphism for $n = 3$ or for $n = 2$ if $\operatorname{cap}(\Gamma) \neq 1$; see e.g., [8,9,11,16,24]. Here $\operatorname{cap}(\Gamma)$ denotes the capacity, or conformal radius, or transfinite diameter of Γ. We therefore assume that, for positive definiteness, $\operatorname{cap}(\Gamma) < 1$ for $n = 2$ which can always be arranged by scaling, if necessary. Consequently, the bilinear form defined by

$$B(u, v) := \langle Vu, v \rangle = \int_\Gamma v(x)Vu(x)\,\mathrm{d}s_x \quad \forall u, v \in \hat{H}^{-1/2}(\Gamma) \tag{3.2}$$

is symmetric, continuous, and coercive on $\hat{H}^{-1/2}(\Gamma) \times \hat{H}^{-1/2}(\Gamma)$. Furthermore, the linear functional defined by

$$F(v) := \langle f, v \rangle = \int_\Gamma f(x)v(x)\,\mathrm{d}s_x \quad \forall v \in \hat{H}^{-1/2}(\Gamma) \tag{3.3}$$

is continuous on $H^{-1/2}(\Gamma)$.

Symm's equation is thus a special model problem of (2.1) and Assumption 1 is satisfied with $H = \hat{H}^{-1/2}(\Gamma)$ and $\|\cdot\|_H = \|\cdot\|_{\hat{H}^{-1/2}(\Gamma)}$. For the Galerkin approximation (2.4), we can use, for example, $S_h = S_h^0$ a space of piecewise constants. The choice of the complementary space S_h^c is quite flexible so long as condition (2.8) is met. The main question remains is whether Assumption 3 holds for

Symm's integral equation. We first show that the constant γ in (2.15) is independent of the mesh size h. Our analysis of the strengthened Cauchy–Schwarz inequality (2.15) follows closely to that of [3,12] for second-order elliptic PDEs. We first cite a lemma from [3].

Lemma 1. *Let $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ denote two inner products on a vector space X. Let $\|\cdot\|$ and $|\cdot|$ denote the corresponding norms. Suppose that there exist positive constants λ_1 and λ_2 such that*

$$0 < \lambda_1 \leqslant \frac{(z,z)}{\langle z,z\rangle} \leqslant \lambda_2 \tag{3.4}$$

for all nonzero $z \in X$. For any nontrivial $x, y \in X$, let

$$\gamma_1 = \frac{(x,y)}{\|x\|\|y\|},$$

$$\gamma_2 = \frac{\langle x,y\rangle}{|x||y|}.$$

Then

$$\left(\frac{\lambda_1}{\lambda_2}\right)^2 (1 - \gamma_2^2) \leqslant 1 - \gamma_1^2. \tag{3.5}$$

Lemma 2. *Let the bilinear form $B(\cdot,\cdot)$ be defined by (3.2), the bilinear form $A(\cdot,\cdot)$ be defined by (2.11), the BE space $S_h \subset S_h^p \subset L^2(\Gamma)$, and the complementary space $S_h^c \subset S_h^q \subset L^2(\Gamma)$ be constructed such that Assumption 2 holds. Then (2.15) holds for the constant $\gamma \in [0,1)$ independent of the mesh size h.*

Proof. For $w \in S_h$ and $v \in S_h^c$, let w_i and v_i be defined by (2.12). Obviously, w_i and $v_i \in \hat{H}^{-1/2}(\Gamma)$. The proof of (2.15) can be reduced to an element-by-element estimate. On each $\Gamma_i \in \mathcal{T}_h$, if $\gamma_i \in [0,1)$ is independent of h such that

$$|B(w_i,v_i)| \leqslant \gamma_i \sqrt{B(w_i,w_i)}\sqrt{B(v_i,v_i)}, \tag{3.6}$$

then

$$
\begin{aligned}
|A(w,v)| &\leqslant \sum_i |B(w_i,v_i)| \\
&\leqslant \sum_i \gamma_i \sqrt{B(w_i,w_i)}\sqrt{B(v_i,v_i)} \\
&\leqslant \gamma \left(\sum_i B(w_i,w_i)\right)^{1/2} \left(\sum_i B(v_i,v_i)\right)^{1/2} \\
&= \gamma \sqrt{A(w,w)}\sqrt{A(v,v)} \quad \forall w \in S_h, \ v \in S_h^c,
\end{aligned}
$$

where

$$\gamma = \max_i \gamma_i.$$

The proof can further be reduced to a reference element. For any element Γ_i, let π_i be an invertible affine mapping

$$\pi_i : \xi \in \Gamma_r \mapsto \pi_i(\xi) = Q_i \xi + \xi_0 \in \Gamma_i,$$

such that

$$\Gamma_i = \pi_i(\Gamma_r),$$

where Γ_r is the reference element. For $n = 2$, we take Γ_r to be an interval $[-a, a]$ with $\mathrm{cap}(\Gamma_r) < 1$ where a is a positive constant. For $n = 3$, since the partition $\mathcal{T}_h$ is regular, the mapping has the following property, see [10],

$$\frac{\sigma_i}{h_r} |\xi| \leqslant |Q_i \xi| \leqslant \frac{h_i}{\sigma_r} |\xi|, \tag{3.7}$$

where h_r and σ_r are parameters of (2.6) in terms of the reference element. Let $S_{h,i}$ and $S_{h,i}^c$ denote the restrictions of S_h and S_h^c, respectively, to the element Γ_i. And let S_r and S_r^c denote some fixed finite-dimensional spaces of polynomials defined on the reference element Γ_r such that the mapping π_i maps S_r onto $S_{h,i}$ and S_r^c onto $S_{h,i}^c$. Using the change of variables, for each element Γ_i inequality (3.6) becomes

$$B(w_i, v_i) = J_i^2 B_{r,i}(w_{r,i}, v_{r,i})$$

$$\leqslant J_i^2 \gamma_i \sqrt{B_{r,i}(w_{r,i}, w_{r,i})} \sqrt{B_{r,i}(v_{r,i}, v_{r,i})}$$

$$= \gamma_i \sqrt{J_i^2 B_{r,i}(w_{r,i}, w_{r,i})} \sqrt{J_i^2 B_{r,i}(v_{r,i}, v_{r,i})}$$

$$= \gamma_i \sqrt{B(w_i, w_i)} \sqrt{B(v_i, v_i)},$$

where $w_{r,i} = w_i \circ \pi_i \in S_r$, $v_{r,i} = v_i \circ \pi_i \in S_r^c$, J_i is the Jacobian of the mapping, and

$$B_{r,i}(w_{r,i}, v_{r,i}) := \int_{\Gamma_r} \left(\int_{\Gamma_r} \frac{-1}{2\pi} \ln \left(\frac{h_i}{2a} |\xi - \eta| \right) w_{r,i}(\eta) \, ds_\eta \right) v_{r,i}(\xi) \, ds_\xi$$

for $n = 2$,

$$B_{r,i}(w_{r,i}, v_{r,i}) := \int_{\Gamma_r} \left(\int_{\Gamma_r} \frac{1}{4\pi |Q_i(\xi - \eta)|} w_{r,i}(\eta) \, ds_\eta \right) v_{r,i}(\xi) \, ds_\xi$$

for $n = 3$. Clearly, $B_{r,i}(\cdot, \cdot)$ defines an inner product on $S_r \oplus S_r^c$. Since $B_{r,i}(\cdot, \cdot)$ exhibits different properties with respect to different kernels, the quantity γ_i being independent of h is proved in two separate cases of $n = 2$ and 3. We first prove for the case of $n = 2$. Since $h_i \leqslant \mathrm{cap}(\Gamma) < 1$, we assume that h_i is small enough such that $h_i/2a < 1$. Let

$$A_1(w_r, v_r) := \int_{\Gamma_r} \left(\int_{\Gamma_r} w_r(\eta) \, ds_\eta \right) v_r(\xi) \, ds_\xi,$$

$$A_2(w_r, v_r) := \int_{\Gamma_r} \left(\int_{\Gamma_r} \frac{-1}{2\pi} \ln |\xi - \eta| w_r(\eta) \, ds_\eta \right) v_r(\xi) \, ds_\xi$$

for all w_r and v_r in $S_r \oplus S_r^c$. Then $A_1(\cdot, \cdot)$ and $A_2(\cdot, \cdot)$ are two inner products independent of h on $S_r \oplus S_r^c$. Note that S_r and S_r^c are fixed and linearly independent subspaces on the reference element Γ_r. Therefore, there exist two constants $\gamma_{r,1}$ and $\gamma_{r,2} \in [0, 1)$ independent of h such that, for $j = 1, 2$,

$$|A_j(w_r, v_r)| \leqslant \gamma_{r,j} A_j(w_r, w_r)^{1/2} A_j(v_r, v_r)^{1/2} \quad \forall w_r \in S_r, \ v_r \in S_r^c. \tag{3.8}$$

We then have

$$|B_{r,i}(w_{r,i}, v_{r,i})|$$

$$= \left| \frac{-1}{2\pi} \ln \frac{h_i}{2a} A_1(w_{r,i}, v_{r,i}) + A_2(w_{r,i}, v_{r,i}) \right|$$

$$\leqslant \left(\frac{-1}{2\pi} \ln \frac{h_i}{2a} \right) |A_1(w_{r,i}, v_{r,i})| + |A_2(w_{r,i}, v_{r,i})|$$

$$\leqslant \frac{-1}{2\pi} \ln \frac{h_i}{2a} \gamma_{r,1} A_1(w_{r,i}, w_{r,i})^{1/2} A_1(v_{r,i}, v_{r,i})^{1/2} + \gamma_{r,2} A_2(w_{r,i}, w_{r,i})^{1/2} A_2(v_{r,i}, v_{r,i})^{1/2}$$

$$\leqslant \gamma_r \left(\frac{-1}{2\pi} \ln \frac{h_i}{2a} A_1(w_{r,i}, w_{r,i}) + A_2(w_{r,i}, w_{r,i}) \right)^{1/2}$$

$$\times \left(\frac{-1}{2\pi} \ln \frac{h_i}{2a} A_1(v_{r,i}, v_{r,i}) + A_2(v_{r,i}, v_{r,i}) \right)^{1/2}$$

$$= \gamma_r \sqrt{B_{r,i}(w_{r,i}, w_{r,i})} \sqrt{B_{r,i}(v_{r,i}, v_{r,i})},$$

where $\gamma_r = \max\{\gamma_{r,1}, \gamma_{r,2}\}$. This concludes that $\gamma_i = \gamma_r$ for all $i = 1, 2, \ldots, N$ and they are independent of h. For the case of $n = 3$, we define

$$A_3(w_r, v_r) := \int_{\Gamma_r} \left(\int_{\Gamma_r} \frac{1}{4\pi|\xi - \eta|} w_r(\eta) \, ds_\eta \right) v_r(\xi) \, ds_\xi$$

for all w_r and v_r in $S_r \oplus S_r^c$. Again, A_3 is an inner product independent of h on $S_r \oplus S_r^c$ with the existence of the corresponding constant $\gamma_{r,3} \in [0, 1)$. That is, (3.8) holds for $j = 3$. For each element Γ_i, inequalities (3.7) imply that

$$\gamma_{1,i} = \frac{\sigma_r}{h_i} \leqslant \frac{B_{r,i}(z, z)}{A_3(z, z)} \leqslant \frac{h_r}{\sigma_i} = \lambda_{2,i}.$$

Therefore, with (2.6), Lemma 1 yields that

$$\gamma_i^2 \leqslant 1 - \left(\frac{\lambda_{1,i}}{\lambda_{2,i}} \right)^2 (1 - \gamma_{r,3}^2)$$

$$= 1 - \left(\frac{\sigma_i \sigma_r}{h_i h_r} \right)^2 (1 - \gamma_{r,3}^2)$$

$$\leqslant 1 - \left(\delta \frac{\sigma_r}{h_r} \right)^2 (1 - \gamma_{r,3}^2)$$

$$< 1$$

and that γ_i is independent of h since $\delta \in (0,1]$, $\sigma_r/h_r \in (0,1]$, and $\gamma_{r,3} \in [0,1)$ are all independent of h. This thus completes the proof. $\square$

This lemma suggests that the construction of S_h^c is very flexible. For example, the hierarchical basis functions can be used in such a way that the shape functions of S_h^c are of the next higher order than that of S_h. The error estimator can thus be used in all h-, p-, and hp-version BEM [23].

To justify estimate (2.17) for Symm's equation, we need to further verify conditions (2.13) and (2.14). The following well-known lemma, see also, e.g., [5,6,15], is essential to establish these conditions and will be used for the next model problem as well. For simplicity, the lemma is restricted to two space dimensions only.

Lemma 3. *Let Γ be partitioned as in (2.5). Then, for $t = \frac{1}{2}$ or $t = \frac{-1}{2}$, there exist two positive constants C_3 and C_4 independent the number of sub-intervals N, i.e., independent of the mesh parameter h, such that*

$$C_3 \sum_{i=1}^{N} \|u\|_{H^t(\Gamma_i)}^2 \leqslant \|u\|_{H^t(\Gamma)}^2, \tag{3.9}$$

$$\|u\|_{\hat{H}^t(\Gamma)}^2 \leqslant C_4 \sum_{i=1}^{N} \|u\|_{\hat{H}^t(\Gamma_i)}^2. \tag{3.10}$$

Theorem 2. *If all assumptions in Lemma 2 hold, then for Symm's boundary integral equation (3.1) in two space dimensions we have estimate (2.17) with the constants $\gamma, C_1,$ and C_2 all independent of h.*

Proof. By Assumption 2, the basis functions for the complementary space S_h^c have supports in their respective subintervals. Moreover, these functions are piecewise polynomials in $L^2(\Gamma)$. Evidently, the spaces $S_{\bar{h}}$ and S_h^c so constructed make inequalities (3.9) and (3.10) hold for $u \in S_{\bar{h}}$ and $u \in S_h^c$, respectively. Applying the equivalence of the norms $\|\cdot\|_H$ and $\|\cdot\|_B$, inequalities (2.13) and (2.14) of Assumption 3 hold. By Lemma 2, the theorem is thus asserted. $\square$

3.2. Hypersingular integral equations

The Neumann problem for the Laplacian is related to the integral equation

$$Wu(x) := -\frac{\partial}{\partial n_x} \int_\Gamma u(y) \frac{\partial G(x,y)}{\partial n_y} \, ds_y = f(x), \quad x \in \Gamma \tag{3.11}$$

for the unknown displacement u on Γ, where $G(x,y) = -(1/2\pi) \ln|x - y|$ for $n = 2$ and $G(x,y) = (1/4\pi|x - y|)$ for $n = 3$, and f is determined by the given Neumann data. The integral in (3.11) is to be understood as a Hadamard finite-part integral. For simplicity, we specifically consider the model problems used in [8], namely, the Γ is a closed curve for $n = 2$ and is an open surface for $n = 3$.

Let

$$H := \{v \in H^{1/2}(\Gamma): \langle 1, v \rangle = 0\},$$

$$H' := \{g \in H^{-1/2}(\Gamma): \langle g, 1 \rangle = 0\}$$

for $n = 2$ and

$$H := \{v \in \hat{H}^{1/2}(\Gamma)\},$$

$$H' := H^{1/2}(\Gamma)$$

for $n = 3$.

We define the bilinear form and the linear functional for (3.11) as

$$B(u, v) := \langle Wu, v \rangle \quad \forall u, v \in H, \tag{3.12}$$

$$F(v) := \langle f, v \rangle \quad \forall v \in H. \tag{3.13}$$

The a priori theory presented in [8,11] suitable for our purposes is summarized in the following lemma for which the proof is therein referred.

Lemma 4. *Assumption* 1 *holds for the variational problem* (2.1) *with the bilinear form and the linear functional given, respectively, by* (3.12) *and* (3.13).

Note that the bilinear form can be written as

$$B(u, v) = \int_{\Gamma} \left(\int_{\Gamma} \frac{-1}{2\pi} \ln(|x - y|) \frac{\partial u(y)}{\partial s_y} \, ds_y \right) \frac{\partial v(x)}{\partial s_x} \, ds_x \tag{3.14}$$

for $n = 2$ and

$$B(u, v) = \int_{\Gamma} \left(\int_{\Gamma} \frac{1}{|x - y|^3} u(y) \, ds_y \right) v(x) \, ds_x$$

for $n = 3$. Hence, the proof of the strengthened Cauchy–Schwarz inequality (2.15) for the hypersingular integral equations proceeds in a similar way as that for Symm's integral equation.

Lemma 5. *Let the bilinear form* $B(\cdot, \cdot)$ *be defined by* (3.12), *the bilinear form* $A(\cdot, \cdot)$ *be defined by* (2.11), *the BE space* $S_h \subset S_h^p \subset H$, *and the complementary space* $S_h^c \subset S_h^q \subset H$ *be constructed such that Assumption* 2 *holds. Then* (2.15) *holds for the constant* $\gamma \in [0, 1)$ *independent of the mesh size* h.

Moreover, with a similar proof of Theorem 2, the following theorem is thus a consequence of this lemma and Lemma 3.

Theorem 3. *If all assumptions in Lemma* 5 *hold, then for the hypersingular integral equation* (3.11) *in two space dimensions we have the estimate* (2.17) *with the constants* $\gamma, C_1,$ *and* C_2 *all independent of* h.

Step 1: Give an initial mesh $\mathcal{T}_j$, $j = 0$, on Γ.
Step 2: Construct a BE space S_h on $\mathcal{T}_j$.
Step 3: Solve (2.4) for $u_h \in S_h$.
Step 4: For each $\Gamma_i \in \mathcal{T}_j$, $i = 1, \ldots, N$,
   Step 4.1: construct a complementary BE space $S_h^c(\Gamma_i)$ on Γ_i,
   Step 4.2: solve (2.16) for $\tilde{e}_i \in S_h^c(\Gamma_i)$, and
   Step 4.3: compute the local error indicator $\eta_i := \|\tilde{e}_i\|_B$.
Step 5: Compute the global error estimator $\|\tilde{e}\|_A = (\sum_i^N \eta_i^2)^{1/2}$, the approximate solution norm
       $\|u_h\|_B$, and the relative error r.e. $= \|\tilde{e}\|_A / \|u_h\|_B$.
Step 6: If r.e. $<$ TOL go to Step 7, where TOL is a preset error tolerance, else
   Step 6.1: do, for $n = 1, \ldots, N$, if $\eta_i \geqslant \sigma \eta_{\max}$ then subdivide the element Γ_i, where $\sigma \in [0, 1]$
       is a given parameter and $\eta_{\max} = \max\{\eta_i\}_1^N$,
   Step 6.2: a new mesh $\mathcal{T}_j$, $j := j + 1$, is thus constructed, and go to Step 2.
Step 7: Stop.

Fig. 1. An adaptive algorithm.

Remark 3. The estimators developed in [8,13,26] are all analyzed on the bases of the dual norm H' for the residual error $f - Vu_h$ or $f - Wu_h$. The dual norm in general is not computable. Consequently, their approaches require more regularity of the residual or equivalently more regularity of the exact solution u in order to measure the estimators in a higher and computable norm such as the $L^2(\Gamma)$ norm.

4. Numerical examples

Three objectives are considered for this section; namely, to justify the effectiveness of the proposed estimator, to show the efficiency of the resulting adaptive scheme, and to illustrate the complementary subspaces S_h^c. A standard h-version, adaptive algorithm based on the proposed error estimator is given in Fig. 1.

The following two examples are related to the Laplacian

$$\Delta u = 0 \quad \text{in } \Omega, \tag{4.1}$$

where the domain Ω is an L-shaped polygon shown in Fig. 2. The boundary condition for (4.1) is

$$u = g_D \quad \text{on } \Gamma \tag{4.2}$$

for Symm's integral equation and is

$$\frac{\partial u}{\partial n} = g_N \quad \text{on } \Gamma \tag{4.3}$$

for the hypersingular integral equation. The functions g_D and g_N are chosen so that the corresponding exact solutions are, in polar coordinates, $u = r^{2/3} \sin 2\theta/3$ for (4.2) and $u = r^{1/7} \sin \theta/7$ for (4.3); see also [8].
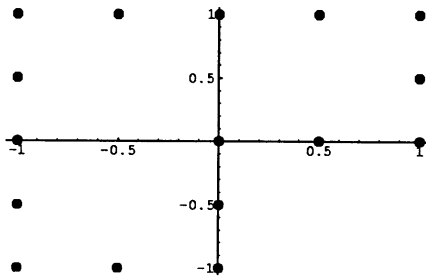
Example 4.1 (Symm's integral equation). As discussed above, one of the key ingredients of the present estimator is the construction of complementary subspaces S_h^c based on the current approximation space S_h. The construction can be done on the reference element.
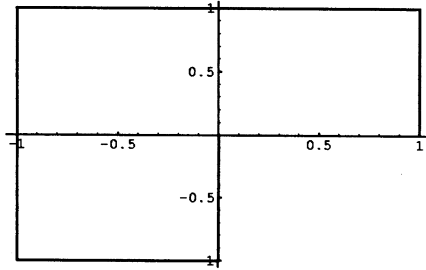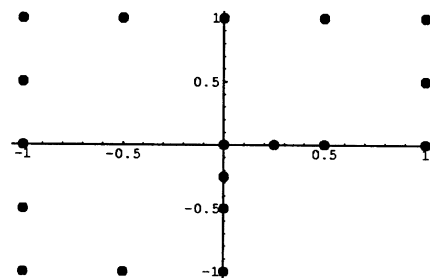
1. *Problem domain.*
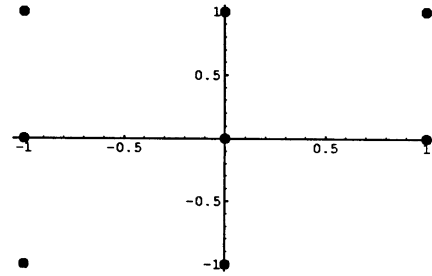
A. *Initial mesh N = 8.*

B. $N = 16$.

C. $N = 18$.

Fig. 2.

For our experiments, the space S_h for Symm's equation is given by piecewise constants. The nodal points corresponding to piecewise constants are, in the sense of the average of quadrature rules, defined at the middle points of elements. To satisfy assumption (2.8), we should avoid the shape functions of S_h^c that are defined with nodal points being in the middle and that will generate a constant function. Hence, we can, for example, construct S_h^c via the mapping of the shape functions

$$\psi_{r,1}(\xi) = \tfrac{1}{2}(1 - \xi)^2, \quad \xi \in [-1, 1],$$
$$\psi_{r,2}(\xi) = \tfrac{1}{2}(1 + \xi)^2, \quad \xi \in [-1, 1] \tag{4.4}$$

defined on the reference element $\Gamma_r = [-1, 1]$, i.e., $S_h^c \subset S_h^2$. Therefore, on each element, we have a 2×2 local system to be solved in Step 4.2 of the above adaptive algorithm.

More specifically, (2.16) leads to systems of 2×2 linear algebraic equations

$$A_k e_k = b_k, \quad k = 1, \dots, N, \tag{4.5}$$

where the four entries of the matrix A_k and two entries of b_k are given as

$$A_k(i,j) = \int_{\Gamma_k} \left(\int_{\Gamma_k} \frac{-1}{2\pi} \ln(|x - y|) \psi_{k,j}(y) \, ds_y \right) \psi_{k,i}(x) \, ds_x, \quad i, j = 1, 2,$$
$$b_k(i) = \int_{\Gamma_k} f(x) \psi_{k,i}(x) \, ds_x, \quad i, j = 1, 2,$$

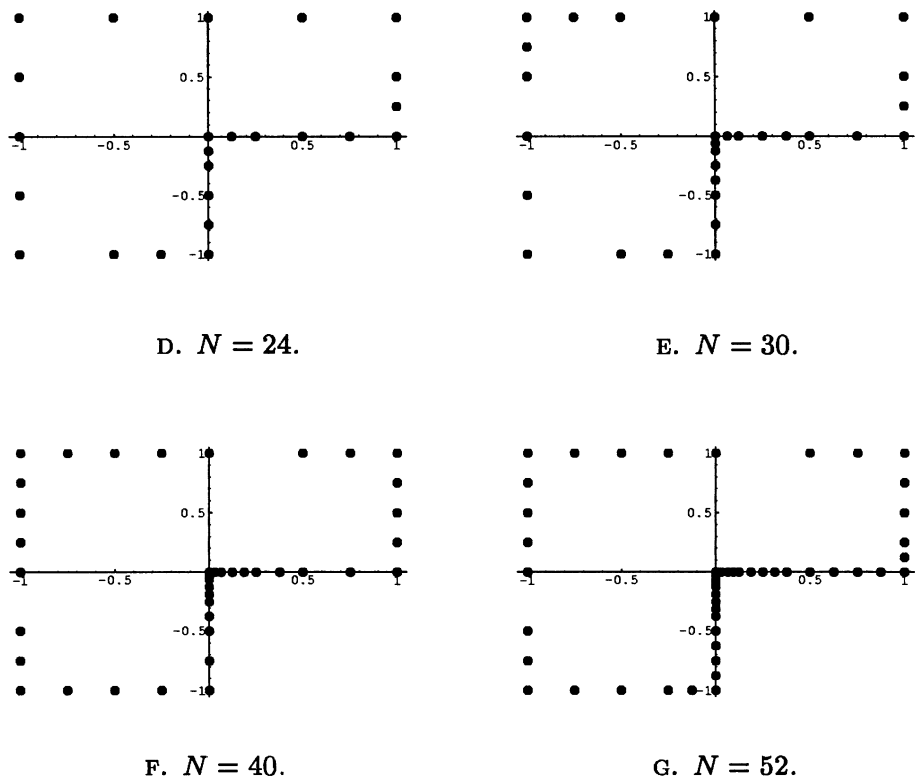D. $N = 24$.　　　　　　　　E. $N = 30$.

F. $N = 40$.　　　　　　　　G. $N = 52$.

Fig. 2. Continued.

where $\psi_{k,i}$, $i = 1, 2$, are shape functions obtained by transforming the two basis functions (4.4) to the element Γ_k. The solution of (4.5) then defines an error indicator for that element as stated in Step 4.3, namely,

$$\tilde{e}_k = \tilde{e}_{k,1}\psi_{k,1} + \tilde{e}_{k,2}\psi_{k,2}, \quad \boldsymbol{e}_k = (e_{k,1}, e_{k,2}).$$

We then proceed to obtain the global error estimator in Step 5.

The mesh diagrams Figs. 2A–2G are showing a typical scenario of adaptive process as the estimator is capturing the point singularity at the origin. The estimator is very effective as shown by the effectivity indices in the last column in Tables 2A and 2B for both uniform and adaptive approaches, where the refinement factor σ is defined in Step 6.2. Moreover, if the relative error was preset to, for instance, 1%, the uniform approach requires about 10 times elements of the adaptive approach. The adaptive method is clearly showing advantageous features for singularly behaved problems.

Example 4.2 (A hypersingular integral equation). For our numerical experiments, we choose $S_h = S_h^1$ a space of linear functions while the complementary space S_h^c is constructed, via the reference element, by

$$\psi = 1 - \xi^2, \quad \xi \in [-1, 1]. \tag{4.6}$$

Table 2A
Example 4.1 using uniform meshes $\sigma = 0$

N	$\|e\|_B$	r.e.	θ
8	0.194	0.130	1.001
16	0.115	0.075	0.910
32	0.073	0.047	0.935
64	0.047	0.030	0.955
128	0.030	0.019	0.975
256	0.019	0.012	0.993
512	0.012	0.008	1.010

Table 2B
Example 4.1 using adaptive meshes $\sigma = 0.1$

N	$\|e\|_B$	r.e.	θ
8	0.194	0.130	1.001
16	0.115	0.075	0.910
18	0.076	0.049	0.944
24	0.049	0.031	0.952
30	0.032	0.020	0.964
40	0.020	0.013	0.972
52	0.013	0.008	0.981

Again, it can be easily verified that $S_h^c \subset S_h^2$ satisfies Assumption 2. On each element, (2.16) corresponds to a single equation. Consequently, our estimator reduces to a residual-type error estimators developed in [8,26,13]. More specifically, using the formulas in [23] to explicitly evaluate (3.14) for (4.6), we obtain

$$
B(\psi_i, \psi_i) = -\frac{1}{2\pi} \int_{\Gamma_i} \left(\int_{\Gamma_i} \ln|x - y| \frac{\partial \psi_i(y)}{\partial s_y} \, ds_y \right) \frac{\partial \psi_i(x)}{\partial s_x} \, ds_x
$$
$$
= -\frac{1}{2\pi} \int_{-1}^{1} \left(\int_{-1}^{1} \ln\left(\frac{h_i}{2} |\xi - \eta| \right) \frac{d\psi(\xi)}{d\xi} \, d\xi \right) \frac{d\psi(\eta)}{d\eta} \, d\eta
$$
$$
= \frac{2}{\pi}.
$$

The error indicator for the element Γ_i is hence

$$
\eta_i = \sqrt{B(\tilde{e}_i, \tilde{e}_i)} = \sqrt{\frac{\pi}{2}} (F(\psi_i) - B(u_h, \psi_i)),
$$

where the last term is a computable residual.

Using only one shape function (4.6) to define the complementary space S_h^c on each element, our numerical results are showing very effective error estimators as presented in Tables 3A and 3B.

Table 3A
Example 4.2 using uniform meshes $\sigma = 0$

N	$\|e\|_B$	r.e.	θ
8	0.138	0.463	0.830
16	0.124	0.380	0.820
32	0.112	0.316	0.828
64	0.101	0.265	0.842
128	0.090	0.224	0.856
256	0.081	0.191	0.871
512	0.073	0.163	0.886

Table 3B
Example 4.2 using adaptive meshes $\sigma = 0.5$

N	$\|e\|_B$	r.e.	θ
8	0.138	0.463	0.830
10	0.128	0.392	0.822
12	0.117	0.331	0.849
14	0.106	0.280	0.865
16	0.097	0.240	0.882
18	0.088	0.206	0.898
20	0.080	0.179	0.915
22	0.073	0.157	0.932
24	0.067	0.138	0.949
26	0.062	0.122	0.965
28	0.057	0.109	0.981
30	0.053	0.099	0.997

Acknowledgements

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