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Sortabilities of Partition Properties

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Abstract. Consider the partition of a set of integers into parts. Various partition properties have been proposed in the literature to facilitate the restriction of the focus of attention to some small class of partitions. Recently, Hwang, Rothblum and Yao defined and studied the sortability of these partition properties as a tool to prove the existence of a partition with such a property in a given family. In this paper we determine the sortability indices of the seven most interesting properties of partitions providing a complete solution to the sortability issue.

Keywords: partition, consecutive partition, nested partition, order-consecutive partition

1. Introduction

Consider the set of integers $I_n = \{1, 2, \ldots, n\}$. A partition π of I_n is a finite collection of disjoint sets $\pi_1, \pi_2, \ldots, \pi_p$ whose union is I_n, p is called the *size* of π and $\pi_1, \pi_2, \ldots, \pi_p$ the *parts* of π. Further, if $n_1, n_2, \ldots, n_p$ are the sizes of $\pi_1, \pi_2, \ldots, \pi_p$, respectively, then $\{n_1, n_2, \ldots, n_p\}$ is called the *shape* of π. Different types of restrictions can be imposed on π. π is referred to as a *shape-partition* if its shape is given, a *size-partition* if p is given, and an *open partition* if neither its shape nor its size is given. We may also write an $\{n_1, n_2, \ldots, n_p\}$-partition or a p-partition to highlight the shape or the size. For a partition π and an element $j \in I_n$, define $\pi(j)$ as the part of π that contains j. Then π can be represented in the form $t(\pi)$ which is the sequence $\pi(1)\pi(2)\cdots\pi(n)$ or $i_1, i_2, \ldots, i_n$ for short where $j \in \pi_{i_j}$ for all $j \in I_n$. For instance, a partition π of I_5 with $\pi_1 = \{1, 3\}, \pi_2 = \{2, 5\}, \pi_3 = \{4\}$ can be represented by $t(\pi)$ as $\pi_1\pi_2\pi_1\pi_3\pi_2$ or 12132 for short.

A partition property is a univariate relation over partitions. A set of partitions is said to *satisfy* a property Q if it contains a partition with property Q. Seven such properties have been proposed in the literature; each occurs in optimal partitions for a corresponding class of problems thereby allowing one to restrict attention to partitions with that property in search for our optimal partition. An underlying relation between two disjoint subsets of I_n

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which characterizes each of these seven properties is: Let I and J be two disjoint subsets of I_n. I is said to *penetrate* J, written $I \to J$, if there exist a, c in J and b in I such that $a < b < c$.

We now define the seven properties: π is

N (nested):	$\pi_i \to \pi_j$ implies $\pi_j \not\to \pi_i$.
F (fully nested):	For all i and j, either $\pi_i \to \pi_j$, $\pi_j \not\to \pi_i$ or vice versa.
A (almost fully nested):	F except that parts of size 1 do not have to penetrate other parts.
C (consecutive):	For all i and j, $\pi_i \not\to \pi_j$.
S (size-consecutive):	C plus the condition that larger elements go to parts of larger sizes.
E (extremal):	A special case of S with $p-1$ parts of size 1.
O (order-consecutive):	Parts can be indexed such that $\pi_i \not\to \bigcup_{j=1}^{i-1} \pi_j$ for all i.

(See (Hwang et al., 1996) for references on N, F, C, O. A was proposed in (Gal and Klots, 1995), S in (Hwang et al., 1985) and E in (Anily and Federgruen, 1991).)

It was shown (Hwang and Mallow, 1995; Kreweras, 1972; Yeh et al., 1998) that the number of open partitions, the number of size partitions for general size and the number of shape partitions for general shape are all exponential in n, but the corresponding numbers of partitions satisfying Q, for any $Q \in \{N, F, A, C, S, E, O\}$, is polynomial. Therefore, in an optimal partition problem it would be very helpful if we can prove the existence of an optimal partition satisfying Q to limit the scope of search. A typical way of proving such an existence is to show that any optimal partition π not satisfying Q can be step-by-step locally sorted into an optimal partition satisfying Q. In this paper, "locally sorted" means that for some fixed k, a set K of k parts of π not satisfying Q are sorted into a partition π', with the same restrictions as π, such that $\pi'_i = \pi_i$ for $\pi_i \notin K$ and the restriction of π' to $\bigcup_{\pi_i \in K} \pi_i$ satisfies Q. We refer to such a sorting as a Q-sorting of K. Note that optimality must be preserved in a local sorting. The reason of doing local sortings instead of one global sorting is because the preservation of optimality is much easier to maneuver at the local level.

Instead of associating the sortability notion with optimality, we associate it with families Π of partitions satisfying certain conditions (the reader can think of these families as families of optimal partitions). Therefore, optimality preservation means keeping the derived partitions after local sorting in the same family. Note that for a given k, there could be many choices of which k parts to sort, and for given k parts, there could be many ways to Q-sort. We define four different levels of coverage: *strong*, *part-specific*, *sort-specific*, *weak*, depending on which local sortings must yield partitions staying in the family. More specifically, for given k and type of restriction, a family Π is called:

(i) *strongly Q-sortable* if Π contains all Q-sorting of all K;
(ii) *part-specific Q-sortable* if Π contains all Q-sorting of a specific K;
(iii) *sort-specific Q-sortable* if Π contains a specific Q-sorting for all K;
(iv) *weakly Q-sortable* if Π contains a specific Q-sorting of a specific K.

Property Q is called *weakly-*, *sort-specific-*, *part-specific-*, *strongly-k sortable* if Π satisfies Q for any family Π satisfying (i), (ii), (iii) and (iv), respectively. We can also describe the sortability of Q by associating with it a triple (l, k, t), where l is the level of coverage ((i), (ii), (iii) and (iv) above) and t is the type of restriction (shape, size, open) on Π. Note that the stronger is the coverage of the family Π, the easier is for Π to satisfy Q. It was shown in (Hwang et al., 1996) that it suffices to find a partition statistic $s(\pi)$ which is either strictly decreasing during Q-sorting and $s(\pi)$ is lower bounded, or strictly increasing and $s(\pi)$ is upper bounded (noting that the number of partitions is finite). In particular, to show the property Q is (i) strongly- (ii) part-specific- (iii) sort-specific- (iv) weakly-k sortable, it suffices to show that $s(\pi)$ is decreasing (or increasing) during (i) all Q-sorting for all K, (ii) all Q-sorting of a specific K, (iii) a specific Q-sorting for all K, (iv) a specific Q-sorting of a specific K. It should be noted that when we sort a subset K of k parts by k-open-sorting into a subset K' of k' parts, all parts in K' must be distinct from the parts in $\pi \backslash K$. The reason is if K' includes a part in $\pi \backslash K$, then it is not clear whether we are doing k-sorting or $(k+1)$-sorting.

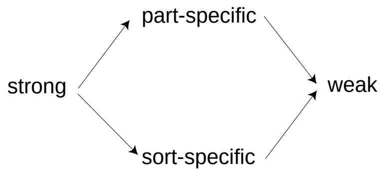
Currently, it is known (Hwang et al., 1996) that

- C is strongly-2-open-sortable.
- N is part-specific-2-open-sortable.
- O is not weakly-2- or 3-shape-sortable.
- F is not part-specific-2-shape-sortable.

Although the sortabilities of C and N were proved in (Hwang et al., 1996) only for "size", the proofs are good for "open". Hence we state them in the more general version. In this paper, we also study the sortability of S, E and A and completely solve the (l, k, t)-sortability issue.

2. Some preliminary results

In talking about the (l, k, t)-sortability, any missing variable in the triple will be interpreted as that the statement is valid for all choices of that variable. It is easily verified the implications among the four types of levels:



Lemma 2.1. *When the level is strong or part-specific, open-sortability implies size-sortability implies shape-sortability.*

Proof: A weak or sort-specific Q-shape-sortable family is also a Q-size- and Q-open-sortable family. Thus any such family not satisfying Q provides an example against shape- or size- or open-sortability all. □

Lemma 2.2. *Let Q and Q' be two partition properties such that $Q \Rightarrow Q'$ ($Q' = Q$ allowed). If every (l, k, t')-Q'-sortable family contains a subfamily which is (l, k, t)-Q-sortable, then Q is (l, k, t)-sortable implies Q' is (l, k, t')-sortable.*

Proof: Let Π' be an (l, k, t')-Q'-sortable family which contains an (l, k, t)-Q-sortable subfamily Π. The (l, k, t)-sortability of Q implies that Π satisfies Q, and hence Π' satisfies Q', since $Q \Rightarrow Q'$. □

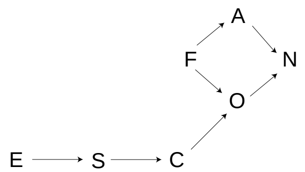
Corollary 2.3. *For l = weak or sort-specific, $(l, k,$ shape$)$-sortability implies $(l, k,$ size$)$-sortability implies $(l, k,$ open$)$ sortability.*

Proof: Set $Q' = Q$ in Lemma 2.2 and note that for the given l, every (l, k, p)-Q-sortable family contains an $(l, k, \{n_1, \ldots, n_p\})$-$Q$-sortable family. □

Corollary 2.4. *For $Q \Rightarrow Q'$, Q' is k-consistent and l = weak or sort-specific, Q is (l, k, t)-sortable implies Q' is (l, k, t)-sortable.*

Proof: Every Q-sorting is a Q'-sorting. Hence for the given l, a Q'-sortable family contains a Q-sortable subfamily. □

The implications among the partition properties N, F, C, O were given in (Hwang et al., 1996). It is easily verified that adding A, S, E yields the following partial order:



The implications among the levels and Lemma 2.1 and Corollary 2.3 immediately lead to

Theorem 2.5. *For fixed k, we have the following implications among sortabilities:*

strong-open ⇒ strong-size ⇒ strong-shape ⇒ sort-sp.-shape ⇒ sort-sp.-size ⇒ sort-sp.-open

⇓ ⇓ ⇓ ⇓ ⇓ ⇓

part-sp.-open ⇒ part-sp.-size ⇒ part-sp.-shape ⇒ weak-shape ⇒ weak-size ⇒ weak-open

In subsequent sections we will present the k-sortability of a property Q with respect to the 12 classifications given in Theorem 2.5 by a 2×6 matrix where cell (i, j) has the level-type modifier as listed above. An entry k (or $\bar{k}$) means Q is (or is not) sortable with respect to that modifier.

Next we give a sortability implication for different k's.

Theorem 2.6. *Let $Q \in \{N, F, A, C, S, E, O\}$. If Q is not strongly (or sort-specific) k-sortable, then Q is not strongly (or sort-specific) k'-sortable for $k' > k$.*

Proof: We only prove the theorem for strong shape-sortability, the other cases are similar. Since Q is not strongly k-shape-sortable, there exists a family Π and a shape $\{n_i\}$ such that Π is weakly k-Q-shape-sortable, but Π does not satisfy Q.

Suppose $Q \in \{N, C, S, E, O\}$. Construct Π' from Π by adding the prefix $(p + k' - k)(p + k' - k - 1) \cdots (p + 1)$ to every $t(\pi) \in \Pi$. Suppose $Q \in \{F, A\}$. Then also add the suffix $(p + 1)(p + 2) \cdots (p + k' - k)$ to each $t(\pi)$. In either case it is easily verified that every $t(\pi') \in \Pi'$ has the same shape and preserves the Q-satisfiability of the corresponding $\pi \in \Pi$; hence Π' does not satisfy Q. Furthermore, Π' is weakly k'-Q-shape-sortable, since π^1 is a k-Q-shape-sorting of π^0 implies that $(\pi^1)'$ is a k-Q-shape-sorting of $(\pi^0)'$. Therefore, Π' is not strongly k'-Q-shape-sortable. □

The following is needed in Section 5. But because its general nature, we give it here.

Theorem 2.7. *Let Π be an l-2-Q-size-sortable family not satisfying Q and Π minimizes $(p, |\Pi|)$ lexicographically. Then no parts can stay put throughout Π.*

Proof: Since Π is minimal, every partition in Π is in a cycle $\pi^1 \to \pi^2 \to \cdots \to \pi^r \to \pi^1$, where $\pi^i \to \pi^{i+1}$ means π^{i+1} is obtained from π^i through 2-Q-size-sorting. Let A be a part which stays throughout Π. If one of these 2-Q-size-sorting involves A, say A with B, then A remains unchanged implies B is so too, violating the definition of 2-sorting. If we remove A from each π^i, then π^i is still 2-Q-size-sortable, and hence still not satisfying Q. Thus, we obtain a new l-2-Q-size-sortable family not satisfying Q but having a smaller p, a contradiction to the definition of Π. □

Note that the argument that no part stays put in a counterexample family is good only for 2-sortability, as the counterexample in Theorem 4.1 has two parts 2 and 3 staying put. Also, the argument is good only for size-sortability, as the counterexample in Theorem 5.5 has part 3 staying put.

3. The sortability of extremalness

For any subset S of I_n, denoted by $\min(S)$ the minimum of S and $\max(S)$ the maximum. Extremalness is a partition property with a single shape once n and p are given. Therefore, it is not defined for shape-sortability.

Theorem 3.3. *E is strongly-k-open-sortable for all $k \geq 2$.*

Proof: For any partition π of I_n, define $s(\pi) = (s_n, s_{n-1}, \ldots, s_1)$ where s_j is the number of non-singleton parts π_i that does not contain element j but contains at least one element less than j. Suppose K is a set of k parts of π that does not satisfy E; and π' is a K-E-open-sorting of π by E-sorting K into K'. Let $\pi'_{i'}$ be the part of K' that contains the largest

Table 1.

all k	all k	not applicable	not applicable	all k	all k
all k	all k	not applicable	not applicable	all k	all k

element in $X = \bigcup_{\pi_i \in K} \pi_i$. Write $s(\pi) = u(\pi) + v(\pi)$, i.e., for each $1 \le j \le n$, $s_j = u_j + v_j$, where u_j (v_j) is the number of non-singleton parts π_i in K (in $\pi \backslash K$) that does not contain j but contains at least one element less than j. Note that $v(\pi) = v(\pi')$; and $u(\pi)$ restricted to X is a non-zero vector but $u(\pi')$ restricted to X is zero. Moreover, $u'_j \le 1$ for any $j \notin X$ with $\min(\pi'_{i'}) < j$ and $u'_j = 0$ otherwise. Also, $u_j \ge 1$ for all $j > \max(\pi'_i)$.

Suppose $u(\pi)$ is lexicographically less than or equal to $u(\pi')$. Since $u(\pi) \ne u(\pi')$, there exists an element j^* in I_n such that $u_{j^*} < u'_{j^*}$ and $u_j = u'_j$ for all $j > j^*$. This is possible only when $u_{j^*} = 0$ and $u'_{j^*} = 1$, which implies $j^* \notin X$ and $\min(\pi'_{i'}) < j^* < \max(\pi'_{i'})$. Since $u_{j^*} = 0$, any element of X less than j^* forms a singleton part in K. As K does not satisfy E, $u_j > 0 = u'_j$ for at least one $j \in X$ with $j > j^*$, a contradiction. Therefore, $u(\pi')$ is lexicographically less than $u(\pi)$; and so $s(\pi')$ is lexicographically less than $s(\pi)$. □

We summarize the sortability of E in Table 1.

4. The sortability of consecutiveness

Recall that $\pi(j)$ is the part element j belongs to π. It was shown (Hwang et al., 1996) that C is strongly-2-open-sortable.

Theorem 4.1. *C is not strongly k-shape-sortable for all $k \ge 3$.*

Proof: Note that by Theorem 2.6, we only need to consider the case of $k = 3$. Let $\Pi = \{\pi^1, \pi^2\} = \{12134, 42131\}$, where $p = 4$ and the shape is $\{2, 1, 1, 1\}$. Then Π is weakly 3-C-shape-sortable, since we can C-sort parts 1, 2, 4 of π^1 into π^2, and parts 1, 3, 4 of π^2 into π^1. But Π contains no partition satisfying C. □

Theorem 4.2. *C is part-specific-k-open-sortable for all $k \ge 2$.*

Proof: Suppose that π does not satisfy C. Define $s(\pi) = \min\{j \in I_n : \pi(j)$ penetrates $\pi(j-1)\}$. Choose two elements x and y such that $x \le s(\pi) - 1 < s(\pi) \le y$ and $K = \{\pi(x), \pi(x+1), \ldots, \pi(y)\}$ is of size k. C-sort K to obtain a partition π'. Clearly, $s(\pi) < s(\pi')$. □

Theorem 4.3. *C is sort-specific-k-shape sortable for all $k \ge 2$.*

Proof: Follows from Corollary 2.4 and the sort-specific-k-shape sortability of E (refer to Theorems 3.3 and 2.5). □

We summarize the sortability of C with Table 2.

Table 2.

$2, \bar{k}$: $k \geq 3$	$2, \bar{k}$: $k \geq 3$	$2, \bar{k}$: $k \geq 3$	all k	all k	all k
all k	all k	all k	all k	all k	all k

5. The sortability of size-consecutiveness

Recall that $\min(\pi_i)$ is the minimum of π_i and $\max(\pi_i)$ the maximum.

Theorem 5.1. *S is strongly k-shape-sortable for all $k \geq 2$.*

Proof: Define $s(\pi) = \sum_{i=1}^{p} \max(\pi_i)$. Suppose that $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_k}$ are k parts not satisfying S. Let π' be any $\{i_1, i_2, \ldots, i_k\}$-$S$-shape-sorting of π. Without loss of generality, assume that

$$\max\left(\pi_{i_1}\right) < \max\left(\pi_{i_2}\right) < \cdots < \max\left(\pi_{i_k}\right), \text{ and}$$
$$\max\left(\pi'_{i_1}\right) < \max\left(\pi'_{i_2}\right) < \cdots < \max\left(\pi'_{i_k}\right).$$

Then it is easily verified that $\max(\pi_{i_t}) \geq \max(\pi'_{i_t})$ for $1 \leq t \leq k$ and there exists at least one t satisfying $\max(\pi_{i_t}) > \max(\pi'_{i_t})$. Hence $\sum_{t=1}^{k} \max(\pi_{i_t}) > \sum_{t=1}^{k} \max(\pi'_{i_t})$. Consequently, $s(\pi)$ is decreasing in any k-S-sorting. □

Theorem 5.2. *S is strongly 2-size-sortable.*

Proof: Suppose to the contrary that there exists a weakly-2-S-size-sortable family not satisfying S. Let $\Pi = \{\pi^1 \to \pi^2 \to \cdots \to \pi^r \to \pi^1\}$ be such a family which minimizes (p, r) lexicographically. Let $\pi^i_{f_i}$ be the part of π^i containing the first element of I_n. By Lemma 2.7, there exists $\pi^t_{f_t}$ such that $\max(\pi^t_{f_t})$ is smallest among all $\max(\pi^i_{f_i})$ and $\pi^t_{f_t} \neq \pi^{t+1}_{f_{t+1}}$ where index $t+1$ is assumed to be modulo r. Assume that π^{t+1} is obtained from π^t by S-sorting parts $\pi^t_{f_t}$ and B. We only need to consider the following two cases:

Case 1. B does not penetrate $\pi^t_{f_t}$.

It must be $|\pi^t_{f_t}| > |B|$ and hence $\max(\pi^{t+1}_{f_{t+1}}) < \max(\pi^t_{f_t})$, a contradiction.

Case 2. B penetrates $\pi^t_{f_t}$ (say b in B and c, d in $\pi^t_{f_t}$ with $c < b < d$).

We note that $\pi^{t+1}_{f_{t+1}}$ contains $\max(\pi^t_{f_t})$ for otherwise $\max(\pi^{t+1}_{f_{t+1}}) < \max(\pi^t_{f_t})$ contrary to the choice of $\pi^t_{f_t}$. It forces $b \in \pi^{t+1}_{f_{t+1}}$. Let $\pi^{t+j}_{f_{t+j}}$ be the first $\pi^i_{f_i}$ after $\pi^{t+1}_{f_{t+1}}$ such that $b \notin \pi^i_{f_i}$. Then π^{t+j} is obtained from π^{t+j-1} by S-sorting $\pi^{t+j-1}_{f_{t+j-1}}$ and a part C. Since $\pi^{t+j}_{f_{t+j}}$ is consecutive in $\pi^{t+j-1}_{f_{t+j-1}} \cup C$, $\max(\pi^{t+j}_{f_{t+j}}) < b < \max(\pi^t_{f_t})$, a contradiction. Thus $b \in \pi^i_{f_i}$ for all i contradicting the assumption that $b \notin \pi^t_{f_t}$. □

Theorem 5.3. *S is not strongly k-size-sortable for all $k \geq 3$.*

Proof: Note that by Theorem 2.6, we only need to consider the case of $k = 3$. Let $\Pi = \{\pi^1, \pi^2\} = \{11233444, 11233244\}$, where $p = 4$. Then Π is weakly 3-S-size-sortable, since we can S-sort parts 1, 2, 4 of π^1 into π^2, and parts 2, 3, 4 of π^2 into π^1. But Π contains no partition satisfying S. □

Theorem 5.4. *S is part-specific-3-size-sortable.*

Proof: Let Π be a sort-specific-3-S-sortable family of p-partitions. Suppose $\pi \in \Pi$ does not satisfy S. We could assume that π satisfies C because we can always use the part-specific-3-size-sortability of C (noting that an S-sorting is also a C-sorting) to obtain a partition satisfying C. So, we need only to look at the subset $\Pi' \subseteq \Pi$ of partitions satisfying C. For $\pi \in \Pi'$, label the parts in increasing order as the elements are increasing (thus π_1 would consist of the smallest elements). We shall prove by induction the claim that for any $\pi \in \Pi'$ and $3 \leq i \leq p$, there exists $\pi' \in \Pi'$ such that $n'_1 \leq n'_2 \leq \cdots \leq n'_i$ and $\pi^i_j = \pi_j$ for $j \geq i + 1$. And thus Π' satisfies S.

For $i = 3$, either $\pi' = \pi$ or a $\{1, 2, 3\}$-S-sorting π' of π is as desired. Suppose the claim holds for i. For any $\pi \in \Pi'$ and $i + 1$, by the induction hypothesis, there exists a $\pi' \in \Pi'$ such that $n'_1 \leq n'_2 \leq \cdots \leq n'_i$ and $\pi'_j = \pi_j$ for $j \geq i + 1$. If $n'_i \leq n'_{i+1}$, then the π' is as desired. Assume $n'_i > n'_{i+1}$. We may assume that π' is chosen such that n'_{i+1} is as large as possible. 3-S-sort $\pi'_{i-1}, \pi'_i, \pi'_{i+1}$ into $\pi''_{i-1}, \pi''_i, \pi''_{i+1}$. Suppose $n''_{i+1} \leq n'_{i+1}$. Then $n''_{i-1} = n'_{i-1} + n'_i + n'_{i+1} - n''_i - n''_{i+1} \geq n'_{i-1} \geq n'_{i-2} = n''_{i-2}$ since $n'_i > n'_{i+1} \geq n''_{i+1} \geq n''_i$. So π'' is as desired. Suppose $n''_{i+1} > n'_{i+1}$. By the induction hypothesis, there exists a $\pi''' \in \Pi'$ with $n'''_1 \leq n'''_2 \leq \cdots \leq n'''_i$ and $\pi'''_{i+1} = \pi''_{i+1}$ and $\pi'''_j = \pi''_j = \pi'_j = \pi_j$ for $j \geq i + 2$. But $n'''_{i+1} = n''_{i+1} > n'_{i+1}$, contradicting the choice of π'. □

Theorem 5.5. *S is not part-specific-2-open-sortable.*

Proof: Let $\Pi = \{\pi^1, \pi^2, \pi^3\}$ be the family of partitions defined in Table 3, which is read as follows. The 0th row specifies subpartitions K of π, e.g., $\overline{\pi_1} = \pi \setminus \{\pi_1\}$ which gives $\overline{\pi^1_1} = \{\pi^1_2, \pi^1_3\}$, $\overline{\pi^2_1} = \{\pi^2_2, \pi^2_3\}$, and $\overline{\pi^3_1} = \{\pi^3_2, \pi^3_3, \pi^3_4\}$; $\overline{\pi_2, \pi_3} = \pi \setminus \{\pi_2, \pi_3\}$ which gives $\overline{\pi^1_2, \pi^1_3} = \{\pi^1_1\}$, $\overline{\pi^2_2, \pi^2_3} = \{\pi^2_1\}$, and $\overline{\pi^3_2, \pi^3_3} = \{\pi^3_1, \pi^3_4\}$. The entry π^j at row π^i and column K means that π^i is K-S-2-open-sorted into π^j. Typically, the first row should be read as: $\pi^1 = 1112233$ and we can $\overline{\pi_3}$-S-2-open-sort π^1 into π^2. It is straightforward to verify that Π is sort-specific-2-S-open-sortable and Π does not satisfy S. □

Table 3.

$K = \pi \setminus \{\pi_i\}$ or $\pi \setminus \{\pi_i, \pi_j\}$	$\overline{\pi_1}$	$\overline{\pi_3}$	$\overline{\pi_2, \pi_3}$
$\pi^1 = 11111222233333$		π^2	
$\pi^2 = 11122222233333$	π^3		
$\pi^3 = 11144222233333$			π^1

Table 4.

$K=\pi\backslash\{\pi_i\}$ or $\pi\backslash\{\pi_i,\pi_j\}$	$\overline{\pi_1}$	$\overline{\pi_2}$	$\overline{\pi_3}$	$\overline{\pi_4}$	$\overline{\pi_3,\pi_4}$	$\overline{\pi_2,\pi_4}$	$\overline{\pi_2,\pi_3}$
$\pi^1=1122333334444$	π^2	π^3					
$\pi^2=1152223334444$					π^4	π^5	π^6
$\pi^3=1122133334444$			π^7	π^1			
$\pi^4=1122223334444$	π^2			π^1			
$\pi^5=1132223334444$	π^2			π^1			
$\pi^6=1142223334444$	π^2	π^5	π^4				
$\pi^7=1122433334444$	π^2	π^3					

Theorem 5.6. *S is not part-specific-3-open-sortable.*

Proof: Consider a family of partitions $\Pi=\{\pi^1,\pi^2,\ldots,\pi^7\}$ defined in Table 4, whose meaning is as described in the proof of Theorem 5.5. It is straightforward to check that Π is sort-specific-3-S-open-sortable and Π does not satisfy S. □

Theorem 5.7. *S is not part-specific-k-size-sortable for all $k\geq 4$.*

Proof: Let ${}^4\Pi=\{{}^4\pi^1,{}^4\pi^2,\ldots,{}^4\pi^{11}\}$ be a family of 5-partitions of I_{15} defined in Table 5, in which π^i stands for ${}^4\pi^i$. It is straightforward to check that ${}^4\Pi$ is sort-specific-S-4-size-sortable and ${}^4\Pi$ does not satisfy S.

Table 5.

$K=\pi\backslash\{\pi_i\}$	$\overline{\pi_0}$	$\overline{\pi_1}$	$\overline{\pi_2}$	$\overline{\pi_3}$	$\overline{\pi_4}$
$\pi^1=001122133334444$	π^2			π^5	π^{11}
$\pi^2=001222233334444$			π^6	π^5	π^{11}
$\pi^3=001102223334444$			π^{10}	π^4	π^{11}
$\pi^4=001122223334444$	π^2	π^3			π^{11}
$\pi^5=001122433334444$	π^2	π^3	π^1		
$\pi^6=001222213334444$	π^2	π^7		π^4	π^{11}
$\pi^7=001022213334444$	π^8		π^9	π^4	π^{11}
$\pi^8=001022233334444$			π^9	π^5	π^{11}
$\pi^9=001122213334444$	π^2			π^4	π^{11}
$\pi^{10}=001132223334444$	π^2	π^3			π^{11}
$\pi^{11}=001122333334444$	π^2	π^3	π^1		

Table 6.

$\bar{k}$: all k	$2, \bar{k}$: $k \geq 3$	all k	all k	all k	all k
$\bar{k}$: all k	$2, 3, \bar{k}$: $k \geq 4$	all k	all k	all k	all k

Note that in this construction, each $t(\pi^i)$ contains 0, i.e., $\pi^i = \{\pi_0^i, \pi_1^i, \pi_2^i, \pi_3^i, \pi_4^i\}$. For $k \geq 5$, we define ${}^k\Pi = \{{}^k\pi^1, {}^k\pi^2, \cdots, {}^k\pi^{2k+3}\}$ from ${}^{k-1}\Pi$ as follows:

$$
\begin{aligned}
{}^k\pi^j &= {}^{k-1}\pi^j \underbrace{k\cdots k}_{k} \text{ for } 1 \leq j \leq 2k+1,\\
{}^k\pi^{2k+2} &= {}^{k-1}\pi^{2k+1}_{0:k-3} \underbrace{(k-2)\cdots(k-2)}_{(k-2)}(k-3)(k-3) \underbrace{(k-1)\cdots(k-1)}_{(k-1)} \underbrace{k\cdots k}_{k},\\
{}^k\pi^{2k+3} &= {}^{k-1}\pi^{2k+1}_{0:k-3} \underbrace{(k-2)\cdots(k-2)}_{(k-2)} \underbrace{(k-1)\cdots(k-1)}_{(k+1)} \underbrace{k\cdots k}_{k},
\end{aligned}
$$

where ${}^{k-1}\pi^{2k+1}_{0:k-3}$ is the shortest prefix subsequence of the sequence ${}^{k-1}\pi^{2k+1}$ that contains all elements in $\{0, 1, 2, 3, \ldots, k-3\}$. For example ${}^4\pi^{11}_{0:2} = 001122$. For $k \geq 5$ it can readily be seen that ${}^{k-1}\pi^{2k+1}_{0:k-3}$ is of the form

$$
0011223334444\cdots \underbrace{(k-3)\cdots(k-3)}_{(k-3)}
$$

with length $3 + \frac{(k-2)(k-3)}{2}$, and hence $|{}^k\pi^j| = 15 + \frac{(k+5)(k-4)}{2}$ for $1 \leq j \leq 2k+3$.

Then it is straightforward to verify by induction on k that ${}^k\Pi$ is sort-specific-k-S-size-sortable and ${}^k\Pi$ does not satisfy S. For instance, if ${}^{k-1}\pi^j$ is $\overline{\pi_i}$-S-$(k-1)$-size-sorted into ${}^{k-1}\pi^{j'}$ in ${}^{k-1}\Pi$, where $1 \leq j \leq 2k+1$ and $0 \leq i \leq k-1$, then ${}^k\pi^j$ is $\overline{\pi_i}$-S-k-size-sorted into ${}^k\pi^{j'}$ in ${}^k\Pi$; if ${}^{k-1}\pi^{2k+1}$ is $\overline{\pi_i}$-S-$(k-1)$-size-sorted into ${}^{k-1}\pi^{j'}$ in ${}^{k-1}\Pi$ for some $0 \leq i \leq k-3$, then ${}^k\pi^{2k+3}$ is $\overline{\pi_i}$-S-k-size-sorted into ${}^k\pi^{j'}$ in ${}^k\Pi$; ${}^k\pi^{2k+2}$ is $\overline{\pi_k}$-S-k-size sorted into ${}^k\pi^{2k+3}$ in ${}^k\Pi$; and ${}^k\pi^{2k+3}$ is $\overline{\pi_{k-2}}$-S-k-size-sorted into ${}^k\pi^{2k+2}$ in ${}^k\Pi$. □

We summarize the sortability of S with Table 6.

6. The sortability of nestedness

Recall that (Hwang et al., 1996) N was shown to be part-specific-2-open-sortable.

Theorem 6.1. *N is not strongly-k-shape-sortable for $k \geq 2$.*

Proof: Note that by Theorem 2.6, we only need to consider the case of $k = 2$. Let $\Pi = \{\pi^1, \pi^2, \pi^3, \pi^4\} = \{1311233222, 1133231222, 1122231332, 1322233112\}$, where $p = 3$ and the shape is $\{3, 4, 3\}$. Then Π is weakly-2-N-shape-sortable, since we can N-sort parts 1, 3 of π^1 into π^2, parts 2, 3 of π^2 into π^3, parts 1, 3 of π^3 into π^4, and parts 1, 2 of π^4 into π^1. But Π contains no partition satisfying N. □

Theorem 6.2. *N is part-specific-k-open-sortable for all $k \geq 2$.*

Proof: Suppose π is a partition which is not nested. A *violation* of π is a quadruple (a, b, c, d) with $a < b < c < d$, where a and c are in a part and b and d in another. Let $s(\pi) = (d(\pi), c(\pi))$ be a lexicographic minimizer of ordered pairs (d, c) where (a, b, c, d) is a violation. Let $\{a(\pi), c(\pi)\} \subseteq \pi_i$ and $\{b(\pi), d(\pi)\} \subseteq \pi_j$.

Consider the *U-restricted penetration* relation on the parts of π using elements in $U = \{1, 2, \ldots, d(\pi) - 1\}$. More precisely, $A \to^U B$ if there exist $x, z \in B$ and $y \in A$ such that $x < y < z < d(\pi)$. By the definition of $s(\pi)$, we have that $\to^U$ is a partial order on the parts of π, or equivalently, $\pi^U = \{\pi_l \cap U : \pi_l \in \pi \text{ and } \pi_l \cap U \neq \emptyset\}$ is a nested partition of U. Let

$$I(\pi) = \{\pi_l \in \pi : \{\pi_l, \pi_j\}^{\{1,2,\ldots,d(\pi)\}} \text{ is not nested}\}.$$

Note that for any $\pi_l \in I(\pi)$, we have $\pi_j \to^U \pi_l$. On the other hand, suppose $\pi_j \to^U \pi_l$. Then $x < y < z < d(\pi)$ for some $x, z \in \pi_l$ and $y \in \pi_j$. Thus $(x, y, z, d(\pi))$ is a violation and so $\pi_l \in I(\pi)$, i.e.,

$$I(\pi) = \{\pi_l \in \pi : \pi_j \to^U \pi_l\}.$$

By the fact "for any two distinct parts penetrated by a common part in a nested partition, one is penetrated by the other" (Lemma 4.5(a) of (Hwang et al., 1996)), we have that $J(\pi) = I(\pi) \cup \{\pi_j\}$ is totally ordered under the relation $\to^U$, i.e.,

$$J(\pi) = \{\pi_j = \pi_{j_1} \to^U \pi_{j_2} \to^U \cdots \to^U \pi_{j_r}\}.$$

Moreover, each $\pi_{j_i} \cap U$ is the disjoint union of two non-empty sets A_i and B_i $(2 \leq i \leq r)$ such that

$$A_r < \cdots < A_3 < A_2 < (\pi_{j_1} \cap U) < B_2 < B_3 < \cdots < B_r < \{d(\pi)\}, \qquad (*)$$

where $A < B$ means $x < y$ for any $x \in A$ and any $y \in B$. Note that using this notation, $\pi_i = \pi_{j_2}$ and $c(\pi) = \min B_2$. Also, $A \not\to^U B$ for any $A \in J(\pi)$ and any $B \in \pi - J(\pi)$.

For the case of $r \geq k$, let $\pi_K = \{\pi_{j_1}, \pi_{j_2}, \ldots, \pi_{j_k}\}$. Clearly, π_K does not U-restricted penetrate any part in $\pi - J(\pi)$ by the definition of $I(\pi)$. For the case of $r < k$, choose $k - r$ parts from $\pi - J(\pi)$ such that any un-chosen part of $\pi - J(\pi)$ is not U-restricted penetrated by any chosen part, e.g., a lower set with respect to the partial order $\to^U$ works. In this case, let π_K be the union of these $k - r$ parts and $J(\pi)$. Then no part in π_K U-restricted penetrates a part in $\pi - \pi_K$.

Now k-N-sort π_K into $\pi'_{K'}$ to get π'. Suppose $s(\pi') = (d(\pi'), c(\pi'))$ is lexicographically less than or equal to $s(\pi)$, where $\{a(\pi'), c(\pi')\} \subseteq \pi'_{i'}$ and $\{b(\pi'), d(\pi')\} \subseteq \pi'_{j'}$. Then, $d(\pi') \leq d(\pi)$. By the minimality of $s(\pi)$, at least one of $\pi'_{i'}$ and $\pi_{j'}$ is in $\pi'_{K'}$. However, as $\pi'_{K'}$ is nested, but $\pi'_{i'}$ and $\pi'_{j'}$ penetrate each other, the other of $\pi'_{i'}$ and $\pi'_{j'}$ is some $\pi_t \in \pi - \pi_K$. We consider two cases.

Table 7.

$\bar{k}$: all k	$\bar{k}$: all k	$\bar{k}$: all k	all k	all k	all k
all k	all k	all k	all k	all k	all k

Case 1. $\{a(\pi'), c(\pi')\} \subseteq \pi'_{r'} \in \pi'_{K'}$ and $\{b(\pi'), d(\pi')\} \subseteq \pi_t \in \pi - \pi_K$.

As $\pi'_{K'}$ is a resorting of π_K, $c(\pi') \in \pi_p \in \pi_K$ and $a(\pi') \in \pi_q \in \pi_K$. Since $\pi_p \to^U \pi_t$, our choice of π_K dictates $r \geq k$ and $\pi_p = \pi_{j_x}, \pi_q = \pi_{j_y}$ and $\pi_t = \pi_{j_z}$ for some $x \leq k < z$ and $y \leq k < z$. Also, $b(\pi') \in A_z$, $d(\pi') \in B_z$ and $a(\pi') \in A_y$ with $y < z$, which contradict the ordering in $(*)$.

Case 2. $\{a(\pi'), c(\pi')\} \subseteq \pi_t \in \pi - \pi_K$ and $\{b(\pi'), d(\pi')\} \subseteq \pi'_{r'} \in \pi'_{K'}$.

As $\pi'_{K'}$ is a resorting of π_K, $b(\pi') \in \pi_p \in \pi_K$ and $d(\pi') \in \pi_q \in \pi_K$. Since $\pi_p \to^U \pi_t$, our choice of π_K dictates $r \geq k$ and $\pi_p = \pi_{j_x}, \pi_q = \pi_{j_y}$ and $\pi_t = \pi_{j_z}$ for some $x \leq k < z$ and $y \leq k < z$. If $y > 1$, the argument for a contradiction is same as Case 1. When $y = 1$, then $d(\pi') = d(\pi)$. But, as $k \geq 2$, we have that $z > 2$ and $c(\pi') \geq \min B_z > \min B_2 = c(\pi)$, contradicting the assumption $s(\pi') < s(\pi)$.

Thus we conclude that $s(\pi')$ is lexicographically greater than $s(\pi)$. □

Theorem 6.3. *N is sort-specific-k-shape-sortable for all $k \geq 2$.*

Proof: Follows from Corollary 2.4 and the sort-specific-k-shape-sortability of E. □

We summarize the sortability of N with Table 7.

7. The sortability of fully nestedness and almost fully nestedness

Recall that (Hwang et al., 1996) F is not part-specific-2-shape-sortable.

Theorem 7.1. *F is not part-specific-k-shape-sortable for all $k \geq 2$.*

Proof: For any $k \geq 2$, let $n = 4k + 2$, $p = k + 1$, $n_1 = 2k + 2$, $n_2 = n_3 = \cdots = n_{k+1} = 2$. Consider the family Π of all $\{n_1, n_2, \ldots, n_p\}$-partitions π of I_n that begins with $\underbrace{1 \cdots 1}_{i}\alpha 1$ or ends with $1\alpha\underbrace{1 \cdots 1}_{i}$ for some $i \geq 1$ and $\alpha = j$ or jj' with $j, j' \geq 2$. It is clear that Π does not satisfy F as for the above π, either π_1 and π_j penetrate each other when $\alpha = j$ or $j \neq j'$, or π_j and some π_i are not fully nested when $\alpha = jj'$ with $j = j'$.

Let $K = \pi \backslash \{\pi_r\}$ be a set of k parts of $\pi \in \Pi$ that does not satisfy F. If $r = 1$, any K-k-F-sorting induces a partition π' in Π, since the beginning (or ending) segment of π' has the same form as that of π except α. Suppose $r \geq 2$, say $\pi_r = \{i < j\}$. Either $i \leq n/2$ or $j \geq n/2$. For the case when $i \leq n/2 = n_1 - 1$, we can k-F-sort K into K' so

Table 8.

$\bar{k}$: all k	$\bar{k}$: all k	$\bar{k}$: all k	all k	all k	all k
$\bar{k}$: all k	$\bar{k}$: all k	$\bar{k}$: all k	all k	all k	all k

that $\{1, 2, \ldots, i-1, i+\beta\} \subseteq \pi'_1$, where $\beta = 1$ if $j > i+1$ and $\beta = 2$ if $j = i+1$. In this case, $\pi' \in \Pi$. Similarly, for the case when $j \geq n/2$, we can K-k-F-sort π into $\pi' \in \Pi$. Thus Π is sort-specific-k-F-sortable but Π does not satisfy F. □

Corollary 7.2. *A is not part-specific-k-shape-sortable for all $k \geq 2$.*

Proof: The above example for F is also an example for A since the given family is also part-specific-k-A-shape sortable but does not satisfy A. □

Theorem 7.3. *F is sort-specific-k-shape-sortable for all $k \geq 2$.*

Proof: Define $\sec\min(\pi_i)$ to be the second smallest element in π_i. Define $s(\pi) = \sum_{i=1}^{p} (\max(\pi_i) - \sec\min(\pi_i))$. Consider the k-F-shape-sorting which recursively fills a part with all the largest element plus the smallest element. Then a k-F-shape-sorting on a set N' of elements is equivalent to a k-C-shape-sorting on the set $N' \backslash \{$the smallest k elements in $N'\}$ and $s(\pi)$ is like the sum of range on the truncated parts (except the innermost part may become empty if it contained a single element). Hence $s(\pi)$ strictly decreases. □

Corollary 7.4. *A is sort-specific-k-shape-sortable for all $k \geq 2$.*

Proof: By Theorem 7.3 and Corollary 2.4. □

We summarize the sortabilities of F and A by Table 8.

8. The sortability of order-consecutiveness

Recall that (Hwang et al., 1996) O is not part-specific-k-shape-sortable for $k = 2, 3$.

Theorem 8.1. *O is not part-specific-k-shape-sortable for all $k \geq 4$.*

Proof: For any $k \geq 4$, let $n = 2k+1$, $p = k+1$, $n_1 = k$, $n_2 = 2$, $n_3 = \cdots = n_{k+1} = 1$. Consider the family Π of all $\{n_1, n_2, \ldots, n_p\}$-partitions π of two patterns as follows. In the first pattern, the first 3 positions of $t(\pi)$ are 232, positions $k+1$ and the last 3 are 1, and from position $k+2$ to position $2k-2$, all but one elements are 1:

$$232\,\underbrace{\text{non 1}}_{k-3}1\,\underbrace{\text{all 1 except for one position}}_{k-3}\,111.$$

Table 9.

$\bar{k}$: all k	$\bar{k}$: all k	$\bar{k}$: all k	all k	all k	all k
$\bar{k}$: all k	$\bar{k}$: all k	$\bar{k}$: all k	all k	all k	all k

The second pattern is a reverse of the first pattern. For any π of the first pattern, let π_i be the part penetrating π_1. Parts $\pi_1, \pi_2, \pi_3, \pi_i$ make π not satisfy O.

Let $K = \pi \setminus \{\pi_r\}$ be a set of k parts not satisfying O. Then $r \in \{4, 5, \ldots, k+1\} \setminus \{i\}$. We can K-k-O-sort π into a partition π' of the second pattern with $\pi'_{i'} = \{r\}$; this can always be done since $n_1 = k$ and r, which stays during the sorting, is in the first k positions. Similarly, for any π of the second pattern and any K not satisfying O, we can sort it into a partition of the first pattern. Therefore, Π is part-specific-k-O-sortable. But neither pattern satisfies O; hence Π does not satisfy O. Theorem 8.1 follows immediately. □

Theorem 8.2. *O is sort-specific-k-shape-sortable for all $k \geq 4$.*

Proof: Follows from Corollary 2.4 and the sort-specific-k-shape-sortability of C. □

We summarize the sortability of O in Table 9.

9. Other types of restriction

Chakravarty et al. (1991) considered the size partition problem except that empty parts are allowed. We will call this the *bounded-size partition problem*. Barnes et al. (1991) considered the shape-partition problem except each n_i lies in a range $b_i \leq n_i \leq u_i$. We will call this the *bounded-shape-partition problem*. Clearly, for $l =$ strong or part-specific, open-sortable implies bounded-size-sortable and size-sortable implies bounded-shape sortable; while for $l =$ weak or sort-specific, bounded-size sortable implies open-sortable and bounded-shape-sortable implies size-sortable. But the reverse implications are also true. For the first case, let Π be an open-Q-sortable family, which does not satisfy Q, and let p be the maximum partition size. Then Π is also an example against the bounded-p sortability. Similarly, let Π be a size-Q-sortable family which does not satisfy Q and whose n_i varies from l_i to u_i. Then Π is also an example against the bounded-$\{l_i, u_i : 1 \leq i \leq p\}$-sortability.

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