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Comments on "Fuzzy programming with nonlinear membership functions ..."

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Abstract

Yang et al., in their paper "Fuzzy programming with nonlinear membership functions ...", published in Fuzzy Sets and Systems 41 (1991), declared that their model can solve a fuzzy program with an S-shaped membership function by adding only one 0–1 variable. This paper indicates that their declaration is correct only for a specific type of S-shape membership functions. We propose another model to treat the fuzzy programs which cannot be solved effectively by Yang et al. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Fuzzy programming; Membership functions

1. Introduction

The fuzzy programming problem discussed by Yang et al. in 1991 [2] is represented as follows:

$$\begin{aligned} &\text{Maximize} \quad \lambda \\ &\text{subject to} \quad \lambda - \mu_s(z_s(X)) \leqslant 0, \quad \text{for } s = 1,2,\ldots,S, \\ &\qquad X \in F \quad \text{(a feasible set)}, \end{aligned} \tag{1}$$

where $\mu_s(X)$ is a membership function of sth objective, which is specified in the following form:

$$\mu_s(z_s(X)) = \begin{cases} 1 & \text{for } z_s(X) \geqslant z_s^*, \\ 1 - \dfrac{z_s^* - z_s(X)}{z_s^* - z_s^-} & \text{for } z_s^- \leqslant z_s(X) \leqslant z_s^*, \\ 0 & \text{for } z_s(X) \leqslant z_s^-, \end{cases} \tag{2}$$

where z_s^* and z_s^- are constants, which represent, respectively, the maximal and minimal levels for the achievement of kth objective.

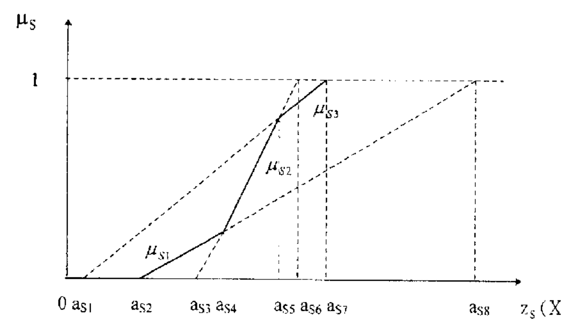

Fig. 1. An S-shaped membership function [2].

An instance of μ_s, represented by Yang et al., is depicted in Fig. 1, where μ_s is an S-shaped membership function approximated by the intersection and union of three ramp-type functions μ_{s1}, μ_{s2} and μ_{s3}. μ_s is

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represented as

$$\mu_s = \mu_{s1} \cup (\mu_{s2} \cap \mu_{s3}) \tag{3}$$

in which $\cup$ means union and $\cap$ means intersection.

Expression (3) can also be rewritten as

$$\mu_s = \mu_{s1} \vee \text{Minimum}\ \{\mu_{s2}, \mu_{s3}\}, \tag{4}$$

in which $\vee$ means "or".

μ_{s1}, μ_{s2} and μ_{s3}, by piecewise approximation, are expressed below:

$$\mu_{s1}(z_s) = \begin{cases} 1 & \text{if } z_s \geqslant a_{s8}, \\ 1 - \dfrac{a_{s8} - z_s}{a_{s8} - a_{s2}} & \text{if } a_{s2} \leqslant z_s < a_{s8}, \\ 0 & \text{otherwise;} \end{cases}$$

$$\mu_{s2}(z_s) = \begin{cases} 1 & \text{if } z_s \geqslant a_{s6}, \\ 1 - \dfrac{a_{s6} - z_s}{a_{s6} - a_{s3}} & \text{if } a_{s3} \leqslant z_s < a_{s6}, \\ 0 & \text{otherwise;} \end{cases} \tag{5}$$

and

$$\mu_{s3}(z_s) = \begin{cases} 1 & \text{if } z_s \geqslant a_{s7}, \\ 1 - \dfrac{a_{s7} - z_s}{a_{s7} - a_{s1}} & \text{if } a_{s1} \leqslant z_s < a_{s7}, \\ 0 & \text{otherwise;} \end{cases}$$

Yang et al. formulate the associated fuzzy programming problem with the membership functions in (4) and (5) as follows:

Yang et al. model:

$$\left.\begin{array}{l} \text{Maximize } \lambda \\ \text{subject to} \\ \quad \lambda \leqslant 1 - \dfrac{a_{s2} - z_s(X)}{a_{s8} - a_{s2}} + M(1-\delta_s), \\ \quad \lambda \leqslant 1 - \dfrac{a_{s3} - z_s(X)}{a_{s6} - a_{s3}} + M\delta_s, \\ \quad \lambda \leqslant 1 - \dfrac{a_{s1} - z_s(X)}{a_{s7} - a_{s1}} + M\delta_s, \\ \quad \delta_s = 0, 1,\ X \in F, \end{array}\right\} \quad s = 1, 2, \ldots, S \tag{6}$$

where M represents a large positive number.

Yang et al. observed that since μ_s in (3) (or (4)) contains one $\cup$ (or one $\vee$) operator, the model in (6) only requires to use one 0–1 variable δ_s for representing μ_s. Yang et al. therefore stated that each union operator in an S-shaped membership function and can be expressed by their model using one 0–1 variable only. This declaration however is not always correct, as analyzed as follows:

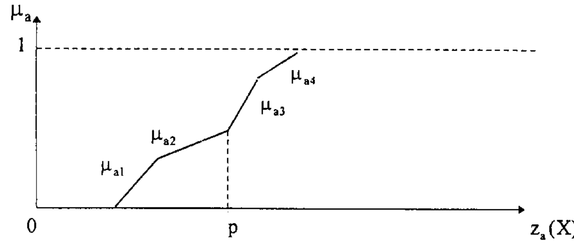

Fig. 2. Type 1 S-shaped membership function.

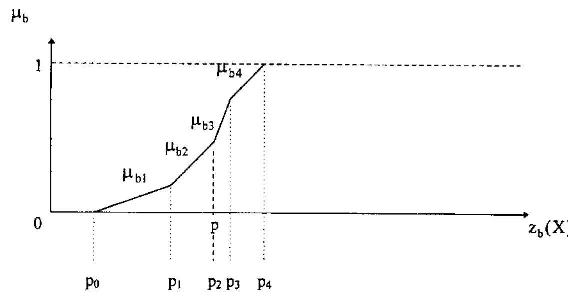

Fig. 3. Type 2 S-shaped membership function.

Consider two S-shaped membership functions μ_a and μ_b appearing in Fig. 2 and Fig. 3, respectively. μ_a can be expressed as

$$\mu_a = (\mu_{a1} \cap \mu_{a2}) \cup (\mu_{a3} \cap \mu_{a4}) \text{ or}$$
$$\mu_a = \text{Minimum}\ \{\mu_{a1}, \mu_{a2}\} \vee \text{Minimum}\ \{\mu_{a3}, \mu_{a4}\}. \tag{7}$$

μ_b can be expressed as

$$\mu_b = (\mu_{b1} \cap \mu_{b2}) \cup (\mu_{b3} \cap \mu_{b4}) \text{ or}$$
$$\mu_b = \text{Maximum}\ \{\mu_{b1}, \mu_{b2}\} \vee \text{Minimum}\ \{\mu_{b3}, \mu_{b4}\}. \tag{8}$$

Both μ_a and μ_b contain only one $\cup$ or $\vee$ operator. However, only μ_a can be represented by the Yang et al. model by adding one 0–1 variable, while μ_b requires to use "two" 0–1 variables if represented by their model. This is checked as follows:

Since $\mu_a = \text{Minimize}\{\mu_{a1}, \mu_{a2}$, for $z_a(X) \leqslant p\}$ and $\mu_a = \text{Minimize}\{\mu_{a3}, \mu_{a4}$ for $z_a(X) \geqslant p\}$, the associated fuzzy program with membership function μ_a can be directly expressed by the Yang et al. model given below:

Model 1 :
Maximize λ
subject to

$$\lambda \leqslant \mu_{a1}(X) + M(1 - \delta_a),$$

$$\lambda \leqslant \mu_{a2}(X) + M(1 - \delta_a),$$

$$\lambda \leqslant \mu_{a3}(X) + M\delta_a,$$

$$\lambda \leqslant \mu_{a4}(X) + M\delta_a,$$

$$\delta_a \in (0, 1), \quad X \in F.$$

It is convenient to check that if $\delta_a = 1$ then $\lambda = \text{Minimum}\{\mu_{a1}, \mu_{a2}\}$, and otherwise $\lambda = \text{Minimum}\{\mu_{a3}, \mu_{a4}\}$,

However, since $\mu_b = \text{Maximize}\{\mu_{b1}, \mu_{b2}$, for $z_b(X) \leqslant p\}$, in the Yang et al. model it is impossible to use one 0–1 variable to express μ_b. In fact, the Yang et al. model requires to use "two" 0–1 variables to treat μ_b, which is formulated below:

Model 2 :
Maximize λ
subject to

$$\lambda \leqslant \mu_{b1}(X) + M(1 - \delta_1),$$

$$\lambda \leqslant \mu_{b2}(X) + M(1 - \delta_2),$$

$$\lambda \leqslant \mu_{b3}(X) + M(\delta_1 + \delta_2),$$

$$\lambda \leqslant \mu_{b4}(X) + M(\delta_1 + \delta_2),$$

$$\delta_1 + \delta_2 \leqslant 1,$$

$$\delta_1, \delta_2 \in (0, 1), \quad X \in F.$$

Model 2 is checked as follows. If $\delta_1 = \delta_2 = 0$ then $\lambda = \text{Minimum}\{\mu_{b3}, \mu_{b4}\}$. If $\delta_1 = 1$ and $\delta_2 = 0$ then $\lambda_1 = \mu_{b1}$; if $\delta_1 = 0$ and $\delta_2 = 1$ then $\lambda = \mu_{b2}$.

We will demonstrate that it is still possible to express μ_b by one 0–1 variable only, as discussed in the following section.

2. Proposed model

We concentrate our discussion on the following two types of S-shaped membership functions, where each type is represented by the union of two groups of functions.

Type 1: (*concave*) ∪ (*concave*). The line segments in both the groups form a set of concave lines as shown in Fig. 2.

Type 2: (*convex*) ∪ (*concave*) *or* (*concave*) ∪ (*convex*). The line segments in both groups form different sets of convex or concave lines. If the first group forms a convex line then the second group forms a concave line, and vice versa, as shown in Fig. 3.

Only the fuzzy programs with Type 1 membership functions can be solved by the Yang et al. model by adding one 0–1 variable. The Yang et al. model, however, needs to use more than one variable to solve the fuzzy programs with Type 2 membership functions.

Here, we propose a model to solve the fuzzy program with Type 2 membership function. Consider Fig. 3 for instance; the associated program is formulated below:

Model 3 :
Maximize $\lambda = -\lambda_1\delta_b + \lambda_2(1 - \delta_b),$

subject to $\lambda_1 \geqslant \mu_{b1}(X) + M(\delta_b - 1),$

$$\lambda_1 \geqslant \mu_{b2}(X) + M(\delta_b - 1),$$

$$\lambda_2 \leqslant \mu_{b3}(X) + M\delta_b,$$

$$\lambda_2 \leqslant \mu_{b4}(X) + M\delta_b,$$

$$X \in F, \quad \delta_b \in (0, 1).$$

Model 3 can be verified as follows:

Case 1: $\delta_b = 1$. In this case, λ_1 needs to be minimized and $\lambda = \text{Maximum}\{\mu_{b1}, \mu_{b2}\}$.

Case 2: $\delta_b = 0$. In this case, $\lambda = \text{Minimum}\{\mu_{b3}, \mu_{b4}\}$.

Since the optimal conditions for both cases are fully formulated, Model 3 is verified.

Model 3 is a non-linear mixed 0–1 program, which can be converted into following linear mixed 0–1 program, based on the linearization procedure developed by Li [1].

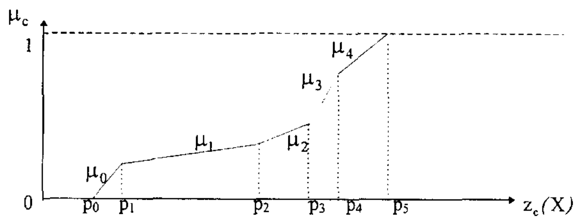

Fig. 4. An S-shaped membership function.

Model 4 :

Minimize $z - \lambda_2$,

subject to
$$z \geqslant \lambda_1 + \lambda_2 + M(\delta_b - 1),$$
$$z \geqslant 0,$$
$$\lambda_1 \geqslant \mu_{b1}(X) + M(\delta_b - 1),$$
$$\lambda_1 \geqslant \mu_{b2}(X) + M(\delta_b - 1),$$
$$\lambda_2 \leqslant \mu_{b3}(X) + M\delta_b,$$
$$\lambda_2 \leqslant \mu_{b4}(X) + M\delta_b,$$
$$X \in F, \quad \delta_b \in (0,1),$$

where variable z is used to replace the polynomial term $\delta_b(\lambda_1 + \lambda_2)$ in Model 3.

Any S-shaped membership function can be regarded as the combination of Type 1 and Type 2 functions. Take Fig. 4 for instance, its membership function can be expressed in the following ways:

$$\mu_c = \mu_0 \cup (\mu_1 \cap \mu_2) \cup (\mu_3 \cap \mu_4) \tag{9}$$

or

$$\mu_c = (\mu_0 \cap \mu_1) \cup \mu_2 \cup (\mu_3 \cap \mu_4). \tag{10}$$

Expression (9) is a (line) ∪ (convex) ∪ (concave) pattern and expression (10) is a (concave) ∪ (line) ∪ (concave) pattern. Both expressions contain 2 union operators, which needs to add two 0–1 variables to formulate a fuzzy programming model.

The associated fuzzy program with a membership function expressed in (9) can be formulated below based on Model 4:

Minimize $\lambda = z - \lambda_0 + \lambda_2$

subject to
$$z \geqslant \lambda_1 + \lambda_2 + M(\delta_2 - 1),$$
$$z \geqslant 0,$$
$$\lambda_0 \leqslant \mu_0(X) + M(1 - \delta_1),$$
$$\lambda_1 \geqslant \mu_1(X) + M(\delta_2 - 1),$$
$$\lambda_1 \geqslant \mu_2(X) + M(\delta_2 - 1),$$
$$\lambda_2 \leqslant \mu_3(X) + M\delta_2,$$
$$\lambda_2 \leqslant \mu_4(X) + M\delta_2,$$
$$\delta_2 + \delta_1 \leqslant 1, \quad X \in F.$$

It is clear to check that if $\delta_1 = 1$ and $\delta_2 = 0$ then $\lambda = \mu_0$; if $\delta_1 = 0$ and $\delta_2 = 1$ then $\lambda = \text{Maximum}\{\mu_1, \mu_2\}$; if $\delta_1 = \delta_2 = 0$ then $\lambda = \text{Minimum}\{\mu_3, \mu_4\}$.

3. Numerical example

Consider the membership function μ_b depicted in Fig. 3, where $(p_0, p_1, p_2, p_3, p_4) = (1, 3, 4, 5, 7)$ and $(\mu(p_0), \mu(p_1), \mu(p_2), \mu(p_3), \mu(p_4)) = (0, 0.1, 0.3, 0.8, 1.0)$. Using the Yang et al. method (Model 1), the optimization program related to Fig. 3 is formulated as below:

Yang et al. model:

Maximize λ

subject to
$$\lambda \leqslant 1 - \frac{21 - z_b(X)}{20} + M(1 - \delta),$$
$$\lambda \leqslant 1 - \frac{7.5 - z_b(X)}{5} + M(1 - \delta),$$
$$\lambda \leqslant 1 - \frac{5.4 - z_b(X)}{2} + M\delta,$$
$$\lambda \leqslant 1 - \frac{7 - z_b(X)}{10} + M\delta,$$
$$\lambda, \quad z_b(X) \geqslant 0.$$

Suppose we add one more constraint $z_b(X) \leqslant 3.5$ to the above program; then the optimal solution found by Yang et al. model is $\mu_b(3.5) = 0.125$, which is located on line μ_{b1}. However, this is incorrect. The correct answer should be $\mu_b(3.5) = 0.2$, which is located on line μ_{b2}.

Solving the same problem by the proposed method is as follows:

The proposed model:

Minimize $Z - \lambda_2$

Subject to $Z \geqslant \lambda_1 + \lambda_2 + M(\theta - 1), \quad Z \geqslant 0,$

$$\lambda_1 \geqslant 1 - \frac{21 - z_b(X)}{20} + M(\theta - 1),$$

$$\lambda_1 \geqslant 1 - \frac{7.5 - z_b(X)}{5} + M(\theta - 1),$$

$$\lambda_2 \leqslant 1 - \frac{5.4 - z_b(X)}{2} + M\theta,$$

$$\lambda_2 \leqslant 1 - \frac{7 - z_b(X)}{10} + M\theta,$$

$$z_b(X) \leqslant 3.5, \lambda_1, \lambda_2, \quad z_b(X) \geqslant 0.$$

This obtained optimal value of $\mu_b(3.5)$ is 0.2, located exactly on the line μ_{b2}. This example demonstrates that the Yang et al. method cannot correctly treat the union of convex function and concave function by using only one 0–1 variable.

4. Conclusions

This paper indicates that the Yang et al. model can only effectively solve the fuzzy program with a specfic S-shaped membership function (the so-called Type 1 function). We propose a new fuzzy programming model to treat the membership function (the so-called Type 2 function) which cannot be handled effectively by the Yang et al. model.

References

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