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The preferable resonators for Kerr-lens mode-locking determined by stability factors of their iterative maps

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Abstract

The generalized stability factor of a general resonator was obtained from its iterative map based on the linear stability analysis. Since a physical system tends to stay with high stability, the preferred resonator for Kerr-lens mode-locking (KLM) is determined by the relative stability between KLM and CW operations. With this criterion the preferable KLM regions agree with the previous experimental self-starting regions when the Kerr medium is located around the center of the resonator. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

In recent years, an approach borrowed from the nonlinear dynamics has been used to study the dynamics of laser resonators [1–3]. By constructing the iterative maps from the beam parameters, the researchers found that the dynamics of the beam parameters have intrinsically complicated behaviors obtained by the bare resonators without invoking the effect of the laser medium [1,2]. In some special resonators the dynamics is very sensitive to nonlinear effect in the geometrically stable region [2]. This approach offers a useful tool to discuss the dynamics of the resonators; in particular, an optical resonator attracts attention on its nonlinearly physical properties.

In a Kerr-lens mode-locking (KLM) resonator, which is a well-known nonlinear resonator for femtosecond pulse generation, the self-focusing effect within a Kerr medium modifies the cavity mode profile [4,5]. This is described as a self-amplitude modulation (SAM) resulting from either decreasing loss through an internal aperture as increasing the KLM intensity (referred to as hard aperturing) [6] or increasing gain by properly adjusting the overlap of the cavity and pump mode profiles (referred to as soft aperturing) [7,8]. Although it was believed that the KLM Ti:sapphire lasers could not start without initial perturbation because of too low nonlinear SAM, several groups had reported self-starting KLM Ti:sapphire lasers without perturbation [6–8]. The self-starting KLM lasers can be achieved by a carefully analytical cavity design [9,10] to optimize dynamic loss modulation for hard aperturing or dynamic gain modulation for soft aperturing. Therefore, the KLM laser is very sensitive to its geometrical structure of resonator. From this characteristic, the iterative map derived from the beam parameters of cavity configuration is suitable for studying the dynamic stability of KLM resonator.

Recently, Hnilo [3] used 4×4 time–space matrix and gain–saturation equation to construct an iterative map on the KLM laser. The map has at least two fixed points corresponding to CW operation and a transform-limited ultrashort pulse. The self-mode-locking region of stable pulse operation was obtained from all the eigenvalues of the maps having moduli smaller than one, but in this region CW operation was also stable. Therefore, he was not able to verify whether a system prefers to operate on

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the basin of the fixed point corresponding to the pulse operation. Here, we present a new method which is simpler and more practical in discussing this problem. Because the Kerr parameter, the beam power over the critical power of self-trapping [11], can be used to distinguish the laser resonators operating at KLM or CW in the spatial domain [9–12], we simplify the map to two-dimensions corresponding to curvature and spot size. On the basis of the nonlinear dynamics, the eigenvalue of the map represents the variant rate of the dynamic system against a small perturbation. Comparing the eigenvalues between KLM and CW operations, we can define the relative stability and obtain the preferred configurations for KLM resonators. We found that the preferable KLM regions agree with that of the previous experiment self-starting regions [6] when the Kerr medium is located around the center of the resonator, even though the maps are constructed from the two-dimensional matrix only.

2. Theory

Now we consider a dynamic system, evolving with an n-dimensional state vector $\boldsymbol{y}$, governed by

$$\dot{\boldsymbol{y}} = \boldsymbol{F}(\boldsymbol{y};k),\ \boldsymbol{y} \in \boldsymbol{R}^n, \tag{1}$$

where k is the dynamical parameter. Using the linear stability analysis, the time evolution of a small perturbation on the state vector, $\boldsymbol{u} = \delta \boldsymbol{y}$, at the fixed point can be written as [13]

$$\dot{\boldsymbol{u}} = \left[D_y \boldsymbol{F}(\boldsymbol{y};k) \right] \boldsymbol{u} + O\left(|\boldsymbol{u}|^2\right), \tag{2}$$

where $D_y\boldsymbol{F}$ is the derivative of $\boldsymbol{F}$ to state vector $\boldsymbol{y}$ and $O(|\boldsymbol{u}|^2)$ denotes the order of norm of $\boldsymbol{u}$ on $\boldsymbol{R}^n$ to the second power. As a result, the dynamic stability at the fixed point is simply determined by solving Eq. (2). This is equivalent to calculate the eigenvalues of the Jacobian matrix at the fixed point for a system governed by an iterative map. The system is dynamically stable when all the moduli of eigenvalues are less than unity and it is unstable if at least one of them is greater than one. Therefore, the stability of the system determined by the dynamic stability of the map is governed by the largest modulus of the eigenvalues. The largest modulus of the eigenvalues, presented as χ, will be focused on and defined as the stability factor of the dynamic system in the following discussion. When the dynamic system $\boldsymbol{F}(\boldsymbol{y};k)$ is subjected to a small increment of dynamical parameter from k to $k+\delta k$, the time evolution of small perturbation on the state vector becomes

$$\dot{\boldsymbol{u}} = \left[D_y \boldsymbol{F}(\boldsymbol{y};k) \right] \boldsymbol{u} + \left\{ \frac{\partial\left[D_y \boldsymbol{F}(\boldsymbol{y};k)\right]}{\partial k} + \sum_i \frac{\partial\left[D_y \boldsymbol{F}(\boldsymbol{y};k)\right]}{\partial y_i} \frac{\partial y_i}{\partial k} \right\} \delta k \boldsymbol{u} + O\left(|\boldsymbol{u}|^2\right). \tag{3}$$

Similarly, the stability of the system with dynamical parameter $k+\delta k$ is determined by solving Eq. (3) for the field or calculating equivalently the eigenvalues of the Jacobian matrix for the map at the fixed point. The former term in the brace of Eq. (3), corresponding to the partial derivative of the stability factor with respect to k at the fixed point, is attributed to adding an external change to the system from increment of k. The latter one corresponds to the system response to the stability factor, due to the variation of the fixed point against increment of k. These two influences will be discussed later by numerical simulation in Section 3.

Moreover, it is worth to note that χ represents the convergent (or divergent) rate of the system against a small perturbation. If the value of χ is smaller, the mode is more stable. Thus, the relative stability between the systems with a small successive increment of k can be defined by

$$\gamma(k) \equiv \frac{\delta\chi}{\delta k} = \frac{\chi(k+\delta k) - \chi(k)}{\delta k}. \tag{4}$$

The dynamic system tends to stay at the lower stability factor, so $\gamma(k) < 0$ represents more stable with increment of k. We will use this concept to deal with the KLM laser with Kerr parameter as a dynamical parameter in the following discussion.

In this paper, a four-mirror KLM laser system, shown in Fig. 1, is used to discuss the stability of the KLM resonator. The iterative map for the resonator configuration is derived from propagating the complex beam parameter q in the cavity. The q-parameter is represented as $1/q = 1/R - i\lambda/(\pi w)^2$, where R is the radius of curvature, w is the spot size and λ is the wavelength of the cavity beam. By adopting the transfer matrix of the q-parameter propagating across the Kerr medium [11], we can obtain all matrices for Gaussian beam across all optical components.

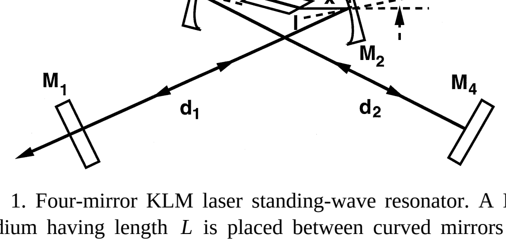

Fig. 1. Four-mirror KLM laser standing-wave resonator. A Kerr medium having length L is placed between curved mirrors M_2 and M_3 with high reflection. M_1 is the output coupler and M_4 are flat mirrors with high reflection.

Then the iterative map is easily derived from *ABCD* law [14]. Assuming that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is the round-trip transfer matrix and the reference plane is chosen to be just after the beam having left the end mirrors M_1. By applying the *ABCD* law, we can relate the q-parameter of the $(n+1)$th round-trip to the nth one as

$$R_{n+1} = \left\{ \mathrm{Re}\left[\frac{C + D\left(\dfrac{1}{R_n} - i\dfrac{\lambda}{\pi w_n^2}\right)}{A + B\left(\dfrac{1}{R_n} - i\dfrac{\lambda}{\pi w_n^2}\right)} \right] \right\}^{-1} \tag{5}$$

and

$$w_{n+1} = \left\{ -\frac{\pi}{\lambda}\mathrm{Im}\left[\frac{C + D\left(\dfrac{1}{R_n} - i\dfrac{\lambda}{\pi w_n^2}\right)}{A + B\left(\dfrac{1}{R_n} - i\dfrac{\lambda}{\pi w_n^2}\right)} \right] \right\}^{-1/2}, \tag{6}$$

where Re and Im represent the real and imaginary parts of a complex number. Then we obtained the two-dimensional iterative map, which time interval is equal to the round-trip time of the resonator. In this system, the fixed point, R_0, w_0, of the map is the self-consistent solution of the geometrical resonator, i.e., the steady-state solution [9,10]. Thus calculating the stability factor of the map at the fixed point is equivalent to determining the dynamic stability of laser cavity.

Solving the eigenvalues of the Jacobian matrix at the fixed point on the map, we get the stability factor

$$\chi = \left(\left(A_r + \frac{B_r}{R_0} + \frac{B_i \lambda}{\pi w_0^2} \right)^2 + \left(A_i + \frac{B_i}{R_0} - \frac{B_r \lambda}{\pi w_0^2} \right)^2 \right)^{-1}, \tag{7}$$

where the subscripts r and i represent the real and imaginary part of the elements in the round-trip transfer matrix. When all optical components of resonator for Gaussian beam transformations are represented as the first-order optical transfer matrices, the above result is the same as the one in Ref. [15], except that the stability factor was defined as $\chi^{-1/2}$. The method in Ref. [15] considered only the variation of q-parameter under a small perturbation, i.e., a one-dimension map is considered. Since a first-order optical transfer system belongs to a linear transfer system, the variation of complex q-parameter is equal to the combination of the variation of the q-parameter's real and imaginary parts. It is expected that the stability factor derived from our method is the same as the result of Ref. [15]. If the Gaussian beam transformation of an optical component cannot be simply represented as a transfer matrix, the variations concerned with real and imaginary parts of q-parameter are usually different. For example, one may use the renormalized q-parameter [12] to study the propagation of q-parameter across Kerr medium in KLM laser. It is more convenient and easier to handle such problem by separating the real and imaginary parts of q-parameter than the method in Ref. [15]. Although the transformation of q-parameter cannot be directly governed by *ABCD* law in the renormalized q-parameter method, the stability factor can also be obtained from the iterative map derived from the variable relationship between before and after one round-trip. In addition, if more effects such as time domain and gain are considered, the stability factors can be obtained from the maps with higher dimension. For sake of simplifying the calculation and obtaining analytical results, we just discuss the dynamic properties by the matrix method with *ABCD* law.

When the *ABCD* law is used and all optical components are represented as the first-order real transfer matrices, we have proved that the dynamical behavior of the map is equivalent to the behavior of the simple harmonic oscillation in Appendix A. The iterative map belongs to a Hamiltonian system. Furthermore, on considering the loss, the loss optical component is usually represented as the transfer matrix having the complex elements. The dynamical behavior becomes a damping oscillation and the imaginary part of the matrix element corresponds to the damping parameter, however, it is still governed by its Hamiltonian [2]. In fact, the real system always has loss such as the mirror having finite extend. A Gaussian function is usually used to taper the mirror with finite extend as a loss component [16] and the taper constant will correspond to the damping parameter. When the loss is included, the stability factor stands for the converge rate of the system against perturbation. Under the same damping parameter, the faster converge rate with the smaller stability factor implies the more stable of the system between neighbor dynamical parameter. Thus, the stability factor can be used to determine the relative stability.

Based on the previous discussion, the system with the variation of the Kerr effect is still equivalent to the simple harmonic oscillation and just has different focusing strength for different K. Whether this 'oscillator' prefers to operate at the CW operation $(K=0)$ or KLM operation $(K>0)$ is determined by the relative stability

$$\gamma_0 = \left.\frac{\delta\chi}{\delta K}\right|_{K=0}. \tag{8}$$

If $\gamma_0 < 0$, KLM operation has faster convergence rate against perturbation than CW one under the same damping parameter. In other words, the KLM operation is more stable than CW one in such resonator structure. We will use the criterion, $\gamma_0 < 0$, to determine the resonator for preferable KLM operation. It is worth to note that the map has only one fixed point associated with the steady-state solution for a fixed K. This fixed point stands for CW operation as $K=0$ and KLM operation as $K>0$ (again, K just stands for the beam power of KLM) [11]. Discussing the stability neighbor $K=0$ is capable to determine the tendency about the resonator preferrence toward KLM or CW operation.

3. Numerical results

For comparing the theoretical results with the experimental data [6], our studies focused on the symmetrically hard aperturing KLM resonator. The resonator's parameters, shown in Fig. 1, are the same as in Ref. [6]. In the symmetric resonator the equal arms d_1 and d_2 are 850 mm, the radii of curvature on the curved mirrors M_2 and M_3 are both 100 mm, and the Brewster-cut Ti:sapphire rod is $L=20$ mm. Considering the astigmatism compensation of Brewster-cut about the rod, the curved mirrors are tilted by $\theta=14.5°$. The separation of the curved mirrors, z, and the distance, x, between the curved mirror M_2 and the rod endface I are the adjustable variables. Moreover, we considered the resonator as two orthogonal astigmatic optical systems corresponding to the tangential and sagittal planes, then we will construct the iterative maps for the corresponding planes. In a hard aperturing KLM laser, one normally inserts a slit near M_1 to constrain the tangential spot size. Thus, the map of tangential plane determines the stability of the resonator from calculating the eigenvalues of the map's Jacobian matrix at the fixed point. The hard aperturing with $\delta<0$ [6], δ denoting the small signal relative spot size variation, represents that the system has the capability to sustain the KLM operation. Of course, the system also has the capability to sustain the CW operation due to the resonator satisfying geometrically stable condition. Whether the resonator prefers the KLM operation is further determined by γ_0 of tangential plane.

From the discussion in Section 2, the dynamical behavior of the system is governed by its Hamiltonian and the loss associates with the damping effect. The preferable operation, which is our main concern in this paper, will not change by varying the tapering constant. The numerical verification is shown in Fig. 2. Fig. 2 shows the relation

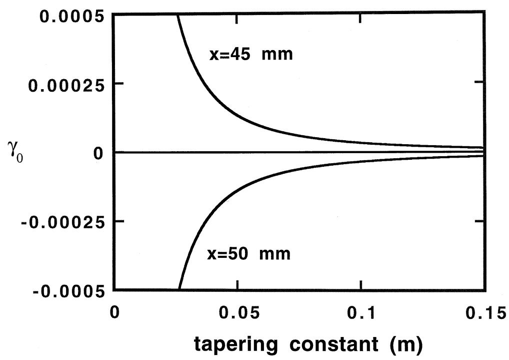

Fig. 2. The relative stability versus the tapering constant.

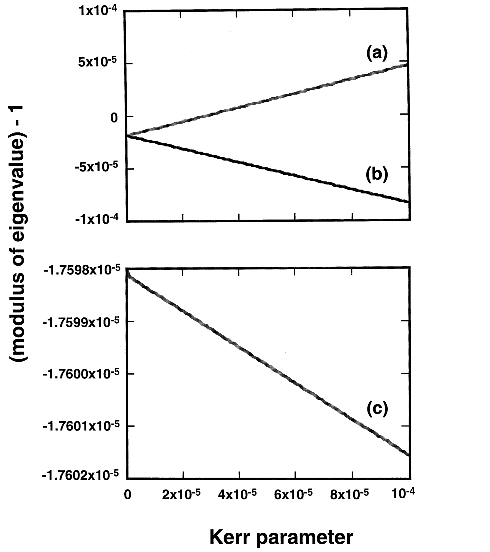

Fig. 3. The modulus of eigenvalue versus the Kerr parameter K. Curve (a) represents the evolution of eigenvalue corresponding to the influence of the former term in the brace of Eq. (3) and curve (b) is obtained from considering the influence of the latter one. The total evolution with respect to K is plotted as curve (c) with extended scale.

between γ_0 and the tapering constant with $z=116.5$ mm. Owing to the tapering constant just corresponding to the damping parameter, we simplified to add a Gaussian tapering at M_4, respectively. From Fig. 2, the tendency is classified into two cases. One is that γ_0 is always greater than zero and a monotonically decreasing function of the tapering constant such as at $x=45$ mm in Fig. 2. The other has contrary change with γ_0 being always less than zero and increasing γ_0 against the tapering constant such as at $x=50$ mm in Fig. 2. Although γ_0 depends on the tapering constant for a resonator, the sign of γ_0 is unchanged, i.e., the preferable operation ($\gamma_0<0$ for KLM or $\gamma_0>0$ for CW) of the resonator is independent of the tapering constant. The result agrees with the previous discussion. Thus the tapering constant is set as 10 cm in the following simulations.

From Eq. (3), there are two types of influence on the stability factor: one is attributed to the partial derivative of the stability factor with respect to K, the other results from the variation of the fixed point against increment K. For the sake of understanding these two effects, we had separately calculated the modulus of eigenvalue about these two types of influence by increasing K. Curve (a) in Fig. 3 represents the modulus of eigenvalue versus K just considering the former influence with $z=116.5$ mm and $x=50$ mm, and curve (b) describes the result corresponding to the latter influence. The combination of these two effects determines the variation of the stability factor with respect to K. The stability factor of the system versus K is plotted as curve (c) in Fig. 3 with extended scale. Obviously, these two effects act in opposite and near balance, so they cancel nearly each other and the variation of the stability factor becomes small. Besides, we found that the cavity configurations in the geometrically stable region have the similar tendency as above mentioned except the edge of this region. At the edge of geometrically stable region, the variation of stability factor is dominated by the influence corresponding to the partial stability factor with respect to K from our numerical simulations. However, the influence on the change of the fixed point cannot be neglected in most cases, so the relative stability γ_0 must be described as the variation of the stability factor. In addition, the stability factor is a monotonically decreasing function of K and the relative stability $\gamma_0 < 0$ at $z = 116.5$ mm and $x = 50$ mm. In other words, the laser system is more stable by appending power to mode-locking rather than to CW and the resonator prefers the KLM operation. In fact, the stability factor is not always a monotonically decreasing function of K for some z and x with the relative stability $\gamma_0 < 0$. They have one minimum stability factor under the reasonable range of K value in experiments $(K < 0.4)$ for our simulation. Owing to the K standing for the beam power of the KLM laser, the above cases represent that the higher power operation is more unstable than the lower power one and these resonators do not easily obtain higher KLM power. This phenomenon had been observed in various experiments, e.g., Ref. [17]. They found that the pulse train became unstable as the pumping power increased above a specific power. But these resonators can still operate at the pulse (KLM) operation at lower power. If we focus on the preferable operation, the γ_0 is capable of determining the resonator configuration preferring to the KLM or CW operation. We will use the γ_0 to study the stability of hard aperturing KLM lasers by changing the configuration variables z and x.

For the hard aperturing KLM lasers, the contour figure of γ_0 in the tangential plane is shown in Fig. 4 where the dot marks are the duplicated self-starting results in Ref. [6] for comparison. From Fig. 4, the resonator configuration with $\gamma_0 < 0$ prefers KLM operation when a mechanism, such as the hard aperturing in this case, has capability to sustain the KLM and CW operations. We find that the regions with $\gamma_0 < 0$ agree with the self-starting regions of Ref. [6] when the Kerr medium is placed around the center of the resonator. However, when the Kerr medium is placed far away from the center of the resonator, the beam waist may be located far away from the center of Kerr medium or outside the material. Then the effects of beam focalization [11] and the efficiencies of extracting power from gain medium must be considered in practice. We think that this is the main reason for the unpredicted results of our method. Besides, the resonator may also be affected by other additional perturbations such as vibration, thermal and turbulence fluctuation, existence and competition of high-order transverse modes, etc. Owing to

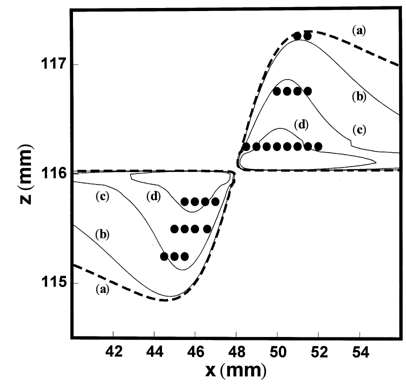

Fig. 4. The simulated KLM preferable region of hard-aperturing KLM lasers. The contour values are (a) $\gamma_0 = 0$, (b) $\gamma_0 = -10^{-6}$, (c) $\gamma_0 = -10^{-5}$, and (d) $\gamma_0 = -5\times10^{-6}$, respectively. The dot marks are the duplicated self-starting results of Ref. [6].

the strength of mechanical tapping which may be beyond that of the intrinsic perturbation discussed above to cause large cavity structure change, our simple approach is not suitable for the KLM initiated by mechanical tapping.

On the other hand, another approach borrowed from the classical mechanics can also be used to verify the previous results. In this approach we do not consider the loss. From Appendix A, we can obtain the Hamiltonian for a resonator, which is function of K and denoted as $H(K)$. Because the Hamiltonian represents the energy of the harmonic oscillation system and the system prefers to stay at the lower energy, $(\delta H(K)/\delta K)|_{K=0} < 0$ stands for the system having lower energy with larger K and it prefers to operate at $K > 0$ (KLM operation). From our numerical experiment, the same regions of the preferable resonators for KLM are obtained from these two approaches. This result also verifies that the dynamical behavior about the preferable operation is governed by the Hamiltonian, however, the system is a loss or lossless one.

Due to the nature that KLM resonator is sensitive to geometrical configuration, the dynamics of Gaussian beam in bare resonator may govern the preferable condition for self-starting KLM. As a result, even though we do not consider the mechanism of the self-starting in KLM resonator, the regions with $\gamma_0 < 0$ agree with the self-starting regions of experiment. Moreover, not only the $\delta < 0$ region contains $\gamma_0 < 0$ but also γ_0 is always greater than zero in the region with $\delta > 0$. From this result, $\gamma_0 < 0$ seems to be a more strict condition than the one with $\delta < 0$. In addition, we can optimize the resonator design by minimizing γ_0. The minimum γ_0 in the whole region is -8.06×10^{-4} at $z = 116.1$ mm and $x = 50$ mm under above mentioned resonator parameters. The optimal hard aperturing KLM laser favors to operate near the confocal edge of the geometrically stable region. This result also agrees with the previous one that the KLM favors to operate at the borders of the stability region [6].

4. Conclusion

By considering a two-dimensional iterative map derived from the propagation of q-parameter, we have generalized the stability factor as the modulus of eigenvalue at fixed point. The system tends to operate at lower stability factor as a successive variation on dynamical parameter k because the stability factor corresponds to converging rate against perturbation. We found that the variation of stability factor with respect to the Kerr parameter provides an available criterion for studying self-starting KLM lasers in the bare resonator with Kerr-lens effect only. As a result, the numerical simulation agrees with previous self-starting experimental data in the hard aperturing KLM lasers. In addition, this effective procedure can be used to study preferable resonator configuration for three-mirror KLM or the other mode competition systems. One can obtain optimal resonator designs based on simple mathematical calculations.

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Appendix A. Hamiltonian for the propagation of Gaussian beam

A.1. Beam propagating in a lenslike medium

Because the laws of transformation of Gaussian beams are identical to the laws of transformation of ray pencils [18], we begin our discussion from the ray tracing. Let us consider a lenslike medium, whose index has the square-law profile on coordinate x' and is $n(x') = n_0[1 - \frac{1}{2}\Omega^2(x')^2]$ where n_0 is the axial index and Ω is the focusing strength. When a ray propagates in the lenslike medium along z', the distance away from the optical axis, denoted by $u(z')$, and the slope of ray, denoted by $\dot{u}$, follow the ray equation [18]

$$\frac{\mathrm{d}u}{\mathrm{d}z'} = \dot{u} \tag{A1}$$

and

$$\frac{\mathrm{d}\dot{u}}{\mathrm{d}z'} = -\Omega^2 u, \tag{A2}$$

where z' is the system axial. We can obtain a Hamiltonian

$$H = \frac{1}{2}(\dot{u}^2 + \Omega^2 u^2), \tag{A3}$$

and the $ABCD$ transfer matrix is

$$\begin{bmatrix} \cos(\Omega z') & \frac{1}{\Omega}\sin(\Omega z') \\ -\Omega\sin(\Omega z') & \cos(\Omega z') \end{bmatrix}. \tag{A4}$$

On the other hand, the lowest mode Gaussian beam parameters can be obtained in terms of guiding ray parameters as [19]

$$w(z') = \sqrt{2}|u(z')| \tag{A5}$$

and

$$R(z') = w(z')/\dot{w}(z'). \tag{A6}$$

From Eqs. (A4), (A5) and (A6), the Hamiltonian of the Gaussian beam propagating in the lenslike medium is

$$H = \frac{1}{4}\left[\frac{w^2(z')}{R^2(z')} + \Omega^2 w^2(z')\right]. \tag{A7}$$

A.2. Beam propagating in a lossless optical resonator

Under unchanging focusing properties, a curved mirror with radius of curvature r can be replaced by a flat mirror and a lens in front of it with focal length $f = r$ [18]. Then we can obtain the equivalent resonator having two flat end-mirrors and take the reference plane at one of these mirrors. Let us denote

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

as the ray transfer matrix of the system from the reference plane to the other end mirror. The elements of matrix are all real numbers in the lossless optical resonator. We get the round-trip matrix $\mathbf{M}_{rt}$ of the resonator and

$$\mathbf{M}_{rt} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} d & b \\ c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad+bc & 2bd \\ 2ac & ad+bc \end{bmatrix}. \tag{A8}$$

Since $\mathbf{M}_{rt}$ is a symmetric matrix, it can be equivalent to Eq. (A4) when we choose

$$\cos(\Omega z') = ad + bc \tag{A9}$$

and

$$\Omega^2 = -\frac{ac}{bd}. \tag{A10}$$

It can be verified under the determinant of transfer matrix equal to unity that the Ω and z' are real number if and only if $|(A' + D')/2| < 1$ (the geometrical stable condition of the resonator). This result shows that the Gaussian beam propagating in the optical resonator after every round-trip has identical dynamical behavior as propagating in the lenslike medium. When we construct the iterative map of the beam parameter for the optical resonator with the interval time equal to a round-trip time, the map belongs to the Hamiltonian one. From Eqs. (A6), (A8) and (A9), the Hamiltonian corresponding to the iterative map described as Eqs. (5) and (6) is

$$H=\frac{1}{4}\left[\frac{w^2(z')}{R^2(z')}+\left(-\frac{cd}{ab}\right)w^2(z')\right]. \tag{A11}$$

The evolution of spot size of the map is governed by the simple harmonic oscillation, the result agrees with the numerical simulation of our previous result in Ref. [2]. Moreover, if we consider the loss effect by the elements of transfer matrix having the imaginary number, the evolution of spot size has the behavior like a damping oscillation [2].

References

[1] C. Palma, Opt. Comm. 129 (1996) 120.
[2] M.-D. Wei, W.-F. Hsieh, C.C. Sung, Opt. Comm. 146 (1998) 201.
[3] A.A. Hnilo, J. Opt. Soc. Am. B 12 (1995) 718.
[4] D. Spence, P. Kean, W. Sibbett, Opt. Lett. 16 (1991) 42.
[5] F. Salin, J. Squier, M. Piche, Opt. Lett. 19 (1994) 1040.
[6] G. Cerullo, S. De Silvestri, V. Magni, Opt. Lett. 19 (1994) 1040.
[7] M. Lai, Opt. Lett. 19 (1994) 772.
[8] J.-M. Shieh, F. Ganikhanov, K.-H. Lin, W.-F. Hsieh, C.L. Pan, J. Opt. Soc. Am. B 12 (1995) 945.
[9] K.-H. Lin, W.-F. Hsieh, J. Opt. Soc. Am. B 11 (1994) 737.
[10] K.-H. Lin, Y.-C. Lai, W.-F. Hsieh, J. Opt. Soc. Am. B 12 (1994) 468.
[11] V. Magni, S. De Silvestri, A. Cybo-Ottone, Opt. Comm. 96 (1993) 348.
[12] H.A. Haus, J.G. Fujimoto, E.P. Ippen, IEEE J. Quantum Electron. 28 (1992) 2086.
[13] S. Wiggins, Introduction to Applied Nonlinear Dynamic Systems and Chaos, Chap. 2, Springer-Verlag, New York, 1990.
[14] E. Seigman, Laser, Chap. 20, Mill Valley, CA, 1986.
[15] L.W. Casperson, IEEE J. Quantum Electron. QE10 (1974) 629.
[16] A. Yariv, Optical Electronics, Appendix A, Saunders College Publishing, 1991.
[17] K. Read, F. Blonigen, N. Riccelli, M. Murnane, H. Kapteyn, Opt. Lett. 21 (1996) 489.
[18] J.A. Arnaud, Beam and Fiber Optics, Chap. 2, Academic Press, New York, 1976.
[19] M. Nazarathy, A. Hardy, J. Shamir, J. Opt. Soc. Am. 72 (1982) 1409.