



# MODELLING THE STATIC AND DYNAMIC BEHAVIOR OF A CONICAL SPRING BY CONSIDERING THE COIL CLOSE AND DAMPING EFFECTS

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A method to calculate the load–deflection relation of the conical spring is proposed and verified by experimental data statically. The non-linearity of the spring rate and coil close effect by influencing the end effect are considered in the formulation. It shows that the maximum error between simulation and experimental results is 4.6%. The dynamic equations for the conical spring are also derived by considering the non-linear spring rate and damping effect due to coil clash. This is the first such modelling for a conical spring. The dynamic equations are solved by perturbation and the numerical methods. It is found that the natural frequency varies with the initial amplitude due to the coil clash in compression. Also, the maximum amplitude in compression is found to be smaller than the maximum amplitude in extension.

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## 1. INTRODUCTION

Conical springs have the advantages of varying natural frequencies and avoiding buckling at large deflections. Also, conical springs can provide variable spring rates. In the early years, most of the studies of conical springs focused on the stress and deflections.

Berry [1] proposed a method to predict the coil close position under a given load for a constant-pitch conical spring. However, there was no experimental verification, and the algorithm of determining the load to make all spring coils in contact was lacking. Lin and Pisano [2] formulated the general dynamic equations of helical springs. Wu *et al.* [3] proposed a method to predict the coil close length of compression helical springs with static verification, but only the cylindrical springs were discussed. Hunt and Crossley [4] and Lankarani and Nikravesh [5] proposed algorithms to determine the damping coefficients of multibody systems in impact.

Very little literature on dynamic investigation of conical springs can be found. In this study, the dynamic equations in P.D.E. form for a general conical spring is derived by considering diameter variation. A method is also presented to predict the coil close length of the conical spring in compression. The static load–deflection relation is also formulated and compared with experimental data.

The dynamic equations in O.D.E. form for a conical spring are also derived here by modelling the non-linear spring rate as a polynomial function in compression stroke, and the damping effects due to the coil clash is also considered, which is the first such modelling for conical springs.

## 2. FORMULATING THE DYNAMIC EQUATION FOR THE CONICAL SPRING

In the formulation of the dynamic equation of the conical spring, two kinetic energy terms and one potential energy term are considered. The dynamic equation can be deduced from Hamilton's principle:

$$\int_{t_2}^{t_1} \delta \left( \sum T_i - \sum V_i \right) dt = 0. \quad (1)$$

The first kinetic energy due to the rotation of the spring wire about its own geometric axis is defined as

$$T_1 = \int_0^L \frac{1}{2} I_{s,m} \left( \frac{\partial \psi(s, t)}{\partial t} \right)^2 ds, \quad (2)$$

where  $I_{s,m}$  is the mass moment of inertia per unit length of spring wire, and the "angle of twist",  $\psi(s, t)$ , defines the absolute rotation of any cross-section with respect to an inertial frame. Moreover,  $L$  is the total helix length.

The second kinetic energy term due to the vertical motion of the spring wire is given as

$$T_2 = \int_0^L \frac{1}{2} m_s \left( \frac{\partial y(s, t)}{\partial t} \right)^2 ds, \quad (3)$$

where  $m_s$  is the mass per unit length of the spring wire and  $y$  is the axial displacement.

The spring's strain energy due to the torsional twist of the spring coil can be obtained as

$$V_1 = \int_0^L \frac{1}{2} G_s J \left( \frac{\partial \psi(s, t)}{\partial s} \right)^2 ds \quad (4)$$

where  $G_s$  denotes the shear modulus of the spring material and  $J$  represents the polar moment of inertial of spring wire cross-section.

Since all energy terms concerned are fully derived, the dynamic equation of a general conical spring turns out to be

$$G_s J (r \partial^2 y / \partial s^2 + r' \partial y / \partial s) = (r I_{s,m} + r^3 m_s) \partial^2 y / \partial t^2, \quad (5)$$

where  $r' = \partial r / \partial s$ .

If the radius change  $r'$  is zero, i.e., the cylindrical spring, the differential equation can be simplified into the well known undamped wave equation:

$$V_w^2 \partial^2 y / \partial t^2 = \partial^2 y / \partial s^2, \quad (6)$$

where  $V_w^2 = G_s d^4 / \rho (8r^2 + d^2)$  and  $\rho$  represent the spring wire density.

## 3. SPRING RATE FOR A GENERAL HELICAL SPRING

In order to derive the spring constant of the general helical spring with variable helix radius and variable pitch angle, a co-ordinate system, as shown in Figure 1, is defined.

A general helix parametrized by arc length can be expressed as

$$\tilde{X}(s) = r(s) \cos \theta(s) \tilde{i} + r(s) \sin \theta(s) \tilde{j} + h(s) \tilde{k}, \quad (7)$$

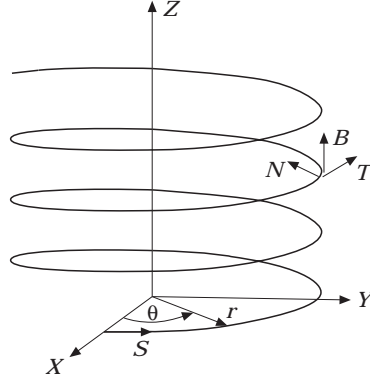


Figure 1. The intrinsic coordinate system.

where mean radius  $r$ , polar angle  $\theta$ , and local helix height  $h$ , are all functions of helical length  $s$ .

Then the tangent of the parametric curve is expressed as the derivative with respect to the helical length  $s$ :

$$\tilde{T}(s) = \tilde{X}'(s) = (r' \cos \theta - \theta' r \sin \theta) \tilde{i} + (r' \sin \theta + \theta' r \cos \theta) \tilde{j} + h' \tilde{k}. \quad (8)$$

The unit tangent is

$$\tilde{t} = \tilde{T}(s) / |\tilde{T}(s)| = (1/W) [(r' \cos \theta - \theta' r \sin \theta) \tilde{i} + (r' \sin \theta + \theta' r \cos \theta) \tilde{j} + h' \tilde{k}], \quad (9)$$

where

$$W = \sqrt{r'^2 + r^2 \theta'^2 + h'^2}.$$

The unit normal vector of the helical curve is perpendicular to its tangent and may be represented by

$$\tilde{n} = (1/\sqrt{r'^2 + r^2 \theta'^2}) [-(r' \sin \theta + \theta' r \cos \theta) \tilde{i} + (r' \cos \theta - \theta' r \sin \theta) \tilde{j}]. \quad (10)$$

The unit vector  $\tilde{b}$  is perpendicular to  $\tilde{t}$  and  $\tilde{n}$ , so

$$\begin{aligned} \tilde{b} = \tilde{t} \cdot \tilde{n} &= \frac{1}{W \times \sqrt{r'^2 + r^2 \theta'^2}} \\ &\times [-h'(r' \cos \theta - \theta' r \sin \theta) \tilde{i} - h'(r' \sin \theta + \theta' r \cos \theta) \tilde{j} + (r'^2 + r^2 \theta'^2) \tilde{k}] \end{aligned} \quad (11)$$

The above three unit vectors  $\tilde{t}$ ,  $\tilde{n}$ , and  $\tilde{b}$  form a right-handed orthonormal co-ordinate. If a static load  $\tilde{P}$  is acting on the spring along the center line. i.e.,

$$\tilde{P} = F \tilde{k}, \quad (12)$$

then the moment at the spring wire becomes:

$$\tilde{M} = Fr \sin \theta \tilde{i} - Fr \cos \theta \tilde{j}. \quad (13)$$

Three components of force  $\tilde{P}$  and moment  $\tilde{M}$  in  $\tilde{t}$ ,  $\tilde{n}$ , and  $\tilde{b}$  directions can be expressed as

$$\tilde{P}_T = \tilde{P} \cdot \tilde{t} = Fh'/W \quad \tilde{P}_B = \tilde{P} \cdot \tilde{b} = F\sqrt{r'^2 + r^2 \theta'^2}/W, \quad \tilde{P}_N = \tilde{P} \cdot \tilde{n} = 0. \quad (14-16)$$

$$\tilde{M}_T = \tilde{M} \cdot \tilde{t} = -Fr^2 \theta' / W, \quad \tilde{M}_B = \tilde{M} \cdot \tilde{b} = Fr^2 h' \theta' / W \sqrt{r'^2 + r^2 \theta'^2} \quad (17-18)$$

$$\tilde{M}_N = \tilde{M} \cdot \tilde{n} = -2Fr r' \sin^2 \theta / \sqrt{r'^2 + r^2 \theta'^2}. \quad (19)$$

The total strain energy  $U_{total}$  is the sum of the strain energy terms from the above components.

$$U_{total} = U_1 + U_2 + U_3 + U_4 + U_5 \quad (20)$$

where

$$U_1 = \frac{1}{2G_s A} \int_0^L P_B^2 ds, \quad U_2 = \frac{1}{2E_s A} \int_0^L P_T^2 ds, \quad U_3 = \frac{1}{2E_s I_B} \int_0^L M_B^2 ds,$$

$$U_4 = \frac{1}{2G_s J} \int_0^L M_T^2 ds, \quad U_5 = \frac{1}{2E_s I_N} \int_0^L M_N^2 ds$$

$G_s$  is the shear modulus of spring material,  $E_s$  is the elastic modulus of spring material,  $A$  is the area of spring wire cross-section,  $I_B$  is moment of inertia of the wire cross-section about  $\tilde{b}$ -axis,  $I_N$  is moment of inertia of the wire cross-section about  $\tilde{n}$ -axis,  $J$  is the polar moment of the inertia of the spring wire, and  $L$  is the coil length. From Castigliano's theorem, the axial deflection  $\delta$  is

$$\delta = \partial U_{total} / \partial F = F/k, \quad (21)$$

where the spring rate  $k$  of a general helical spring is equivalent to

$$k = 1/(C_a + C_b + C_c + C_d + C_e), \quad (22)$$

where

$$C_a = \frac{1}{G_s A} \int_0^L \frac{(r'^2 + r^2\theta'^2)}{W^2} ds, \quad C_b = \frac{1}{E_s A} \int_0^L \frac{h'^2}{W^2} ds,$$

$$C_c = \frac{1}{E_s I_B} \int_0^L \frac{r^4 h^2 \theta'^2}{W^2(r'^2 + r^2\theta'^2)} ds,$$

$$C_d = \frac{1}{G_s J} \int_0^L \frac{r^4 \theta'^2}{W^2} ds, \quad C_e = \frac{1}{E_s I_N} \int_0^L \frac{4r^2 r'^2 \sin^2 \theta}{W^2(r'^2 + r^2\theta'^2)} ds. \quad (23)$$

For a conical spring with constant pitch,  $p(s, t)$ , is a constant,

$$h' = \sin p, \quad r(s) = r_1 + [(r_2 - r_1)/L]s, \quad r'(s) = (r_2 - r_1)/L,$$

$$\theta(s) = \int_0^s \cos p/r(\zeta) d\zeta, \quad \theta'(s) = \cos p/r(s), \quad L = \pi(r_1 + r_2)n,$$

where  $n$  is the original number of active coils,  $r_1$  is the radius of the smallest coil on top, and  $r_2$  is the radius of the largest coil at the bottom.

Then

$$C_a = \frac{L(r'^2 + \cos^2 p)}{G_s A(1 + r'^2)}, \quad C_b = \frac{\sin^2 p L}{E_s A(1 + r'^2)}, \quad C_d = \frac{L(r_2^2 + r_1 r_2 + r_1^2) \cos^2 p}{3G_s J(1 + r'^2)},$$

$$C_e = \frac{4r'^2 L(r_2^2 + r_1 r_2 + r_1^2) \sin^2 p}{3E_s I_N (r'^2 + \cos^2 p)}, \quad C_c = \frac{L(r_2^2 + r_1 r_2 + r_1^2) \sin^2 p \cos^2 p}{E_s I_B (1 + r'^2)(r'^2 + \cos^2 p)}. \quad (24)$$

4. STATIC EXPERIMENTAL VERIFICATION

Since the diameter of each active coil is different in a conical spring, a larger diameter makes the coils more flexible. The phenomenon of coil close must be considered. The most common type of end turns employed in helical compression springs are ends squared and ground or forged, as shown in Figure 2. Since the pitch angles at two ends are smaller in general, the effect of the end turns can be estimated to achieve accurate determination of coil close in the conical springs.

By considering the end effect, where the spring coils are not uniformly spaced at the top and bottom coils, let  $h(s)$  be the distance from the bottom edge of the first active coil to the upper edge of the fixed coil. Then

$$h(s) = \int_0^s \sin p(\zeta) d\zeta - x(s), \quad 0 \leq s \leq L_a, \quad (25)$$

where  $L_a$  is the coil length of the first coil, and  $x(s)$  denotes the vertical thickness of the fixed coil. Wu *et al.* [3] proposed the relationship to find the total compression deflection  $y$  needed for the coil with distance  $h(s)$  starting to contact with the fixed coil as

$$y = Y_{close} + Y_{active} = h(s) + (K_1 / K_2)h(s), \quad (26)$$

where  $Y_{close}$  is the deflection necessary for the coil of distance  $h(s)$  to have the coil close,  $Y_{active}$  is the deflection of the remaining coil length  $L - s$  under the same load,  $K_1$  is the spring rate for the coil close length  $s$  and  $K_2$  is the spring rate for the remaining coil length  $L - s$ .

Let  $Y_s$  be the total deflection to compress the spring solid and  $n$  is the original number of active coils. Then the uniform pitch  $h_c$  to the next coil is

$$h_c = (Y_s - [h(L_a) + x(L_a)]) / (n - 1) = \text{constant}, \quad (27)$$

where

$$Y_s = \int_0^L \sin p(\zeta) d\zeta - (n - 1)d. \quad (28)$$

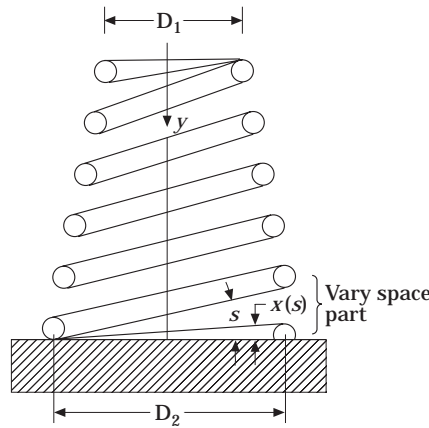


Figure 2. The conical spring.

Once  $D_y$  is specified, the coil length  $L_{close}$  and the coil number  $n_y$  to have coil close can be calculated as

$$L_{close} = (D_2 - D_y)/2r', \quad n_y = 2L_{close} / \pi(D_y + D_2) - 1 \quad (29, 30)$$

The next coil's diameter  $D_{yn}$  for the diameter  $D_y$  is estimated by

$$D_{yn} = (D_{y1} + D_{y2})/2, \quad (31)$$

where

$$D_{y1} = D_y + 2drL_{c1}, \quad D_{y2} = D_y + 2drL_{c2}, \quad L_{c1} = \pi D_y, \quad L_{c2} = \pi D_{y1},$$

According to equations (22) and (24), the spring rate  $k_c$  for a conical spring with one coil can be calculated when the diameter changes from  $D_y$  to  $D_{yn}$  with length  $L_{c3} = \pi(D_y + D_{yn})/2$ , then the relation to find the load  $F_y$  needed for the coil with diameter  $D_y$  starting to contact with the neighboring coil for the uniform pitch  $h_c$  can be calculated as

$$F_y = h_c \times k_c, \quad (32)$$

Also, the spring rate  $K_2$  for the remaining coil length  $L - L_{close}$  can be determined. Therefore, the deflections under load  $F_y$  can be expressed as two parts:

$$Y_{close} = (n_y/n)Y_s + h(L_a), \quad Y_{active} = F_y/K_2 \quad (33, 34)$$

The spring load–deflection relationship then becomes

$$y = Y_{close} + Y_{active} = F_y/K_2 + (n_y/n)Y_s + h(L_a). \quad (35)$$

Parameters of a conical spring are calibrated and listed in Table 1 to verify load–deflection relation of equation (35).

The experimental data are taken on an universal testing machine which records the compressive force under the given displacement. The results of equation (35) and experimental data are shown in Figure 3. The maximum error between experiment and simulation is about 0.6 kg.

TABLE 1  
*Parameters for the conical spring*

| Symbol                      | Description                                 | Value |
|-----------------------------|---|-------|
| $D_2$ (mm)                  | The largest diameter of the conical spring  | 56.2  |
| $D_1$ (mm)                  | The smallest diameter of the conical spring | 27.3  |
| $d$ (mm)                    | Wire diameter                               | 3     |
| $n$                         | Original active coils                       | 5.4   |
| $G$ (GPa)                   | Shear modulus of spring material            | 69.23 |
| $p$ (°)                     | pitch angle                                 | 6     |
| $E_s$ (GPa)                 | Elastic modulus of spring material          | 180   |
| $\nu$                       | Poisson ratio                               | 0.3   |
| $\rho$ (kg/m <sup>3</sup> ) | The density of spring wire                  | 7920  |

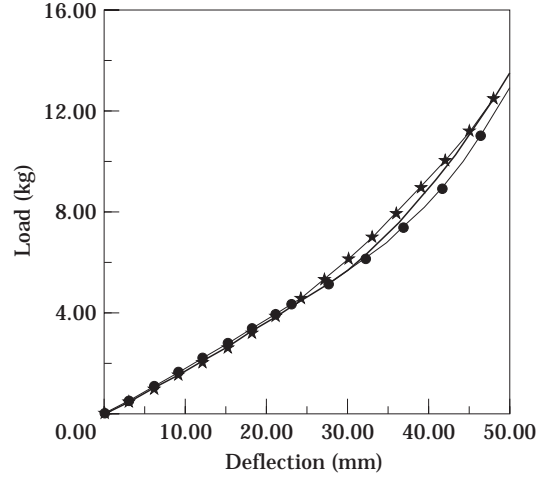


Figure 3. The load–deflection diagram for a conical spring. Key: —, experiment; —●—, simulation; —★—,  $F = ky + 0.0004y^3$ .

## 5. DYNAMIC EQUATIONS IN O.D.E. FORM

### 5.1. NON-LINEAR SPRING RATE MODELLING

In order to investigate the dynamic behavior of the conical spring, the non-linear spring rate is expressed as a third order polynomial,  $F = 0.16y + 0.00004y^3$ , as shown in Figure 3. Because of the coils close, the spring rate of the conical spring is considered as non-linear only in compression. From the above experimental data and curve fitting results, the force–displacement relation is expressed as

$$F = ky + \epsilon y^3, \quad \text{for } y \geq 0, \quad (36)$$

In the case of the spring with the parameters set out in Table 1,  $k = 0.16 \text{ kg/mm}$ ,  $\epsilon = 0.00004 \text{ kg/mm}^3$ . Then the dynamic equations of motion for the conical spring can be written as

$$m\ddot{y} + ky + \epsilon y^3 = 0, \quad \text{for } y \geq 0, \quad (37)$$

$$m\ddot{y} + ky = 0, \quad \text{for } y < 0 \quad (38)$$

for the initial condition  $y = A_m = \text{constant}$ ,  $\dot{y} = 0$  at time  $t = 0$  and  $m = 1 \text{ kg}$ .

The closed form solution of the differential equations governing the behavior of the dynamic system is difficult to solve. It is possible to obtain approximate solutions of the differential equations of the system in the form of a power series with a small parameter  $\epsilon$  by perturbation techniques [6, 7], since  $\epsilon$  is far smaller than  $k$ .

If the solution  $y$  of equation (37) and the frequency are written in the form of the power series  $\epsilon$ ,

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots, \quad \omega^2 = \omega_0^2 + \epsilon \omega_1^2 + \epsilon^2 \omega_2^2 + \dots \quad (39, 40)$$

which leads to the system equations:

$$\ddot{y}_0 + \omega_0^2 y_0 = 0, \quad \ddot{y}_1 + \omega_1^2 y_1 = \omega_1^2 y_0 - y_0^3, \quad \ddot{y}_2 + \omega_2^2 y_2 = \omega_2^2 y_0 + \omega_1^2 y_1 - 3y_0^2 y_1. \quad (41, 42, 43)$$

Those equations can be solved in sequence. The solution of equation (42) is the same as the linear equation

$$y_0 = A_m \cos \omega t \quad \text{and} \quad \omega = \omega_0 = \sqrt{k/m}, \quad (44)$$

Introducing  $y_0$  into equation (42) one obtains the solution,

$$y_1 = -(A_m^3/32\omega^2) (\cos \omega t - \cos 3\omega t) + (\omega_1^2 A_m - \frac{3}{4} A_m^3) (t/2\omega) \sin \omega t. \quad (45)$$

It is not difficult to see that the second term on the right side of equation (46) is a secular term, becoming infinitely large as  $t$  is infinite. So let the coefficient of the secular term be zero. i.e.,

$$\omega_1^2 = \frac{3}{4} A_m^2 \quad (46)$$

Similarly, introducing  $y_0$  and  $y_1$  into equation (43) leads to

$$y_2 = -(A_m^5/1024\omega^4) (\cos \omega t - \cos 5\omega t) \quad \text{and} \quad \omega_2^2 = -\frac{3}{128} A_m^4 / \omega^2. \quad (47, 48)$$

Then the approximate solution of the differential equation with three terms can be obtained in the form

$$y = A_m \cos \omega t - \frac{\epsilon A_m^3}{32\omega^2} (\cos \omega t - \cos 3\omega t) - \frac{\epsilon^2 A_m^5}{1024\omega^4} (\cos \omega t - \cos 5\omega t) \quad (49)$$

and

$$\omega^2 = \omega_0^2 + \epsilon \frac{3A_m^2}{4} - \epsilon^2 \frac{3A_m^4}{128\omega^2}, \quad (50)$$

then

$$\omega = [\{(\omega_0^2 + \frac{3}{4} \epsilon A_m^2) + \sqrt{\omega_0^4 + \frac{3}{2} \omega_0^2 \epsilon A_m^2 + \frac{15}{32} \epsilon^2 A_m^4}\}/2]^{1/2}. \quad (51)$$

For  $y < 0$  and  $\pi/2\omega \leq t \leq \pi/2\omega + \pi/\omega_0$ ,

$$y = A_L \cos \omega_0 t. \quad (52)$$

where

$$A_L = (A_m \omega - \epsilon A_m^3 / 8\omega + \epsilon^2 A_m^5 / 256\omega^3) / \omega_0 \sin \omega_0 \pi / 2\omega. \quad (53)$$

Thus the effect of the spring non-linearity is reflected in both the amplitude and frequency of the periodic motion. In the linear system, the frequency is independent of the input amplitude  $A_m$  (Figure 4).

Since the parameter  $\epsilon = 0.00004$  is very small and  $\omega$  is larger than  $\omega_0$ , the maximum amplitude of  $A_L$  is larger than the input amplitude  $A_m$ , as shown in Figure 5 and Figure 6. Because of the coils close, the spring rate of the conical spring in the compression stroke is larger than the spring rate in the extension stroke. Thus the maximum amplitude in compression is smaller than the input amplitude. According to equation (51), the frequency increases while the input amplitude  $A_m$  increases, as shown in Figure 5. The differential equation is also solved numerically by a computer software, Matlab. The approximate frequency from equation (51) is larger than the numerical solution shown in Figure 7, and the maximum error is about 0.055 rad/s.

## 5.2. THE DAMPING EFFECT DUE TO THE COIL CLASH

Because the coil impact happens at fairly low velocities, and the time is very short during impact, some additional assumptions are included.



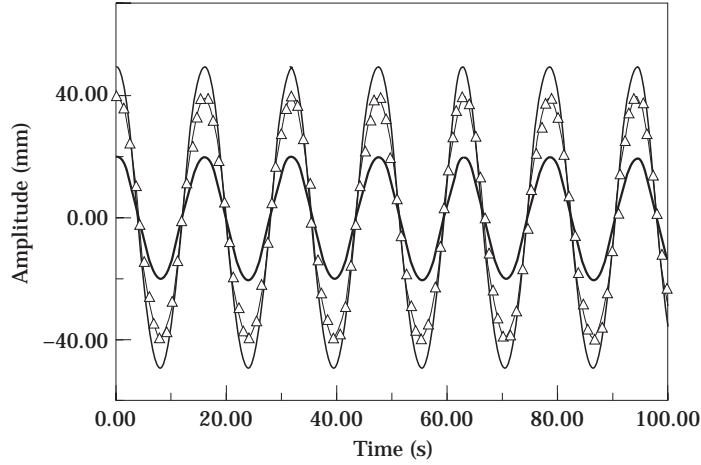


Figure 4. Vibration of the linear spring rate. Key: —,  $A = 50$  mm;  $-\triangle-$ ,  $A = 40$  mm; —,  $A = 20$  mm.

(1) The effect of reflected elastic shock waves on the forces in the zone of contact is not considered.

(2) The indentation velocity is equal to the impact velocity, and the impact velocity  $V_i$  is equal to the average velocity of the coils clash length  $L_{close}$ , i.e.,

$$V_i = \frac{1}{2} (n_y / n) \dot{y}. \quad (54)$$

(3) The damping force is proportional to the impact velocity and the damping coefficient. i.e.,

$$f_{damping} = cV_i. \quad (55)$$

The damping coefficient  $c$  during impact derived by Hunt and Crossley [4] is

$$c = \frac{3}{2} \alpha k_c y_m^{3/2}. \quad (56)$$

for steel,  $\alpha$  is about 0.08 s/m and

$$k_c = [2E_s / 3(1 - \nu^2) (4/d)^{-2}], \quad (57)$$

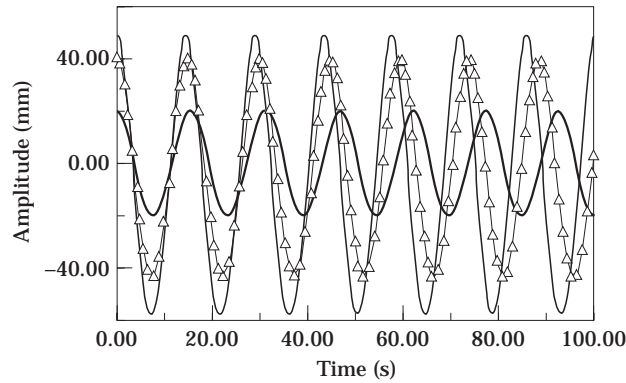


Figure 5. Vibration of the conical spring without damping. Key as for Figure 4.

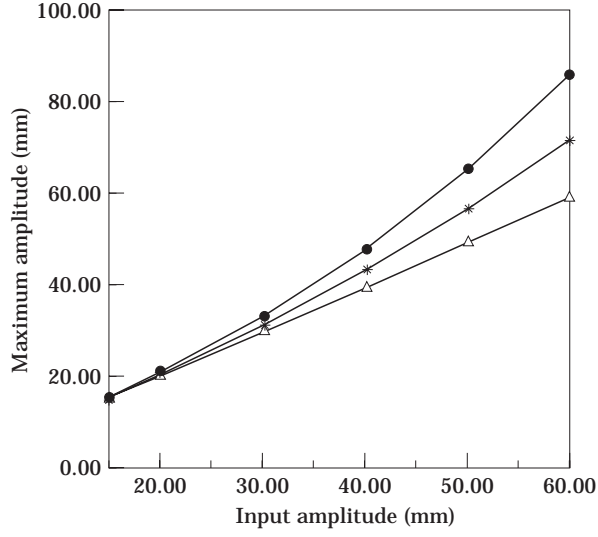


Figure 6. The input amplitude versus maximum amplitude. Key:  $-\triangle-$ ,  $A_{max}$  for  $y > 0$ ;  $-\star-$ , numerical  $A_{max}$  for  $y < 0$ ;  $-\bullet-$ , analytical  $A_{max}$  for  $y < 0$ .

where  $E_s$  is the elastic modulus of the spring material,  $\nu$  is the Poisson ratio,  $d$  is the wire diameter, and the maximum approaching distance  $y_m$  is written as

$$y_m = [m_{close} V_i^2 (g + 1)/2k_c]^{1/(g+1)} \quad (58)$$

where  $g = 3/2$  and  $m_{close}$  is the mass of the coils clash length,

$$m_{close} = L_{close} \times \pi d^2/4 \times \rho, \quad (59)$$

where  $\rho$  is the spring wire's density.

From equation (54) to equation (59), one obtains

$$f_{damping} = \frac{3}{2} \alpha k_c [(L_{close} \times \pi d^2/4 \times \rho \times \frac{1}{4} (n_y^2/n^2) \dot{y}^2 \times \frac{5}{2} / 2k_c)^{2/5}]^{3/2} \times [n_y / 2n(\dot{y})] \quad (60)$$

$$= \frac{3}{4} (n_y^{11/5} / n) \alpha k_c (5\pi L_{close} \times d^2 \times \rho \times \dot{y}^2 / 64k_c n^2)^{3/5} \dot{y}. \quad (60)$$

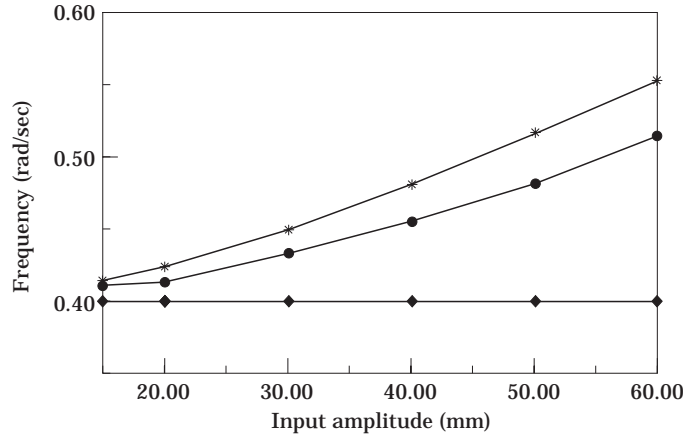


Figure 7. The input amplitude versus frequency. Key:  $-\diamond-$ , linear;  $-\star-$ , analytical;  $-\bullet-$ , numerical.

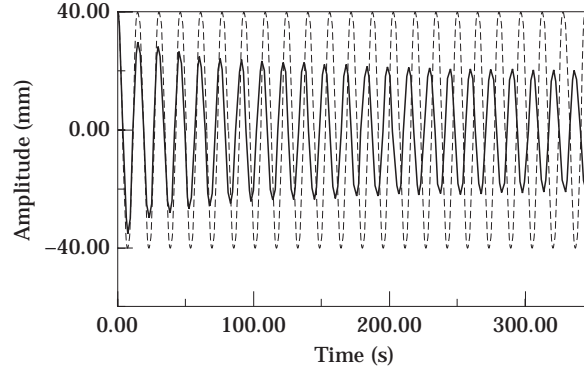


Figure 8. Vibration of the conical spring with impact damping. Key: —, with damping effect; - - - , without damping effect.

Since  $L_{close}$  and  $n_y$  are a function of compression  $y$ , the damping force is a function of  $y$  and  $\dot{y}$ . By considering the damping force, the dynamic equations of the conical spring become

$$m\ddot{y} + ky + \epsilon y^3 + \frac{3}{4} (n_y^{11/5} / n) \alpha k_c (5\pi L_{close} d^2 \rho \dot{y}^2 / 64 k_c n^2)^{3/5} \dot{y} = 0; \quad \text{for } y \geq 0 \quad (61)$$

$$m\ddot{y} + ky = 0, \quad \text{for } y < 0 \quad (62)$$

The frequency of the vibration differs from the frequency of the undamped system by about 1.5% for input amplitude  $A = 40$  mm. The damping effect due to the coils clash cannot be ignored when the amplitude is large. When the  $L_{close}$  becomes shorter with the decreasing amplitude, the damping effect due to coil clash is not as evident then, as shown in Figure 8.

## 6. CONCLUSIONS

To consider the diameter change, three energy terms are considered in deriving the dynamic equation of the conical spring. An algorithm to find the static load–deflection relation for conical springs is proposed here with static verification, and the method to predict the coil close length is more general than Berry's method which only caters for uniform spacing of the conical springs. The dynamic equations in O.D.E. form for a conical spring are also derived by considering the nonlinearity of the spring rate in compression stroke and the damping effects due to the coil clash which is the first of such modeling in this area. Because of the coil clash in the compression stroke, the frequency varies with the input amplitude, and maximum amplitude in the compression stroke is smaller than the maximum amplitude in the extension stroke.

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