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A study of the total chromatic number of equibipartite graphs

Bor-Liang Chen[a,*] Chun-Kan Cheng[b], Hung-Lin Fu[b], Kuo-Ching Huang[c]

[a] *Department of Mathematics, Tunghai University, Taichung, Taiwan, R.O.C.*
[b] *Department of Applied Mathematics, National Chiao Tung University, Hsienchu, Taiwan, R.O.C.*
[c] *Department of Applied Mathematics, Providence University, Taichung Hsien, Taiwan, R.O.C.*

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Abstract

The total chromatic number $\chi_t(G)$ of a graph G is the least number of colors needed to color the vertices and edges of G so that no adjacent vertices or edges receive the same color, no incident edges receive the same color as either of the vertices it is incident with. In this paper, we obtain some results of the total chromatic number of the equibipartite graphs of order $2n$ with maximum degree $n-1$. As a part of our results, we disprove the biconformability conjecture.

1. Introduction

A total coloring of a graph G is a mapping $\pi: V(G) \cup E(G) \to C$ such that no incident or adjacent pair of elements of $V(G) \cup E(G)$ receive the same color. Thus a total coloring of G incorporates both a vertex coloring and an edge coloring of G, and satisfies the additional condition that no vertex receives the same color as an edge incident with the vertex. The *total chromatic number* $\chi_t(G)$ is the least value of $|C|$ for which G has a total coloring.

A well-known conjecture of Behzad [1], and independently of Vizing [8] is that $\Delta(G)+1 \leqslant \chi_t(G) \leqslant \Delta(G)+2$. The lower bound here is easy to see, but whether the upper bound holds is still unknown. This is also called the total coloring conjecture (TCC).

If the conjecture is proved to be true for a class of graphs, then the graphs G having $\chi_t(G) = \Delta(G)+1$ are *type* 1 graphs, and the other graphs are *type* 2, i.e., $\chi_t(G) = \Delta(G)+2$.

* Corresponding author. E-mail: blchen@s867.thu.edu.tw.

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In [7], Rosenfeld proved that a bipartite graph satisfies TCC, which is also immediate from the result of König which states that a bipartite graph is of class 1 [3]. Thus we can study the classification problem of bipartite graphs. The following results are known.

Theorem 1.1 (Behzad et al. [2]). *A complete bipartite graph $K_{m,n}$ is type 2 if and only if $m = n$.*

Theorem 1.2 (Hilton [6]). *Let J be a subgragh of $K_{n,n}$, $e(J) = |E(J)|$, and $m(J)$ be the maximum size of a matching in J. Then $\chi_t(K_{n,n} \backslash E(J)) = n + 2$ if and only if $e(J) + m(J) \leqslant n - 1$.*

In what follows we shall focus on the bipartite graph $G = (A, B)$ where $|A| = |B| = n$. Such a graph is also called an *equibipartite graph*. It can be seen that Theorem 1.2 is mainly concerned with equibipartite graphs of order $2n$ with maximum degree n. In this paper, we shall study the equibipartite graphs of order $2n$ with maximum degree $n - 1$.

For equibipartite graphs, it is convenient to present a total coloring by using an array with its sideline and headline. Let $G = (A, B)$ be an equibipartite graph of order $2n$ where $A = \{x_1, x_2, \ldots, x_n\}$ and $B = \{y_1, y_2, \ldots, y_n\}$. If π is a total coloring of G, then it has an $n \times n$ array M such that $M(i,j) = \pi(x_i y_j)$ where $x_i y_j \in E(G)$, and the sideline (and headline) of M represents the vertex coloring of A (and B) with respect to π. Let M^* be M with its sideline and headline. Then M^* will be referred to as a total coloring array of G. (Fig. 1. is an example.)

Note that if G is type 1, then M will be a partial latin square of order n, furthermore each row including the sideline contains distinct elements, so does each column including the headline. Clearly, in order to be a total coloring array of an equibipartite graph, M^* has to satisfy some further conditions on vertex coloring.

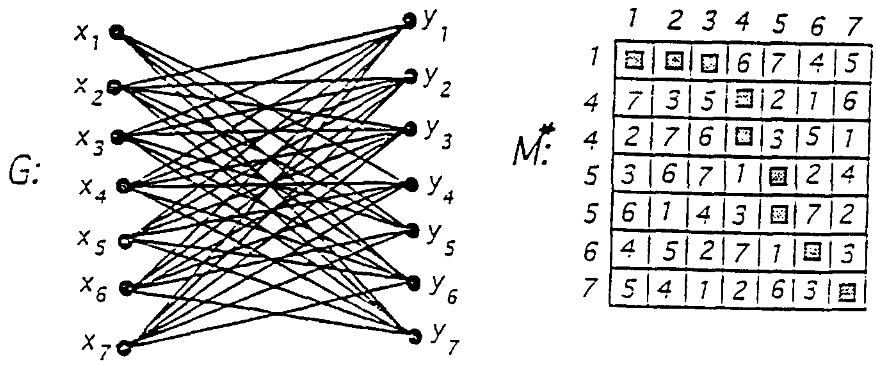


Fig. 1.

A graph G is biconformable if G is equibipartite and G has a vertex coloring $\varphi : V(G) \to \{c_1, c_2, \ldots, c_{\Delta(G)+1}\}$ such that the following conditions hold: (i) $\mathrm{def}(G) = \sum_{v \in V(G)}(\Delta(G) - deg_G(v)) \geqslant \sum_{i=1}^{\Delta(G)+1} |a_i - b_i|$, and (ii) $|V_{<\Delta}(A \backslash A_j)| \geqslant b_j - a_j$ and $|V_{<\Delta}(B \backslash B_j)| \geqslant a_j - b_j$ for each $j \in \{1, 2, \ldots, \Delta(G) + 1\}$, where $|V_{<\Delta}(S)|$ is the number of vertices in $S \subseteq V(G)$ which have degree less than $\Delta(G)$, $A_j = \varphi^{-1}(c_j) \cap A$, $B_j = \varphi^{-1}(c_j) \cap B$, $a_j = |A_j|$, and $b_j = |B_j|$.

The following result shows that biconformability is a necessary condition for the equibipartite graph to be type 1.

Lemma 1.3. *Let $G = (A, B)$ be an equibipartite graph. If G is type 1, then G is biconformable. Equivalently, if G is not biconformable, then G is type 2.*

Proof. Since G is type 1, there exists a total coloring π which uses $\Delta(G) + 1$ colors. Let these colors be $c_1, c_2, \ldots, c_{\Delta(G)+1}$. Therefore, $\pi|_{V(G)}$ is a vertex coloring of G which uses the colors $c_1, c_2, \ldots, c_{\Delta(G)+1}$. For each color c_j, let a_j and b_j be the number of vertices in A_j and B_j, respectively, which are colored with c_j. Since G is a bipartite graph, for each $j = 1, 2, \ldots, \Delta(G) + 1$, there are at least $|a_j - b_j|$ vertices of G in which the color c_j is missing on the edges which are incident with these $|a_j - b_j|$ vertices. This implies that there are at least $|a_j - b_j|$ vertices which have deficiency one. Thus $\mathrm{def}(G) \geqslant \sum_{i=1}^{\Delta(G)+1} |a_i - b_i|$, $|V_{<\Delta}(A \backslash A_j)| \geqslant b_j - a_j$ and $|V_{<\Delta}(B \backslash B_j)| \geqslant a_j - b_j$ follow by the same reason. This concludes the proof. □

For the equibipartite graphs of order $2n$ with maximum degree n, we can obtain a necessary and sufficient condition for being biconformable.

Lemma 1.4. *Let J be a subgraph of $K_{n,n}$ which has at least one isolated vertex. Then $G = K_{n,n} \backslash E(J)$ is biconformable if and only if $e(J) + m(J) \geqslant n$.*

Proof. If G is biconformable, then there exists a vertex coloring $\varphi : V(G) \to C = \{c_1, c_2, \ldots, c_{\Delta(G)+1}\}$ such that $\mathrm{def}(G) \geqslant \sum_{i=1}^{\Delta(G)+1} |a_i - b_i|$. Let t be the number of independent edges of J in which two end vertices have the same color. Clearly, $t \leqslant m(J)$. Also, $\sum_{i=1}^{\Delta(G)+1} |a_i - b_i| \geqslant 2n - 2t$. By the fact that J contains an isolated vertex, we have $\Delta(G) = n$ and $2e(J) = \mathrm{def}(G) \geqslant \sum_{i=1}^{\Delta(G)+1} |a_i - b_i| \geqslant 2n - 2t \geqslant 2n - 2m(J)$. Hence, $e(J) + m(J) \geqslant n$.

Conversely, by Theorem 1.2, G is type 1 and then by Lemma 1.3, G is biconformable. □

We note here that if $G = K_{n,n} \backslash E(J)$, where J contains at least one isolated vertex, then by Lemma 1.4, G is type 1 if and only if G is biconformable. But this conclusion is not true in general. A well-known example, the Möbius band of order 14, M_{14} (Fig. 2) is biconformable and it is a type 2 graph. In [5], Hamilton et al. posed the so-called biconformability conjecture in order to obtain a clear picture for the classification of bipartite graphs with respect to total coloring.

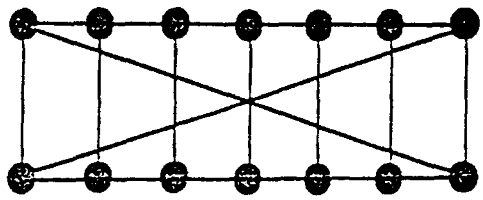


Fig. 2.

Conjecture 1.5 (*Biconformability Conjecture*). Let G be a bipartite graph with $\Delta(G) \geqslant (3/14)(|V(G)|+1)$. Then G is type 2 if and only if G contains an induced equibipartite subgraph H with $\Delta(H)=\Delta(G)$ which is not biconformable.

In this paper, we study the total coloring of equibipartite graphs of order $2n$ with maximum degree $n-1$ and we obtain some sufficient conditions for the graphs to be type 1. Clearly, one of the requirement is 'biconformable'. As long as the biconformability itself is not enough to ensure that the graph is type 1, then there is a possibility of obtaining a counterexample to Conjecture 1. We shall mention a class of counterexamples in Section 4. Finally, following our results, we pose a conjecture in the direction of solving the classification problem of the equibipartite graph of order $2n$ with maximum degree $n-1$.

2. The basic lemma

It is easy to see that if H is a subgraph of G such that $\Delta(H)=\Delta(G)$ and G is type 1, then H is also a type 1 graph. In other words, if we delete some edges from a type 1 graph G without changing the maximum degree, then the graph obtained is also type 1. Therefore, it suffices to study the *maximal* one which has degree $\Delta(G)$. A vertex is called a *major vertex* of G if the degree of this vertex is $\Delta(G)$, and the vertex which is not major vertex is called a *minor vertex*. A graph is *maximal* if all the minor vertices are mutually adjacent. Now the following lemma is easy to see.

Lemma 2.1. *If G is a maximal subgraph of $K_{n,n}$ with $\Delta(G)=n-1$. Then $J=K_{n,n} \backslash E(G)$ is a (vertex) disjoint union of stars.*

Proof. Suppose not. Since $\Delta(G)=n-1$, the degree of each vertex of J is at least one. Therefore J is a spanning subgraph of $K_{n,n}$. If J contains a cycle, J must be an even cycle, and hence there exists a pair of minor vertices in G which are not adjacent. This is not possible for a maximal graph. Therefore J is a spanning forest. Now if there exists a component of J which is not a star, then there are two adjacent vertices in the component which are of degree at least two. This implies that in G, there are

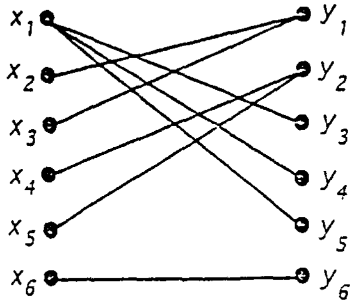


Fig. 3. $\langle 3;2,23;1\rangle$ graph.

two minor vertices which are not adjacent. Again it is a contradiction. Hence we have the proof. □

Let $G=(A,B)=K_{n,n}\backslash E(J)$ be a maximal graph with $\Delta(G)=n-1$. Since J is a disjoint union of stars, hence denote J by an $(s+t+1)$-tuple $\langle m_1,m_2,\dots,m_s;$ $n_1,n_2,\dots,n_t;\ r\rangle_n$ where m_i, $i=1,2,\dots,s$, is the degree of $x_i\in A$ such that $\deg_J(x_i)\geqslant 2$, n_j is the degree of $y_j\in B$, $j=1,2,\dots,t$, such that $\deg_J(y_i)\geqslant 2$, and r is the number of independent edges in J. Without loss of generality, we may assume that $m_1\geqslant m_2\geqslant\cdots\geqslant m_s$, $n_1\geqslant n_2\geqslant\cdots\geqslant n_t$, and $\deg_J(x_k)=\deg_J(y_k)=1$ for each $k\geqslant t+\sum_{i=1}^{s} m_i = s+\sum_{j=1}^{t} n_j$. Therefore we also have

$$\deg_J(x_{s+1})=\deg_J(x_{s+2})=\cdots=\deg_J(x_{s+\sum_{j=1}^{t} n_j})=1$$

$$\deg_J(y_{t+1})=\deg_J(y_{t+2})=\cdots=\deg_J(y_{t+\sum_{i=1}^{s} m_i})=1,$$

and $n=t+r+\sum_{i=1}^{s} m_i=s+r+\sum_{j=1}^{t} n_j$. For clarity, we give an example in Fig. 3.

The following result characterizes the biconformability of the maximal equibipartite graph of order $2n$ with maximal degree $n-1$.

Proposition 2.2. *Let $J=\langle m_1,m_2,\dots,m_s;n_1,n_2,\dots,n_t;r\rangle_n$ and $G=K_{n,n}\backslash E(J)$. Then G is biconformable if and only if*

$$s+\left\lceil\frac{\sum_{i=1}^{s}\max\{m_i-s,o\}}{s}\right\rceil+t+\left\lceil\frac{\sum_{j=1}^{t}\max\{n_j-t,o\}}{t}\right\rceil+r\leqslant n. \qquad (1)$$

Proof. If G is biconformable, then there exists a vertex coloring φ using the colors $c_1,c_2,\dots,c_n$ which satisfies the biconformability. Let $N_J(x)$ denote the neighbor of x in J. If $x\in A$ (resp. B) is a center of a star of size at least 2, and c_i is a color which occurs in $N_J(x)$, then clearly only x and the vertices which in B (resp. A) can be colored with c_i. On the other hand, if two centers of A (resp. B) are colored with c_i, then c_i cannot occur in B (resp. A). This implies that for each star of size $l\geqslant 2$, it contributes a quantity either $l-1$ or l to $\sum_{i=1}^{n}|a_i-b_i|$ depending on whether this star contains a vertex of degree one which uses a common color with the center or not. Thus by the fact that $\mathrm{def}(G)=\sum_{i=1}^{s}(m_i-1)+\sum_{i=1}^{t}(n_i-1)\geqslant\sum_{i=1}^{n}|a_i-b_i|$, we conclude that no two centers of stars of size $\geqslant 2$ can receive a common color and in each star of size $\geqslant 2$ the color that occurs in the center x also occurs in $N_J(x)$. Therefore, if $\varphi(x)=c_i$, $x\in A$ (resp. B), c_i occurs in B (resp. A) at most

s (resp. t)times, if x is the center of a star of size $\geqslant s$ (resp. t);

l times, if x is the center of a star of size l, $2\leqslant l<s$; and

1 times, if x is incident with an independent edge.

Furthermore, if c_i does not occur in A (resp. B), then c_i can occur at most s times in B (resp. A). Now (1) is a direct result of the vertex coloring using at most n colors.

Conversely, if (1) is true, then the biconformable vertex coloring can be obtained by assigning the colors to the vertices following the processes: (i) if xy is an independent edge of J, then $\varphi(x)=\varphi(y)$, and for each independent edge one color is used. (ii) All centers of stars of size at least 2 are colored differently. (iii) If a center $x\in A$ (resp. B) of a star of size l is colored with c_i, then color the vertices of $N_G(x)$ with $c_i\min\{s,l\}$ times (resp. $\min\{t,l\}$ times). (iv) If there are s_1 and s_2 vertices in A and B respectively which are not colored yet, then use $\lceil s_1/t\rceil$ and $\lceil s_2/s\rceil$ colors to color them respectively. As explained in the necessity part, the coloring obtained by the above processes is biconformable. □

It is easy to see that if H is a subgraph of a biconformable graph G such that $V(H)=V(G)$ and $\Delta(H)=\Delta(G)$, then H is also biconformable. But if G is not biconformable, we may still have a subgraph H of G with $\Delta(H)=\Delta(G)$ and H is biconformable. In what follows, we obtain a necessary and sufficient condition for an equibipartite maximal graph G which contains a subgraph H such that H is not biconformable and $\Delta(H)=\Delta(G)$.

Proposition 2.3. *Let $G=K_{n,n}\backslash E(J)$, where $J=\langle m_1,\ldots,m_s;n_1,\ldots,n_t;r\rangle_n$. Then G contains an equibipartite subgraph H with $\Delta(H)=\Delta(G)$ which is not biconformable if and only if either $n\leqslant m_1+n_1$, or (1) is not true.*

Proof. Assume that H is an equibipartite subgraph of G such that $\Delta(H)=\Delta(G)$ which is not biconformable. First, if $V(H)=V(G)$, then clearly G is not biconformable either. By Proposition 2.2, (1) is not true. On the other hand, if $V(H)\subsetneq V(G)$, then $|V(H)|=2(n-1)$ *and* $\Delta(H)=\Delta(G)=n-1$. Let $J_H=K_{n-1,n-1}\backslash E(H)$ and u,v be two vertices in $V(G)$ such that H is a maximal subgraph of $H_1=G\backslash\{u,v\}$. Furthermore, Let u' and v' be two vertices in $V(G)$ such that $\deg_J(u')=m_1$, $\deg_J(v')=n_1, H'=G\backslash\{u',v'\}$ and $J_{H'}=K_{n-1,n-1}\backslash E(H')$. Now we have

$$e(J_H)+m(J_H)\geqslant e(J_{H_1})+m(J_{H_1})\geqslant e(J_{H'})+m(J_{H'})=2n-m_1-n_1-2.$$

Since H is not biconformable, H is type 2. By Theorem 1.2, $\chi_t(H)=n+1(\Delta(H)=n-1)$ if and only if $e(J_H)+m(J_H)\leqslant n-2$. This implies that $2n-m_1-n_1-2\leqslant n-2$, and therefore $n\leqslant m_1+n_1$.

Conversely, in the case that (1) is not true, then G is not biconformable. Hence the existence of H is obvious. Assume that $n\leqslant m_1+n_1$, $\deg_G(x_1)=m_1$, and $\deg_G(y_1)=n_1$. Let $H=G\backslash\{x_1,y_1\}$ and $J_H=J\backslash\{x_1,y_1\}$. Again

$$e(J_H)+m(J_H)=2n-m_1-n_1-2\leqslant n-2.$$

Since H is of order $2(n-1)$ and $\Delta(H)=n-1$. By Theorem 1.2, H is a type 2 subgraph of G, and hence not biconformable. □

Corollary 2.4. *Let $G=K_{n,n}\backslash E(J)$, where $J=\langle m_1,\ldots,m_s;n_1,\ldots,n_t;r\rangle_n$ and G is biconformable. Then every equibipartite subgraph H with $\Delta(H)=\Delta(G)$ is biconformable if and only if $n>m_1+n_1$.*

Proof. It is a direct result of Propositions 2.2 and 2.3. □

We note here that, from Corollary 2.4, if we can find a type 2, biconformable graph $G=K_{n,n}\backslash E(J)$ where $J=\langle m_1,\ldots,m_s;n_1,\ldots,n_t;r\rangle_n$ and $n>m_1+n_1$, then the biconformability conjecture can be disproved. Not surprisingly, we shall see that the graph in Fig. 3 is one of this kind.

3. The problem of distributing colored balls (DCB)

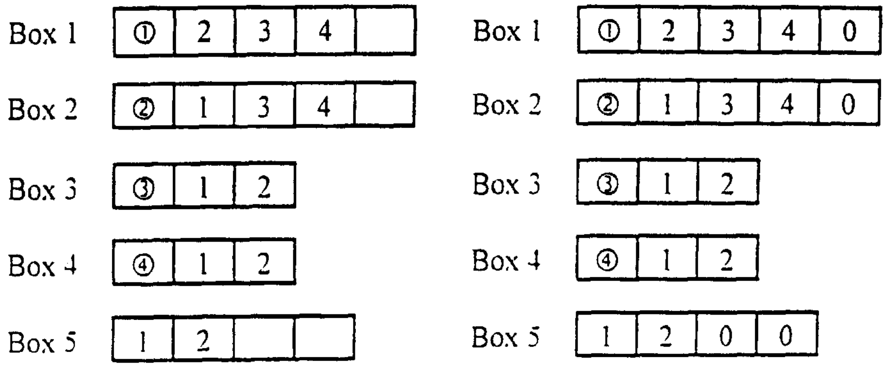
In order to obtain a good necessary condition for a type 1 maximal equibipartite graph with maximal degree $n-1$ (hopefully this condition is also sufficient), we introduce a problem which is formulated by biconformable total colorings. The details will be explained in next section.

DCB Problem. Suppose that we have t different colored balls and there are n_i balls of the ith color, $i=1,2,\ldots,t$. Without loss of generality, let $n_1\geqslant n_2\geqslant\cdots\geqslant n_t$. The DCB problem is to determine the minimum number of boxes which are needed to distribute all the balls given

(i) the ith box contains exactly one ball of the ith color and in total at most n_i balls, $i=1,2,\ldots,t$;
(ii) the jth box contains at most t balls for each $j>t$; and
(iii) every box consists of different colored balls.

Let $N=(n_1,n_2,\ldots,n_t)$ and $b(N)$ denote the minimum number of boxes we need to distribute the colored balls properly into different boxes. In order to find $b(N)$ we need the Fulkerson's theorem on digraphical sequence.

Theorem 3.1 (Fulkerson [4]). *A sequence $(s_1,t_1),(s_2,t_2),\ldots,(s_p,t_p)$ of ordered pairs of nonnegative integers with s_i the outdegree, t_i the indegree, and $s_1\geqslant s_2\geqslant\cdots\geqslant s_p$,*



(a) (b)

Fig. 4(a) and (b).

is digraphical if and only if

(i) $s_i \leqslant p-1$ *and* $t_i \leqslant p-1$ *for* $1 \leqslant i \leqslant p$;

(ii) $\sum_{i=1}^{p} s_i = \sum_{i=1}^{p} t_i$;

(iii) $\sum_{i=1}^{n} s_i \leqslant \sum_{i=1}^{n} \min\{n-1, t_i\} + \sum_{i=n+1}^{p} \min\{n, t_i\}$ *for* $1 \leqslant n < p$.

Before we prove the main lemma, we shall use an example to explain our idea. In Fig. 4(a), we have $N=(5,5,3,3)$ and $b(N)=5$. The numbers respresent the colors of the colored balls.

Since there are three boxes in Fig. 4(a) which are not full, we can fill in some dummy balls with color 0 without changing the minimum number of boxes. Fig. 4(b) is such an adjustment.

Now we can define a digraph G by way of Fig. 4(b). Let $V(G)=\{v_1, v_2, v_3, v_4, u_1, w_1, w_2, w_3, w_4\}$ where v_i represents the box which contains the ith color ball, $i=1,2,3,4$, and u_1 represents the extra box in which we can distribute at most 4 distinct colored balls. Finally, let the w_i's represent the dummy balls (one for each). The arcs of G can be seen in Fig. 5, the indegree of v_i is $n_i - 1$ which represents that except for the ith color ball, there are $n_i - 1$ balls in the box. Furthermore, if the extra box u_1 contains an ith colored ball, then (v_i, u_1) is an arc of G, and (w_k, v_i) is an arc of G provided that ith box contains a dummy color ball w_k. Clearly, G has a digraphical sequence: $(4,4),(4,4),(2,2),(2,2),(0,4),(1,0),(1,0),(1,0),(1,0)$. Since the sequence is digraphical, the property (iii) in Theorem 3.1 holds and we shall use (iii) to find $b(N)$.

Proposition 3.2. *Let* $N=(n_1, n_2, \ldots, n_t)$ *where* $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_t$ *are positive integers. Then*

$$b(N) = \max_{1 \leqslant k \leqslant t} \left\lceil \frac{\sum_{i=1}^{k} n_i - \sum_{i=1}^{k} \min\{k, n_i\} - \sum_{i=k+1}^{t} \min\{k, n_i - 1\}}{k} \right\rceil + t.$$

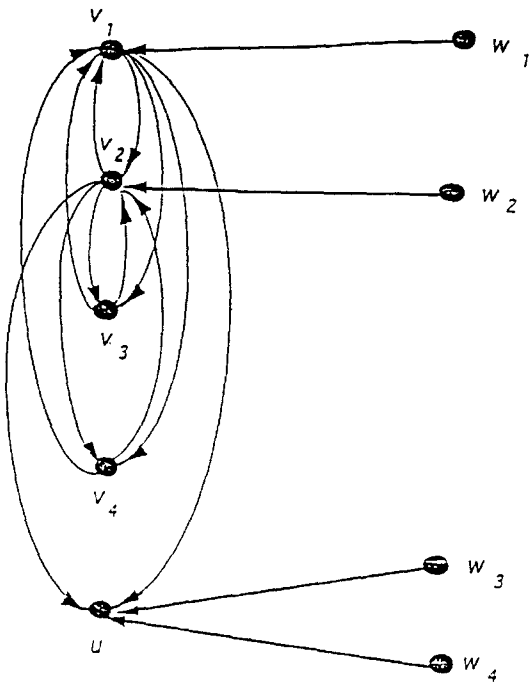


Fig. 5.

Proof. By the definition of the DCB problem, $b(N)\geqslant t$. Let γ be the minimum number of extra boxes we need, i.e., $b(N)=t+\gamma$. Then we can define a digraph G similar to Fig. 5 with $V(G)=\{v_1,v_2,\ldots,v_t,u_1,u_2,\ldots,u_\gamma,w_1,w_2,\ldots,w_{\gamma t}\}$, and the degree sequence:

$$(n_1-1,n_1-1),(n_2-1,n_2-1),\ldots,(n_t-1,n_t-1),$$

$$\underbrace{(0,t),\ldots,(0,t)}_{\gamma\text{ times}},\ \underbrace{(1,0),\ldots,(1,0)}_{\gamma t\text{ times}}.$$

(The number of dummy balls are decided by the number of extra boxes.) Since the sequence is digraphical, by Theorem 3.1, we have

$$\sum_{i=1}^{k}(n_i-1)\leqslant\sum_{i=1}^{k}\min\{k-1,n_i-1\}+\sum_{i=k+1}^{|V(G)|}\min\{k,t_i\}$$

for each $1\leqslant k\leqslant t$ where t_i is the indegree of a vertex. Hence

$$\left(\sum_{i=1}^{k}n_i\right)-k\leqslant\sum_{i=1}^{k}\min\{k-1,n_i-1\}+\sum_{i=k+1}^{t}\min\{k,t_i\}+k\gamma,$$

i.e.

$$k\gamma \geqslant \sum_{i=1}^{k} n_i - \left(\sum_{i=1}^{k} \min\{k-1, n_i-1\} + k\right) - \sum_{i=k+1}^{t} \min\{k, t_i\}.$$

This implies that

$$\gamma \geqslant \left(\sum_{i=1}^{k} n_i - \sum_{i=1}^{k} \min\{k, n_i\} - \sum_{i=k+1}^{t} \min\{k, n_i-1\}\right) \Big/ k$$

for each $1 \leqslant k \leqslant t$. Thus

$$b(N) \geqslant \max_{1\leqslant k\leqslant t} \left\lceil \frac{\sum_{i=1}^{k} n_i - \sum_{i=1}^{k} \min\{k, n_i\} - \sum_{i=k+1}^{t} \min\{k, n_i-1\}}{k} \right\rceil + t.$$

Now let

$$\gamma' = \max_{1\leqslant k\leqslant t} \left\lceil \frac{\sum_{i=1}^{k} n_i - \sum_{i=1}^{k} \min\{k, n_i\} - \sum_{i=k+1}^{t} \min\{k, n_i-1\}}{k} \right\rceil.$$

Conversely, we see that (iii) holds for $1 \leqslant k < t + \gamma' t + \gamma'$. Therefore $b(N) \leqslant \gamma' + t$. ($b(N)$ is a minimum.) And we have the proof. □

Now we are ready for the main theorem.

4. The main results

Let $G = K_{n,n} \backslash E(J)$ where $J = \langle m_1, m_2, \ldots, m_s;\ n_1, n_2, \ldots, n_t;\ r\rangle_n$. Then J can be decomposed into three edge-disjoint induced subgraphs; H_1 is induced by the stars with centers $x_1, x_2, \ldots, x_s$, H_2 is induced by the stars with centers $y_1, y_2, \ldots, y_t$, and H_3 is induced by all the independent edges. (Following the notations in Section 2.) Let φ_1, φ_2 and φ_3 be the vertex colorings of G restricted on $V(H_1)$, $V(H_2)$ and $V(H_3)$, respectively, such that the mages of φ_1, φ_2 and φ_3 are mutually disjoint. Clearly, the union of φ_1, φ_2 and φ_3 is a vertex coloring of G using $|\varphi_1(V(H_1))| + |\varphi_2(V(H_2))| + |\varphi_3(V(H_3))|$ colors. Now if $n \geqslant b(M) + b(N) + r$ where $M = (m_1, m_2, \ldots, m_s)$ and $N = (n_1, n_2, \ldots, n_t)$, then we can reserve $b(M)$ colors for φ_1, $b(N)$ colors for φ_2 and r colors for φ_3. A biconformable vertex coloring φ can be obtained by the following assignment: (i) in $V(H_1)$, let $\varphi(x_i) = \alpha_i$; for the vertices in $N_G(x_i)$, at most s vertices can be colored with α_i, $i = 1, 2, \ldots, s$, and each occurs at most s times; (ii) in $V(H_2)$, let $\varphi(y_i) = \beta_i$; for the vertices in $N_G(y_i)$, at most t vertices can be colored with β_i, $i = 1, 2, \ldots, t$, and each occurs at most t times; and (iii) in $V(H_3)$, for each an independent edge xy, let $\varphi(x) = \varphi(y) = \gamma_i$. Then we have the following lemma.

Lemma 4.1. *Let $G = K_{n,n} \backslash E(J)$ where $J = \langle m_1, m_2, \ldots, m_s;\ n_1, n_2, \ldots, m_s;\ r\rangle_n$. Also let $M = (m_1, m_2, \ldots, m_s)$ and $N = (n_1, n_2, \ldots, n_t)$. Then G is biconformable provided that $n \geqslant b(M) + b(N) + r$.*

The above inequality plays an important role in the classification of maximal equibipartite graphs with maximum degree $n-1$.

Proposition 4.2. *Let $G=K_{n,n}\backslash E(J)$ where $J=\langle m_1,m_2,\ldots,m_s;\ n_1,n_2,\ldots,n_t,\ r\rangle_n$. Let $M=(m_1,m_2,\ldots,m_s)$ and $N=(n_1,n_2,\ldots,n_t)$. If G is type* 1, *then $n \geqslant b(M)+b(N)+r$.*

Proof. Following the notation before Lemma 4.1, let $V(G)=V(H_1)\cup V(H_2)\cup V(H_3)$. Since G is type 1, G is biconformable. Therefore, for each type 1 total coloring φ of G, we need at least $b(M)$ colors to color $V(H_1)$ and at least $b(N)+r$ colors to color the rest vertices of G. In order to prove the proposition, it suffices to show that no color that occurs in $V(H_1)$ can occur in $V(H_2)\cup V(H_3)$ and no color that occurs in $V(H_2)$ can occur in $V(H_1)\cup V(H_3)$. The proof of the second statement is similar to the first. Thus we prove the first statement only.

First, we see that in the total coloring φ each color occurs at most $n+1$ times (either on vertices or edges), and in total there are $2n$ vertices and $n^2-(2n-s-t-r)$ edges in G. Furthermore, only the colors occurs on the center of stars or the vertices of independent edges can occur $n+1$ times, hence there are $s+t+r$ colors which occurs $n+1$ times and n for each of the other colors. Now, if x is one of the $b(M)$ colors for $V(H_1)$ which does not occur on the centers, then it occurs on the major vertices in $V(H_1)$. As a consequence, x cannot occur on $V(H_2)\cup V(H_3)$. On the other hand, if x is a color on a center of a star in H_1, then x cannot be used in coloring $V(H_2)\cup V(H_3)$. For otherwise, x occurs only n times in G. This concludes the proof. □

Now we obtain a class of maximal equibipartite graphs with maximum degree $n-1$ which are type 2.

Proposition 4.3. *Let $J=\langle m_1,m_2,\ldots,m_s;\ n_1,n_2,\ldots,n_t;\ r\rangle_n$. If $n_1=n_{r+1}=t\geqslant r+1$, then $G=K_{n,n}\backslash E(J)$ is type* 2.

Proof. Let $M=(m_1)$ and $N=(t,t,\ldots,t,n_{r+2},\ldots,n_t)$. Then $b(M)=m_1$ and $b(N)\geqslant t$. This implies that $b(M)+b(N)+r\geqslant m_1+t+r$. By Proposition 4.2, if $b(N)>t$, then $b(M)+b(N)+r>n$ and G is type 2. We are done. Thus assume that $b(N)=t$ and G is type 1. Let φ be a type 1 total coloring of G and we use the colors $c_1,c_2,\ldots,c_n$. Let the centers of stars (in J) of size at least 2 be $x_1; y_1,y_2,\ldots,y_t$. Since φ is biconformable, the colors that occur on $N_J(x_1)$ are all distinct such that there is one vertex colored with $\varphi(x_1)$, and for $1\leqslant i\leqslant r+1$, the color that occurs on $N_J(y_i)$ is exactly the same as $\varphi(y_i)$. (Note that by Lemma 4.1, $V(H_1)$, $V(H_2)$, and $V(H_3)$ use different colors.) Without loss of generality, let the colors that occur on $N_J(x_1)$ be $c_1,c_2,\ldots,c_{m_1}$ and the colors that occur on $\bigcup_{i=1}^{r+1} N(y_i)$ be $c_{m_1+1},c_{m_1+2},\ldots,c_{m_1+r+1}$. Now in B, except for $y_1,y_2,\ldots,y_t$, there are $n-t$ major vertices which are of degree $n-1$ and c_{m_1+i}, does not occur on these vertices for each $1\leqslant i\leqslant r+1$. Therefore, there exists a matching T_i which is incident with these vertices and each edge of T_i is colored with c_{m_1+i}. Since c_{m_1+i} occurs on $N_J(y_i)$, T_i is incident with x_1, i.e., there exists an edge

$x_1 y_{t+j_i}$, $1 \leqslant j_i \leqslant r$, which is colored with c_{m_1+i}. Now, in total, we have $r+1$ edges of the form $x_1 y_{t+j_i}$ which have distinct colors. This is not possible. Thus G must be type 2.

As a special case of the graphs in Proposition 4.3, let $G = K_{n,n} \backslash E(J)$ where $J = \langle m_1;\ t,t,\dots,t;\ r\rangle_n$, $1 \leqslant r < t$ and $n = t^2 + r + 1$. Then G is type 2. Furthermore, by Proposition 2.4, since $1 + (m_1 - 1) + t = r = m_1 + t = r = n$, G is biconformable. Also, $n > m_1 + t$, every equibipartite subgraph H with $\Delta(G) = \Delta(H)$ is biconformable. Thus we have the following result which shows that the biconformability conjecture is not true in general. $\square$

Proposition 4.4. *Let $G = K_{n,n} \backslash E(J)$ where $J = \langle m_1;\ t,t,\dots,t;\ r\rangle_n$, $1 \leqslant r < t$ and $n = t^2 + r + 1$. Then G is a counterexample to Conjecture* 1.5.

Note that, by Propositions 4.2 and 4.4, the condition $b(M) + b(N) + r \leqslant n$ for a type 1 graph is necessary but not sufficient. On the other hand, the biconformability conjecture is true for the case when $\Delta(G) = n$ as mentioned in Lemma 1.4. Hence the counterexample obtained here is sharp with respect to $\Delta(G)$.

With the work we have done so far we believe the following Conjecture might be true.

Conjecture 4.5. Let $G = K_{n,n} \backslash E(J)$ where $J = \langle m_1, m_2, \dots, m_s;\ n_1, n_2, \dots, n_t;\ r\rangle_n$. Then G is type 2 if and only if either

$$n < b(m_1, m_2, \dots, m_s) + b(n_1, n_2, \dots, n_t) + r, \quad \text{or } s = 1 \text{ and } n_1 = n_{r+1} = t \geqslant r + 1.$$

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