ELSEVIER
Discrete Mathematics 183 (1998) 103–117
DISCRETE MATHEMATICS

# The Doyen–Wilson theorem for maximum packings of $K_n$ with 4-cycles

H.-L. Fu[a], C.C. Lindner[b,*,1]

[a]Department of Applied Mathematics, National Chiao-Tung University Hsin-Chu, Taiwan, ROC
[b]Department of Discrete and Statistical Sciences, 120 Math Annex, Auburn University, Auburn, ALA 36849-5307, USA

Received 29 May 1996; received in revised form 25 March 1997; accepted 31 March 1997

**Abstract**

Necessary and sufficient conditions are given to embed a maximum packing of $K_m$ with 4-cycles into a maximum packing of $K_n$ with 4-cycles, both when the leave of the given packing is preserved, and when the leave of the given packing is not necessarily preserved.

## 1. Introduction

A *Steiner triple system* (or simply *triple system*) of *order* $n$ is a pair $(S,T)$, where $T$ is an edge-disjoint collection of triangles (triples) which partition the edge set of $K_n$ (the complete undirected graph on $n$ vertices) with vertex set $S$. It has been known forever (= since 1847 [5]) that the spectrum for triple systems (= the set of all $n$ such that a triple system of order $n$ exists) is precisely the set of all $n \equiv 1$ or $3 \pmod 6$. In this case $|T| = n(n-1)/6$.

The triple system $(S_1,T_1)$ is said to be *embedded* in the triple system $(S_2,T_2)$ provided $S_1 \subseteq S_2$ and $T_1 \subseteq T_2$. We also say that $(S_1,T_1)$ is a *subsystem* of $(S_2,T_2)$. It is trivial to show that if $(S_1,T_1)$ is a proper subsystem of $(S_2,T_2)$ then $2|S_1|+1 \leqslant |S_2|$. Now, a quite natural question to ask is: given integers $m \equiv 1$ or $3 \pmod 6$ and $n \equiv 1$ or $3 \pmod 6$ with $2m+1 \leqslant n$, does there exist a triple system of order $n$ containing a subsystem of order $m$? In 1973 the celebrated work of Jean Doyen and Richard Wilson [2] showed that this is in fact the case.

* Corresponding author. E-mail: lindncc@mail.auburn.edu.
[1] Supported by NSF grant DMS-9300959.


PII S0012-365X(97)00080-0

**The Doyen and Wilson Theorem [2].** Let $2m+1 \leqslant n$, $m \equiv 1$ or $3 \pmod 6$, and $n \equiv 1$ or $3 \pmod 6$. Then there exists a triple system of order $n$ containing a subsystem of order $m$.

Over the years, any problem involving trying to prove a similar result for a given combinatorial structure has come to be called a *Doyen–Wilson problem*, and (not too surprisingly) any solution a *Doyen–Wilson type theorem*. The Doyen–Wilson Theorem has spawned a cottage industry with respect to just about any combinatorial design you can think of. The history of work along these lines is much too extensive to go into here. The interested reader is referred to [1] for an excellent history of this problem. Now it does not take a lot of imagination to think of one Doyen–Wilson type problem that is a natural generalization of the original result: maximum packings of $K_n$ with triples. Without going into details, this problem has been settled (in two different ways) by the combined work in [3,4,7].

The object of this paper is the *complete solution* of the Doyen–Wilson problem for maximum packings of $K_n$ with 4-cycles.

## 2. Statement of the problem

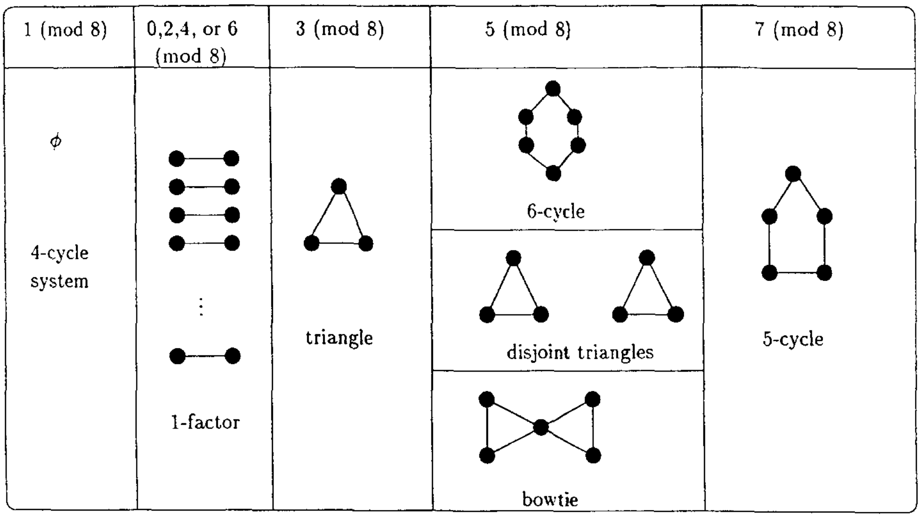
A 4-*cycle system* of *order n* is a pair $(S, C)$, where $C$ is an edge disjoint collection of 4-cycles which partition the edge set of $K_n$ with vertex set $S$. It is a well known [6] that the spectrum for 4-cycle systems (= the set of all $n$ such that a 4-cycle system of order $n$ exists) is *precisely* the set of all $n \equiv 1 \pmod 8$. It is trivial to prove the Doyen–Wilson Theorem for 4-cycle systems.

**Theorem 2.1.** *Let $m < n$ and $m$, $n \equiv 1 \pmod 8$. Then there exists a 4-cycle system of order $n$ containing a 4-cycle system of order $m$.*

**Proof.** Let $(\{\infty\} \cup X, C_1)$ be a 4-cycle system of order $m$ and $(\{\infty\} \cup Y, C_2)$ a 4-cycle system of order $n-m+1$. Let $S = \{\infty\} \cup X \cup Y$ and define a collection of 4-cycles $C_3$ as follows: (1) $C_1 \subseteq C_3$, (2) $C_2 \subseteq C_3$, and (3) decompose the complete bipartite graph with parts $X$ and $Y$ into 4-cycles (this is easy, see [9] for example) and place these 4-cycles in $C_3$. Then $(S, C_3)$ is a 4-cycle system of order $n$ containing a subsystem of order $m$. □

A *packing* of $K_n$ with 4-cycles (or a *partial* 4-cycle system) is an ordered triple $(S, P, L)$, (pair $(S, P)$), where $S$ is the vertex set of $K_n$, $P$ is a collection of edge disjoint 4-cycles of the edge set of $K_n$, and $L$ is the set of edges in $K_n$ not belonging to a 4-cycle in $P$. The *number n* is called the *order* of the packing (partial 4-cycle system) and the set of edges in $L$ is called the *leave*.

If $|P|$ is as large as possible (or $|L|$ is as small as possible) the packing is said to be *maximum* (MPC). For example, a 4-cycle system is a maximum packing with leave the




Leaves of a maximum packings

Fig. 1. Leaves of a maximum packings.

empty set. When $n \not\equiv 1 \pmod 8$ it is easy to see that the leave of a MPC is: a 1-factor if $n \equiv 0, 2, 4,$ or $6 \pmod 8$, a 3-cycle if $n \equiv 3 \pmod 8$, a 5-cycle if $n \equiv 7 \pmod 8$, and a graph of even degree with 6 edges if $n \equiv 5 \pmod 8$ (so a 6-cycle, or a pair of disjoint 3-cycles (triangles), or a *bowtie* (= a pair of 3-cycles (triangles) having a vertex in common)) (see Fig. 1).

It is well known (see [8] for example) that MPCs can be constructed for every $n$ *and* with all possible leaves for $n \equiv 5 \pmod 8$, except when $n = 5$. In this case only a bowtie is possible as a leave, since a 6-cycle and 2 disjoint triangles use 6 vertices.

The MPC $(S_1, P_1, L_1)$ is said to be embedded in the MPC $(S_2, P_2, L_2)$ provided $S_1 \subseteq S_2$ and $P_1 \subseteq P_2$. Now a bit of reflection shows that there are two types of embeddings possible: one that *preserves* the leave $L_1$; i.e., $L_1 \subseteq L_2$; and one that does not necessarily preserve the leave. In the case where the leave $L_1$ is not necessarily preserved, the MPC $(S_1, P_1, L_1)$ is treated as a partial 4-cycle system $(S_1, P_1)$. This second type of embedding is considerably more complex and difficult than the embedding where the leave is preserved. Never-the-less we give a complete solution of the Doyen–Wilson problem for both types of embeddings; i.e., we determine necessary and sufficient conditions for a MPC of order $m$ to be embedded in a MPC of order $n$ as a partial 4-cycle system and with the leave preserved. We will handle the more difficult type of embedding first, since the embeddings with the leave preserved are trivial modifications of the embedding where the leave is not necessarily preserved.

We will organize the embeddings where the leave is *not necessarily preserved* into four sections: even into even, even into odd, odd into even, and odd into odd. The first of these sections is quite trivial. The remaining three are far from trivial! In

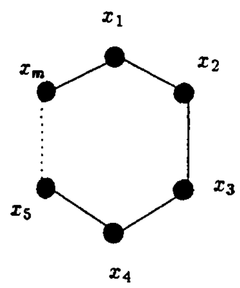



Fig. 2.

what follows we will denote the $m$-cycle ($m=3,4,5$, or 6) by $(x_1,x_2,x_3,\ldots,x_m)$ or $(x_1,x_m,x_{m-1},\ldots,x_2)$ (see Fig. 2).

## 3. Even into even

This is by far the easiest case to handle, and so a good place to start!

**Lemma 3.1.** *A MPC of order* $2n$ *can be embedded in a MPC of order* $2n+2t$ *if and only if* $t\geqslant 1$.

**Proof.** The necessary condition is less than trivial. Let $(S_1,P_1,L_1)$ and $(S_2,P_2,L_2)$ be MPCs of orders $2n$ and $2t$, respectively (so the leaves are 1-factors), where $S_1\cap S_2=\phi$. Let $S_3=S_1\cup S_2$ and define a collection of 4-cycles $P_3$ by: (1) $P_1\subseteq P_3$, (2) $P_2\subseteq P_3$, and (3) for each $\{a,b\}\in L_1$ and each $\{c,d\}\in L_2$, the 4-cycle $(a,c,b,d)$ *or* $(a,d,b,c)\in P_3$. Then $(S_3,P_3,L_3)$ is a MPC of order $2n+2t$, where $L_3=L_1\cup L_2$ and contains both $(S_1,P_1)$ and $(S_2,P_2)$ as subsystems. □

**Remark.** It is *important* to note that the containing MPC $(S_3,P_3,L_3)$ preserved the leaves $L_1$ and $L_2$. Hence Lemma 3.1 also takes care of the embedding with the leave preserved.

## 4. Even into odd

The following construction is the principal tool used in the embeddings of a MPC of even order into a MPC of odd order.

*The Fundamental Construction.* Let $(S_1,P_1,L_1)$ be a MPC of order $2t$ (so $L_1$ is a 1-factor containing $t$ edges) and $X$ a set of odd size $x$ such that $S_1\cap X=\phi$. Let $K_x$ be the complete graph of order $x$ with vertex set $X$ and $G$ a subgraph of $K_x$ with each vertex having even degree and such that $E(K_x)\backslash E(G)$ can be partitioned in a collection of 4-cycles $P_2$. Form any one of the following graphs in Fig. 3.

In each of (2), (3), (4), (5), and (6) the graph between $(S_1, P_1, L_1)$ and $K_x$ is called the *connecting graph*. Now let $\alpha$ be a 1–1 mapping from $E(G)$ onto the edges of $L_1$ *not* belonging to the connecting graph. Define a collection of 4-cycles $C$ as follows:

(i) For each edge $\{a,b\} \in E(G)$ such that $\{a,b\}\alpha = \{c,d\}$, place *exactly one* of the 4-cycles $(a,b,c,d)$, $(a,b,d,c)$ in $C$.

(ii) For each vertex $a \in K_x$ let $X_a = \{b \in S_1 \mid \text{the edge } \{a,b\} \text{ is in a cycle of type (i)}$ or is an edge belonging to the connecting graph$\}$. Now some of the sets $X_a$ may be empty, but the nonempty sets $X_a$ each contain an even number of vertices and partition the set $S_1$. For each nonempty $X_a$ partition the complete bipartite graph with parts $X_a$ and $X\backslash\{a\}$ into 4-cycles and place these 4-cycles in $C$.

Set $S_3 = S_1 \cup X$, $P_3 = P_1 \cup P_2 \cup C$, and $L_3 =$ the connecting graph. Then $(S_3, P_3, L_3)$ is a MPC of order $2t + x$ with leave the connecting graph and, of course, it contains $(S_1, P_1)$. □

With the above construction in hand the embedding of MPCs of even order into MPCs of odd order goes quite smoothly.

**Lemma 4.1.** *A necessary and sufficient condition to embed a MPC of order* $2t$ *in a MPC of* odd *order* $2t+x$ *is*: $\binom{x}{2} \geqslant t, t-1, t-2$, *or* $t-1$ *if and only if* $2t + x \equiv 1, 3, 5$, *or* $7 \pmod 8$, *respectively.*

**Proof.** We begin with the necessary conditions. So let $(S_1, P_1, L_1)$ be a MPC of order $2t$ and $(S_2, P_2, L_2)$ a MPC of odd order $2t + x$ containing $(S_1, P_1)$. If $2t + x \equiv 1, 3, 5$, or $7 \pmod 8$ then *at most* $0, 1, 2$, or $1$ edges of $L_1$, can belong to the leave $L_2$. The other (at least) $t$, $t-1$, $t-2$, or $t-1$ edges must belong to 4-cycles each of which contains *exactly one* edge in $S_2\backslash S_1$. Since $|S_2\backslash S_1| = x$ we must have $\binom{x}{2} > t$, $t-1$, $t-2$, or $t-1$ respectively. In the following constructions, the cases $t = 1$ and $2$ are trivial and so in every case we assume $t \geqslant 3$.

Now let $(S_1, P_1, L_1)$ be a MPC of order $2t$ and $2t + x \equiv 1, 3, 5$, or $7 \pmod 8$ and $\binom{x}{2} \geqslant t$, $t-1$, $t-2$, or $t-1$ as the case may be. We break the proof up into four cases:

$2t + x \equiv 1 \pmod 8$. Let $(S_2, P_2, L_2)$ be a MPC of order $x$: If $x \equiv 1 \pmod 8$ use part (1) of the Fundamental Construction (FC) with $G$ consisting of any $t/4$ 4-cycles of $P_2$. If $x \equiv 3 \pmod 8$ use part (1) of the FC with $G$ consisting of $L_2$ along with any $(t-3)/4$ 4-cycles of $P_2$. If $x \equiv 5 \pmod 8$ use part (1) of the FC with $G$ consisting of $L_2$ along with any $(t-6)/4$ 4-cycles of $P_2$. If $x \equiv 7 \pmod 8$ use part (1) of the FC with $G$ consisting of $L_2$ along with any $(t-5)/4$ 4-cycles of $P_2$.

$2t + x \equiv 3 \pmod 8$. Let $(S_2, P_2, L_2)$ be a MPC of order $x$. We use part (2) of the FC. If $x \equiv 1 \pmod 8$ take $G$ to be any $(t-1)/4$ 4-cycles of $P_2$. If $x \equiv 3 \pmod 8$ take $G$ to be $L_2$ along with any $(t-4)/4$ 4-cycles of $P_2$. If $x \equiv 5 \pmod 8$ take $G$ to consist of $L_2$ (take $L_2$ to be a bowtie in the case where $x = 5$) along with any $(t-7)/4$ 4-cycles of $P_2$. If $x \equiv 7 \pmod 8$ take $G$ to consist of $L_2$ along with any $(t-6)/4$ 4-cycles of $P_2$.

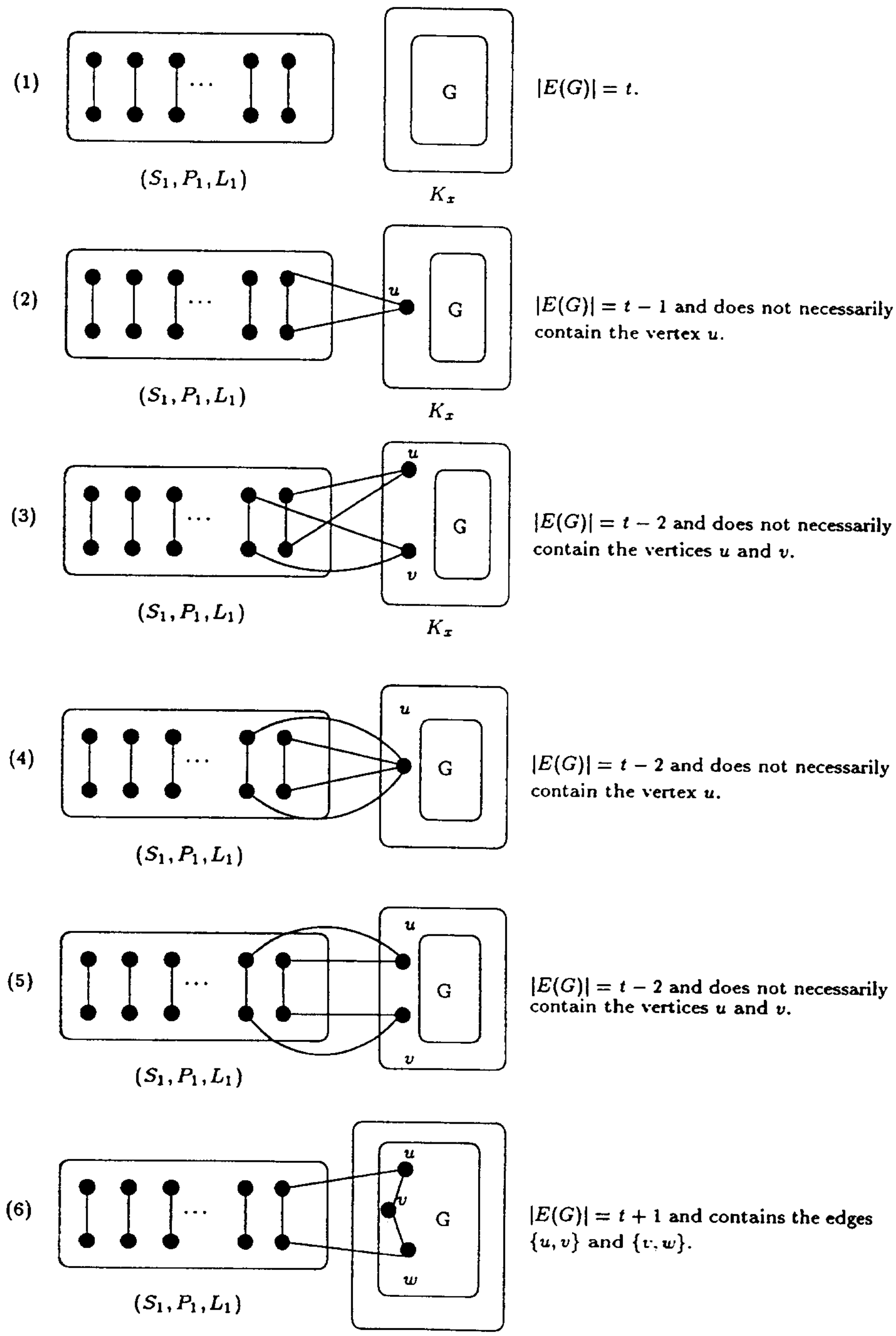

(1)
G
(S1, P1, L1)
Kx
|E(G)| = t.
(2)
u
G
(S1, P1, L1)
Kx
|E(G)| = t − 1 and does not necessarily contain the vertex u.
(3)
u
v
G
(S1, P1, L1)
Kx
|E(G)| = t − 2 and does not necessarily contain the vertices u and v.
(4)
u
G
(S1, P1, L1)
|E(G)| = t − 2 and does not necessarily contain the vertex u.
(5)
u
v
G
(S1, P1, L1)
|E(G)| = t − 2 and does not necessarily contain the vertices u and v.
(6)
u
v
w
G
(S1, P1, L1)
|E(G)| = t + 1 and contains the edges {u, v} and {v, w}.

Fig. 3.

$2t+x \equiv 5 \pmod 8$. Let $(S_2,P_2,L_2)$ be a MPC of order $x$. We use part (3), (4), and (5) of the FC. If $x \equiv 1 \pmod 8$ take $G$ to be any $(t-2)/4$ 4-cycles of $P_2$. If $x \equiv 3 \pmod 8$ take $G$ to be $L_2$ along with along $(t-5)/4$ 4-cycles of $P_2$. If $x \equiv 5 \pmod 8$ take $G$ to be $L_2$ (take $L_2$ to be a bowtie in the case where $x=5$) along with any $(t-8)/4$ 4-cycles of $P_2$. If $x \equiv 7 \pmod 8$ take $G$ to be $L_2$ plus any $(t-7)/4$ 4-cycles of $P_2$.

$2t+x \equiv 7 \pmod 8$. Let $(S_2,P_2,L_2)$ be a MPC of order $x$. We use part (6) of the FC. If $x \equiv 1 \pmod 8$ take $G$ to be any $(t+1)/4$ 4-cycles of $P_2$ and use a path of length 2 in one of these 4-cycles for the connecting graph. If $x \equiv 3 \pmod 8$ take $G$ to be $L_2$ plus any $(t-2)/4$ 4-cycles of $P_2$ and use a path of length 2 in $L_2$ in the connecting graph. If $x \equiv 5 \pmod 8$ take $G$ to be $L_2$ plus any $(t-5)/4$ 4-cycles of $P_2$ using a path of length 2 belonging to $L_2$ for the connecting graph. If $x \equiv 7 \pmod 8$ take $G$ to be $L_2$ plus any $(t-4)/4$ 4-cycles of $P_2$ and use a path of length 2 belonging to $L_2$ for the connecting graph.

Combining the above four cases completes the proof. □

## 5. Odd into even

We divide these constructions into four parts.

**Lemma 5.1.** *A MPC of order* $m \equiv 1 \pmod 8$ (= *4-cycle system*) *can be embedded in a MPC of order* $2t$ *if and only if* $2t \geqslant 2m$.

**Proof.** The necessary condition is obvious. Now let $(S_1,P_1,L_1=\phi)$ be a MPC of order $m \equiv 1 \pmod 8$. Set $S_2 = S_1 \times \{1,2\}$ and define a collection of 4-cycles $P_2$ as follows: for each 4-cycle $(a,b,c,d) \in P_1$ place the four 4-cycles $((a,1),(b,1),(c,1),(d,1))$, $((a,2),(b,2),(c,2),(d,2))$, $((a,1),(b,2),(c,1),(d,2))$, and $((a,2),(b,1),(c,2),(d,1))$ in $P_2$. Then $(S_2,P_2,L_2)$ is a MPC of order $2m$ with leave the 1-factor $L_2 = \{\{(a,1),(a,2)\} \mid a \in S_1\}$. By Lemma 3.1 $(S_2,P_2,L_2)$ can be embedded in a MPC of order $2t$ for every $2t \geqslant 2m$. □

**Lemma 5.2.** *A MPC of order* $m \equiv 3 \pmod 8$ *can be embedded in a MPC of order* $2t$ *if and only if* $2t \geqslant 2m-2$.

**Proof.** Let $(S_1,P_1,L_1)$ be a MPC of order $m \equiv 3 \pmod 8$ and $(S_2,P_2,L_2)$ a MPC of order $2t$ containing $(S_1,P_1)$. Since $L_1$ is a triangle and $L_2$ is a 1-factor, *at most* one edge of $L_1$, can be used in the leave $L_2$. Hence at least $m-2$ vertices of $S_1$ must be covered by edges in $L_2$ with one vertex in $S_1$ and one vertex in $S_2 \backslash S_1$. It follows that $2t \geqslant m+(m-2)=2m-2$.

Now let $(S_1,P_1,L_1)$ be a MPC of order $m \equiv 3 \pmod 8$ with leave $L_1=(a,b,c)$ and let $X = S_1 \backslash \{b,c\}$. Let $S_2 = \{b,c\} \cup (X \times \{1,2\})$ and define a collection of 4-cycles $P_2$ as follows: (1) define a copy of $(S_1,P_1)$ on $\{b,c\} \cup (X \times \{1\})$ and place these 4-cycles in $P_2$; (2) $(b,(a,1),c,(a,2)) \in P_2$; (3) partition the complete bipartite graph with parts $\{b,c\}$ and $(X\backslash\{a\}) \times \{2\}$ into 4-cycles and place these 4-cycles in $P_2$; (4) let $(X,P_2^*)$ be a 4-cycle system and for each 4-cycle $(d,e,f,g) \in P_2^*$, place the three 4-cycles $((d,2),(e,2),(f,2),(g,2))$, $((d,2),(e,1),(f,2),(g,1))$, and $((d,1),(e,2),(f,1),(g,2))$ in $P_2$. Then $(S_2,P_2,L_2)$ is a MPC of order $2m-2$ with leave $L_2 = \{\{b,c\}\} \cup \{\{(x,1),(x,2)\} \mid x \in X\}$. By Lemma 3.1 $(S_2,P_2,L_2)$ can be embedded in a MPC of order $2t$ for every $2t \geqslant 2m-2$. □

**Lemma 5.3.** *A MPC of order $m \equiv 5 \pmod 8$ can be embedded in a MPC of order $2t$ if and only if* (i) $2t \geqslant 2m-4$ *if the leave is a bowtie or a pair of disjoint triangles, and* (ii) $2t \geqslant 2m-6$ *if the leave is a 6-cycle.*

**Proof.** We break the proof into three parts depending on the leave.

*Leave a bowtie.* Let $(S_1,P_1,L_1)$ be a MPC of order $m \equiv 5 \pmod 8$ with leave the bowtie $(a,b,\infty)$, $(c,d,\infty)$. Any MPC of order $2t$ containing $(S_1,P_1)$ can use at most 2 edges of $L_1$ in the leave and so $2t \geqslant 2m-4$.

Let $X = S_1\backslash\{a,b,c,d\}$ and set $S_2 = \{a,b,c,d\} \cup (X \times \{1,2\})$. Define a collection of 4-cycles $P_2$ as follows:
(1) Define a copy of $(S_1,P_1)$ on $\{a,b,c,d\} \cup (X \times \{1\})$ and place these 4-cycles in $P_2$;
(2) $(a,(\infty,1),b,(\infty,2))$ and $(c,(\infty,1),d,(\infty,2)) \in P_2$;
(3) partition the complete bipartite graph with parts $\{a,b,c,d\}$ and $(X\backslash\{\infty\}) \times \{2\}$ into 4-cycles and place these 4-cycles in $P_2$; and
(4) let $(X,P_2^*)$ be a 4-cycle system and for each 4-cycle $(e,f,g,h) \in P_2^*$, place the three 4-cycles $((e,2),(f,2),(g,2),(h,2))$, $((e,2),(f,1),(g,2),(h,1))$, and $((e,1),(f,2),(g,1),(h,2))$ in $P_2$.

Then $(S_2,P_2,L_2)$ is a MPC of order $2m-4$ with leave $L_2 = \{\{a,b\},\{c,d\}\} \cup \{\{(x,1),(x,2)\} \mid x \in X\}$. By Lemma 3.1 $(S_2,P_2,L_2)$ can be embedded in a MPC of order $2t$ for every $2t \geqslant 2m-4$. □

*Leave 2 disjoint triangles.* Let $(S_1,P_1,L_1)$ be MPC of order $m \equiv 5 \pmod 8$ with leave the pair of disjoint triangles $(a,b,\infty_1)$ and $(c,d,\infty_2)$. As in the bowtie case, any MPC of order $2t$ containing $(S_1,P_1)$ can contain at most 2 edges of $L_1$ in the leave and so $2t \geqslant 2m-4$.

Set $X = S_1\backslash\{a,b,c,d\}$ and $S_2 = \{a,b,c,d\} \cup (X \times \{1,2\})$. Define a collection of 4-cycles $P_2$ as follows:
(1) Define a copy of $(S_1,P_1)$ on $\{a,b,c,d\} \cup (X \times \{1\})$ and place these 4-cycles in $P_2$;
(2) $(a,(\infty_1,1),b,(\infty_1,2))$ and $(c,(\infty_2,1),d,(\infty_2,2)) \in P_2$;
(3) partition the complete bipartite graph with parts $\{a,b\}$ and $(X\backslash\{\infty_1\}) \times \{2\}$ into 4-cycles and place these 4-cycles in $P_2$;
(4) partition the complete bipartite graph with parts $\{c,d\}$ and $(X\backslash\{\infty_2\}) \times \{2\}$ into 4-cycles and place these 4-cycles in $P_2$; and
(5) the same as (4) when the leave is a bowtie.

Then $(S_2,P_2,L_2)$ is a MPC of order $2m-4$ with leave $L_2=\{\{a,b\},\{c,d\}\}\cup\{\{(x,1),(x,2)\}\mid x\in X\}$. By Lemma 3.1, $(S_2,P_2,L_2)$ can be embedded in a MPC of order $2t$ for every $2t\geqslant 2m-4$.

*Leave a 6-cycle.* Now if the leave is a 6-cycle then $m\geqslant 13$. Trivially, if $(S_1,P_1,L_1)$ is a MPC of order $m$ with leave a 6-cycle, any MPC of order $2t$ containing $(S_1,P_1)$ can use at most 3 edges of $L_1$ in the leave. Hence $2t\geqslant 2m-6$.

We handle the case $m=13$ separately. So let $(S_1,P_1,L_1)$ be a MPC of order 13, where $S_1=\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ and $L_1=(1,2,3,4,5,6)$. Let $X=\{7,8,9,10,11,12,13\}$, $S_2=\{1,2,3,4,5,6\}\cup(X\times\{1,2\})$, and define a collection of 4-cycles $P_2$ as follows:

(1) Define a copy of $(S_1,P_1)$ on $\{1,2,3,4,5,6\}\cup(X\times\{1\})$, with leave $(1,2,3,4,5,6)$, and place these 4-cycles in $P_2$; and

(2) place the following 27 4-cycles in $P_2$:
$((7,2),(10,2),(8,2),(11,2))$, $((7,2),(12,2),(8,2),(13,2))$,
$((9,2),(12,2),(10,2),(13,2))$, $((7,2),1,(8,2),2)$,
$((7,2),3,(8,2),4)$, $((7,2),5,(8,2),6)$, $((9,2),1,(10,2),2)$,
$((9,2),3,(10,2),4)$, $((9,2),5,(10,2),6)$, $((11,2),1,2,(12,2))$,
$((12,2),3,4,(13,2))$, $((11,2),5,6,(13,2))$, $((12,2),1,(13,2),5)$,
$((11,2),2,(13,2),3)$, $((11,2),4,(12,2),6)$, $((7,2),(8,1),(11,2),(9,2))$,
$((7,2),(10,1),(9,2),(8,2))$, $((9,2),(12,1),(11,2),(10,2))$,
$((7,1),(8,2),(10,1),(11,2))$, $((7,2),(9,1),(8,2),(11,1))$,
$((7,2),(12,1),(8,2),(13,1))$, $((9,2),(7,1),(10,2),(8,1))$,
$((9,2),(11,1),(10,2),(13,1))$, $((10,2),(9,1),(13,2),(12,1))$,
$((11,2),(9,1),(12,2),(13,1))$, $((12,2),(7,1),(13,2),(8,1))$,
$((12,2),(10,1),(13,2),(11,1))$.

Then $(S_2,P_2,L_2)$ is a MPC of order 20 with leave the 1-factor $L_2=\{\{1,6\},\{2,3\},\{4,5\},\{(7,1),(7,2)\},\{(8,1),(8,2)\},\{(9,1),(9,2)\},\{(10,1),(10,2)\},\{(11,1),(11,2)\},\{(12,1),(12,2)\},\{(13,1),(13,2)\}\}$.

We can now consider $(S_1,P_1,L_1)$ to be a MPC of order $m\equiv 5 \pmod 8$ where $m\geqslant 21$. Without loss in generality we can consider $L_1=(1,2,3,4,5,6)$ and $S_1=\{1,2,3,4,5,6\}\cup X\cup Y$, where $X=\{7,8,9,10,11,12,13\}$. Now set $S_2=\{1,2,3,4,5,6\}\cup(Z\times\{1,2\})$, $Z=X\cup Y$, and define a collection of 4-cycles $P_2$ as follows:

(1) Define a copy of $(S_1,P_1)$ on $\{1,2,3,4,5,6\}\cup(Z\times\{1\})$ and place these 4-cycles in $P_2$;

(2) place the 27 4-cycles in (2) for the case $m=13$ above in $P_2$;

(3) let $\infty\in X$ and let $(\{\infty\}\cup Y,P_2^*)$ be a 4-cycle system and for each 4-cycle $(e,f,g,h)\in P_2^*$, place the three 4-cycles $((e,2),(f,2),(g,2),(h,2))$, $((e,2),(f,1),(g,2),(h,1))$, and $((e,1),(f,2),(g,1),(h,2))$ in $P_2$; and

(4) partition the complete bipartite graph with parts $(X\backslash\{\infty\})\times\{2\}$ and $Y\times\{1\}$, (ii) $\{1,2,3,4,5,6\}\cup((X\backslash\{\infty\})\times\{1\})$ and $Y\times\{2\}$, and (iii) $(X\backslash\{\infty\})\times\{2\}$ and $Y\times\{2\}$ into 4-cycles and place these 4-cycles in $P_2$.

Combining (1)–(4) gives a MPC $(S_2,P_2,L_2)$ of order $2m-6$ with leave the union of the leave in part (2) of the $m=13$ case along with $\{(y,1),(y,2)\mid y\in Y\}$.

Lemma 3.1 can be used to extend $(S_2, P_2, L_2)$ to a MPC of order $2t$ for every $2t \geqslant 2m-6$. □

**Lemma 5.4.** *A MPC of order $m \equiv 7 \pmod 8$ can be embedded in a MPC of order $2t$ if and only if $2t \geqslant 2m-4$.*

**Proof.** Let $(S_1, P_1, L_1)$ be a MPC of order $m \equiv 7 \pmod 8$ with leave a 5-cycle. Then any MPC of order $2t$ containing $(S_1, P_1)$ can use at most 2 edges of $L_1$ in the leave. Hence $2t \geqslant 2m-4$. We begin by handling the case $m=7$ separately. Let $(S_1, P_1, L_1)$ be a MPC of order 7, where $S_1 = \{1,2,3,4,5,6,7\}$ and $L_1 = (1,3,4,5,2)$. Let $X = \{5,6,7\}$, $S_2 = \{1,2,3,4\} \cup (X \times \{1,2\})$, and define a collection of 4-cycles $P_2$ as follows:

(1) Define a copy of $(S_1, P_1, L_1)$ on $\{1,2,3,4\} \cup (X \times \{1\})$, with leave $(1,3,4,(5,1),2)$ and place these 4-cycles in $P_2$; and

(2) place the following six 4-cycles in $P_2$: $((6,2),(7,2),3,1)$, $(4,(5,1),2,(5,2))$, $((6,2),(5,2),(7,2),(5,1))$, $((7,2),4,(6,2),2)$, $(1,(5,2),(6,1),(7,2))$, and $((5,2),3,(6,2),(7,1))$.

Then $(S_2, P_2, L_2)$ is a MPC of order 10 with leave the 1-factor

$$L_2 = \{\{1,2\},\{3,4\},\{(5,1),(5,2)\},\{(6,1),(6,2)\},\{(7,1),(7,2)\}\}.$$

We now consider $(S_1, P_1, L_1)$ to be a MPC of order $m \equiv 7 \pmod 8$ where $m \geqslant 15$. Without loss in generality we can consider $L_1 = (1,3,4,5,2)$ and $S_1 = \{1,2,3,4\} \cup X \cup Y$, where $X = \{5,6,7\}$. Let $S_2 = \{1,2,3,4\} \cup (Z \times \{1,2\})$, $Z = X \cup Y$, and define a collection of 4-cycles $P_2$ as follows:

(1) Define a copy of $(S_1, P_1)$ on $\{1,2,3,4\} \cup (Z \times \{1\})$ and place these 4-cycles in $P_2$;

(2) place the six 4-cycles in (2) in the case $m=7$ above in $P_2$;

(3) let $(\{5\} \cup Y, P_2^*)$ be a 4-cycle system and for each 4-cycle $(e,f,g,h) \in P_2^*$, place the three 4-cycles $((e,2),(f,2),(g,2),(h,2))$, $((e,2),(f,1),(g,2),(h,1))$, and $((e,1),(f,2),(g,1),(h,2))$ in $P_2$; and

(4) partition the complete bipartite graph with parts (i) $(X \backslash \{5\}) \times \{2\}$ and $Y \times \{1\}$, (ii) $\{1,2,3,4\} \cup \{(6,1),(7,1)\}$ and $Y \times \{2\}$, and (iii) $\{(6,2),(7,2)\}$ and $Y \times \{2\}$ into 4-cycles and place these 4-cycles in $P_2$.

Combining (1), (2), (3), and (4) gives a MPC $(S_2, P_2, L_2)$ of order $2m-4$ with leave the union of the leave in part (2) of the $m=7$ case along with $\{\{(y,1),(y,2)\} \mid y \in Y\}$. Lemma 3.1 can be used to extend $(S_2, P_2, L_2)$ to a MPC of order $2t$ for every $2t \geqslant 2m-4$. □

## 6. Odd into odd

We will show that a MPC of odd order $m$ can always be embedded in a MPC of order $m+2$, with the *one exception* when $m \equiv 5 \pmod 8$ *and* the leave is a pair of disjoint triangles. In this case the best possible result is $m+4$. Combining this with

Lemma 3.1 shows that a MPC of odd order $m$ can be embedded in a MPC of order $m+2t$ for every $t\geqslant 1$, except when $m\equiv 5 \pmod 8$ and the leave is a pair of disjoint triangles. In this case we can obtain an embedding of size $m+2t$ for every $t\geqslant 2$.

The necessary conditions are transparent in every case except $m\equiv 5 \pmod 8$ with leave a pair of disjoint triangles. A routine computation takes care of this. We will break the proof into four parts.

**Lemma 6.1.** *A MPC of order $m\equiv 1 \pmod 8$ can be embedded in a MPC of order $m+2$.*

**Proof.** Let $(S_1,P_1,L_1=\phi)$ be a MPC of order $m\equiv 1 \pmod 8$. Let $S_2=\{\infty_1,\infty_2\}\cup S_1$ and define a collection of 4-cycles $P_2$ as follows: (1) $P_1\subseteq P_2$; and (2) partition the complete bipartite graph with parts $\{\infty_1,\infty_2\}$ and $S_1\backslash\{\infty\}$ (where $\infty$ is any element in $S_1$) into 4-cycles and place these 4-cycles in $P_2$.

Then $(S_2,P_2,L_2)$ is a MPC of order $m+2$ with leave the triangle $(\infty,\infty_1,\infty_2)$. □

**Lemma 6.2.** *A MPC of order $m\equiv 3 \pmod 8$ can be embedded in a MPC of order $m+2$ having each of the three possible leaves, except for the case $m=3$. In this case only a bowtie is possible.*

**Proof.** We break up the proof into three parts depending on the leave. Trivially, only a bowtie is possible when $m=3$. So we assume that $m\geqslant 11$ in what follows.

*Leave a bowtie.* Let $(S_1,P_1,L_1)$ be a MPC of order $m\equiv 3 \pmod 8$ with leave the triangle $(a,b,c)$. Let $S_2=\{\infty_1,\infty_2\}\cup S_1$ and define a collection of 4-cycles as follows: (1) $P_1\subseteq P_2$; and (2) partition the complete bipartite graph with parts $\{\infty_1,\infty_2\}$ and $S_1\backslash\{a\}$ into 4-cycles. Then $(S_2,P_2,L_2)$ is a MPC of order $m+2$ with leave the bowtie $(\infty_1,\infty_2,a)$, $(a,b,c)$.

*Leave 2 disjoint triangles.* Exactly the same as the bowtie case except partition the complete bipartite graph with parts $\{\infty_1,\infty_2\}$ and $S_1\backslash\{d\}$, $d\notin\{a,b,c\}$, into 4-cycles. The leave is then the pair of disjoint triangles $(\infty_1,\infty_2,d)$ and $(a,b,c)$.

*Leave a 6-cycle.* Let $(S_1,P_1,L_1)$ be a MPC of order $m\equiv 3 \pmod 8$ with leave the triangle $(a,b,c)$. Let $S_2=\{\infty_1,\infty_2\}\cup S_1$, $d\neq e\in S_1\backslash\{a,b,c\}$, and define a collection of 4-cycles as follows:

(1) $P_1\subseteq P_2$;
(2) $(\infty_1,\infty_2,a,c)$ and $(\infty_1,e,\infty_2,b)\in P_2$; and
(3) partition the complete bipartite graph with parts $\{\infty_1,\infty_2\}$ and $S_1\backslash\{a,b,c,d,e\}$ into 4-cycles and place these 4-cycles in $P_2$. Then $(S_2,P_2,L_2)$ is a MPC of order $m+2$ with leave the 6-cycle $(\infty_1,a,b,c,\infty_2,d)$.

Combining the above three constructions completes the proof. □

**Lemma 6.3.** *A MPC of order $m\equiv 5 \pmod 8$ with leave a bowtie or 6-cycle can be embedded in a MPC of order $m+2$. If the leave is a pair of disjoint triangles it can be embedded in a MPC of order $m+4$.*

**Proof.** Quite naturally we break the proof up into three cases depending on the leave.

*Leave a bowtie.* Let $(S_1,P_1,L_1)$ be a MPC of order $m\equiv 5 \pmod 8$ with leave the bowtie $(a,b,\infty)$, $(c,d,\infty)$. Set $S_2=\{\infty_1,\infty_2\}\cup S_1$ and define a collection of 4-cycles as follows: (1) $P_1\subseteq P_2$; (2) place the two 4-cycles $(\infty_1,\infty,c,d)$ and $(\infty_2,\infty,a,b)$ in $P_2$; and (3) partition the complete bipartite graph with parts $\{\infty_1,\infty_2\}$ and $S_1\backslash\{\infty,a,c\}$ into 4-cycles and place these 4-cycles in $P_2$. Then $(S_2,P_2,L_2)$ is a MPC of order $m+2$ with leave the 5-cycle $(\infty_1,b,\infty,d,\infty_2)$.

*Leave a 6-cycle.* Let $(S_1,P_1,L_1)$ be a MPC of order $m\equiv 5 \pmod 8$ with leave the 6-cycle $(a,b,c,d,e,f)$. Set $S_2=\{\infty_1,\infty_2\}\cup S_1$ and define a collection of 4-cycles as follows: (1) $P_1\subseteq P_2$; (2) place the two 4-cycles $(\infty_2,c,d,e)$ and $(\infty_1,e,f,a)$ in $P_2$; and (3) partition the complete bipartite graph with parts $\{\infty_1,\infty_2\}$ and $S_1\backslash\{c,e,f\}$ into 4-cycles and place these 4-cycles in $P_2$. Then $(S_2,P_2,L_2)$ is a MPC of order $m+2$ with leave the 5-cycle $(\infty_1,\infty_2,a,b,c)$.

*Leave two disjoint triangles.* Let $(S_1,P_1,L_1)$ be a MPC of order $m\equiv 5 \pmod 8$ with leave the pair of disjoint triangles $(1,2,3)$ and $(4,5,6)$. Let $S_2=\{a,b,c,d\}\cup S_1$ and define a collection of 4-cycles $P_2$ as follows: (1) $P_1\subseteq P_2$; (2) the seven 4-cycles $(a,1,3,b)$, $(c,4,6,d)$, $(a,3,d,4)$, $(1,b,5,d)$, $(c,3,2,1)$, $(b,4,5,6)$, and $(a,2,c,6)\in P_2$; and (3) partition the complete bipartite graph with parts $S_1\backslash\{1,2,3,4,6\}$ and $\{a,c\}$ into 4-cycles and place these 4-cycles in $P_2$; and (4) partition the complete bipartite graph with parts $S_1\backslash\{1,3,4,5,6\}$ and $\{b,d\}$ into 4-cycles and place these 4-cycles in $P_2$. Then $(S_2,P_2,L_2)$ is a MPC of order $m+4$ with leave the empty set (= a 4-cycle system).

Putting the above three arguments together completes the proof of the lemma. □

**Lemma 6.4.** *A MPC of order* $m\equiv 7 \pmod 8$ *can be embedded in a MPC of order* $m+2$ *(4-cycle system).*

**Proof.** Let $(S_1,P_1,L_1)$ be a MPC of order $m\equiv 7 \pmod 8$ with leave $L_1=(a,b,c,d,e)$. Let $S_2=\{\infty_1,\infty_2\}\cup S_1$ and define a collection of 4-cycles as follows: (1) $P_1\subseteq P_2$; (2) place the three 4-cycles $(\infty_1,a,b,c)$, $(\infty_2,a,e,d)$, and $(\infty_1,\infty_2,c,d)$ in $P_2$; and (3) partition the complete bipartite graph with parts $\{\infty_1,\infty_2\}$ and $S_1\backslash\{a,b,c\}$ into 4-cycles and place these 4-cycles in $P_2$. Then $(S_2,P_2,L_2)$ is a MPC of order $m+2$ (= a 4-cycle system) with leave the empty set. □

We can now combine Lemmas 6.1, 6.2, 6.3, and 6.4 along with Lemma 3.1 into the following corollary.

**Corollary 6.5.** *A MPC of odd order m can be embedded in an MPC of order* $m+2t$ *with all possible leaves for every* $t\geqslant 1$ *with the following exceptions. If* $m=3$ *and* $t=1$ *only a bowtie is possible. If* $m\equiv 5 \pmod 8$ *with leave a pair of disjoint triangles, the embedding is all* $m+2t$ *for every* $t\geqslant 2$ *(all possible leaves).*

**Proof.** The proof is immediate in all cases except for $m\equiv 5 \pmod 8$ with leave a pair of disjoint triangles. In this case we first obtain an embedding of size $m+4$ and then go up by 2s using a leave other than a pair of disjoint triangles when $m + 2t \equiv 5 \pmod 8$. □

## 7. Summary of embeddings as a partial 4-cycle system

Combining the results in Sections 2–6 gives the following theorem.

**Theorem 7.1.** *The necessary conditions to embed a MPC of order m (considered as partial 4-cycle system) in a MPC of order n are sufficient.*

Table 1

| $m \equiv i \pmod 8$ | Necessary and sufficient conditions for a MPC of order $m$ (considered as a partial 4-cycle system) to be embedded in a MPC of order $n$ |
|---|---|
| $m$ even | all even $n \geq m + 2$ |
| | all $n \equiv 1 \pmod 8 \geq m + x \equiv 1 \pmod 8$ where $x$ is the smallest possible integer such that $\binom{x}{2} \geq t$. |
| | all $n \equiv 3 \pmod 8 \geq m + x \equiv 3 \pmod 8$ where $x$ is the smallest positive integer such that $\binom{x}{2} \geq t - 1$. |
| | all $n \equiv 5 \pmod 8 \geq m + x \equiv 5 \pmod 8$ where $x$ is the smallest positive integer such that $\binom{x}{2} \geq t - 2$ (all possible leaves). |
| | all $n \equiv 7 \pmod 8 \geq m + x \equiv 7 \pmod 8$ where $x$ is the smallest positive integer such that $\binom{x}{2} \geq t - 1$. |
| 1 | all even $n \geq 2m$ |
| | all odd $n \geq m + 2$ (all possible leaves for $n \equiv 5 \pmod 8$) |
| 3 | all even $n \geq 2m - 2$ |
| | all odd $n \geq m + 2$ (all possible leaves for $n \equiv 5 \pmod 8$ except for $n = 5$, when only a bowtie is possible) |
| 5 | all even $n \geq 2m - 4$ if the leave is a bowtie or pair of disjoint triangles, and all even $n \geq 2m - 6$ if the leave is a 6-cycle. |
| | all odd $n \geq m + 2$ if the leave is a bowtie or a pair of disjoint triangles, and all odd $n \geq m + 4$ if the leave is a pair of disjoint triangles (all possible leaves for $n \equiv 5 \pmod 8$) |
| 7 | all even $n \geq 2m - 4$ |
| | all odd $m + 2$ (all possible leaves for $n \equiv 5 \pmod 8$). |

Table 1 gives a quick easy to read summary of all of the embeddings for MPCs considered as partial 4-cycle systems; i.e., the leave is not necessarily preserved.

## 8. Embeddings with the leave preserved

As mentioned in Section 2, there are two types of Doyen–Wilson theorems for MPCs: one where the MPC is considered as a partial 4-cycle system (and so we don't care if we keep the leave) and the other where the leave is preserved. The second type of embedding follows immediately from the first with just a few observations. In the case where the leave is preserved, the only possibilities are even into even and odd into odd. The following table summarizes the Doyen–Wilson Theorem for MPCs with the leave preserved.

The following lemma along with Theorem 7.1 is all that is necessary to verify the information in Table 2.

**Lemma 8.1.** *A MPC of odd order $m$ can be embedded in a MPC of order $m+8t$ with the leave preserved for every $t \geqslant 1$.*

**Proof.** In the proof of Theorem 2.1 replace $(\{\infty\} \cup X, C_1)$ with a MPC $(\{\infty\} \cup X, P_1, L_1)$. □

**Theorem 8.2.** *The information in Table 2 is correct.*

**Proof.** The necessary conditions are transparent. The case where $m$ is even follows from the remark after the proof of Lemma 3.1. The case $m \equiv 1 \pmod 8$ is identical with the case $m \equiv 1 \pmod 8$ in Theorem 7.1. If $m \equiv 3 \pmod 8$ Lemma 8.1 gives an embedding for all $n \equiv 3 \pmod 8$. If $m = 3$ only a bowtie is possible for $n = m + 2 = 5$.

Table 2

| $m \equiv i$ (mod 8) | Necessary and sufficient conditions for a MPC of order $m$ to be embedded in a MPC of order $n$ with the leave preserved. |
|---|---|
| even | all even $n \geq m+2$ |
| 1 | all odd $n \geq m+2$ (all possible leaves for $n \equiv 5 \pmod 8$) |
| 3 | all $m \equiv 3 \pmod 8$ and all $n \equiv 5 \pmod 8$ (with leaves a bowtie or 2 disjoint triangles, except when $n = 5$ where only a bowtie is possible) |
| 5 | all $n \equiv 5 \pmod 8$ |
| 7 | all $n \equiv 7 \pmod 8$ |

If $m \equiv 3 \pmod 8 \geqslant 11$, Lemma 6.2 gives an embedding of order $m+2 \equiv 5 \pmod 8$ with the leave preserved into a MPC with leave a bowtie or two disjoint triangles. Lemma 8.1 extends this to all $n \equiv 5 \pmod 8 \geqslant m+2$. The cases $m \equiv 5$ or $7 \pmod 8$ follow immediately from Lemma 8.1. □

## References


[1] D.E. Bryant, C.A. Rodger, On the Doyen–Wilson Theorem for $m$-cycle systems, J. Combin. Designs 4 (1994) 253–271.

[2] Jean Doyen, R.M. Wilson, Embeddings of Steiner triple systems, Discrete Math. 5 (1973) 229–239.

[3] H.L. Fu, C.C. Lindner, C.A. Rodger, The Doyen–Wilson Theorem for minimum coverings of $K_n$ with triples, submitted.

[4] A. Hartman, Partial triple systems and edge colourings, Discrete Math. 62 (1986) 183–196.

[5] T.P. Kirkman, On a problem in combinations, Cambridge and Dublin Math. J. 2 (1847) 191–204.

[6] A. Kotzig, On the decomposition of complete graphs into $4k$-gons, Mat.-Fyz. Časopis Sloven. Akad. Vied. 15 (1965) 229–233.

[7] E. Mendelsohn, A. Rosa, Embedding maximal packings of triples, Congr. Numer. 40 (1983) 235–247.

[8] J. Schönheim, A. Bialostocki, Packing and covering the complete graph with 4-cycles, Can. Math. Bull. 18 (1975) 703–708.

[9] D. Sotteau, Decomposition of $K_{m,n}(K^*_{m,n})$ into cycles (circuits) of length $2k$, J. Combin. Theory (B) 30 (1981) 75–81.