

Optimal 1-hamiltonian graphs¹

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Abstract

In this paper, we present a family of 3-regular, planar, and hamiltonian graphs. Any graph in this family remains hamiltonian if any node or any edge is deleted. Moreover, the diameter of any graph in this family is $O(\sqrt{p})$ where p is the number of nodes. © 1998 Elsevier Science B.V.

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1. Introduction

An interconnection network connects the processors of the parallel computer. Its architecture can be represented as a graph in which the nodes correspond to the processors and the edges to the communication links. Hence, we use graph and network interchangeably. There are lot of mutually conflicting requirements in designing the topology of computer networks. It is almost impossible to design a network which is optimum from all aspects. One has to design a suitable network depending on the requirements and their properties. The hamiltonian properties is one of the major requirements in designing the topology of network. For example, “Token Passing” approach is used in some distributed operation systems. Interconnection network requires the presence of hamiltonian cycles in the structure to meet this approach. Fault tolerant is also desirable in massive parallel systems that have a relatively high probability of failure. A number of fault tolerant designs for specific multiprocessor architectures have been proposed based on graph theoretic models in which the processor-to-processor interconnection structure is represented by a graph.

In this paper, a network is represented as an undirected graph. For the graph theoretical definition and notation we follow [1]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(a, b) \mid (a, b) \text{ is an unordered pair of } V\}$. We say that V is the node set and E is the edge set of G . Let $p = |V|$ and $q = |E|$. Two nodes a and b are adjacent if $(a, b) \in E$. A path is a sequence of nodes such that two consecutive nodes are adjacent. A path is delimited by $\langle x_0, x_1, x_2, \dots, x_{n-1} \rangle$. We use P^{-1} to denote the path $\langle x_{n-1}, \dots, x_2, x_1, x_0 \rangle$ if P is the path $\langle x_0, x_1, x_2, \dots, x_{n-1} \rangle$. A path is called a hamiltonian path if its nodes are distinct and they span V . A

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cycle is a path of at least three nodes such that the first node is the same as the last node. A cycle is called a *hamiltonian cycle* if its nodes are distinct except the first node and the last node and they span V . Let k be a positive integer. A graph G is *k-hamiltonian* if $G - V' - E'$ is hamiltonian for any set $V' \subset V$ and $E' \subset E$ with $|V'| + |E'| \leq k$. Obviously, every *k-hamiltonian* graph has at least $k + 3$ nodes. Moreover, the degree of every node in a *k-hamiltonian* graph is at least $k + 2$. A p -node *k-hamiltonian* graph is *optimal* if it contains the least number of edges among all p -node *k-hamiltonian* graphs. The design of *k-hamiltonian* graph is equivalent to *k-fault-tolerant* designs for token ring. Mukhopadhyaya and Sinha [5] worked on *1-hamiltonian graphs*. Sung et al. [7] studied optimal *k-hamiltonian graphs* with $k = 2$ and $k = 3$.

Mukhopadhyaya and Sinha [5] proposed a family of optimal *1-hamiltonian* planar graphs. The diameter of the graph in the family is $\lfloor p/6 \rfloor + 2$ if p is even and $\lfloor p/8 \rfloor + 3$ if p is odd. Similar problem has been discussed by Harary and Hayes [2,3]. A graph G is *k-node hamiltonian* if $G - V'$ is hamiltonian for any $V' \subset V$ with $|V'| = k$, and a graph G is *k-edge hamiltonian* if $G - E'$ is hamiltonian for any $E' \subset E$ with $|E'| = k$. A p -node *k-node hamiltonian* graph is *optimal* if its number of edges is the smallest, and a p -node *k-edge hamiltonian* graph is *optimal* if its number of edges is the smallest. In [3], Harary and Hayes presented a family of optimal p -node *k-node hamiltonian* graphs for every positive integer k , and in whereas [2], Harary and Hayes presented a family of optimal p node *k-edge hamiltonian* graphs for every positive integer k . For the special case $k = 1$, the family of optimal *1-node hamiltonian* graphs proposed in [3] is actually the family of optimal *1-edge hamiltonian* graphs proposed in [2]. Hence this family of graphs is *1-hamiltonian*. Any single graph in this family is planar and of diameter $\lfloor (p + 1)/3 \rfloor$.

In [3], Harary and Hayes were not sure that their proposed optimal *1-node hamiltonian* graphs are of only such optimal graphs and asked for the determination of all such graphs. With the family of graphs proposed by Mukhopadhyaya and Sinha mentioned above, we can easily solve Harary and Hayes' problem with a counterexample. Note that a *1-node hamiltonian* graph may not be necessarily hamiltonian. A graph is *hypohamiltonian* if G is not hamiltonian but $G - v$ is hamiltonian for every $v \in V$. There are a lot of studies on *hypohamiltonian* graphs [4,6,8]. It seems that the problem of determination of all optimal *1-node hamiltonian* graphs is very difficult.

In this paper, we propose a family of optimal *1-hamiltonian* graphs. Each graph in this family is planar and of diameter $O(\sqrt{p})$. The degree of each node in any graph of this family is 3.

2. Definitions

In this section, we are going to present a family of graphs $G(k)$ for every positive integer k . There are $2k + 1$ disjoint isomorphic induced subgraphs, $G(k, i)$ with $0 \leq i \leq 2k$, in $G(k)$. For $0 \leq i \leq 2k$, $G(k, i)$ is the graph $(V(k, i), E(k, i))$ where

$$\begin{aligned} V(k, i) &= \{x_{i,j}^r \mid 1 \leq j \leq k\} \cup \{x_{i,j}^l \mid 1 \leq i \leq k\} \cup \{y_i, z_i\}, \quad \text{and} \\ E(k, i) &= \{(x_{i,j}^l, x_{i,j-1}^l) \mid 1 < j \leq k\} \cup \{(x_{i,j}^r, x_{i,j+1}^r) \mid 1 \leq j < k\} \\ &\quad \cup \{(x_{i,j}^l, x_{i,j}^r) \mid 1 \leq j \leq k\} \cup \{(x_{i,1}^l, y_i), (y_i, z_i), (y_i, x_{i,1}^r)\}. \end{aligned}$$

In Fig. 1, we illustrate the graph $G(k, i)$.

Then, we define the graph $G(k) = (V(k), E(k))$ with

$$\begin{aligned} V(k) &= \bigcup_{0 \leq i \leq 2k} V(k, i), \quad \text{and} \\ E(k) &= \bigcup_{0 \leq i \leq 2k} E(k, i) \cup \{(z_i, z_{i+1 \bmod (2k+1)}) \mid 0 \leq i \leq 2k\} \cup \{(x_{i,k}^l, x_{i+1 \bmod (2k+1), k}^r) \mid 0 \leq i \leq 2k\}. \end{aligned}$$

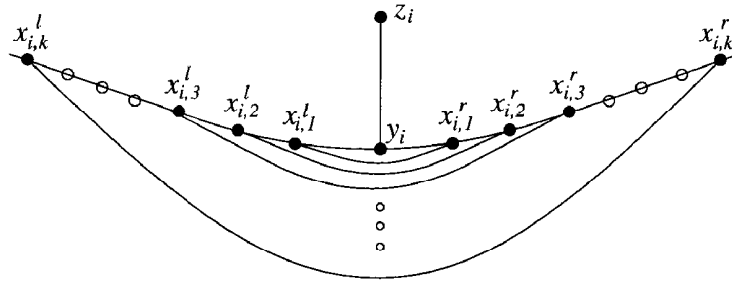


Fig. 1.

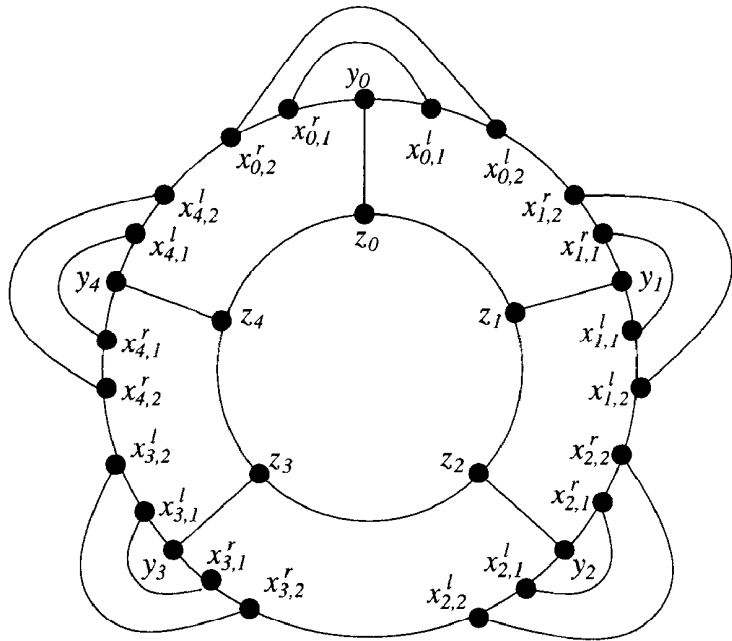


Fig. 2.

In Fig. 2, we illustrate the graph $G(2)$.

Obviously, the graph $G(k)$ is planar and with $p = 4k^2 + 6k + 2$ nodes. Each vertex of $G(k)$ is of degree 3. Assume that x is in $V(k, i)$ and y is in $V(k, j)$. It is easy to see that the distance between x and z_i is at most $k + 1$, the distance between z_i and z_j is at most k and the distance between z_j and y is at most $k + 1$. Hence the diameter of $G(k)$ is $O(\sqrt{p})$.

3. Hamiltonian properties

Lemma 1. $G(k, i)$ has a hamiltonian path joining z_i with $x_{i,k}^r$ for every positive integer k, i with $0 \leq i \leq 2k$.

Proof. Suppose that k is an odd integer.

$$\langle z_i, y_i, x_{i,1}^l, x_{i,1}^r, x_{i,2}^r, x_{i,2}^l, \dots, x_{i,k-1}^r, x_{i,k}^l, x_{i,k}^r \rangle$$

is a hamiltonian path joining z_i with $x_{i,k}^r$. Suppose that k is an even integer.

$$\langle z_i, y_i, x_{i,1}^r, x_{i,1}^l, x_{i,2}^l, x_{i,2}^r, \dots, x_{i,k-1}^l, x_{i,k}^r, x_{i,k}^l \rangle$$

is a hamiltonian path joining z_i with $x_{i,k}^l$. \square

We use $P(i, z, r)$ to denote the hamiltonian path joining z_i with $x_{i,k}^r$. By the symmetric property of $G(k, i)$, there is also a hamiltonian path joining z_i with $x_{i,k}^l$. We use $P(i, z, l)$ to denote the hamiltonian path joining z_i with $x_{i,k}^l$. We also use $Q(i, r, l)$ to denote the path $\langle x_{i,k}^r, x_{i,k-1}^r, \dots, x_{i,1}^r, y_i, x_{i,1}^l, x_{i,2}^l, \dots, x_{i,k}^l \rangle$. With these notations, it is easy to see that

$$\begin{aligned} \langle z_0 \xrightarrow{P(0,z,l)} x_{0,k}^l, x_{1,k}^r \xrightarrow{Q(1,r,l)} x_{1,k}^l, x_{2,k}^r \xrightarrow{Q(2,r,l)} x_{2,k}^l, \dots, \\ x_{2k-1,k}^r \xrightarrow{Q(2k-1,r,l)} x_{2k-1,k}^l, x_{2k,k}^r \xrightarrow{P^{-1}(2k,z,r)} z_{2k}, z_{2k-1}, \dots, z_0 \rangle \end{aligned}$$

form a hamiltonian cycle of $G(k)$. By the symmetric property of $G(k)$, it is easy to see that $G(k)$ is 1-edge fault tolerant.

To discuss the 1-node fault tolerant property of $G(k)$, we need the following lemma.

Lemma 2. *For any positive integer j with $1 \leq j \leq k$, $G(k, i) - \{x_{i,j}^r\}$ has a hamiltonian path joining z_i with $x_{i,k}^r$ or $x_{i,k}^l$.*

Proof. We first assume that j is an odd integer and k is an even integer. It is easy to see that

$$\langle z_i, y_i, x_{i,1}^l, x_{i,1}^r, x_{i,2}^l, x_{i,2}^r, x_{i,3}^l, \dots, x_{i,j-1}^l, x_{i,j}^l, x_{i,j+1}^r, x_{i,j+2}^l, \dots, x_{i,k}^l, x_{i,k}^r \rangle$$

is a hamiltonian path joining z_i with $x_{i,k}^r$. Other cases can be similarly discussed. Hence the lemma is proved. \square

We may assume the faulty node of $G(k)$ is y_0, z_0 , or $x_{0,j}^r$ for $1 \leq j \leq k$.

Assume the fault node is z_0 . It is easy to see that

$$\begin{aligned} \langle x_{0,k}^r \xrightarrow{Q(0,r,l)} x_{0,k}^l, x_{1,k}^r \xrightarrow{P^{-1}(1,z,r)} z_1, z_2 \xrightarrow{P(2,z,l)} x_{2,k}^l, x_{3,k}^r \xrightarrow{P^{-1}(3,z,r)} z_3, \dots, \\ x_{2i-1,k}^r \xrightarrow{P^{-1}(2i-1,z,r)} z_{2i-1}, z_{2i} \xrightarrow{P(2i,z,l)} x_{2i,k}^l, \dots, z_{2k} \xrightarrow{Q(2k,r,l)} x_{2k,k}^l, x_{0,k}^r \rangle \end{aligned}$$

form a hamiltonian cycle in $G(k) - \{z_0\}$.

Assume the faulty node is y_0 . It can be checked that

$$\begin{aligned} \langle x_{0,k}^r, x_{0,k-1}^r, \dots, x_{0,1}^r, x_{0,1}^l, x_{0,2}^l, \dots, x_{0,k}^l, x_{1,k}^r \xrightarrow{Q(1,r,i)} x_{1,k}^l, x_{2,k}^r \xrightarrow{Q(2,r,i)} x_{2,k}^l, \dots, \\ x_{2k-2,k}^r \xrightarrow{Q(2k-2,r,i)} x_{2k-2,k}^l, x_{2k-1,k}^r \xrightarrow{P^{-1}(2k-1,z,r)} z_{2k-1}, z_{2k-2}, \dots, z_0, z_{2k} \xrightarrow{Q(2k,r,l)} x_{2k,k}^l, x_{0,k}^r \rangle \end{aligned}$$

form a hamiltonian cycle in $G(k) - \{y_0\}$

Finally, we assume that the faulty node is $x_{0,j}^r$ for $1 \leq j \leq k$. It follows from Lemma 2 and the symmetric property of $G(k)$, we may assume that there exists a hamiltonian path R joining z_0 with $x_{0,k}^l$ in $G(k, 0) - \{x_{0,j}^r\}$. Suppose that k is even. Then

$$\langle z_0 \xrightarrow{R} x_{0,k}^l, x_{1,k}^r \xrightarrow{Q(1,r,l)} x_{1,k}^l, x_{2,k}^r \xrightarrow{Q(2,r,l)} x_{2,k}^l, \dots, \\ x_{2k-1,k}^r \xrightarrow{Q(2k-1,r,l)} x_{2k-1,k}^l, x_{2k,k}^r \xrightarrow{P^{-1}(2k,z,r)} z_{2k}, z_{2k-1}, z_{2k-1}, \dots, z_1, z_0 \rangle$$

is a hamiltonian cycle.

Similarly, we can discuss the case that k is odd. Thus $G(k)$ is 1-node fault tolerant. Since the degree of any node in $G(k)$ is 3, $G(k)$ is an optimal 1-hamiltonian graph.

Theorem 3. $G(k)$ is an optimal 1-hamiltonian graph with diameter $O(\sqrt{p})$ where p is the number of nodes in $G(k)$.

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