二維度動態最佳化模型之特徵

Characterizations of a Two-Dimensional Dynamic Optimization Model

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碩 士 論 文

A Thesis

Submitted to Department of Applied Mathematics

National Chiao Tung University

in partial Fulfillment of the Requirements

for the Degree of

Master

in

Applied Mathematics

July 2008

Hsinchu, Taiwan, Republic of China

中華民國九十七年七月

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摘　　要

由於簡化動態最佳化模型的結構簡單，所以被廣泛地應用在許多領域，如：經濟學。Mitra [1]曾經討論過狀態空間為一維度的動態模型，本文將討論狀態空間為二維度的模型，並且用兩種方法得出兩個最佳解的特徵。

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ABSTRACT

Dynamic optimization models are currently in use in a number of different areas in economics. The reduced-form model is widely used because of its simple mathematical structure; one keeps track of the transition over time of only the state variable, from one state to another. Mitra [1] discussed the reduced-form model with one-dimensional state space. We discuss the reduced-form model with two-dimensional state space, and obtain two characterizations of its optimal programs by using primal and dual approaches.

誌　　謝

本篇論文的完成，首先要感謝我的指導老師——李明佳教授。老師在這兩年來提供給我許多寶貴的意見，也耐心地和我一起討論。除了論文上的協助，我的看書效率也因為老師的指導與要求提升了許多；我明白了，原來書不一定得從第一頁開始翻起。

在新竹兩年來，承蒙許多師長、同學的照顧和幫忙。感謝呂明杰學長、其研究室和其附近研究室裡的學長姐們陪我做口試練習。感謝研究室 126 裡的夥伴們：陳子鴻、龔柏任、吳偉帆和我的同門兄弟——曾世忠，讓我擁有這充實的兩年。由於要感謝的人太多了，所以也感謝老天在我身邊安排這麼多朋友。

最後，謹以感恩之心，感激之情將此論文獻給我生命中的第一個教育團隊——爸爸、媽媽、阿姨和哥哥。沒有他們的鼓勵、付出與犧牲，本人必定無法完成今天的學業。

目　　　録

1 Introduction

The reduced-form model is the standard form used to describe dynamic optimization problems arising in economics. And a wide variety of dynamic optimization problems in economics can be reduced to this form. Mitra [1] discussed the reduced-form model with one-dimensional state space. We first introduce it below:

A *state space* $Y = [0, b]$ is given, where $b > 0$. Time is measured in discrete periods $t \in \{0, 1, 2, ...\} \equiv \mathbf{N}$. At each time t the state of the economic system is described by a point $y_t \in Y$.

The *reduced form model with one-dimensional state space* Y is described as a triplet (Ω, u, δ) satisfying the following conditions:

(i) $\Omega \subseteq Y \times Y$ is a closed and convex set containing $(0, 0)$, and $\forall y \in Y\ \exists z \in Y$ such that $(y, z) \in \Omega$.

(ii) $u : \Omega \to \mathbb{R}$ is a bounded, concave and upper semicontinuous function.

(iii) $\delta \in (0, 1)$.

Given (Ω, u, δ) and $y \in Y$, we are concerned with the optimization problem:

$$\max_{(y_t)_0^\infty \in A_y} F_y((y_t)_0^\infty)$$

where $A_y = \{(y_t)_0^\infty\ ;\ \forall t \in \mathbf{N}\ (y_t, y_{t+1}) \in \Omega \text{ and } y_0 = y\}$ and $F_y : A_y \to \mathbb{R}$ is defined by

$$F_y((y_t)_0^\infty) = \sum_{t=0}^{\infty} \delta^t u(y_t, y_{t+1})$$

In the description of the above problem, δ is to be interpreted as a *discounted factor*, u as a *utility function*, Ω as the *transition possibility set*, and $y \in Y$ as the *initial state*. $(y_t)_0^\infty \in A_y$ is called a *program* from y. A solution, $(y_t^*)_0^\infty$, to the optimization problem is referred to as an *optimal program* from y.

We now give an example and show how it can be converted to its reduced-form.

An agent has a plot of land (normalized to unity), which is good for growing a certain type of tree. The growth process for the tree is as follows:

After saplings are planted, the tree grows in timber content for two years,

- After one year the timber content is $a \in (0, 1)$.
- After two years the timber content is 1.

Denote y_t to be the amount of the land with two-year old trees at the end of period t, and $1 - y_t$ to be the amount of the land with one-year old trees at the end of period t.

The one-year old trees which are not cut down at the end of periods t become the two-year old trees at the end of period $t+1$.

Hence, the total timber content of all trees cut down at the end of period t is, given by

$$c_t = y_t + a[(1 - y_t) - y_{t+1}].$$

The agent's optimization problem can be written as:

$$\begin{array}{rcl} & & \max_{(c_t)_0^\infty} \sum_{t=0}^{\infty} \delta^t R(c_t) \\ \text{Subject to} \quad c_t & = & y_t + a[(1 - y_t) - y_{t+1}] \text{ for } t \in \mathbf{N}, \\ y_{t+1} & \leq & 1 - y_t \text{ for } t \in \mathbf{N}, \\ 0 & \leq & y_t \leq 1 \text{ for } t \in \mathbf{N}, \\ y_0 & = & y \geq 0, \end{array}$$

where $R(\cdot)$ is the agent's return function.

To convert this problem to its reduced-form, we define

- the state space to be $Y = [0, 1]$,
- the transition possibility set, Ω, to be:

$$\Omega = \{(y, z) \in Y \times Y;\ z \leq 1 - y\},$$

- the utility function, $u : \Omega \to \mathbb{R}$, to be:

$$u(y, z) = R(a + (1 - a)y - az).$$

Thus, the optimization problem becomes:

$$\max_{(y_t)_0^\infty \in A_y} \sum_{t=0}^{\infty} \delta^t u(y_t, y_{t+1}) \text{ (or } \max_{(y_t)_0^\infty \in A_y} F_y((y_t)_0^\infty))$$

where $A_y = \{(y_t)_0^\infty\ ;\ \forall t \in \mathbf{N}\ (y_t, y_{t+1}) \in \Omega \text{ and } y_0 = y\}$.

2 The Reduced-Form Model with Two-Dimensional State Space

Now, we turn to the reduced-form model with two-dimensional state space and characterize its optimal program.

Given the *state space,* $X = [0, b] \times [0, b] \subset \mathbb{R}^2$, where $b > 0$. Time is measured in discrete periods $t \in \{0, 1, 2, ...\} \equiv \mathbf{N}$. At each time t the state of the economic system is described by a point $x_t \in X$.

Definition 1 *The* reduced form model *is described as a triplet* (Ω, u, δ) *satisfying the following conditions:*

(i) $\Omega \subseteq X \times X$ *is a closed and convex set containing* $(0, 0)$*, where* $0 = \binom{0}{0} \in X$*, and* $\forall x \in X$ $\exists y \in X$ *such that* $(x, y) \in \Omega$.

(ii) $u : \Omega \to \mathbb{R}$ *is a bounded, concave and upper semicontinuous function.*

(iii) $\delta \in (0, 1)$.

Given (Ω, u, δ) and $x \in X$, we will be concerned with the optimization problem

$$\max_{(x_t)_0^\infty \in A_x} F_x((x_t)_0^\infty) \tag{1}$$

where $A_x = \{(x_t)_0^\infty\ ;\ \forall t \in \mathbf{N}\ (x_t, x_{t+1}) \in \Omega \text{ and } x_0 = x\}$ and $F_x : A_x \to \mathbb{R}$ is defined by

$$F_x((x_t)_0^\infty) = \sum_{t=0}^{\infty} \delta^t u(x_t, x_{t+1})$$

As described above, δ is to be interpreted as a *discounted factor*, u as a *utility function*, Ω as the *transition possibility set*, and $x \in X$ as the *initial state*. $(x_t)_0^\infty \in A_x$ is called a *program* from x. A solution to (1) is referred to as an *optimal program* from x.

Theorem 2 *For every initial state* $x \in X$*, there exists an optimal program from* x*.*

Proof. Let $x \in X$. Since u is bounded, we can find $B > 0$ such that $|u(x, y)| \leq B$ for all $(x, y) \in \Omega$. Thus, for every $(x_t)_0^\infty \in A_x$,

$$\sum_{t=0}^{\infty} \delta^t u(x_t, x_{t+1})$$

is an absolutely convergent series, and hence a convergent series, with

$$\sum_{t=0}^{\infty} \delta^t u(x_t, x_{t+1}) \leq \frac{B}{1-\delta}$$

Let $S_x = \sup\{\sum_{t=0}^{\infty} \delta^t u(x_t, x_{t+1})\ ;\ (x_t)_0^\infty \in A_x\}$. We will now show that this supremum is actually attained by some $(x_t)_0^\infty \in A_x$. By definition of S_x, there is a sequence of programs $(x_t^n)_{t=0}^\infty$, $n = 1, 2, 3, ...$, starting from the initial state x, such that

$$\sum_{t=0}^{\infty} \delta^t u(x_t^n, x_{t+1}^n) \geq S_x - \frac{1}{n}$$

By the Cantor diagonal process, we can find a subsequence n' (of n) such that for each $t \in \mathbf{N}$, we have a number x_t^0, such that

$$x_t^{n'} \to x_t^0 \text{ as } n' \to \infty \tag{2}$$

Since Ω is closed, $(x_t^0, x_{t+1}^0) \in \Omega$ for $t \in \mathbf{N}$, and $x_0^0 = x_0^{n'} = x$. Thus $(x_t^0)_{t=0}^\infty$ is also a program starting from x. We claim that

$$\sum_{t=0}^{\infty} \delta^t u(x_t^0, x_{t+1}^0) = S_x \tag{3}$$

If the claim were not true, then we could find $\epsilon > 0$, such that

$$\sum_{t=0}^{\infty} \delta^t u(x_t^0, x_{t+1}^0) \leq S_x - \epsilon \tag{4}$$

Pick T large enough so that $B\delta^{T+1}/(1-\delta) \leq \epsilon/4$. For $t \in \{0, 1, ..., T\}$, we can use (2) and the upper semicontinuity of u to get

$$\limsup_{n' \to \infty} u(x_t^{n'}, x_{t+1}^{n'}) \leq u(x_t^0, x_{t+1}^0)$$

We can now pick an integer $N > 4/\epsilon$, such that for $t \in \{0, 1, ..., T\}$, we have

$$u(x_t^{n'}, x_{t+1}^{n'}) \leq u(x_t^0, x_{t+1}^0) + \frac{\epsilon(1-\delta)}{4}$$

whenever $n' \geq N$. Then, for $n' \geq N$, we can obtain the following inequalities:

$$\begin{aligned}
\sum_{t=0}^{\infty} \delta^t u(x_t^0, x_{t+1}^0) &\geq \sum_{t=0}^{T} \delta^t u(x_t^0, x_{t+1}^0) - \frac{\epsilon}{4} \\
&\geq \sum_{t=0}^{T} \delta^t u(x_t^{n'}, x_{t+1}^{n'}) - \frac{\epsilon}{2} \\
&\geq \sum_{t=0}^{\infty} \delta^t u(x_t^{n'}, x_{t+1}^{n'}) - \frac{3\epsilon}{4} \\
&\geq S_x - \frac{1}{n'} - \frac{3\epsilon}{4} \\
&> S_x - \epsilon
\end{aligned}$$

which contradicts (4) and establishes our claim (3). This means that $(x_t^0)_0^\infty$ is an optimal program from x. ■

3 The Value Function

Since Theorem 2 gives the existence of an optimal program from x, for every $x \in X$, we can define the *value function* of (Ω, u, δ):

Definition 3 *The* value function, $V : X \to \mathbb{R}$, *of* (Ω, u, δ) *is defined as follows:*

$$V(x) = F_x((x_t^*)_0^\infty) \ \left(or \ = \sum_{t=0}^{\infty} \delta^t u(x_t^*, x_{t+1}^*) \ \right)$$

where $(x_t^*)_0^\infty$ *is an optimal program from* x *corresponding to the initial state* $x \in X$.

Lemma 4 *If* $(x_t^*)_0^\infty$ *is an optimal program from* $x = x_0^*$*, then* $(x_t^*)_T^\infty$ *is an optimal program from* $x = x_T^*$.

Proof. Suppose not, let $(y_t)_0^\infty$ be a program from $y_0 = x_T^*$ such that

$$\sum_{t=0}^{\infty} \delta^t u(y_t, y_{t+1}) > \sum_{t=T}^{\infty} \delta^{t-T} u(x_t^*, x_{t+1}^*)$$

Then the sequence $(x_0^*, ..., x_T^*, y_1, y_2, ...)$ is also a program from x_0^*, and

$$\sum_{t=0}^{T-1} \delta^t u(x_t^*, x_{t+1}^*) + \sum_{t=T}^{\infty} \delta^t u(y_t, y_{t+1}) > \sum_{t=0}^{\infty} \delta^t u(x_t^*, x_{t+1}^*)$$

which contradicts the optimality of $(x_t^*)_0^\infty$ from $x = x_0^*$. so, $(x_t^*)_T^\infty$ is an optimal program from $x = x_T^*$. ■

The following result summarizes the basic properties of the value function and gives the primal approach.

Theorem 5 *The value function,* V*, of* (Ω, u, δ) *has the following properties:*

(i) V *is a concave and continuous function on* X.

(ii) V *satisfies the following functional equation*

$$V(x) = \max_{y \in \Omega_x} \{u(x, y) + \delta V(y)\} \tag{5}$$

for all $x \in X$ *where* $\Omega_x = \{y \in X \ ;(x, y) \in \Omega\}$.

(iii) $(x_t)_0^\infty$ *is an optimal program from* x_0 *if and only if*

$$V(x_t) = u(x_t, x_{t+1}) + \delta V(x_{t+1}) \ for \ t \in \mathbf{N} \tag{6}$$

Proof. (i) Let x, $x' \in X$ and let $0 < \lambda < 1$. Let $(x_t)_0^\infty$, $(x'_t)_0^\infty$ be optimal programs from x and x' respectively. Define $x" = \lambda x + (1-\lambda)x'$, and let $(x"_t)_0^\infty$ be an optimal program from $x"$. By the convexity of Ω , the sequence $(\lambda x_t + (1-\lambda)x'_t)_0^\infty$ is a program from $x"$. Thus, we have

$$\begin{aligned} V(x") &= \sum_{t=0}^{\infty} \delta^t u(x"_t, x"_{t+1}) \\ &\geq \sum_{t=0}^{\infty} \delta^t u(\lambda x_t + (1-\lambda)x'_t, \lambda x_{t+1} + (1-\lambda)x'_{t+1}) \\ &\geq \sum_{t=0}^{\infty} \delta^t [\lambda u(x_t, x_{t+1}) + (1-\lambda)u(x'_t, x'_{t+1})] \\ &= \lambda V(x) + (1-\lambda)V(x') \end{aligned}$$

the second inequality following from the concavity of u on Ω. Thus, V is concave on X.

In order to establish the continuity of V on X, we first establish its upper semicontinuity:

If V were not upper semicontinuous on X, we can find $x^n \in X$ for $n = 1, 2, 3, ...,$ with

$$\lim_n x^n = x^0 \text{ and } \lim_n V(x^n) = V > V(x^0)$$

Denote by (x_t^n) an optimal program from x^n for $n = 0, 1, 2, ...$ and ϵ by $V - V(x^0)$. We can find T large enough so that $B\delta^{T+1}/(1-\delta) \leq (\epsilon/4)$, where $B > 0$ such that $|u(x, y)| \leq B$ for all $(x, y) \in \Omega$.

Clearly, we can find a subsequence n' (of n) such that for $t = 0, 1, 2, ..., T$

$$x_t^{n'} \to x_t^0 \text{ as } n' \to \infty$$

Then, by the upper semicontinuity of u, we can find N such that

$$u(x_t^{n'}, x_{t+1}^{n'}) \leq u(x_t^0, x_{t+1}^0) + \frac{\epsilon(1-\delta)}{4} \text{ for } t = 0, 1, 2, ..., T$$

whenever $n' \geq N$. Thus, for $n' \geq N$, we obtain

$$\begin{aligned} V(x^0) &\geq \sum_{t=0}^{\infty} \delta^t u(x_t^0, x_{t+1}^0) \\ &\geq \sum_{t=0}^{T} \delta^t u(x_t^0, x_{t+1}^0) - \frac{\epsilon}{4} \\ &\geq \sum_{t=0}^{T} \delta^t u(x_t^{n'}, x_{t+1}^{n'}) - \frac{\epsilon}{2} \\ &\geq \sum_{t=0}^{\infty} \delta^t u(x_t^{n'}, x_{t+1}^{n'}) - \frac{3\epsilon}{4} \\ &= V(x^{n'}) - \frac{3\epsilon}{4} \end{aligned}$$

Since $V(x^{n'}) \to V$, we have $V(x^0) \geq V - 3\epsilon/4 > V - \epsilon = V(x^0)$, a contradiction. Thus, V is upper semicontinuous on X.

V is concave on X, and hence continuous on $int(X)$ (the interior of X).

Now, we consider V on ∂X (the boundary of X).

Since X is boundedly polyhedral and V is boundedly concave on the interior of X. By Gale [2], V can be extended in a unique way to a continuous concave function on X.

Thus, let $V^* : X \to \mathbb{R}$ be the extension of $V : X \to \mathbb{R}$, then for each $x \in int(X)$, $V(x) = V^*(x)$.

Let $y = \binom{0}{c}$, $z = \binom{b}{c} \in \partial X$, $0 < c < b$, $x^n = \binom{d^n}{c} \in int(X)$, and $x^n \to y$.Then

$$
\begin{aligned}
V(x^n) &= V((1 - \frac{d^n}{b})y + \frac{d^n}{b}z) \\
&\geq (1 - \frac{d^n}{b})V(y) + \frac{d^n}{b}V(z)
\end{aligned}
$$

Hence,

$$
\liminf_n V(x^n) \geq V(y)
$$

On the other hand, since V is upper semicontinuous on X, $\lim_n \sup V(x^n) \leq V(y)$. So, $\lim_n V(x^n) = V(y)$.

Since $V(x^n) = V^*(x^n)$,

$$
\lim_n V(x^n) = V(y) = \lim_n V^*(x^n) = V^*(y)
$$

Similarly, we can show that $V = V^*$ on $\{0, b\} \times (0, b) \cup (0, b) \times \{0, b\}$. Since V^* is continuous on $\{0, b\} \times (0, b) \cup (0, b) \times \{0, b\}$, so does V.

Now, let $y^n = \binom{w^n}{b}$, $0 < w^n < b$, and $y^n \to \binom{0}{b}$. Then we can use the same argument to show that V is continuous at $\binom{0}{b}$.

Similarly, we can show that V is continuous at $\binom{b}{b}$, $\binom{0}{0}$, and $\binom{b}{0}$. So, V is continuous on ∂X. Hence, V is continuous on X.

(ii) Let $y \in \Omega_x$, and let $(y_t)_0^\infty$ be an optimal program from y. Then $(x, y_0, y_1, ...)$ is a program from x, and hence, by the definition of V,

$$
\begin{aligned}
V(x) &\geq u(x, y_0) + \sum_{t=1}^{\infty} \delta^t u(y_{t-1}, y_t) \\
&= u(x, y_0) + \delta \sum_{t=1}^{\infty} \delta^{t-1} u(y_{t-1}, y_t) \\
&= u(x, y_0) + \delta \sum_{t=0}^{\infty} \delta^t u(y_t, y_{t+1}) \\
&= u(x, y_0) + \delta V(y)
\end{aligned}
$$

So, we have established that

$$
V(x) \geq u(x, y) + \delta V(y) \text{ for all } y \in \Omega_x \tag{7}
$$

Next, let $(x_t)_0^\infty$ be an optimal program from x,and note that

$$\begin{aligned} V(x) &= u(x_0,x_1)+\delta\sum_{t=1}^{\infty}\delta^{t-1}u(x_t,x_{t+1}) \\ &= u(x_0,x_1)+\delta\sum_{t=0}^{\infty}\delta^{t}u(x_{t+1},x_{t+2}) \\ &\leq u(x_0,x_1)+\delta V(x_1) \end{aligned}$$

Using (7), choose $y=x_1$, we have

$$V(x)=u(x_0,x_1)+\delta V(x_1) \tag{8}$$

Now, (7) and (8) establish (5).

(iii) If $(x_t)_0^\infty$ satisfies (6), then we get for any $T\geq 1$,

$$V(x)=\sum_{t=0}^{T}\delta^t u(x_t,x_{t+1})+\delta^{T+1}V(x_{T+1})$$

Since $|V(x)|\leq B/(1-\delta)$ for all $x\in X$ and $\delta^{T+1}\to 0$ as $T\to\infty$, we have

$$V(x)=\sum_{t=0}^{\infty}\delta^t u(x_t,x_{t+1})$$

Then, by the definition of V, $(x_t)_0^\infty$ is an optimal program from x.

Conversely, let $(x_t^*)_0^\infty$ be an optimal program from $x=x_0^*$. Then, by Lemma 4, $(x_T^*,x_{T+1}^*,...)$ is also an optimal program from $x=x_T^*$. Now, using the result (8) in (ii) above, we have $V(x_t^*)=u(x_t^*,x_{t+1}^*)+\delta V(x_{t+1}^*)$ for $t\in\mathbf{N}$. ■

4 Monotonicity of the Value Function

We now consider the state space X with an order relation and require the reduced-form model (Ω, u, δ) satisfying an additional condition. And we would obtain the monotonicity of the value function.

Definition 6 *Let* $x = \binom{x^1}{x^2}, y = \binom{y^1}{y^2} \in X$*, we define*

(i) $x \geq y$ *by* $x^1 \geq y^1$ *and* $x^2 \geq y^2$

(ii) $x > y$ *by* $x \geq y$ *and* $x \neq y$*.*

Condition 7 *If* $(x, z) \in \Omega$ *and* $x' \in X$*,* $x' \geq x$*,* $z \geq z' \geq 0$*, then* $(x', z') \in \Omega$ *and* $u(x, z) \leq u(x', z')$*.*

Theorem 8 (Monotonicity of the Value Function) *Suppose that* (Ω, u, δ) *satisfies Condition 7. If* $x, y \in X$ *and* $x \geq y$*, then* $V(x) \geq V(y)$*.*

Proof. Let $x, y \in X, x \geq y$ and $(y_t)_0^\infty$ be an optimal program from y . Choose $(x_t)_0^\infty$ with $x_0 = x$ and $x_t = y_t$ for $t \geq 1$.

By Condition 7 $(x_0, x_1) \in \Omega$ and $u(y_0, y_1) \leq u(x_0, x_1)$. Thus $(x_t)_0^\infty$ is a program from x and

$$V(y) = \sum_{t=0}^{\infty} \delta^t u(y_t, y_{t+1}) \leq \sum_{t=0}^{\infty} \delta^t u(x_t, x_{t+1}) \leq V(x)$$

■

5 Dual Approach Characterization

We now develop the dual approach. This approach is developed by mathematical induction. Lemma 9 and Lemma 11 establish the base case and the inductive step of the mathematical induction respectively. Theorem 12 is the conclusion of the induction, which states the necessary conditions of an optimal program from the initial state in the interior of the state space. And Theorem 13 can be considered as the converse statement of Theorem 12.

Lemma 9 *Suppose that* (Ω, u, δ) *satisfies Condition 7. If* $x \in int(X)$*, then there exists* $p \in \mathbb{R}^2$ *such that* $\langle p, x\rangle \geq 0$ *and* $V(x) - \langle p, x\rangle \geq V(y) - \langle p, y\rangle$ *for all* $y \in X$*.*

Proof. Since x is an interior point of X and V is concave, V has a support at x. That is, there exists $p \in \mathbb{R}^2$ such that

$$V(x) + \langle p, y - x\rangle \geq V(y) \text{ for all } y \in int(X)$$

or

$$\langle p, y - x\rangle \geq V(y) - V(x) \text{ for all } y \in int(X)$$

Now, consider $z \in \partial X$, we could choose a sequence (y^n) in $int(X)$ such that $y^n \to z$. Then

$$\langle p, y^n - x\rangle \geq V(y^n) - V(x)$$

and

$$\langle p, z - x\rangle \geq V(z) - V(x)$$

since V is continuous on X. Hence, $V(x) - \langle p, x\rangle \geq V(y) - \langle p, y\rangle$ for all $y \in X$. Now, we pick $y = rx$ with $r > 1$ such that $rx \in X$ and $rx \geq x$. Then

$$\langle p, rx - x\rangle = \langle p, (r-1)x\rangle \geq V(rx) - V(x) \geq 0$$

By the monotonicity of V, $\langle p, x\rangle \geq 0$. ■

We now add an additional condition to (Ω, u, δ).

Condition 10 *There exists* $(\widetilde{x}, \widetilde{z}), (\widehat{x}, \widehat{z})$ *in* Ω *with* $\{\widetilde{z}, \widehat{z}\}$ *to be linearly independent.*

Lemma 11 *Suppose that* (Ω, u, δ) *satisfies Condition 7 and Condition 10. Let* (x_t) *be an optimal program from* $\overline{x} \in X$*. If for some* $t \in \mathbf{N}$ *such that there exists* $p_t \in \mathbb{R}^2$ *satisfying*

$$\delta^t V(x_t) - \langle p_t, x_t \rangle \geq \delta^t V(x) - \langle p_t, x \rangle \qquad \text{for all } x \in X \tag{9}$$

and

$$\langle p_t, x_t \rangle \geq 0$$

then there exists $p_{t+1} \in \mathbb{R}^2$ *satisfying*

$$\delta^{t+1} V(x_{t+1}) - \langle p_{t+1}, x_{t+1} \rangle \geq \delta^{t+1} V(x) - \langle p_{t+1}, x \rangle \qquad \text{for all } x \in X$$

and

$$\langle p_{t+1}, x_{t+1} \rangle \geq 0,$$

furthermore,

$$\delta^t u(x_t, x_{t+1}) + \langle p_{t+1}, x_{t+1} \rangle - \langle p_t, x_t \rangle \geq \delta^t u(x, y) + \langle p_{t+1}, y \rangle - \langle p_t, x \rangle$$
$$\text{for all } (x, y) \in \Omega$$

Proof. Since (x_t) is an optimal program, by Theorem 5 (ii), we have

$$V(x) \geq u(x, y) + \delta V(y) \qquad \text{for all } (x, y) \in \Omega$$

and by Theorem 5 (iii),

$$V(x_t) = u(x_t, x_{t+1}) + \delta V(x_{t+1}) \qquad \text{for all } t \in \mathbf{N}$$

Using these in (9), we get

$$\theta_{t+1} = \delta^t u(x_t, x_{t+1}) + \delta^{t+1} V(x_{t+1}) - \langle p_t, x_t \rangle \geq \delta^t u(x, y) + \delta^{t+1} V(y) - \langle p_t, x \rangle \qquad \text{for all } (x, y) \in \Omega$$

Thus, we have

$$\theta_{t+1} - \delta^t u(x, y) + \langle p_t, x \rangle \geq \delta^{t+1} V(y) \qquad \text{for all } (x, y) \in \Omega \tag{10}$$

Define two sets A and B as follows:

$$A = \left\{ \begin{pmatrix} w \\ y \end{pmatrix} \in \mathbb{R} \times \mathbb{R}^2;\ (x, y) \in \Omega \text{ and } w > \theta_{t+1} - \delta^t u(x, y) + \langle p_t, x \rangle \right\}$$
$$B = \left\{ \begin{pmatrix} w \\ y \end{pmatrix} \in \mathbb{R} \times \mathbb{R}^2;\ y \in X \text{ and } w \leq \delta^{t+1} V(y) \right\}$$

Clearly, A and B are nonempty,disjoint and convex sets with $int(B) \neq \phi$ (since u is concave on Ω and V is concave on X). By the separation theorem for convex sets [3] in $\mathbb{R}^3$, there exists $\begin{pmatrix} \mu \\ v \end{pmatrix} \in \mathbb{R}^3$ with $\begin{pmatrix} \mu \\ v \end{pmatrix} \neq 0$ and $\alpha \in \mathbb{R}$ such that

$$\left\langle \begin{pmatrix} \mu \\ v \end{pmatrix}, \begin{pmatrix} w \\ y \end{pmatrix} \right\rangle = \mu w + \langle v, y \rangle \geq \alpha \qquad \text{for all } \begin{pmatrix} w \\ y \end{pmatrix} \in A \tag{11}$$

$$\left\langle \begin{pmatrix} \mu \\ v \end{pmatrix}, \begin{pmatrix} w \\ y \end{pmatrix} \right\rangle = \mu w + \langle v, y \rangle \leq \alpha \quad \text{for all} \begin{pmatrix} w \\ y \end{pmatrix} \in B \tag{12}$$

Let $\begin{pmatrix} w_1 \\ y \end{pmatrix} \in A$ and $\begin{pmatrix} w_2 \\ y \end{pmatrix} \in B$, $w_1 > w_2$, we have $\mu w_1 \geq \alpha - \langle v, y \rangle \geq \mu w_2$. Hence, $\mu \geq 0$.

Define $q_{t+1} = -v$. Then, using (11) and (12),

$$\mu[\theta_{t+1} - \delta^t u(x, y) + \langle p_t, x \rangle] - \langle q_{t+1}, y \rangle \geq \mu \delta^{t+1} V(y') - \langle q_{t+1}, y' \rangle \tag{13}$$

$$\text{for all } (x, y) \in \Omega,\ y' \in X$$

Substituting $x = x_t$ and $y = x_{t+1}$ in (13),

$$\mu \delta^{t+1} V(x_{t+1}) - \langle q_{t+1}, x_{t+1} \rangle \geq \mu \delta^{t+1} V(y') - \langle q_{t+1}, y' \rangle \text{ for all } y' \in X \tag{14}$$

Substituting $y' = x_{t+1}$ in (13),

$$\begin{aligned} &\mu[\delta^t u(x_t, x_{t+1}) - \langle p_t, x_t \rangle] + \langle q_{t+1}, x_{t+1} \rangle \\ &\geq \mu[\delta^t u(x, y) - \langle p_t, x \rangle] + \langle q_{t+1}, y \rangle \\ &\text{for all } (x, y) \in \Omega \end{aligned} \tag{15}$$

We, now, claim that $\mu \neq 0$. For if $\mu = 0$,then by (14), we have

$$\langle q_{t+1}, x_{t+1} \rangle \leq \langle q_{t+1}, y' \rangle \text{ for all } y' \in X,$$

while by (15)

$$\langle q_{t+1}, x_{t+1} \rangle \geq \langle q_{t+1}, y \rangle \text{ for all } y \text{ such that } (x, y) \in \Omega \text{ for some } x \in X$$

Thus

$$\langle q_{t+1}, x_{t+1} \rangle = \langle q_{t+1}, y \rangle \text{ for all } y \text{ such that } (x, y) \in \Omega \text{ for some } x \in X$$

Since $(\widehat{x}, \widehat{z}) \in \Omega$ with $\widehat{z} > 0$ and $(\widehat{x}, 0) \in \Omega$ by Condition10, we have

$$\langle q_{t+1}, x_{t+1} \rangle = \langle q_{t+1}, \widehat{z} \rangle = \langle q_{t+1}, 0 \rangle = 0$$

and, similarly, since $(\widetilde{x}, \widetilde{z}) \in \Omega$ with $\widetilde{z} > 0$ and $(\widetilde{x}, 0) \in \Omega$, we have

$$\langle q_{t+1}, x_{t+1} \rangle = \langle q_{t+1}, \widetilde{z} \rangle = \langle q_{t+1}, 0 \rangle = 0$$

So, $q_{t+1} = -v = 0$, since $\widehat{z}$ and $\widetilde{z}$ are linearly independent. Hence $\begin{pmatrix} \mu \\ v \end{pmatrix} = 0$, a contradiction. Thus, $\mu \neq 0$, and since $\mu \geq 0$,we have $\mu > 0$.

Define $p_{t+1} = q_{t+1}/\mu$ and use (14), (15) to get

$$\delta^{t+1} V(x_{t+1}) - \langle p_{t+1}, x_{t+1} \rangle \geq \delta^{t+1} V(y') - \langle p_{t+1}, y' \rangle \tag{16}$$

$$\text{for all } y' \in X$$

$$\delta^t u(x_t, x_{t+1}) + \langle p_{t+1}, x_{t+1}\rangle - \langle p_t, x_t\rangle \geq \delta^t u(x, y) + \langle p_{t+1}, y\rangle - \langle p_t, x\rangle \qquad (17)$$

for all $(x, y) \in \Omega$

Now, show that $\langle p_{t+1}, x_{t+1}\rangle \geq 0$.

If $x_{t+1} \in [0, b) \times [0, b)$,then by choosing $y' \in X$, $y > x_{t+1}$ in (16), we get

$$\langle p_{t+1}, y' - x_{t+1}\rangle \geq \delta^{t+1}(V(y') - V(x_{t+1})) \geq 0$$

Choose $y' = rx_{t+1}, r > 1$ such that $y' \in X$. By the monotone property of V,

$$(r-1)\langle p_{t+1}, x_{t+1}\rangle \geq \delta^{t+1}(V(rx_{t+1}) - V(x_{t+1})) \geq 0.$$

So, $\langle p_{t+1}, x_{t+1}\rangle \geq 0$.

If $x_{t+1} \in \{b\} \times [0, b] \cup [0, b] \times \{b\}$,then by choosing $(x, y) = (x_t, y)$ with $0 \leq y \leq x_{t+1}$ in (17), we get

$$\langle p_{t+1}, x_{t+1}\rangle \geq \langle p_{t+1}, y\rangle$$

. That is, $\langle p_{t+1}, x_{t+1} - y\rangle \geq 0$

If $y = 0 \in X$, we get $\langle p_{t+1}, x_{t+1}\rangle \geq 0$. ■

Theorem 12 *Suppose that (Ω, u, δ) satisfies Condition 7 and Condition 10. Let $(x_t)_0^\infty$ be an optimal program from $\overline{x} \in int(X)$. Then there exists $(p_t)_0^\infty$ with $p_t \in \mathbb{R}^2$ such that for all $t \in \mathbf{N}$*

(i) $\langle p_t, x_t\rangle \geq 0$

(ii) $\delta^t u(x_t, x_{t+1}) + \langle p_{t+1}, x_{t+1}\rangle - \langle p_t, x_t\rangle \geq \delta^t u(x, y) + \langle p_{t+1}, y\rangle - \langle p_t, x\rangle$ *for all* $(x, y) \in \Omega$

(iii) $\delta^t V(x_t) - \langle p_t, x_t\rangle \geq \delta^t V(x) - \langle p_t, x\rangle$ *for all* $x \in X$

(iv) $\lim_{t\to\infty} \langle p_t, x_t\rangle = 0$

Proof. Using Lemma 9, there exists $p_0 \in \mathbb{R}^2$ such that

$$\langle p_0, x_0\rangle \geq 0 \qquad \text{and}$$

$$V(x_0) - \langle p_0, x_0\rangle \geq V(x) - \langle p_0, x\rangle \text{ for all } x \in X$$

Using Lemma 11, there exists a sequence (p_t) with $p_t \in \mathbb{R}^2$ for $t \in \mathbf{N}$ satisfying (i)-(iii).

Using $x = 0$ in (iii), we get

$$\delta^t(V(x_t) - V(0)) \geq \langle p_t, x_t\rangle \text{ for all } t \in \mathbf{N}.$$

Since $|V(x)|$ is bounded on X , and $\delta^t \to 0$ as $t \to \infty$ and $\langle p_t, x_t\rangle \geq 0$ for all $t \in \mathbf{N}$. We have $\lim_{t\to\infty} \langle p_t, x_t\rangle = 0$. ■

We now state Theorem 13. We can find that it is almost the converse statement of Theorem 12, herein, however, we do not require that the initial state to be in the interior of the state space and the reduced-form model satisfies Condition 7 and Condition 10.

Theorem 13 *If* $(x_t)_0^\infty$ *is a program from* $\overline{x}\in X$ *and there is* $(p_t)_0^\infty$ *such that* $p_t \in \mathbb{R}^2$ *for all* $t \in \mathbf{N}$ *and the following holds:*

(i) $\langle p_t, x_t\rangle \geq 0$

(ii) $\delta^t u(x_t, x_{t+1}) + \langle p_{t+1}, x_{t+1}\rangle - \langle p_t, x_t\rangle \geq \delta^t u(x, y) + \langle p_{t+1}, y\rangle - \langle p_t, x\rangle$ *for all* $(x, y) \in \Omega$

(iii) $\lim_{t\to\infty} \langle p_t, x_t\rangle = 0$

then $(x_t)_0^\infty$ *is an optimal program from* $\overline{x}\in X$ *.*

Proof. Let $(x'_t)_0^\infty$ be any program from $\overline{x}$. Using (ii) , we have for $t \geq 0$,

$$\delta^t[u(x'_t, x'_{t+1}) - u(x_t, x_{t+1})] \leq [\langle p_{t+1}, x_{t+1}\rangle - \langle p_t, x_t\rangle] - \left[\langle p_{t+1}, x'_{t+1}\rangle - \langle p_t, x'_t\rangle\right] .$$

Then

$$\begin{aligned} & \sum_{t=0}^{T} \delta^t[u(x'_t, x'_{t+1}) - u(x_t, x_{t+1})] \\ \leq \quad & [\langle p_{T+1}, x_{T+1}\rangle - \langle p_0, x_0\rangle] - [\langle p_{T+1}, x'_{T+1}\rangle - \langle p_0, x'_0\rangle] \\ = \quad & \langle p_{T+1}, x_{T+1}\rangle - \langle p_{T+1}, x'_{T+1}\rangle \leq \langle p_{T+1}, x_{T+1}\rangle \end{aligned}$$

Let $T \to \infty$, we get

$$\sum_{t=0}^{\infty} \delta^t u(x'_t, x'_{t+1}) - \sum_{t=0}^{\infty} \delta^t u(x_t, x_{t+1}) \leq \lim_{T\to\infty} \langle p_{T+1}, x_{T+1}\rangle = 0$$

which proves $(x_t)_0^\infty$ is optimal from $\overline{x}$. ■

6 Appendix

Definition 14 (Convex Set) *Let U be a subset of a linear space L. We say that U is convex if $x, y \in U$ implies that $z = [\lambda x + (1-\lambda)y] \in U$ for all $\lambda \in [0,1]$.*

Definition 15 (Concave Function) *Let f be a real-valued function defined on a convex set $U \subset L$. We say that f is concave on U if for $x_1, x_2 \in U, \alpha \in (0,1)$, $f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha f(x_1) + (1-\alpha)f(x_2)$.*

Definition 16 (Boundedly Polyhedral) *A subset is called boundedly polyhedral provided that its intersection with any polytope is a polytope.*

Definition 17 (Upper Semicontinuous Function) *Suppose f is a concave function defined on a polytope $D \subseteq \mathbb{R}^n$. f is upper semicontinuous on D; that is, corresponding to each $x_0 \in D$ and each $\epsilon > 0$, there is a $\delta > 0$ such that $\|x - x_0\| < \delta$ implies $f(x) - f(x_0) < \epsilon$.*

Definition 18 (Support of Concave Functions) *Let U be a convex subset of a normed linear space L. We say a concave function $f : U \to \mathbb{R}$ has support at $x_0 \in U$ if there is an affine function $A : L \to \mathbb{R}$ such that $A(x_0) = f(x_0)$ and $A(x) \geq f(x)$ for every $x \in U$.*

Theorem 19 (Basic Separation Theorem for Convex Sets in $\mathbb{R}^n$) *Let U and W be convex sets in $\mathbb{R}^n$ with $int(U) \neq \phi$, $int(U) \cap W = \phi$. Then there is a hyperplane that separates U and W [3, Theorem 32B].*

Theorem 20 (Fundamental Theorem on the Support of a Concave Function) *The function f is concave on an open convex set U of a normed linear space L if and only if f has support at each point of U [3, Theorem 43B].*

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