國立交通大學

應用數學系

碩 士 論 文

在擁有部分資訊且需付手續費的財務模型中之最佳投資策略

Optimal Trading Strategy with Transaction Cost in a Partial Information Model

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指導教授：吳慶堂　副教授

中 華 民 國 九十七 年 七 月

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在擁有部分資訊且需付手續費的財務模型中之最佳投資策略

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摘　　要

本論文介紹當投資者在財務市場做交易的時候需要付一筆固定比率的手續費時，投資人應該如何決定最佳投資策略。在這篇論文裡對於離散時間的財務模型我們分別給風險中立、風險趨避兩種投資人一些結果。另一方面，本篇論文也討論當投資人在市場裡只能觀察到部分資訊時，投資人又該如何決定最佳投資策略。

Optimal Trading Strategy with Transaction Cost in a Partial Information Model

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ABSTRACT

In this thesis, we study that when the investor needs to pay a constant proportional transaction cost at each trading in a financial market how he (or) she decides the optimal trading strategy. We give some main results for the risk-neutral and risk-averse investors, respectively, in discrete financial model. And we also discuss the optimal trading strategy for the investor when he (or she) can only observe partial information in the financial market.

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此篇論文能夠完成，首先，我最想感謝的是恩師也就是指導教授—吳慶堂教授。自從大五那年旁聽了吳慶堂教授的高等微積分之後，便為我建立了穩固的數學分析能力；因為有了老師的教導，讓我如願的考上交通大學的應用數學系研究所，有機會當個研究生。之後，我懇求吳慶堂教授能夠擔任我的指導教授；於是，展開了我的研究生活。在這段時間中，老師給了我許多的指導，好讓我對於在財務數學的領域中有了更多的認識；此外，更要感謝指導教授不辭辛苦、不畏沉悶花時間陪伴著我 meeting，聽著一遍又一遍老師已經熟透的內容，並適時的給予我想法，加深我對於相關內容的認知。另外，我還想感謝陳育慈學姊，提供了我許多關於財金方面的知識，讓我對於金融市場的現況有更多的了解；以及陳冠羽學長給我數值模擬的建議，以完成數值結果。最後，還想感謝蔡明誠學長、蔡明耀學長以及陳偉國學長，在我有課業問題時，能夠跟我討論，提供我許多的想法。

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目　　　錄

CHAPTER 1

Introduction

In a classic paper, Merton (1971) developed optimal portfolio and consumption rules for an investor managing a portfolio of risky assets whose prices evolve as geometric Brownian motions. In Merton's model, it assumed that investors trade costless. However, investors in real capital markets face nontrivial transaction costs, so it is interested to discuss the effect on trading strategies when the assumption of frictionless is removed.

Magill and Constantinides (1976) got optimal trading policies which are more reasonable in continuous time theory formulated by Merton. Since they introduced transaction costs in the model, the investors traded at suitable disjoint intervals of time rather than trading at anytime. Davis and Norman (1990) investigated the optimal consumption and investment decisions with transaction costs equal to a fixed proportion of the amount transacted.

For the perspectives of the investors, we invest in some assets in discrete time, thus, the information that we observed from the market is collected in discrete time. Here we face a problem that if we only can observe the information which is collected in discrete time, what decisions will be the best? What trading strategies will make the maximal profit? Or under what decisions we will not be bankrupt in finite time?

Moreover, what trading strategies will make the maximal profit if we only can observe the stock prices in discrete time model? Since in real capital market the drift term b and the noise term B in the equation (2.1) are not observable. This problem has been studied widely, for example, Karatzas and Xue (1991), Lakner (1995, 1998), Bouchard and Pham (2003), and Xiong and Yang (2005). In such a situation, we call the model with "partial information".

In this thesis, we assume that the stock price is governed by a simple discrete model similar to the Black-Scholes model with the interest rate 0. In Chapter 2, we assume all coefficients in the model are deterministic and the noise term is Gaussian. And we also assume an investor only need to pay constant proportional transaction cost when he (or she) sells some stocks. We discuss two different utilities, says risk-neutral and risk-averse, here. And for the risk-averse utility, we will find a "no trading" interval and give some numerical results. In Chapter 3, we consider the appreciation return of stock b as a random variable and assume b is Gaussian, then use the method (optional projection theorem) in Huang (2007) to rewrite the stock price model. Finally, we give a risk-averse utility example. In last chapter, we give some ideas for the future work.

CHAPTER 2

Optimal Strategy with Transaction Cost in Discrete Model

The basic problem for the investor in the financial mathematics is to reach the maximal profit via trading in the financial market. The trading strategy plays an important role for every investor in a financial market. The main question is how to find the best strategies in different cases. In this chapter we consider the case of the market model with one risky asset (stock) and one riskless asset (bond).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

2.1. Model Setup

Let $(S_n)_{n\geq 0}$ be stock prices in a financial market, where $S_0 \in \mathbb{R}^+$ is given, and we assume that the interest rate is identical to 0. At time n, suppose that the investor can only observe the stock prices up to time n. Thus, the information the investor observe is $\mathcal{G}_n$, the natural filtration generated by $S_0, S_1, ..., S_n$.

Assume that the stock prices follows the relation

$$S_{n+1} - S_n = S_n[b_n + \sigma_n(B_{n+1} - B_n)], \quad n \geq 0, \tag{2.1}$$

where b_n is the appreciation return of the stock, σ_n is volatility of the stock, and (B_n) is a noise.

Assumption 1.

(1) All the coefficients b_n and σ_n are assumed as deterministic.

(2) $(B_{n+1} - B_n)_{n\geq 0}$ is a Gaussian process with mean 0 and variance 1, and is (totally) independent for all $n \geq 0$.

Under the Gaussian assumption it is not easy to know the distribution of S_n. However, if we consider the stock return

$$X_{n+1} = \frac{S_{n+1} - S_n}{S_n},$$

then the stock price equation (2.1) will be rewritten as

$$X_{n+1} = b_n + \sigma_n(B_{n+1} - B_n). \tag{2.2}$$

Remark 2.

(1) (X_{n+1}) is a Gaussian process with mean b_n and variance σ_n^2.

(2) $\{X_{n+1}, n \geq 0\}$ are totally independent for all $n \geq 0$.

We denote $\mathcal{G}_n^*$ the natural filtration generated by $\{X_0, X_1, \cdots, X_n\}$ with $X_0 = S_0$. The following lemma tells us that $X_0, X_1, \cdots, X_n$ and $S_0, S_1, \cdots, S_n$ generate the same filtration for all n.

Lemma 3. $\mathcal{G}_n^* = \mathcal{G}_n$ for all $n \geq 0$.

PROOF. Due to $S_{n+1} = S_n X_{n+1} + S_n$,

(1) When $n = 0$, $\mathcal{G}_0^* = \mathcal{G}_0$.

(2) When $n = k$, assume that $\mathcal{G}_k^* = \mathcal{G}_k$.

(3) When $n = k+1$, we have

$$X_{n+1} = \frac{S_{n+1} - S_n}{S_n} \in \mathcal{G}_{k+1} \qquad \text{and} \qquad S_{n+1} = S_n X_{n+1} + S_n \in \mathcal{G}_{k+1}^*.$$

By mathematical inductions, we have $\mathcal{G}_n^* = \mathcal{G}_n$ for all $n \geq 0$.

Remark 4. Since $\mathcal{G}_0$ is the σ-field generated by S_0 and $S_0 \in \mathbb{R}^+$ is given, we have $\mathcal{G}_0 = \{\emptyset, \Omega\}$. Then for any integrable random variable X, we have

$$E(X|\mathcal{G}_0) = E(X).$$

Remark 5.

(1) By (2.2), $B_{n+1} - B_n$ is independent of $\mathcal{G}_n^*$ for all $n \geq 0$.

(2) By Lemma 3, $B_{n+1} - B_n$ is also independent of $\mathcal{G}_n$ for $n \geq 0$.

2.2. Trading Strategy with Transaction Cost and Backward Inductions

In this section we introduce the trading strategy with transaction cost in discrete time model, which is derived from the model specified in Kabanov (2002). Suppose that the random variables ξ_{n+1} and η_{n+1} describes the number of shares of assets invested in stock and bond at time n (after the trading), respectively. Thus the wealth process at time n is given by

$$V_n = \xi_{n+1} S_n + \eta_{n+1}. \tag{2.3}$$

Moreover, if the initial endowment is given by x, then the initial wealth $x = \xi_0 S_0 + \eta_0$.

Remark 6. (ξ, η) is called a trading strategy if both of ξ_n and η_n are predictable with respect to the filtration $(\mathcal{G}_n)$, i.e., ξ_n is the number of shares of the stock between the $n-1$ (after the trading) and the time n (before the trading). Thus, our wealth at time n is V_n (after the trading) defined by (2.3).

We assume that the investors need to pay the transaction cost when they sell stocks and we denote the transfers from the stock to the bond by $\overline{L}_n^{10}$ (amount by money) at time n. Moreover, we consider a model with constant proportional transaction costs and the proportion is $\lambda^{10} \in (0,1)$. Thus, the wealth process (after the trading) is given by

$$V_n = \xi_{n+1} S_n + \eta_{n+1} = V_{n-1} + \xi_n (S_n - S_{n-1}) - \lambda^{10} \overline{L}_n^{10} \qquad n \geq 1,$$

$$V_0 = x - \lambda^{10} \overline{L}_0^{10}.$$

And, by (2.3) we have

$$\eta_{n+1} = \eta_n + (\xi_n - \xi_{n+1}) S_n - \lambda^{10} \overline{L}_n^{10}.$$

Remark 7. In Kabanov (2002), it introduces the following model with transaction cost. Let V_n^1 denote the total value (amount by money) of the stock at time n under transaction cost, V_n^0 denote the total value of the bond at time n under transaction cost. And ξ_k represents the number of shares of the stock the investor holds at time $k-1$ (after the trading).

The portfolio value evolves according to the equation

$$V_n^1 = v^1 + \sum_{k=0}^{n} \xi_k (S_k - S_{k-1}) + \sum_{k=0}^{n} \overline{L}_k^{01} - \sum_{k=0}^{n} (1 + \lambda^{10}) \overline{L}_k^{10}$$

$$V_n^0 = v^0 - \sum_{k=0}^{n} \overline{L}_k^{01} + \sum_{k=0}^{n} \overline{L}_k^{10}$$

where $\overline{L}_n^{ij}$ represents transfers from the ith to the jth asset at time n under transaction costs, v^0 and v^1 are initial endowments in the bond and the stock respectively. It is easy to verify that our model is equivalent to that introduced by Kabanov, since

$$\begin{aligned} V_n^1 + V_n^0 &= v^0 + v^1 + \sum_{k=0}^{n} \xi_k (S_k - S_{k-1}) - \sum_{k=0}^{n} \lambda^{10} \overline{L}_k^{10} \\ &= V_{n-1} + \xi_n (S_n - S_{n-1}) - \lambda^{10} \overline{L}_n^{10}. \end{aligned}$$

Problem 8. For some future time $n = N$, in order to make the most profit, how do we invest at the beginning under the model in a small investor perspectives?

Due to the argument in game theory, we know that the optimal decision can be constructed by the backward induction, see, e.g., Fudenberg and Tirole (1991).

Definition 9 (Backward Induction). This is a mathematical technique for finding the optimal choice in each step in a game. The idea is to start by solving the optimal strategy of the last step, and then work backward to compute the optimal strategy before.

At time $N-1$ our goal is to find the optimal strategy ξ_N^* such that

$$\max_{\xi_N} E[U(V_{N-1} + \xi_N(S_N - S_{N-1}))|\mathcal{G}_{N-1}], \tag{2.4}$$

for given strategy $(\xi_n, \eta_n)_{n \le N-1}$, i.e., we aim to solve (2.4) to get the optimal solution ξ_N^*. Moreover, at time $m \le N-2$ we choose the optimal ξ_{m+1}^* satisfying

$$\max\{E[U(V_m + \sum_{k=m+1}^{N} \xi_k(S_k - S_{k-1}) - \sum_{k=m+1}^{N-1} \lambda^{10}\overline{L}_k^{10})|\mathcal{G}_m]\},$$

for given strategy $(\xi_n, \eta_n)_{n \le m}$ and $\xi_k = \xi_k^*$ for $k \ge m+2$.

Remark 10. If the terminal time is 1, the optimal trading strategy solved by the backward induction is that solved by the same as the maximal expected utility.

2.3. Optimal Strategy with Transaction Cost Under Risk-Neutral Utility

In mathematical finance, utility function is a measure of the relative satisfaction gained by consuming different bundles of goods and services. In our model utility is a measure of the relative satisfaction gained by making different profits.

In this section we consider the risk-neutral utility function denoted by $U(x) = x$. First we give a one-period example under the risk-neutral utility function and show some results.

Example 11. If the terminal time is 1, by the maximal expected utility we want to compute

$$E[U(V_0 + \xi_1(S_1 - S_0) - x)] = E[\xi_1(S_1 - S_0) - \lambda^{10}\overline{L}_0^{10}].$$

(1) If we buy some stocks at time 0 ($\xi_0 < \xi_1$) and ξ_1 is $\mathcal{G}_0$-measurable, then

$$\begin{aligned} E[\xi_1(S_1 - S_0) - \lambda^{10}\overline{L}_0^{10}] &= E[\xi_1(S_1 - S_0)] \\ &= \xi_1 S_0 E[b_0 + \sigma_0(B_1 - B_0)]. \end{aligned}$$

Due to the Gaussian assumption we get

$$\xi_1 S_0 E[b_0 + \sigma_0(B_1 - B_0)] = \xi_1 S_0 b_0.$$

Hence,

$$E[U(V_0 + \xi_1(S_1 - S_0) - x)] = \xi_1 S_0 b_0. \tag{2.5}$$

(2) If we sell some stocks at time 0 ($\xi_0 > \xi_1$), then

$$\begin{aligned} & E[\xi_1(S_1 - S_0) - \lambda^{10}\overline{L}_0^{10}] \\ = \; & E[\xi_1 S_0[b_0 + \sigma_0(B_1 - B_0)] - \lambda^{10} S_0(\xi_0 - \xi_1)] \\ = \; & -\lambda^{10} S_0(\xi_0 - \xi_1) + \xi_1 S_0 b_0 + E[\xi_1 S_0 \sigma_0(B_1 - B_0)]. \end{aligned}$$

Because of the Gaussian assumption, we get

$$E[\xi_1(S_1 - S_0) - \lambda^{10}\overline{L}_0^{10}] = -\lambda^{10} S_0(\xi_0 - \xi_1) + \xi_1 S_0 b_0$$

Hence,

$$E[U(V_0 + \xi_1(S_1 - S_0) - x)] = -\lambda^{10} S_0(\xi_0 - \xi_1) + \xi_1 S_0 b_0 \le \xi_1 S_0 b_0, \tag{2.6}$$

since $\xi_0 > \xi_1$. From (2.5), (2.6), in the case without any restrictions on the trading strategy, the investors trade according the sign of b_0. If $b_0 \geq 0$, the investor should buy the stocks as much as possible. If $b_0 + \lambda^{10} < 0$, the investors should make short-sell as much as possible. In the case with constraint, for example, $0 \leq \xi_1 \leq x/S_0$, no short-sell and no loan, the optimal strategy is given by

$$\xi_1^* = \begin{cases} x/S_0, & \text{if } b_0 \geq 0, \\ 0, & \text{if } b_0 + \lambda^{10} < 0 \end{cases}.$$

2.4. Optimal Strategy with Transaction Cost Under Risk-Averse Utility

We consider the risk-averse utility function given by

$$U(x) = -\frac{1}{\theta}\exp(-\theta x),$$

where $\theta > 0$ is the absolute risk aversion and be a constant.

First we show a one-period example below.

Example 12. If the terminal time is 1, by the maximal expected utility we have to compute

$$\begin{aligned}
& E\left[-\frac{1}{\theta}\exp(-\theta(V_0 + \xi_1(S_1 - S_0) - x))\right] \\
= \; & E\left[-\frac{1}{\theta}\exp(-\theta(-\lambda^{10}\overline{L}_0^{10} + \xi_1(S_1 - S_0)))\right] \\
= \; & E\left[-\frac{1}{\theta}\exp(\theta\lambda^{10}\overline{L}_0^{10} - \theta\xi_1(S_1 - S_0))\right] \\
= \; & -\frac{1}{\theta}E\left[\exp(\theta\lambda^{10}\overline{L}_0^{10} - \theta\xi_1 S_0[b_0 + \sigma_0(B_1 - B_0)])\right] \\
= \; & -\frac{1}{\theta}\exp(-\theta\xi_1 S_0 b_0)E\left[\exp(\theta\lambda^{10}\overline{L}_0^{10} - \theta\xi_1 S_0\sigma_0(B_1 - B_0))\right].
\end{aligned}$$

We have to discuss it in two cases.

(1) If we buy stocks at time 0, then

$$
\begin{aligned}
& -\frac{1}{\theta}\exp(-\theta\xi_1 S_0 b_0) E\left[\exp(\theta\lambda^{10}\overline{L}_0^{10} - \theta\xi_1 S_0\sigma_0(B_1 - B_0))\right] \\
= \quad & -\frac{1}{\theta}\exp(-\theta\xi_1 S_0 b_0) E[\exp(-\theta\xi_1 S_0\sigma_0(B_1 - B_0))].
\end{aligned}
$$

Due to the Gaussian assumption, we get

$$
-\frac{1}{\theta}\exp(-\theta\xi_1 S_0 b_0) E[\exp(-\theta\xi_1 S_0\sigma_0(B_1 - B_0))] = -\frac{1}{\theta}\exp(-\theta\xi_1 S_0 b_0 + \frac{1}{2}\theta^2\xi_1^2 S_0^2\sigma_0^2).
$$

Hence,

$$
E[-\frac{1}{\theta}\exp(-\theta(V_0 + \xi_1(S_1 - S_0) - x))] = -\frac{1}{\theta}\exp(-\theta\xi_1 S_0 b_0 + \frac{1}{2}\theta^2\xi_1^2 S_0^2\sigma_0^2).
$$

Thus from the fundamental calculus, we maximize the conditional expectation to get the optimal ξ_1 and from $x = \xi_0 S_0 + \eta_0$ we have

$$
\xi_1 = \frac{b_0}{\theta S_0\sigma_0^2} > \xi_0 = \frac{x - \eta_0}{S_0}. \tag{2.7}
$$

(2) If we sell stocks at time 0, then

$$
\begin{aligned}
& E\left[-\frac{1}{\theta}\exp(-\theta(V_0 + \xi_1(S_1 - S_0) - x))\right] \\
= \quad & -\frac{1}{\theta}\exp(-\theta\xi_1 S_0 b_0) E[\exp(\theta\lambda^{10} S_0(\xi_0 - \xi_1) - \theta\xi_1 S_0\sigma_0(B_1 - B_0))].
\end{aligned}
$$

Due to the Gaussian assumption of $B_1 - B_0$, we have

$$
E[\exp(\theta\lambda^{10} S_0(\xi_0 - \xi_1) - \theta\xi_1 S_0\sigma_0(B_1 - B_0))] = \exp(\theta\lambda^{10} S_0(\xi_0 - \xi_1) + \frac{1}{2}\theta^2\xi_1^2 S_0^2\sigma_0^2).
$$

Hence,

$$
\begin{aligned}
& E\left[-\frac{1}{\theta}\exp(-\theta(V_0 + \xi_1(S_1 - S_0) - x))\right] \\
= \quad & -\frac{1}{\theta}\exp\left(-\theta\xi_1 S_0 b_0 + \theta\lambda^{10} S_0(\xi_0 - \xi_1) + \frac{1}{2}\theta^2\xi_1^2 S_0^2\sigma_0^2\right).
\end{aligned}
$$

Thus from the fundamental calculus, we maximize the conditional expectation to get the optimal ξ_1 and from $x = \xi_0 S_0 + \eta_0$ we have

$$\xi_1 = \frac{b_0 + \lambda^{10}}{\theta S_0 \sigma_0^2} < \xi_0 = \frac{x - \eta_0}{S_0}. \tag{2.8}$$

From (2.7), (2.8), we conclude that the investors will neither buy nor sell stocks when their initial wealth (amount by money) is in the interval

$$\left[\frac{b_0}{\theta\sigma_0^2} + \eta_0, \frac{b_0 + \lambda^{10}}{\theta\sigma_0^2} + \eta_0\right].$$

Moreover, if one wants to buy stocks he may choose the strategy $\xi_1 = \dfrac{b_0}{\theta S_0 \sigma_0^2}$ at time 0 to reach the maximum profit, and if one wants to sell stocks he may choose the strategy $\xi_1 = \dfrac{b_0 + \lambda^{10}}{\theta S_0 \sigma_0^2}$ at time 0 to reach the maximum profit.

Remark 13. Now the "no trading" interval is

$$\left[\frac{b_0}{\theta\sigma_0^2} + \eta_0, \frac{b_0 + \lambda^{10}}{\theta\sigma_0^2} + \eta_0\right].$$

And if one wants to buy (sell) stocks he may choose the strategy $\xi_1 = \dfrac{b_0}{\theta S_0 \sigma_0^2}$ ($\xi_1 = \dfrac{b_0 + \lambda^{10}}{\theta S_0 \sigma_0^2}$) at time 0. We find some phenomenons from this example.

(1) For fixed b_0 and σ_0, if the initial stock price S_0 is too high, we will be conservative for our strategy in the stock.

(2) For fixed σ_0 and S_0, if the appreciation rate of the stock be positive and grows up, we invest the number of shares of the stock more.

(3) For fixed b_0 and S_0, if the volatility of the stock grows up, we are conservative when investing the stock.

(4) The length of the "no trading" interval is $\dfrac{\lambda^{10}}{\theta\sigma_0^2}$, if λ^{10} increases or σ_0^2 decreases, the "no trading" interval becomes larger.

Example 14. If the terminal time is 2, by backward induction we figure out this situation in two steps.

Step1 : Compute the case from time 1 to time 2.

$$
\begin{aligned}
& E\left[-\frac{1}{\theta}\exp(-\theta\{V_1+\xi_2(S_2-S_1)-V_0-\xi_1(S_1-S_0)\})\middle|\mathcal{G}_1\right] \\
= \; & E\left[-\frac{1}{\theta}\exp(-\theta\{\xi_2(S_2-S_1)-\lambda^{10}\overline{L}_1^{10}\})\middle|\mathcal{G}_1\right] \\
= \; & E\left[-\frac{1}{\theta}\exp(-\theta\{\xi_2 S_1[b_1+\sigma_1(B_2-B_1)]-\lambda^{10}\overline{L}_1^{10}\})\middle|\mathcal{G}_1\right].
\end{aligned}
$$

(1) If we sell stocks at time 1, then

$$
\begin{aligned}
& E\left[-\frac{1}{\theta}\exp(-\theta\{\xi_2 S_1[b_1+\sigma_1(B_2-B_1)]-\lambda^{10}\overline{L}_1^{10}\})\middle|\mathcal{G}_1\right] \\
= \; & E\left[-\frac{1}{\theta}\exp(-\theta\xi_2 S_1[b_1+\sigma_1(B_2-B_1)]+\theta\lambda^{10}(\xi_1-\xi_2)S_1)\middle|\mathcal{G}_1\right].
\end{aligned}
$$

Since (B_2-B_1) is independent of $\mathcal{G}_1$ and is normally distributed, we have

$$
\begin{aligned}
& E\left[-\frac{1}{\theta}\exp(-\theta\xi_2 S_1[b_1+\sigma_1(B_2-B_1)]+\theta\lambda^{10}(\xi_1-\xi_2)S_1)\middle|\mathcal{G}_1\right] \\
= \; & -\frac{1}{\theta}\exp\left(-\theta\xi_2 S_1 b_1+\theta\lambda^{10}(\xi_1-\xi_2)S_1+\frac{1}{2}\theta^2\xi_2^2 S_1^2\sigma_1^2\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& E\left[-\frac{1}{\theta}\exp(-\theta\{V_1+\xi_2(S_2-S_1)-V_0-\xi_1(S_1-S_0)\})\middle|\mathcal{G}_1\right] \\
= \; & -\frac{1}{\theta}\exp\left(-\theta\xi_2 S_1 b_1+\theta\lambda^{10}(\xi_1-\xi_2)S_1+\frac{1}{2}\theta^2\xi_2^2 S_1^2\sigma_1^2\right).
\end{aligned}
$$

Thus from the fundamental calculus, we maximize the conditional expectation to get the optimal ξ_2 and from the assumption $\xi_2<\xi_1$ we have

$$
\xi_2=\frac{b_1+\lambda^{10}}{\theta S_1\sigma_1^2}<\xi_1.
$$

(2) If we buy stocks at time 1, then using the similar argument in Example 1 we have

$$\xi_2 = \frac{b_1}{\theta S_1 \sigma_1^2} > \xi_1.$$

Thus, for given ξ_1, the optimal solution is given by

$$\overline{\xi}_2 = \frac{b_1 + \lambda^{10}}{\theta S_1 \sigma_1^2} I_{\{\xi_1 > \frac{b_1+\lambda^{10}}{\theta S_1 \sigma_1^2}\}} + \frac{b_1}{\theta S_1 \sigma_1^2} I_{\{\xi_1 < \frac{b_1}{\theta S_1 \sigma_1^2}\}} + \xi_1 I_{\{\frac{b_1}{\theta S_1 \sigma_1^2} \leq \xi_1 \leq \frac{b_1+\lambda^{10}}{\theta S_1 \sigma_1^2}\}}. \tag{2.9}$$

Step2 : Consider the trading period from time 0 to time 1, we replace the ξ_2 by $\overline{\xi}_2$ which is given in (2.9) and compute the conditional expectation

$$\begin{aligned}
& E\left[-\frac{1}{\theta}\exp(-\theta\{V_1 + \overline{\xi}_2(S_2 - S_1) - x\}) \middle| \mathcal{G}_0\right] \\
= \; & E\left[-\frac{1}{\theta}\exp(-\theta\{V_0 + \xi_1(S_1 - S_0) - \lambda^{10}\overline{L}_1^{10} + \overline{\xi}_2(S_2 - S_1) - x\}) \middle| \mathcal{G}_0\right] \\
= \; & E\left[-\frac{1}{\theta}\exp(-\theta\{-\lambda^{10}\overline{L}_0^{10} - \lambda^{10}\overline{L}_1^{10} + \xi_1(S_1 - S_0) + \overline{\xi}_2(S_2 - S_1)\}) \middle| \mathcal{G}_0\right].
\end{aligned}$$

Similar as the argument as the trading period from time 1 to time 2. we have to separate it into two cases.

(1) If we sell stocks at time 0, i.e., $\overline{L}_0^{10} = (\xi_0 - \xi_1)S_0$, then we have

$$\begin{aligned}
& E\left[-\frac{1}{\theta}\exp(-\theta\{-\lambda^{10}\overline{L}_0^{10} - \lambda^{10}\overline{L}_1^{10} + \xi_1(S_1 - S_0) + \overline{\xi}_2(S_2 - S_1)\}) \middle| \mathcal{G}_0\right] \\
= \; & E\Bigg[-\frac{1}{\theta}\exp(-\theta\{-\lambda^{10}(\xi_0 - \xi_1)S_0 - \lambda^{10}\overline{L}_1^{10}(I_{\{\xi_1 > \frac{b_1+\lambda^{10}}{\theta S_1 \sigma_1^2}\}} + I_{\{\xi_1 < \frac{b_1}{\theta S_1 \sigma_1^2}\}} + I_{\{\frac{b_1+\lambda^{10}}{\theta S_1 \sigma_1^2} \geq \xi_1 \geq \frac{b_1}{\theta S_1 \sigma_1^2}\}}) \\
& + \xi_1(S_1 - S_0) + \overline{\xi}_2(S_2 - S_1)\}) \; \Bigg| \mathcal{G}_0\Bigg].
\end{aligned}$$

By some calculation, we get

$$
\begin{aligned}
& E\left[-\frac{1}{\theta}\exp\left(\theta\lambda^{10}(\xi_0-\xi_1)S_0-\theta\xi_1(S_1-S_0)+\{\theta\lambda^{10}(\xi_1-\frac{b_1+\lambda^{10}}{\theta S_1\sigma_1^2})S_1\right.\right.\\
&\quad -\theta\frac{b_1+\lambda^{10}}{\theta S_1\sigma_1^2}(S_2-S_1)\}I_{\{S_1>\frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}\}}+\{-\theta\frac{b_1}{\theta S_1\sigma_1^2}(S_2-S_1)\}I_{\{S_1<\frac{b_1}{\theta\xi_1\sigma_1^2}\}}\\
&\quad \left.\left.+\{-\theta\xi_1(S_2-S_1)\}I_{\{\frac{b_1}{\theta\xi_1\sigma_1^2}\le S_1\le\frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}\}}\right)\ \Big|\mathcal{G}_0\right]\\
= & E[-\frac{1}{\theta}\exp(\theta\lambda^{10}(\xi_0-\xi_1)S_0-\theta\xi_1(S_1-S_0)\\
&\quad +\{\theta\lambda^{10}\xi_1S_1-\frac{(b_1+\lambda^{10})^2}{\sigma_1^2}-\frac{b_1+\lambda^{10}}{\sigma_1}(B_2-B_1)\}I_{\{S_1>\frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}\}}\\
&\quad +\{-\frac{b_1^2}{\sigma_1^2}-\frac{b_1}{\sigma_1}(B_2-B_1)\}I_{\{S_1<\frac{b_1}{\theta\xi_1\sigma_1^2}\}}\\
&\quad +\{-\theta\xi_1S_1b_1-\theta\xi_1S_1\sigma_1(B_2-B_1)\}I_{\{\frac{b_1}{\theta\xi_1\sigma_1^2}\le S_1\le\frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}\}})|\mathcal{G}_0]
\end{aligned}
$$

Using the tower property of the conditional expectation, we may rewrite the above equation in the following form

$$
\begin{aligned}
& E[-\frac{1}{\theta}\exp(\theta\lambda^{10}(\xi_0-\xi_1)S_0-\theta\xi_1(S_1-S_0)+\{\theta\lambda^{10}\xi_1S_1-\frac{(b_1+\lambda^{10})^2}{\sigma_1^2}\}I_{\{S_1>\frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}\}}\\
&\quad -\frac{b_1^2}{\sigma_1^2}I_{\{S_1<\frac{b_1}{\theta\xi_1\sigma_1^2}\}}-\theta\xi_1S_1b_1I_{\{\frac{b_1}{\theta\xi_1\sigma_1^2}\le S_1\le\frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}\}})E[\exp(\{-\frac{b_1+\lambda^{10}}{\sigma_1}I_{\{S_1>\frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}\}}\\
&\quad -\frac{b_1}{\sigma_1}I_{\{S_1<\frac{b_1}{\theta\xi_1\sigma_1^2}\}}-\theta\xi_1S_1\sigma_1I_{\{\frac{b_1}{\theta\xi_1\sigma_1^2}\le S_1\le\frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}\}}\}(B_2-B_1))|\mathcal{G}_1]|\mathcal{G}_0].
\end{aligned}
$$

Since $(B_2 - B_1)$ is independent of $\mathcal{G}_1$ and is normally-distributed, we get

$$
\begin{aligned}
& E[-\frac{1}{\theta}\exp(\{\theta\lambda^{10}(\xi_0-\xi_1)S_0-\theta\xi_1(S_1-S_0)+\theta\lambda^{10}\xi_1 S_1-\frac{(b_1+\lambda^{10})^2}{2\sigma_1^2}\}I_{\{S_1>\frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}\}} \\
& \quad +\{\theta\lambda^{10}(\xi_0-\xi_1)S_0-\theta\xi_1(S_1-S_0)-\frac{b_1^2}{2\sigma_1^2}\}I_{\{S_1<\frac{b_1}{\theta\xi_1\sigma_1^2}\}} \\
& \quad +\{\theta\lambda^{10}(\xi_0-\xi_1)S_0-\theta\xi_1(S_1-S_0)-\theta\xi_1 S_1 b_1+\frac{1}{2}\theta^2\xi_1^2 S_1^2\sigma_1^2\}I_{\{\frac{b_1}{\theta\xi_1\sigma_1^2}\le S_1\le\frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}\}})] \\
= \; & -\frac{1}{\theta}\{\int_{\frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}}^{\infty}\exp(\theta\lambda^{10}(\xi_0-\xi_1)S_0-\theta\xi_1(x-S_0)+\theta\lambda^{10}\xi_1 x-\frac{(b_1+\lambda^{10})^2}{2\sigma_1^2})f(x)dx \\
& \quad +\int_{-\infty}^{\frac{b_1}{\theta\xi_1\sigma_1^2}}\exp(\theta\lambda^{10}(\xi_0-\xi_1)S_0-\theta\xi_1(x-S_0)-\frac{b_1^2}{2\sigma_1^2})f(x)dx \\
& \quad +\int_{\frac{b_1}{\theta\xi_1\sigma_1^2}}^{\frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}}\exp(\theta\lambda^{10}(\xi_0-\xi_1)S_0-\theta\xi_1(x-S_0)-\theta\xi_1 x b_1+\frac{1}{2}\theta^2\xi_1^2 x^2\sigma_1^2)f(x)dx\},
\end{aligned}
$$

where $f(x)$ is the probability density function of S_1, i.e.,

$$
f(x)=\frac{1}{\sqrt{2\pi}S_0\sigma_0}\exp(-\frac{(x-S_0-S_0 b_0)^2}{2S_0^2\sigma_0^2}). \tag{2.10}
$$

Due to the first order condition with respect to ξ_1, we have

(2.11)

$$
\begin{aligned}
& \int_{\frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}}^{\infty}(-\theta\lambda^{10}S_0+\theta\lambda^{10}x-\theta(x-S_0))\exp(\theta\lambda^{10}(\xi_0-\xi_1)S_0+\theta\lambda^{10}\xi_1 x \\
& \qquad -\frac{(b_1+\lambda^{10})^2}{2\sigma_1^2}-\theta\xi_1(x-S_0))f(x)dx \\
& +\int_{-\infty}^{\frac{b_1}{\theta\xi_1\sigma_1^2}}(-\theta\lambda^{10}S_0-\theta(x-S_0))\exp(-\frac{b_1^2}{2\sigma_1^2}+\theta\lambda^{10}(\xi_0-\xi_1)S_0-\theta\xi_1(x-S_0))f(x)dx \\
& +\int_{\frac{b_1}{\theta\xi_1\sigma_1^2}}^{\frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}}(-\theta b_1 x+\theta^2 x^2\sigma_1^2\xi_1-\theta\lambda^{10}S_0-\theta(x-S_0))\exp(-\theta\xi_1 b_1 x+\frac{1}{2}\theta^2\xi_1^2 x^2\sigma_1^2 \\
& \qquad +\theta\lambda^{10}(\xi_0-\xi_1)S_0-\theta\xi_1(x-S_0))f(x)dx=0.
\end{aligned}
$$

(2) If we buy stocks at time 0, i.e., $\overline{L}_0^{10} = 0$, then we have

$$
\begin{aligned}
& E[-\frac{1}{\theta}\exp(-\theta\{V_1 + \xi_2(S_2 - S_1) - x\})|\mathcal{G}_0] \\
= \;& E[-\frac{1}{\theta}\exp(-\theta\{-\lambda^{10}\overline{L}_1^{10}(I_{\{\xi_1 > \frac{b_1+\lambda^{10}}{\theta S_1\sigma_1^2}\}} + I_{\{\xi_1 < \frac{b_1}{\theta S_1 \sigma_1^2}\}} + I_{\{\frac{b_1+\lambda^{10}}{\theta S_1\sigma_1^2} \geq \xi_1 \geq \frac{b_1}{\theta S_1\sigma_1^2}\}}) \\
& \quad +\xi_1(S_1 - S_0) + \overline{\xi}_2(S_2 - S_1)\})|\mathcal{G}_0].
\end{aligned}
$$

By some calculation, we get

$$
\begin{aligned}
& E[-\frac{1}{\theta}\exp(-\theta\xi_1(S_1 - S_0) + \{\theta\lambda^{10}(\xi_1 - \frac{b_1+\lambda^{10}}{\theta S_1\sigma_1^2})S_1 - \theta\frac{b_1+\lambda^{10}}{\theta S_1\sigma_1^2}(S_2 - S_1)\}I_{\{S_1 > \frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}\}} \\
& \quad +\{-\theta\frac{b_1}{\theta S_1\sigma_1^2}(S_2 - S_1)\}I_{\{S_1 < \frac{b_1}{\theta\xi_1\sigma_1^2}\}} + \{-\theta\xi_1(S_2 - S_1)\}I_{\{\frac{b_1}{\theta\xi_1\sigma_1^2} \leq S_1 \leq \frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}\}})|\mathcal{G}_0].
\end{aligned}
$$

A similar argument as in case (1), we have

$$
\begin{aligned}
& E[-\frac{1}{\theta}\exp(-\theta\xi_1(S_1 - S_0) + \{\theta\lambda^{10}\xi_1 S_1 - \frac{(b_1+\lambda^{10})^2}{\sigma_1^2}\}I_{\{S_1 > \frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}\}} \\
& \quad -\frac{b_1^2}{\sigma_1^2}I_{\{S_1 < \frac{b_1}{\theta\xi_1\sigma_1^2}\}} - \theta\xi_1 S_1 b_1 I_{\{\frac{b_1}{\theta\xi_1\sigma_1^2} \leq S_1 \leq \frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}\}})E[\exp(\{-\frac{b_1+\lambda^{10}}{\sigma_1}I_{\{S_1 > \frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}\}} \\
& \quad -\frac{b_1}{\sigma_1}I_{\{S_1 < \frac{b_1}{\theta\xi_1\sigma_1^2}\}} - \theta\xi_1 S_1\sigma_1 I_{\{\frac{b_1}{\theta\xi_1\sigma_1^2} \leq S_1 \leq \frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}\}}\}(B_2 - B_1))|\mathcal{G}_1]|\mathcal{G}_0].
\end{aligned}
$$

Since $(B_2 - B_1)$ is independent of $\mathcal{G}_1$ and the Gaussian assumption we get

$$
\begin{aligned}
& E[-\frac{1}{\theta}\exp(\{-\theta\xi_1(S_1 - S_0) + \theta\lambda^{10}\xi_1 S_1 - \frac{(b_1+\lambda^{10})^2}{2\sigma_1^2}\} I_{\{S_1 > \frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}\}} \\
& \quad + \{-\theta\xi_1(S_1 - S_0) - \frac{b_1^2}{2\sigma_1^2}\} I_{\{S_1 < \frac{b_1}{\theta\xi_1\sigma_1^2}\}} \\
& \quad + \{-\theta\xi_1(S_1 - S_0) - \theta\xi_1 S_1 b_1 + \frac{1}{2}\theta^2\xi_1^2 S_1^2\sigma_1^2\} I_{\{\frac{b_1}{\theta\xi_1\sigma_1^2} \le S_1 \le \frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}\}})] \\
= & -\frac{1}{\theta}\{\int_{\frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}}^{\infty} \exp(-\theta\xi_1(x - S_0) + \theta\lambda^{10}\xi_1 x - \frac{(b_1+\lambda^{10})^2}{2\sigma_1^2}) f(x)dx \\
& + \int_{-\infty}^{\frac{b_1}{\theta\xi_1\sigma_1^2}} \exp(-\theta\xi_1(x - S_0) - \frac{b_1^2}{2\sigma_1^2}) f(x)dx \\
& + \int_{\frac{b_1}{\theta\xi_1\sigma_1^2}}^{\frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}} \exp(-\theta\xi_1(x - S_0) - \theta\xi_1 x b_1 + \frac{1}{2}\theta^2\xi_1^2 x^2\sigma_1^2) f(x)dx\},
\end{aligned}
$$

where $f(x)$ is the probability density function of S_1, as in (2.10).

Due to the first order condition with respect to ξ_1, we have

$$
\begin{aligned}
(2.12) \quad & \int_{\frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}}^{\infty} (\theta\lambda^{10}x - \theta(x - S_0)) \exp(\theta\lambda^{10}\xi_1 x - \frac{(b_1+\lambda^{10})^2}{2\sigma_1^2} - \theta\xi_1(x - S_0)) f(x)dx \\
& + \int_{-\infty}^{\frac{b_1}{\theta\xi_1\sigma_1^2}} (-\theta(x - S_0)) \exp(-\frac{b_1^2}{2\sigma_1^2} - \theta\xi_1(x - S_0)) f(x)dx \\
& + \int_{\frac{b_1}{\theta\xi_1\sigma_1^2}}^{\frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}} (-\theta b_1 x + \theta^2 x^2\sigma_1^2\xi_1 - \theta(x - S_0)) \\
& \qquad \exp(-\theta\xi_1 b_1 x + \frac{1}{2}\theta^2\xi_1^2 x^2\sigma_1^2 - \theta\xi_1(x - S_0)) f(x)dx = 0.
\end{aligned}
$$

Conclusion : By the Backward Induction, we observe that when the investors sell stocks at time 0 (i.e., $\xi_1 < \xi_0$), the optimal trading strategy ξ_1^* (sell) at time 0 satisfies the equation (2.11); when the investors buy stocks at time 0 (i.e., $\xi_1 > \xi_0$), the optimal trading strategy ξ_1^* (buy) at time 0 satisfies the equation (2.12). And, the optimal ξ_2^* at time 1 is chosen as in (2.9)

Remark 15. In Example 14, it is difficult to find the closed form for ξ_1 in equation (2.11) and (2.12). However, we can find some property of the equation (2.11) and (2.12). For example, considering the equation (2.12), if we let

$$F(\xi_1) = \int_{\frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}}^{\infty} (\theta\lambda^{10}x - \theta(x - S_0)) \exp(\theta\lambda^{10}\xi_1 x - \frac{(b_1+\lambda^{10})^2}{2\sigma_1^2} - \theta\xi_1(x - S_0))f(x)dx$$
$$+ \int_{-\infty}^{\frac{b_1}{\theta\xi_1\sigma_1^2}} (-\theta(x - S_0)) \exp(-\frac{b_1^2}{2\sigma_1^2} - \theta\xi_1(x - S_0))f(x)dx$$
$$+ \int_{\frac{b_1}{\theta\xi_1\sigma_1^2}}^{\frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}} (-\theta b_1 x + \theta^2 x^2\sigma_1^2\xi_1 - \theta(x - S_0))$$
$$\exp(-\theta\xi_1 b_1 x + \frac{1}{2}\theta^2\xi_1^2 x^2\sigma_1^2 - \theta\xi_1(x - S_0))f(x)dx,$$

then

$$F'(\xi_1) = \int_{\frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}}^{\infty} (\theta\lambda^{10}x - \theta(x - S_0))^2 \exp(\theta\lambda^{10}\xi_1 x - \frac{(b_1+\lambda^{10})^2}{2\sigma_1^2} - \theta\xi_1(x - S_0))f(x)dx$$
$$+ \int_{-\infty}^{\frac{b_1}{\theta\xi_1\sigma_1^2}} (-\theta(x - S_0))^2 \exp(-\frac{b_1^2}{2\sigma_1^2} - \theta\xi_1(x - S_0))f(x)dx$$
$$+ \int_{\frac{b_1}{\theta\xi_1\sigma_1^2}}^{\frac{b_1+\lambda^{10}}{\theta\xi_1\sigma_1^2}} (-\theta b_1 x + \theta^2 x^2\sigma_1^2\xi_1 - \theta(x - S_0))^2$$
$$\exp(-\theta\xi_1 b_1 x + \frac{1}{2}\theta^2\xi_1^2 x^2\sigma_1^2 - \theta\xi_1(x - S_0))f(x)dx.$$

So, $F'(\xi_1) \geq 0$ for all ξ_1, which implies that either the equation (2.12) has a solution or the optimal ξ_1^* happens at the two end points. And a similar result for the equation (2.11). Then we can conclude that there is still a "no trading" interval in two period case.

2.5. Numerical Results

In this section, we give some numerical results for the equation (2.11), (2.12) and try to find the corresponding "no trading" intervals.

In Example 14, we consider a two period model and get a result that if we buy some stocks ($\xi_0 < \xi_1$), then the optimal strategy ξ_1 must satisfy the equation (2.12) and if we sell some stocks ($\xi_0 > \xi_1$), then the optimal strategy must satisfy the equation (2.11).

Here we give some numerical results and observe how these parameters affects the "no trading" interval.

2.5.1. The relation between b_0 and nontrading interval. First, we fix $S_0 = 1$, $\theta = 1$, $b_1 = 1$, $\lambda^{10} = 0.5$, $\sigma_0 = 1$, $\sigma_1 = 1$, and discuss the relation between b_0 and the "no trading" interval from (2.11), (2.12).

b_0	1	2	3	4	5	6	7
ξ_1 (buy)	0.66	1.79	3.62	5.36	6.99	8.53	10.02
ξ_1 (sell)	1.36	2.88	4.43	5.93	7.39	8.82	10.22
nontrading interval	0.70	1.09	0.81	0.57	0.40	0.29	0.20

From the table of the relation between b_0 and ξ_1, we observe that ξ_1 (buy) and ξ_1 (sell) are increasing when b_0 becomes bigger. And from the Figure 2.1, the size of the "no trading" interval may reach a maximum when $b_0 \in [1, 3]$.

In practice, the proportion of the transaction cost is usually 0.003, the appreciation return of the stock b and the volatility of the stock σ have values between 0.5 and 1.5. So, we give another data here.

We fix $S_0 = 1$, $\theta = 1$, $b_1 = 1$, $\lambda^{10} = 0.003$, $\sigma_0 = 0.5$, $\sigma_1 = 0.5$, and discuss the relation between b_0 and the "no trading" interval from (2.11), (2.12).

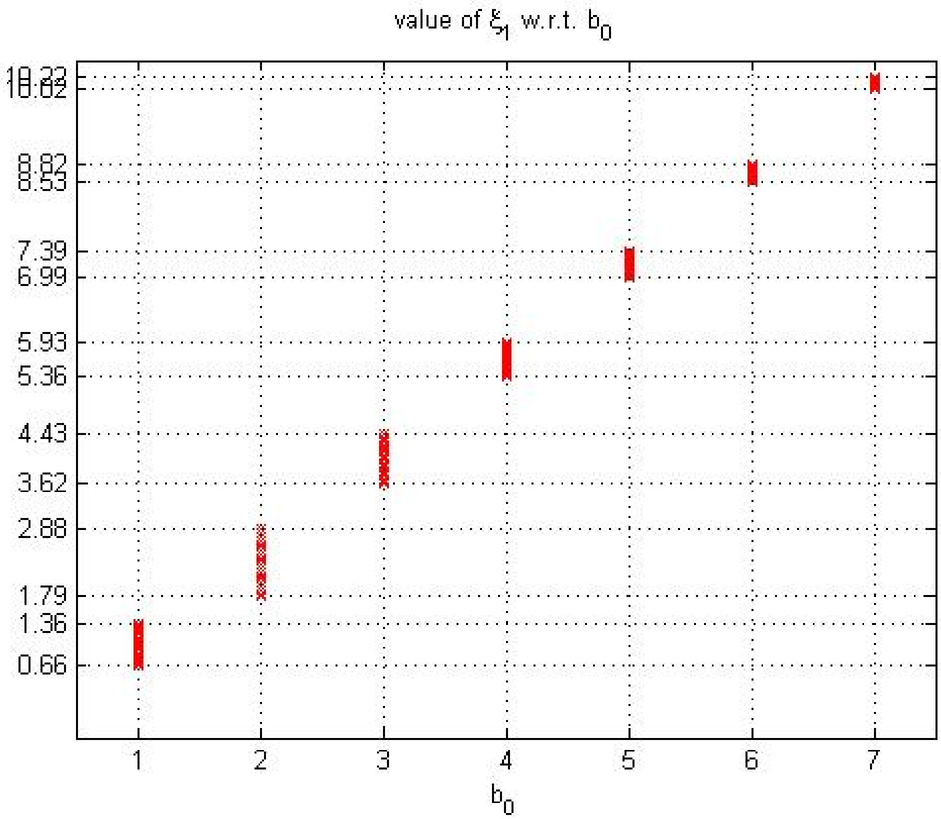

FIGURE 2.1. The relation between b_0 and ξ_1

b_0	0.5	0.6	0.7	0.8	0.9	1.0	1.1
ξ_1 (buy)	1.99	2.39	2.79	3.19	3.59	3.99	4.39
ξ_1 (sell)	2.01	2.41	2.80	3.20	3.60	4.00	4.41
nontrading interval	0.02	0.02	0.01	0.01	0.01	0.01	0.02

From the table of the relation between b_0 and ξ_1, we observe that ξ_1 (buy) and ξ_1 (sell) are increasing when b_0 becomes bigger. From the Figure 2.2, the size of "no trading" interval may reach a minimum when $b_0 \in [0.6, 1.1]$.

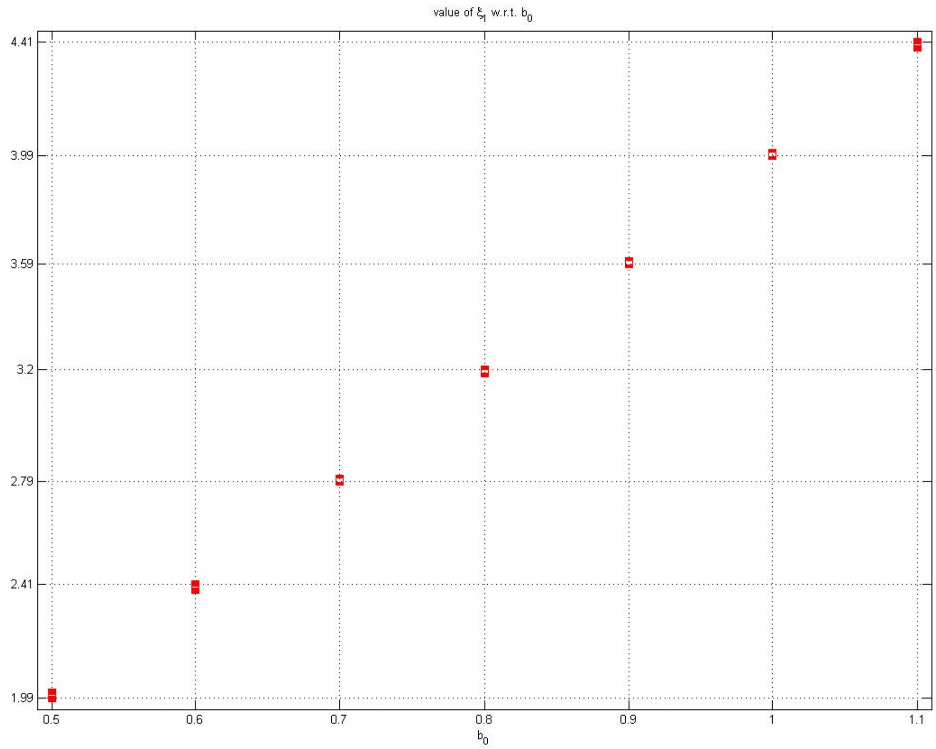

FIGURE 2.2. The relation between b_0 and ξ_1

2.5.2. The relation between b_1 and nontrading interval. Second, we fix $S_0 = 1$, $\theta = 1$, $b_0 = 1$, $\lambda^{10} = 0.5$, $\sigma_0 = 1$, $\sigma_1 = 1$, and discuss the relation between b_1 and the "no trading" interval.

b_1	1	2	3	4	5	6	7
ξ_1 (buy)	0.66	0.89	0.98	0.99	0.99	0.99	0.99
ξ_1 (sell)	1.36	1.40	1.45	1.48	1.49	1.50	1.50
nontrading interval	0.70	0.51	0.47	0.49	0.50	0.51	0.51

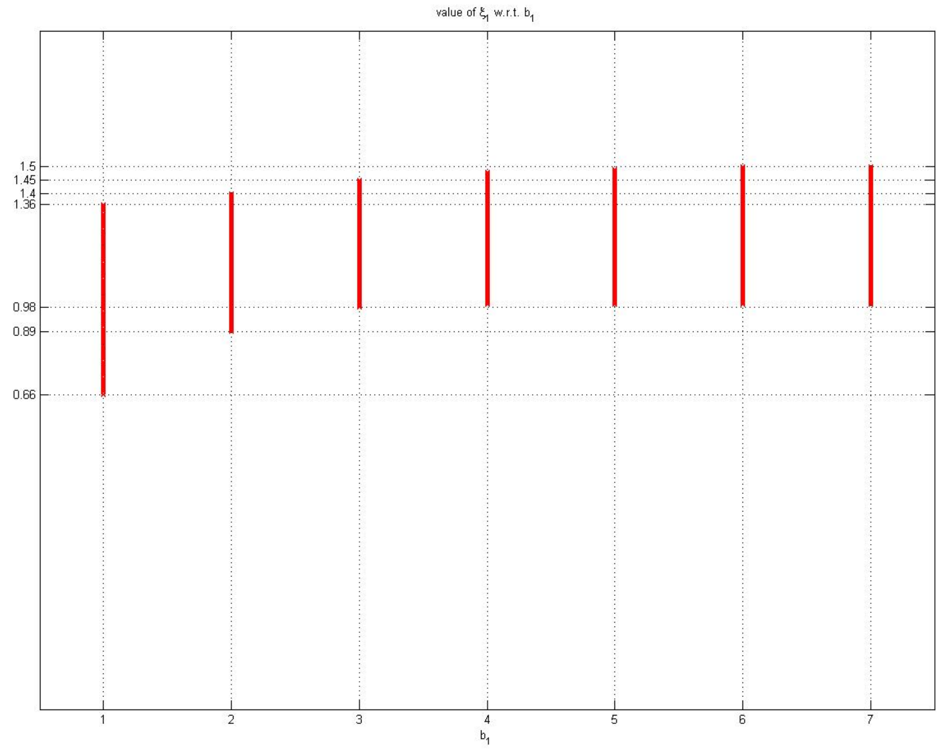

FIGURE 2.3. The relation between b_1 and ξ_1

From the table of the relation between b_1 and ξ_1, we observe that ξ_1 (buy) and ξ_1 (sell) increase slowly when b_1 becomes bigger. And from the Figure 2.3, the size of the "no trading" interval has a very small change even b_1 gets bigger.

In practice, the proportion of the transaction cost is usually 0.003, the appreciation return of the stock b and the volatility of the stock σ have values between 0.5 and 1.5. So, we give another data here.

We fix $S_0 = 1$, $\theta = 1$, $b_0 = 0.5$, $\lambda^{10} = 0.003$, $\sigma_0 = 0.5$, $\sigma_1 = 0.5$, and discuss the relation between b_1 and the "no trading" interval.

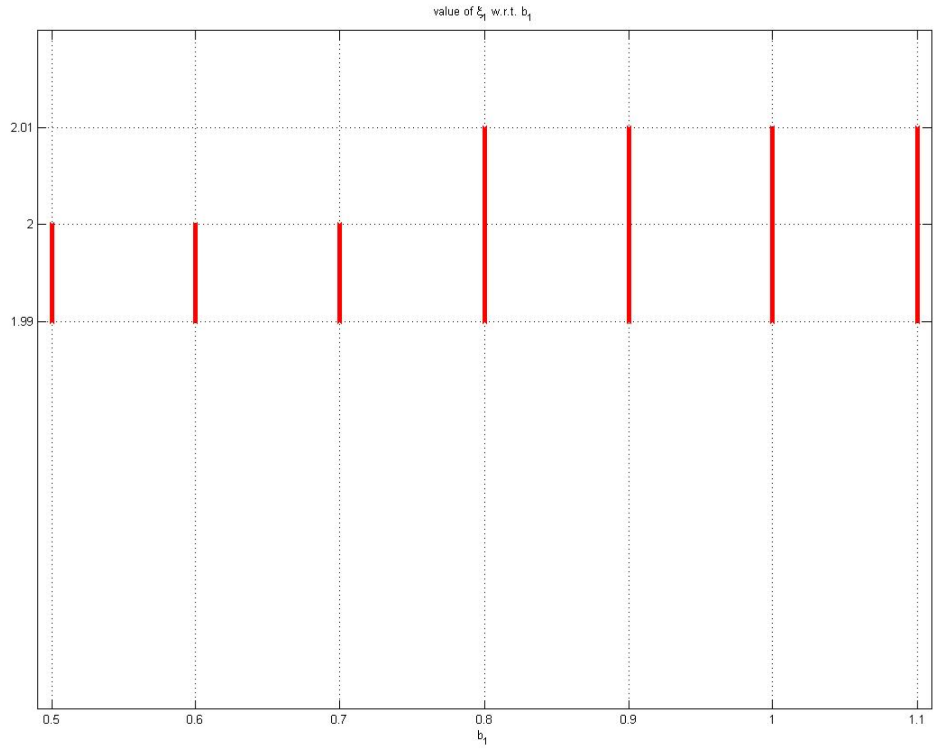

FIGURE 2.4. The relation between b_1 and ξ_1

b_1	0.5	0.6	0.7	0.8	0.9	1.0	1.1
ξ_1 (buy)	1.99	1.99	1.99	1.99	1.99	1.99	1.99
ξ_1 (sell)	2.00	2.00	2.00	2.01	2.01	2.01	2.01
nontrading interval	0.01	0.01	0.01	0.02	0.02	0.02	0.02

From the table of the relation between b_1 and ξ_1, we observe that ξ_1 (buy) and ξ_1 (sell) increase slowly when b_1 becomes bigger. From the Figure 2.4, the size of the "no trading" interval has a very small change even b_1 gets bigger.

2.5.3. The relation between σ_0 and nontrading interval. Third, we fix $S_0 = 0.05$, $\theta = 1$, $b_1 = 1$, $\lambda^{10} = 0.5$, $b_0 = 1$, $\sigma_1 = 1$, and discuss the relation between σ_0 and the "no trading" interval.

σ_0	1	2	3	4	5	6	7
ξ_1 (buy)	13.39	4.72	2.21	1.25	0.79	0.55	0.40
ξ_1 (sell)	27.31	6.86	3.24	1.86	1.19	0.83	0.61
nontrading interval	13.92	2.14	1.03	0.61	0.40	0.28	0.21

From the table of the relation between σ_0 and ξ_1, we observe that ξ_1 (buy) and ξ_1 (sell) decrease very quickly when σ_0 becomes bigger. And from the Figure 2.5, the size of the "no trading" interval decrease very rapidly when σ_0 gets bigger.

In practice, the proportion of the transaction cost is usually 0.003, the appreciation return of the stock b and the volatility of the stock σ have values between 0.5 and 1.5. So, we give another data here.

We fix $S_0 = 1$, $\theta = 1$, $b_1 = 0.5$, $\lambda^{10} = 0.003$, $b_0 = 0.5$, $\sigma_1 = 0.5$, and discuss the relation between σ_0 and the "no trading" interval.

σ_0	0.5	0.6	0.7	0.8	0.9	1.0	1.1
ξ_1 (buy)	1.994	1.386	1.019	0.780	0.617	0.499	0.413
ξ_1 (sell)	2.006	1.394	1.025	0.785	0.620	0.502	0.415
nontrading interval	0.012	0.008	0.006	0.005	0.003	0.003	0.002

From the table of the relation between σ_0 and ξ_1, we observe that ξ_1 (buy) and ξ_1 (sell) decrease very quickly when σ_0 becomes bigger. From the Figure 2.6, the size of the "no trading" interval decrease when σ_0 gets bigger.

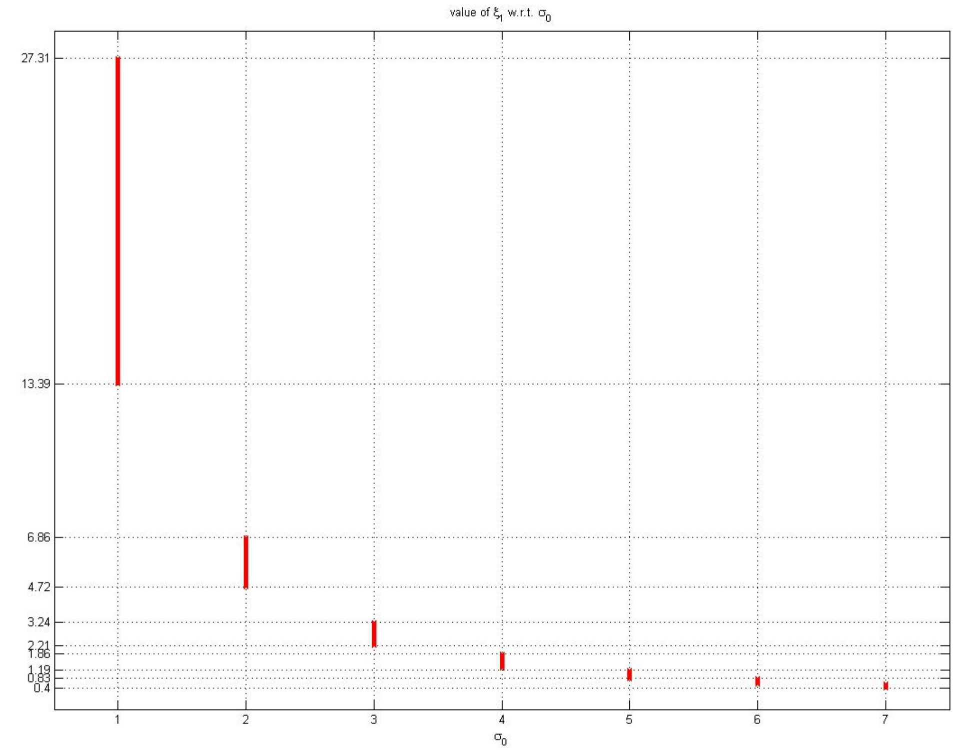

FIGURE 2.5. The relation between σ_0 and ξ_1

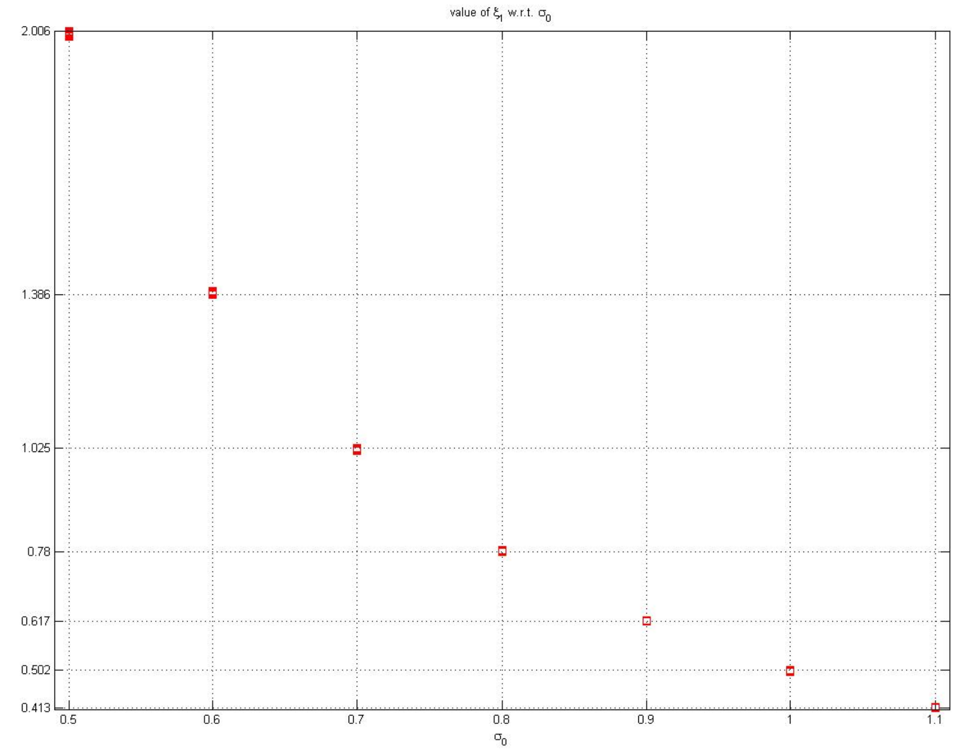

FIGURE 2.6. The relation between σ_0 and ξ_1

CHAPTER 3

Optimal Strategy with Transaction Cost in a Partial Information Discrete Model

In this chapter, we relax the restriction on the drift term b_n and we use the method introduced by Huang (2007).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

3.1. Model Setup

Let $(S_n)_{n\geq 0}$ be the stock prices in the market, where $S_0 \in \mathbb{R}^+$ is given. The stock price follows the recursive relation

$$S_{n+1} - S_n = S_n[b_n + \sigma_n(B_{n+1} - B_n)], \tag{3.1}$$

where b_n is the appreciation return of the stock, σ_n is volatility of the stock, and (B_n) is a noise.

Assumption 16.

(1) σ_n is assumed to be deterministic for all n and $(b_n)_{n\geq 0}$ is assumed as a sequence of random variables.

(2) $(B_{n+1} - B_n)$ is a Gaussian process with mean 0 and variance 1, and they are (totally) independent for all $n \geq 0$.

(3) The processes $(B_{n+1} - B_n)$ and (b_n) are independent.

At time n, we observe the stock prices up to time n, so we let $(\mathcal{G}_n)_{n\geq 0}$ be the natural filtration generated by $S_0, S_1, \cdots, S_n$.

Consider the stock return

$$X_{n+1} = \frac{S_{n+1} - S_n}{S_n},$$

then the model will be rewritten by

$$X_{n+1} = b_n + \sigma_n(B_{n+1} - B_n).$$

Also recall that we have $\mathcal{G}_n^* = \mathcal{G}_n$ as in Chapter 2, for $\mathcal{G}_n^*$ is the natural filtration generated by $X_0, X_1, \cdots, X_n$.

3.2. Optional Projection and The Gaussian Case

Denote by $\hat{b}$ the optional projection, with respect to the filtration $\mathcal{G}_n$, of the process b_n and we have

$$\hat{b}_n = E[b_n|\mathcal{G}_n].$$

Consider the $\mathcal{G}_n$-measurable process

$$\hat{L}_n = S_n - S_0 - \sum_{k=0}^{n-1} \hat{b}_k S_k. \tag{3.2}$$

From some computation we have $(\hat{L}_n)$ is a martingale with respect to $(\mathcal{G}_n)$.

Proposition 17 (Huang (2007), Proposition 28)**.** Let

$$D_n^2 = \sigma_n^2 + E[b_n - \hat{b}_n]^2$$

and assume that D_n is bounded away from 0 for $n \geq 0$. There exists a martingale process $(\hat{B}_n)_{n\geq 0}$ with respect to the filtration $(\mathcal{G}_n)$ such that

$$\hat{L}_n = \sum_{k=0}^{n-1} D_k S_k(\hat{B}_{k+1} - \hat{B}_k), \tag{3.3}$$

where $\hat{B}_{n+1} - \hat{B}_n$ has mean 0 and variance 1 for all n.

From (3.1),(3.2) and (3.3), our model is described by

$$S_{n+1} - S_n = S_n[b_n + \sigma_n(B_{n+1} - B_n)]$$
$$= S_n[\hat{b}_n + D_n(\hat{B}_{n+1} - \hat{B}_n)].$$

Thus,

$$D_n(\hat{B}_{n+1} - \hat{B}_n) = (b_n - \hat{b}_n) + \sigma_n(B_{n+1} - B_n).$$

Assumption 18. We assume that b_n is a Gaussian process and $\{b_n\}$ is independent of $\{B_{n+1} - B_n\}$.

Recall

$$X_{n+1} = \frac{S_{n+1} - S_n}{S_n} = b_n + \sigma_n(B_{n+1} - B_n)$$

and $\mathcal{G}_n = \mathcal{G}_n^*$ for all $n \geq 0$. Let

$$\mathcal{L}_n = \mathcal{L}_{X,n} = \{c_0X_0 + c_1X_1 + \cdots + c_nX_n \text{ where } c_j \in \mathbb{R} \text{ for } 0 \leq j \leq n\}.$$

Then we have the following two Lemma.

Lemma 19 (Huang (2007), Lemma 30)**.** Let $\check{b} = E[b_n|\mathcal{L}_n]$. Then there exists a sequence of coefficients $\check{c}_j$ for $0 \leq j \leq n$ such that

$$\check{b}_n = \check{c}_0X_0 + \check{c}_1X_1 + \cdots + \check{c}_nX_n.$$

Lemma 20 (Huang (2007), Lemma 31)**.** $\hat{b}_n = E[b_n|\mathcal{G}_n] = E[b_n|\mathcal{L}_n]$.

Recall

$$D_n(\hat{B}_{n+1} - \hat{B}_n) = (b_n - \hat{b}_n) + \sigma_n(B_{n+1} - B_n),$$

the following proposition is given.

Proposition 21 (Huang (2007), Proposition 32)**.** $(\hat{B}_{n+1} - \hat{B}_n)$ is a Gaussian process with mean 0 and variance 1 and they are independent with each other for $n \geq 0$.

From the Huang's work above, we can rewrite our model as follows:

$$S_{n+1} - S_n = S_n[\hat{b}_n + D_n(\hat{B}_{n+1} - \hat{B}_n)]$$

and recall our wealth process (after the trading)

$$V_n = \xi_{n+1} S_n + \eta_{n+1} = V_{n-1} + \xi_n(S_n - S_{n-1}) - \lambda^{10}\overline{L}_n^{10} \qquad \forall n \geq 0$$

with the initial wealth $x = \xi_0 S_0 + \eta_0 > 0$. We will show the risk-neutral and risk-averse utility result. Our decision rule is also the backward induction. We first give an example for the Gaussian assumption with the utility

$$U(x) = x,$$

and then give an example for the utility

$$U(x) = -\frac{1}{\theta}\exp(-\theta x),$$

where $\theta > 0$ is the absolute risk aversion and be a constant.

3.3. Examples for Risk-Neutral and Risk-Averse Utility Functions

Example 22. If the terminal time is 1, by the maximal expected utility we want to compute

$$E[U(V_0 + \xi_1(S_1 - S_0) - x)] = E[\xi_1(S_1 - S_0) - \lambda^{10}\overline{L}_0^{10}].$$

(1) If we buy some stocks at time 0 ($\xi_0 < \xi_1$) and ξ_1 is $\mathcal{G}_0$-measurable, then

$$\begin{aligned} E[\xi_1(S_1 - S_0) - \lambda^{10}\overline{L}_0^{10}] &= E[\xi_1(S_1 - S_0)] \\ &= \xi_1 S_0 E[\hat{b}_0 + D_0(\hat{B}_1 - \hat{B}_0)]. \end{aligned}$$

By Proposition 21, we get

$$\xi_1 S_0 E[\hat{b}_0 + D_0(\hat{B}_1 - \hat{B}_0)] = \xi_1 S_0 \hat{b}_0.$$

Hence,

$$E[U(V_0 + \xi_1(S_1 - S_0) - x)] = \xi_1 S_0 \hat{b}_0. \tag{3.4}$$

(2) If we sell some stocks at time 0 ($\xi_0 > \xi_1$), then

$$\begin{aligned} & E[\xi_1(S_1 - S_0) - \lambda^{10}\overline{L}_0^{10}] \\ = \ & E[\xi_1 S_0[\hat{b}_0 + D_0(\hat{B}_1 - \hat{B}_0)] - \lambda^{10} S_0(\xi_0 - \xi_1)] \\ = \ & -\lambda^{10} S_0(\xi_0 - \xi_1) + \xi_1 S_0 \hat{b}_0 + E[\xi_1 S_0 D_0(\hat{B}_1 - \hat{B}_0)]. \end{aligned}$$

By Proposition 21, we get

$$E[\xi_1(S_1 - S_0) - \lambda^{10}\overline{L}_0^{10}] = -\lambda^{10} S_0(\xi_0 - \xi_1) + \xi_1 S_0 \hat{b}_0$$

Hence,

$$E[U(V_0 + \xi_1(S_1 - S_0) - x)] = -\lambda^{10} S_0(\xi_0 - \xi_1) + \xi_1 S_0 \hat{b}_0 \leq \xi_1 S_0 \hat{b}_0, \tag{3.5}$$

since $\xi_0 > \xi_1$. From (3.4), (3.5), in the case without any restrictions on the trading strategy, the investors trade according the sign of $\hat{b}_0$. If $\hat{b}_0 \geq 0$, the investor should buy the stocks as much as possible. If $\hat{b}_0 + \lambda^{10} < 0$, the investors should make short-sell as much as possible. In the case with constraint, for example, $0 \leq \xi_1 \leq x/S_0$, no short-sell and no loan, the optimal strategy is given by

$$\xi_1^* = \begin{cases} x/S_0, & \text{if } \hat{b}_0 \geq 0, \\ 0, & \text{if } \hat{b}_0 + \lambda^{10} < 0 \end{cases}.$$

Example 23. If the terminal time is 1, by the maximal expected utility we have to compute

$$\begin{aligned}
&E[-\frac{1}{\theta}\exp(-\theta(V_0 + \xi_1(S_1 - S_0) - x))] \\
=&E[-\frac{1}{\theta}\exp(-\theta(-\lambda^{10}\overline{L}_0^{10} + \xi_1(S_1 - S_0)))] \\
=&E[-\frac{1}{\theta}\exp(\theta\lambda^{10}\overline{L}_0^{10} - \theta\xi_1(S_1 - S_0))] \\
=&-\frac{1}{\theta}E[\exp(\theta\lambda^{10}\overline{L}_0^{10} - \theta\xi_1 S_0[\hat{b}_0 + D_0(\hat{B}_1 - \hat{B}_0)])] \\
=&-\frac{1}{\theta}\exp(-\theta\xi_1 S_0\hat{b}_0)E[\exp(\theta\lambda^{10}\overline{L}_0^{10} - \theta\xi_1 S_0 D_0(\hat{B}_1 - \hat{B}_0))].
\end{aligned}$$

(1) If we buy stocks at time 0, then

$$\begin{aligned}
&-\frac{1}{\theta}\exp(-\theta\xi_1 S_0\hat{b}_0)E[\exp(\theta\lambda^{10}\overline{L}_0^{10} - \theta\xi_1 S_0 D_0(\hat{B}_1 - \hat{B}_0))] \\
= &-\frac{1}{\theta}\exp(-\theta\xi_1 S_0\hat{b}_0)E[\exp(-\theta\xi_1 S_0 D_0(\hat{B}_1 - \hat{B}_0))].
\end{aligned}$$

Since the Gaussian assumption, we get

$$-\frac{1}{\theta}\exp(-\theta\xi_1 S_0\hat{b}_0)E[\exp(-\theta\xi_1 S_0 D_0(\hat{B}_1 - \hat{B}_0))] = -\frac{1}{\theta}\exp(-\theta\xi_1 S_0\hat{b}_0 + \frac{1}{2}\theta^2\xi_1^2 S_0^2 D_0^2).$$

Hence,

$$E[-\frac{1}{\theta}\exp(-\theta(V_0 + \xi_1(S_1 - S_0) - x))] = -\frac{1}{\theta}\exp(-\theta\xi_1 S_0\hat{b}_0 + \frac{1}{2}\theta^2\xi_1^2 S_0^2 D_0^2).$$

Thus from the fundamental calculus, we maximize the conditional expectation to get the optimal ξ_1 and from $x = \xi_0 S_0 + \eta_0$ we have

$$\xi_1 = \frac{\hat{b}_0}{\theta S_0 D_0^2} > \xi_0 = \frac{x - \eta_0}{S_0}. \tag{3.6}$$

(2) If we sell stocks at time 0, then

$$\begin{aligned}&E[-\frac{1}{\theta}\exp(-\theta(V_0+\xi_1(S_1-S_0)-x))]\\ =\ & -\frac{1}{\theta}\exp(-\theta\xi_1 S_0\hat{b}_0)E[\exp(\theta\lambda^{10}S_0(\xi_0-\xi_1)-\theta\xi_1 S_0 D_0(\hat{B}_1-\hat{B}_0))].\end{aligned}$$

Since the Gaussian assumption, we get

$$E[\exp(\theta\lambda^{10}S_0(\xi_0-\xi_1)-\theta\xi_1 S_0 D_0(\hat{B}_1-\hat{B}_0))]=\exp(\theta\lambda^{10}S_0(\xi_0-\xi_1)+\frac{1}{2}\theta^2\xi_1^2 S_0^2 D_0^2).$$

Hence,

$$E[-\frac{1}{\theta}\exp(-\theta(V_0+\xi_1(S_1-S_0)-x))]=-\frac{1}{\theta}\exp(-\theta\xi_1 S_0\hat{b}_0+\theta\lambda^{10}S_0(\xi_0-\xi_1)+\frac{1}{2}\theta^2\xi_1^2 S_0^2 D_0^2).$$

Thus from the fundamental calculus, we maximize the conditional expectation to get the optimal ξ_1 and from $x=\xi_0 S_0+\eta_0$ we have

$$\xi_1=\frac{\hat{b}_0+\lambda^{10}}{\theta S_0 D_0^2}<\xi_0=\frac{x-\eta_0}{S_0}. \tag{3.7}$$

From (3.6), (3.7), we conclude that the investors will neither buy nor sell stocks when their initial wealth (amount by money) is in the interval $\left[\frac{\hat{b}_0}{\theta D_0^2}+\eta_0,\frac{\hat{b}_0+\lambda^{10}}{\theta D_0^2}+\eta_0\right]$. Moreover, if one wants to buy stocks he may choose the strategy $\xi_1=\frac{\hat{b}_0}{\theta S_0 D_0^2}$ at time 0 to reach the maximum profit, and if one wants to sell stocks he may choose the strategy $\xi_1=\frac{\hat{b}_0+\lambda^{10}}{\theta S_0 D_0^2}$ at time 0 to reach the maximum profit.

Remark 24. Now the "no trading" interval is

$$\left[\frac{\hat{b}_0}{\theta D_0^2}+\eta_0,\frac{\hat{b}_0+\lambda^{10}}{\theta D_0^2}+\eta_0\right].$$

And if one wants to buy (sell) stocks he may choose the strategy $\xi_1=\frac{\hat{b}_0}{\theta S_0 D_0^2}$ ($\xi_1=\frac{\hat{b}_0+\lambda^{10}}{\theta S_0 D_0^2}$) at time 0. We find some phenomenons from this example.

(1) For fixed $\hat{b}_0$ and D_0. If the initial stock price S_0 is too high, we will be conservative for our strategy in the stock.

(2) For fixed D_0 and S_0, if $\hat{b}_n$ is positive and grows up, we invest the number of shares of the stock more.

(3) For fixed $\hat{b}_0$ and S_0, if the volatility of the stock grows up, we are conservative when investing the stock.

(4) The length of the "no trading" interval is $\frac{\lambda^{10}}{\theta D_0^2}$, if λ^{10} increases or D_0^2 decreases, the "no trading" interval becomes larger.

Example 25. If the terminal time is 2, by backward induction we figure out this situation in two steps.

Step1 : Compute the one-period case

$$\begin{aligned}
& E[-\frac{1}{\theta}\exp(-\theta\{V_1+\xi_2(S_2-S_1)-V_0-\xi_1(S_1-S_0)\})|\mathcal{G}_1] \\
= \ & E[-\frac{1}{\theta}\exp(-\theta\{\xi_2(S_2-S_1)-\lambda^{10}\overline{L}_1^{10}\})|\mathcal{G}_1] \\
= \ & E[-\frac{1}{\theta}\exp(-\theta\{\xi_2 S_1[\hat{b}_1+D_1(\hat{B}_2-\hat{B}_1)]-\lambda^{10}\overline{L}_1^{10}\})|\mathcal{G}_1].
\end{aligned}$$

(1) If we sell stocks at time 1, then

$$\begin{aligned}
& E[-\frac{1}{\theta}\exp(-\theta\{\xi_2 S_1[\hat{b}_1+D_1(\hat{B}_2-\hat{B}_1)]-\lambda^{10}\overline{L}_1^{10}\})|\mathcal{G}_1] \\
= \ & E[-\frac{1}{\theta}\exp(-\theta\xi_2 S_1[\hat{b}_1+D_1(\hat{B}_2-\hat{B}_1)]+\theta\lambda^{10}(\xi_1-\xi_2)S_1)|\mathcal{G}_1].
\end{aligned}$$

Since $(\hat{B}_2-\hat{B}_1)$ is independent of $\mathcal{G}_1$ and the Gaussian assumption we have

$$\begin{aligned}
& E[-\frac{1}{\theta}\exp(-\theta\xi_2 S_1[\hat{b}_1+D_1(\hat{B}_2-\hat{B}_1)]+\theta\lambda^{10}(\xi_1-\xi_2)S_1)|\mathcal{G}_1] \\
= \ & -\frac{1}{\theta}\exp(-\theta\xi_2 S_1\hat{b}_1+\theta\lambda^{10}(\xi_1-\xi_2)S_1+\frac{1}{2}\theta^2\xi_2^2 S_1^2 D_1^2).
\end{aligned}$$

Hence,

$$\begin{aligned} & E[-\frac{1}{\theta}\exp(-\theta\{V_1+\xi_2(S_2-S_1)-V_0-\xi_1(S_1-S_0)\})|\mathcal{G}_1] \\ = & -\frac{1}{\theta}\exp(-\theta\xi_2 S_1\hat{b}_1+\theta\lambda^{10}(\xi_1-\xi_2)S_1+\frac{1}{2}\theta^2\xi_2^2 S_1^2 D_1^2). \end{aligned}$$

Thus from the fundamental calculus, we maximize the conditional expectation to get the optimal ξ_2 and from the assumption $\xi_2 < \xi_1$ we have

$$\xi_2 = \frac{\hat{b}_1+\lambda^{10}}{\theta S_1 D_1^2} < \xi_1.$$

(2) If we buy stocks at time 1, then using the similar argument in Example 1 we have

$$\xi_2 = \frac{\hat{b}_1}{\theta S_1 D_1^2} > \xi_1.$$

Step2 : Choose

$$\overline{\xi}_2 = \frac{\hat{b}_1+\lambda^{10}}{\theta S_1 D_1^2} I_{\{\xi_1 > \frac{\hat{b}_1+\lambda^{10}}{\theta S_1 D_1^2}\}} + \frac{\hat{b}_1}{\theta S_1 D_1^2} I_{\{\xi_1 < \frac{\hat{b}_1}{\theta S_1 D_1^2}\}} + \xi_1 I_{\{\frac{\hat{b}_1}{\theta S_1 D_1^2} \le \xi_1 \le \frac{\hat{b}_1+\lambda^{10}}{\theta S_1 D_1^2}\}}.$$

We replace the ξ_2 by $\overline{\xi}_2$ and compute

$$\begin{aligned} & E[-\frac{1}{\theta}\exp(-\theta\{V_1+\overline{\xi}_2(S_2-S_1)-x\})|\mathcal{G}_0] \\ = & E[-\frac{1}{\theta}\exp(-\theta\{V_0+\xi_1(S_1-S_0)-\lambda^{10}\overline{L}_1^{10}+\overline{\xi}_2(S_2-S_1)-x\})|\mathcal{G}_0] \\ = & E[-\frac{1}{\theta}\exp(-\theta\{-\lambda^{10}\overline{L}_0^{10}-\lambda^{10}\overline{L}_1^{10}+\xi_1(S_1-S_0)+\overline{\xi}_2(S_2-S_1)\})|\mathcal{G}_0]. \end{aligned}$$

(1) If we sell stocks at time 0, i.e., $\overline{L}_0^{10} = (\xi_0-\xi_1)S_0$, then we have

$$\begin{aligned} & E[-\frac{1}{\theta}\exp(-\theta\{-\lambda^{10}\overline{L}_0^{10}-\lambda^{10}\overline{L}_1^{10}+\xi_1(S_1-S_0)+\overline{\xi}_2(S_2-S_1)\})|\mathcal{G}_0] \\ = & E[-\frac{1}{\theta}\exp(-\theta\{-\lambda^{10}(\xi_0-\xi_1)S_0-\lambda^{10}\overline{L}_1^{10}(I_{\{\xi_1>\frac{\hat{b}_1+\lambda^{10}}{\theta S_1 D_1^2}\}}+I_{\{\xi_1<\frac{\hat{b}_1}{\theta S_1 D_1^2}\}}+I_{\{\frac{\hat{b}_1+\lambda^{10}}{\theta S_1 D_1^2}\ge\xi_1\ge\frac{\hat{b}_1}{\theta S_1 D_1^2}\}}) \\ & +\xi_1(S_1-S_0)+\overline{\xi}_2(S_2-S_1)\})|\mathcal{G}_0]. \end{aligned}$$

By some calculation, we get

$$
\begin{aligned}
& E[-\frac{1}{\theta}\exp(\theta\lambda^{10}(\xi_0-\xi_1)S_0-\theta\xi_1(S_1-S_0)+\{\theta\lambda^{10}(\xi_1-\frac{\hat{b}_1+\lambda^{10}}{\theta S_1 D_1^2})S_1 \\
& \quad -\theta\frac{\hat{b}_1+\lambda^{10}}{\theta S_1 D_1^2}(S_2-S_1)\}I_{\{S_1>\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1 D_1^2}\}}+\{-\theta\frac{\hat{b}_1}{\theta S_1 D_1^2}(S_2-S_1)\}I_{\{S_1<\frac{\hat{b}_1}{\theta\xi_1 D_1^2}\}} \\
& \quad +\{-\theta\xi_1(S_2-S_1)\}I_{\{\frac{\hat{b}_1}{\theta\xi_1 D_1^2}\le S_1\le\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1 D_1^2}\}})|\mathcal{G}_0] \\
= & E[-\frac{1}{\theta}\exp(\theta\lambda^{10}(\xi_0-\xi_1)S_0-\theta\xi_1(S_1-S_0) \\
& \quad +\{\theta\lambda^{10}\xi_1 S_1-\frac{(\hat{b}_1+\lambda^{10})^2}{D_1^2}-\frac{\hat{b}_1+\lambda^{10}}{D_1}(\hat{B}_2-\hat{B}_1)\}I_{\{S_1>\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1 D_1^2}\}} \\
& \quad +\{-\frac{\hat{b}_1^2}{D_1^2}-\frac{\hat{b}_1}{D_1}(\hat{B}_2-\hat{B}_1)\}I_{\{S_1<\frac{\hat{b}_1}{\theta\xi_1 D_1^2}\}} \\
& \quad +\{-\theta\xi_1 S_1\hat{b}_1-\theta\xi_1 S_1 D_1(\hat{B}_2-\hat{B}_1)\}I_{\{\frac{\hat{b}_1}{\theta\xi_1 D_1^2}\le S_1\le\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1 D_1^2}\}})|\mathcal{G}_0]
\end{aligned}
$$

Using the conditional expectation property, we have

$$
\begin{aligned}
& E[-\frac{1}{\theta}\exp(\theta\lambda^{10}(\xi_0-\xi_1)S_0-\theta\xi_1(S_1-S_0)+\{\theta\lambda^{10}\xi_1 S_1-\frac{(\hat{b}_1+\lambda^{10})^2}{D_1^2}\}I_{\{S_1>\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1 D_1^2}\}} \\
& \quad -\frac{\hat{b}_1^2}{D_1^2}I_{\{S_1<\frac{\hat{b}_1}{\theta\xi_1 D_1^2}\}}-\theta\xi_1 S_1\hat{b}_1 I_{\{\frac{\hat{b}_1}{\theta\xi_1 D_1^2}\le S_1\le\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1 D_1^2}\}})E[\exp(\{-\frac{\hat{b}_1+\lambda^{10}}{D_1}I_{\{S_1>\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1 D_1^2}\}} \\
& \quad -\frac{\hat{b}_1}{D_1}I_{\{S_1<\frac{\hat{b}_1}{\theta\xi_1 D_1^2}\}}-\theta\xi_1 S_1 D_1 I_{\{\frac{\hat{b}_1}{\theta\xi_1 D_1^2}\le S_1\le\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1 D_1^2}\}}\}(\hat{B}_2-\hat{B}_1))|\mathcal{G}_1]|\mathcal{G}_0].
\end{aligned}
$$

Since $(\hat{B}_2 - \hat{B}_1)$ is independent of $\mathcal{G}_1$ and the Gaussian assumption we get

$$
\begin{aligned}
& E[-\frac{1}{\theta}\exp(\{\theta\lambda^{10}(\xi_0-\xi_1)S_0-\theta\xi_1(S_1-S_0)+\theta\lambda^{10}\xi_1S_1-\frac{(\hat{b}_1+\lambda^{10})^2}{2D_1^2}\}I_{\{S_1>\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1D_1^2}\}} \\
& \quad +\{\theta\lambda^{10}(\xi_0-\xi_1)S_0-\theta\xi_1(S_1-S_0)-\frac{\hat{b}_1^2}{2D_1^2}\}I_{\{S_1<\frac{\hat{b}_1}{\theta\xi_1D_1^2}\}} \\
& \quad +\{\theta\lambda^{10}(\xi_0-\xi_1)S_0-\theta\xi_1(S_1-S_0)-\theta\xi_1S_1\hat{b}_1+\frac{1}{2}\theta^2\xi_1^2S_1^2D_1^2\}I_{\{\frac{\hat{b}_1}{\theta\xi_1D_1^2}\le S_1\le\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1D_1^2}\}})] \\
= & -\frac{1}{\theta}\{\int_{\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1D_1^2}}^{\infty}\exp(\theta\lambda^{10}(\xi_0-\xi_1)S_0-\theta\xi_1(x-S_0)+\theta\lambda^{10}\xi_1x-\frac{(\hat{b}_1+\lambda^{10})^2}{2D_1^2})f(x)dx \\
& \quad +\int_{-\infty}^{\frac{\hat{b}_1}{\theta\xi_1D_1^2}}\exp(\theta\lambda^{10}(\xi_0-\xi_1)S_0-\theta\xi_1(x-S_0)-\frac{\hat{b}_1^2}{2D_1^2})f(x)dx \\
& \quad +\int_{\frac{\hat{b}_1}{\theta\xi_1D_1^2}}^{\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1D_1^2}}\exp(\theta\lambda^{10}(\xi_0-\xi_1)S_0-\theta\xi_1(x-S_0)-\theta\xi_1\hat{b}_1x+\frac{1}{2}\theta^2\xi_1^2x^2D_1^2)f(x)dx\},
\end{aligned}
$$

where $f(x)$ is the probability density function of S_1, i.e.,

(3.8)
$$f(x)=\frac{1}{\sqrt{2\pi}S_0D_0}\exp(-\frac{(x-S_0-S_0\hat{b}_0)^2}{2S_0^2D_0^2}).$$

Due to the first order condition with respect to ξ_1, we have

(3.9)

$$
\begin{aligned}
& \int_{\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1D_1^2}}^{\infty}(-\theta\lambda^{10}S_0+\theta\lambda^{10}x-\theta(x-S_0))\exp(\theta\lambda^{10}(\xi_0-\xi_1)S_0+\theta\lambda^{10}\xi_1x \\
& \qquad -\frac{(\hat{b}_1+\lambda^{10})^2}{2D_1^2}-\theta\xi_1(x-S_0))f(x)dx \\
& +\int_{-\infty}^{\frac{\hat{b}_1}{\theta\xi_1D_1^2}}(-\theta\lambda^{10}S_0-\theta(x-S_0))\exp(-\frac{\hat{b}_1^2}{2D_1^2}+\theta\lambda^{10}(\xi_0-\xi_1)S_0-\theta\xi_1(x-S_0))f(x)dx \\
& +\int_{\frac{\hat{b}_1}{\theta\xi_1D_1^2}}^{\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1D_1^2}}(-\theta\hat{b}_1x+\theta^2x^2D_1^2\xi_1-\theta\lambda^{10}S_0-\theta(x-S_0))\exp(-\theta\xi_1\hat{b}_1x+\frac{1}{2}\theta^2\xi_1^2x^2D_1^2 \\
& \qquad +\theta\lambda^{10}(\xi_0-\xi_1)S_0-\theta\xi_1(x-S_0))f(x)dx=0,
\end{aligned}
$$

where $f(x)$ is given in (3.8).

(2) If we buy stocks at time 0, i.e., $\overline{L}_0^{10} = 0$, then we have

$$
\begin{aligned}
& E[-\frac{1}{\theta}\exp(-\theta\{V_1+\xi_2(S_2-S_1)-x\})|\mathcal{G}_0] \\
= \quad & E[-\frac{1}{\theta}\exp(-\theta\{-\lambda^{10}\overline{L}_1^{10}(I_{\{\xi_1>\frac{\hat{b}_1+\lambda^{10}}{\theta S_1 D_1^2}\}}+I_{\{\xi_1<\frac{\hat{b}_1}{\theta S_1 D_1^2}\}}+I_{\{\frac{\hat{b}_1+\lambda^{10}}{\theta S_1 D_1^2}\geq\xi_1\geq\frac{\hat{b}_1}{\theta S_1 D_1^2}\}}) \\
& \quad +\xi_1(S_1-S_0)+\overline{\xi}_2(S_2-S_1)\})|\mathcal{G}_0].
\end{aligned}
$$

By some calculation, we get

$$
\begin{aligned}
& E[-\frac{1}{\theta}\exp(-\theta\xi_1(S_1-S_0)+\{\theta\lambda^{10}(\xi_1-\frac{\hat{b}_1+\lambda^{10}}{\theta S_1 D_1^2})S_1-\theta\frac{\hat{b}_1+\lambda^{10}}{\theta S_1 D_1^2}(S_2-S_1)\}I_{\{S_1>\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1 D_1^2}\}} \\
& \quad +\{-\theta\frac{\hat{b}_1}{\theta S_1 D_1^2}(S_2-S_1)\}I_{\{S_1<\frac{\hat{b}_1}{\theta\xi_1 D_1^2}\}}+\{-\theta\xi_1(S_2-S_1)\}I_{\{\frac{\hat{b}_1}{\theta\xi_1 D_1^2}\leq S_1\leq\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1 D_1^2}\}})|\mathcal{G}_0].
\end{aligned}
$$

Using the conditional expectation property, we have

$$
\begin{aligned}
& E[-\frac{1}{\theta}\exp(-\theta\xi_1(S_1-S_0)+\{\theta\lambda^{10}\xi_1 S_1-\frac{(\hat{b}_1+\lambda^{10})^2}{D_1^2}\}I_{\{S_1>\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1 D_1^2}\}} \\
& \quad -\frac{\hat{b}_1^2}{D_1^2}I_{\{S_1<\frac{\hat{b}_1}{\theta\xi_1 D_1^2}\}}-\theta\xi_1 S_1\hat{b}_1 I_{\{\frac{\hat{b}_1}{\theta\xi_1 D_1^2}\leq S_1\leq\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1 D_1^2}\}})E[\exp(\{-\frac{\hat{b}_1+\lambda^{10}}{D_1}I_{\{S_1>\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1 D_1^2}\}} \\
& \quad -\frac{\hat{b}_1}{D_1}I_{\{S_1<\frac{\hat{b}_1}{\theta\xi_1 D_1^2}\}}-\theta\xi_1 S_1 D_1 I_{\{\frac{\hat{b}_1}{\theta\xi_1 D_1^2}\leq S_1\leq\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1 D_1^2}\}}\}(\hat{B}_2-\hat{B}_1))|\mathcal{G}_1]|\mathcal{G}_0].
\end{aligned}
$$

Since $(\hat{B}_2 - \hat{B}_1)$ is independent of $\mathcal{G}_1$ and the Gaussian assumption we get

$$E[-\frac{1}{\theta}\exp(\{-\theta\xi_1(S_1 - S_0) + \theta\lambda^{10}\xi_1 S_1 - \frac{(\hat{b}_1+\lambda^{10})^2}{2D_1^2}\}I_{\{S_1 > \frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1 D_1^2}\}}$$

$$+\{-\theta\xi_1(S_1 - S_0) - \frac{\hat{b}_1^2}{2D_1^2}\}I_{\{S_1 < \frac{\hat{b}_1}{\theta\xi_1 D_1^2}\}}$$

$$+\{-\theta\xi_1(S_1 - S_0) - \theta\xi_1 S_1\hat{b}_1 + \frac{1}{2}\theta^2\xi_1^2 S_1^2 D_1^2\}I_{\{\frac{\hat{b}_1}{\theta\xi_1 D_1^2} \le S_1 \le \frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1 D_1^2}\}})]$$

$$= \quad -\frac{1}{\theta}\{\int_{\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1 D_1^2}}^{\infty}\exp(-\theta\xi_1(x - S_0) + \theta\lambda^{10}\xi_1 x - \frac{(\hat{b}_1+\lambda^{10})^2}{2D_1^2})f(x)dx$$

$$+\int_{-\infty}^{\frac{\hat{b}_1}{\theta\xi_1 D_1^2}}\exp(-\theta\xi_1(x - S_0) - \frac{\hat{b}_1^2}{2D_1^2})f(x)dx$$

$$+\int_{\frac{\hat{b}_1}{\theta\xi_1 D_1^2}}^{\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1 D_1^2}}\exp(-\theta\xi_1(x - S_0) - \theta\xi_1\hat{b}_1 x + \frac{1}{2}\theta^2\xi_1^2 x^2 D_1^2)f(x)dx\},$$

where $f(x)$ is the probability density function of S_1, as in (3.8).

Due to the first order condition with respect to ξ_1, we have

(3.10)

$$\int_{\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1 D_1^2}}^{\infty}(\theta\lambda^{10}x - \theta(x - S_0))\exp(\theta\lambda^{10}\xi_1 x - \frac{(\hat{b}_1+\lambda^{10})^2}{2D_1^2} - \theta\xi_1(x - S_0))f(x)dx$$

$$+\int_{-\infty}^{\frac{\hat{b}_1}{\theta\xi_1 D_1^2}}(-\theta(x - S_0))\exp(-\frac{\hat{b}_1^2}{2D_1^2} - \theta\xi_1(x - S_0))f(x)dx$$

$$+\int_{\frac{\hat{b}_1}{\theta\xi_1 D_1^2}}^{\frac{\hat{b}_1+\lambda^{10}}{\theta\xi_1 D_1^2}}(-\theta\hat{b}_1 x + \theta^2 x^2 D_1^2\xi_1 - \theta(x - S_0))\exp(-\theta\xi_1\hat{b}_1 x + \frac{1}{2}\theta^2\xi_1^2 x^2 D_1^2 - \theta\xi_1(x - S_0))f(x)dx = 0,$$

where $f(x)$ is given in (3.8).

Conclusion : We get a similar result as in Example 14.

CHAPTER 4

Future Works

In above chapters, we consider the discrete model with the transaction costs, and we find that "no trading" intervals and the optimal strategies corresponding to selling assets and buying assets respectively. But many results are still open. We will illustrate as follows :

(a) First, if the investor is a "large" investor, i.e., the amount of the investment would influence the stock price, how can the investors reach their optimal strategies?

(b) Second, in the discrete model, if the drift term b_n is random and b_n is "not" independent of the noise term, can we find the "no trading" intervals and the optimal strategies corresponding to selling assets and buying assets respectively?

(c) In our model, the stock price is driven by the recursive relation

$$S_{n+1} - S_n = S_n[b_n + \sigma_n(B_{n+1} - B_n)],$$

so the stock price may be negative. However, it will become confused if we do not assume that S_n is positive for all n. Because we will have the conclusion that we sell the stocks at time 0 when $b_0 > \theta\sigma_0^2(x - \eta_0) - \lambda^{10}$ and buy the stocks at time 0 when $b_0 < \theta\sigma_0^2(x - \eta_0)$. So if we introduce another stock price model in which the stock price S_n is positive for all n, what strategy will be suggested? Simply, we can give the model setup as follows :

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let $(S_n)_{n\geq 0}$ be the stock prices in the market and is driven by

$$S_{n+1} = i_n S_n,$$

where i_n is a nonnegative random variable and $u_n := \ln i_n$ is normally distributed, and $S_0 \in \mathbb{R}^+$ is given. Hence similar problems will be discussed as follows :

(1) First, if our market is frictionless (no transaction cost), then for a given risk-neutral or risk-averse utility, what strategy will be find under the backward induction in this model?

(2) Second, if there is transaction cost in this model, What is the result?

Example 26. Here we give a one period model result for (1). If the terminal time is 1 and the utility function is $U(x) = \ln x$, by backward induction we have to compute

$$\begin{aligned} E[U(V_0 + \xi_1(S_1 - S_0))|\mathcal{G}_0] =& E[\ln\{x + \xi_1(S_1 - S_0)\}|\mathcal{G}_0] \\ =& E[\ln\{x + \xi_1 S_0(\exp(u_0) - 1)\}|\mathcal{G}_0]. \end{aligned}$$

Since $\mathcal{G}_0 = \{\phi, \Omega\}$ and assume that u_0 is normally distributed with mean 0 variance 1, then we get

$$\begin{aligned} & E[\ln\{x + \xi_1 S_0(\exp(u_0) - 1)\}|\mathcal{G}_0] \\ =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \ln\{x + \xi_1 S_0(\exp(y) - 1)\} \exp(-\frac{y^2}{2}) dy. \end{aligned}$$

Due to the first order condition with respect to ξ_1, we have

$$\int_{-\infty}^{\infty} \frac{S_0(\exp(y) - 1)}{x + \xi_1 S_0(\exp(y) - 1)} \exp(-\frac{y^2}{2}) dy = 0$$

By the change of variable, we get

$$\int_0^\infty \frac{S_0(\exp(y)-1)}{x\exp(y)-\xi_1 S_0(\exp(y)-1)}\exp(-\frac{y^2}{2})dy$$
$$=\int_0^\infty \frac{S_0(\exp(y)-1)}{x+\xi_1 S_0(\exp(y)-1)}\exp(-\frac{y^2}{2})dy.$$

Then we choose the optimal $\xi_1 = \frac{x}{2S_0}$.

This example gives a result for one period case in the frictionless market model. But how about the two period case or even for any finite terminal time N? And how is the result for the same problem in the model with transaction cost?

(d) If there are more than one risky assets in our model, what is the optimal trading strategy?

(e) Finally, extending our discrete model to continuous time model and solve the similar problems by using other mathematical tools.

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