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NOTE

A NOTE ON THE ASCENDING SUBGRAPH DECOMPOSITION PROBLEM

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Let G be a graph with $\binom{n+1}{2}$ edges. We say G has an ascending subgraph decomposition (ASD) if the edge set of G can be partitioned into n sets generating graphs $G_1, G_2, \ldots, G_n$ such that $|E(G_i)| = i$ (for $i = 1, 2, \ldots, n$) and G_i is isomorphic to a subgraph of G_{i+1} for $i = 1, 2, \ldots, n-1$.

In this note, we prove that if G is a graph of maximum degree $d \leq \lfloor (n+1)/2 \rfloor$ on $\binom{n+1}{2}$ edges, then G has an ASD. Moreover, we show that if $d \leq \lfloor (n-1)/2 \rfloor$, then G has an ASD with each member a matching. Subsequently, we also verify that every regular graph of degree a prime power has an ASD.

1. Introduction

In [1] the authors give the following decomposition conjecture.

Conjecture. Let G be a graph with $\binom{n+1}{2}$ edges. Then the edge set of G can be partitioned into n sets generating graphs $G_1, G_2, \ldots, G_n$ such that $|E(G_i)| = i$ (for $i = 1, 2, \ldots, n$) and G_i is isomorphic to a subgraph of G_{i+1} for $i = 1, 2, \ldots, n-1$.

A graph G that can be decomposed as described in the conjecture will be said to have an ascending subgraph decomposition (abbreviated ASD). The graphs $G_1, G_2, \ldots, G_n$ are said to be members of such a decomposition.

In [1, 2], the conjecture has been verified for star forests. Also, in [2] it is proved that if G is a graph of maximum degree d on $\binom{n+1}{2}$ edges and $n \geq 4d^2 + 6d + 3$, then G has an ASD with each member a matching.

In this note, we prove that if G is a graph of maximum degree $d \leq \lfloor (n+1)/2 \rfloor$ on $\binom{n+1}{2}$ edges, then G has an ASD. Moreover, we show that if $d \leq \lfloor (n-1)/2 \rfloor$, then G has an ASD with each member a matching. As a special case we also verify that every regular graph of degree a prime power has an ASD.

2. Main results

Let N be the set $\{1, 2, \ldots, n\}$, and $A_1, A_2, \ldots, A_k$ be mutually disjoint subsets of N such that $\bigcup_{i=1}^{k} A_i = N$. Let $s(A_i)$ be the sum of all elements in

$A_i(s(\phi)=0)$. We will say that N can be decomposed into subsets of type $\langle s_1, s_2, \ldots, s_k\rangle$ if there exists a collection of mutually disjoint subsets of N, $A_1, A_2, \ldots, A_k$, such that their union is N and $s(A_i)=s_i$, $i=1,2,\ldots,k$. Obviously, $\sum_{i=1}^k s_i = \binom{n+1}{2}$. For example $\{1,2,\ldots,6\}$ can be decomposed into subsets of type $\langle 3,5,6,7\rangle$. ($A_1=\{3\}$, $A_2=\{1,4\}$, $A_3=\{6\}$, $A_4=\{2,5\}$.)

An edge-coloring of a graph is an assignment of colors to its edges so that no two incident edges have the same color. If a graph G has an edge-coloring with k colors, then G is called k-colorable. (Let δ_i denote the number of edges in G which are colored c_i, $i=1,2,\ldots,k$.) After a bit of reflection, we have the following proposition. (Unless stated otherwise, we assume that G has $\binom{n+1}{2}$ edges and that the number of edges that are colored c_i is δ_i.)

Proposition 1. *Let G be a k-colorable graph. If N can be decomposed into subsets of type $\langle\delta_1, \delta_2, \ldots, \delta_k\rangle$, then G has an* ASD *with each member a matching.*

Proof. Since N can be decomposed into subsets of type $\langle\delta_1, \delta_2, \ldots, \delta_k\rangle$, it follows that $s(A_i)=\delta_i$, $i=1,2,\ldots,k$. We can choose G_i as the collection of i edges that are colored c_j if $i\in A_j$. □

We call an edge-coloring equalized if $|\delta_i-\delta_j|\leq 1$ $(1\leq i<j\leq k)$. McDiarmid [3] and de Werra [5] independently proved that if a graph has an edge-coloring with k colors then it has an equalized edge-coloring with k colors. We can easily prove the following result by using the above fact.

Proposition 2. *Let G be a graph with maximum degree $d\leq\lfloor(n-1)/2\rfloor$, then G has an* ASD *with each member a matching.*

Proof. By Vizing's Theorem [4] G has edge chromatic number $\chi'(G)$ at most $\lfloor(n-1)/2\rfloor+1$. Hence we can color G with $n/2$ or $(n+1)/2$ colors depending on whether n is even or odd. By the theorem of McDiarmid and de Werra, we obtain an equalized edge-coloring with $n/2$ or $(n+1)/2$ colors as the case may be. If n is even, then each color occurs $n+1$ times. Since, N can be decomposed into subsets of type $\langle n+1, n+1, \ldots, n+1\rangle$ ($n/2$-tuple), we conclude that G has an ASD with each member a matching by Proposition 1. Similarly, if n is odd, then each color occurs n times. Since N can be decomposed into subsets of type $\langle n, n, \ldots, n\rangle$ ($(n+1)/2$-tuple), we have the proof. □

As a matter of fact, if G is of class one, i.e. $\chi'(G)=d$, then we can let $d\leq\lfloor(n+1)/2\rfloor$ in Proposition 2. Actually, if we simply want to prove that G has an ASD, we can improve the upper bound of d a bit.

Proposition 3. *Let G be a graph with maximum degree $d\leq\lfloor(n+1)/2\rfloor$, then G has an* ASD.

Proof. From Proposition 2, the only cases left are $d = n/2$ (n is even) and $d = (n+1)/2$ (n is odd). If n is even, then G is $(n/2+1)$-colorable. Since we have an equalized edge-coloring, hence we can color the edges by the way: $n/2$ colors occur $n-1$ times and one color occurs n times. Since N can be decomposed into subsets of type $\langle n-1, n-1, \ldots, n-1, n\rangle$ ($(n/2+1)$-tuple), we are done. For the case when n is odd, G is $((n+1)/2+1)$-colorable. Similarly, we can color the edges in the following way: $(n-3)/2$ colors occur $(n-2)$ times and 3 colors occur $(n-1)$ times. Without loss of generality, we let those three colors which occur $(n-1)$ times be c_1, c_2, and c_3. It is not difficult to see $\{1, 2, \ldots, n-3\}$ can be decomposed into subsets of type $\langle n-2, n-2, \ldots, n-2\rangle$ ($(n-3)/2$-tuple), therefore we can choose $G_1, G_2, \ldots, G_{n-3}$ subsequently. We conclude the proof by letting G_{n-2} be the collection of edges colored c_1 except for one edge e, G_{n-1} be the collection of edges colored c_2, and G_n be the collection of those edges colored c_3 and e. □

From Proposition 3, it is easy to see every regular graph of degree a prime power has an ASD.

Proposition 4. *Every regular graph of degree a prime power has an* ASD.

Proof. Let the degree and order of G be d and v respectively. Then $d \cdot v = n \cdot (n+1)$. Hence we have $d \mid n(n+1)$. Since d is a prime power and the common divisor of n and $n+1$ is 1, $d \mid n$ or $d \mid n+1$. If $d < n$, then $d \leq (n+1)/2$. By Proposition 3, G has an ASD. If $d = n$, then $G = K_{n+1}$. The theorem follows from the fact that K_{n+1} has an ASD. □

As we have seen above, if the maximum degree of the graph is not too large, it has an ASD. In what follow we suggest a slightly different approach to the problem.

A vertex covering in a graph is any set of vertices such that each edge of the graph has at least one of its end vertices in the set. We will say $\langle \beta_1, \beta_2, \ldots, \beta_k\rangle$ is a covering pattern for a graph G, if we can find a vertex covering $\{v_1, v_2, \ldots, v_k\}$ such that there are β_i edges incident with the vertex v_i, $i = 1, 2, \ldots, k$ and each edge can be counted only once. For example, *Fig. 1* has a covering pattern $\langle 5, 4, 3, 3\rangle$.

Since the following proposition is easy to see, it will be stated without proof.

Proposition 5. *Let G be a graph with a covering pattern $\langle \beta_1, \beta_2, \ldots, \beta_k\rangle$. If N can be decomposed into subsets of type $\langle \beta_1, \beta_2, \ldots, \beta_k\rangle$, then G has an* ASD *with each member a star.*

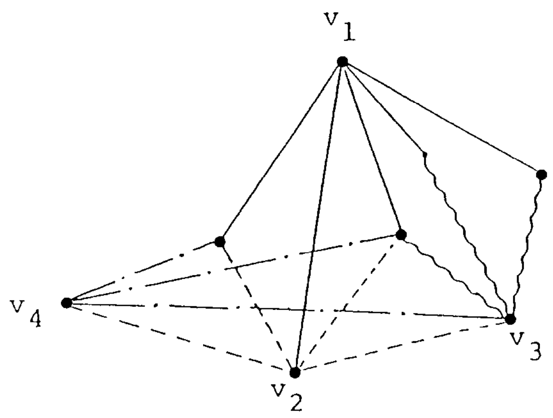

Fig. 1.

The following proposition is also easy to prove, we simply state it.

Proposition 6. *If a graph can be partitioned into edge disjoint paths of length* $r_1, r_2, \ldots, r_k$ *respectively, and the set N can be decomposed into subsets of type* $\langle r_1, r_2, \ldots, r_k \rangle$, *then G has an* ASD *with each member a path.*

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