國立交通大學應用數學系
博 士 論 文

矩陣指數跳躍擴散的

最佳停止問題

Optimal stopping problems for matrix-exponential

jump-diffusion processes

研究生：蔡明耀

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摘要

在本論文之中， 我們在矩陣指數型跳躍擴散隨機過程以及一般報酬函數之下，考慮最佳停止問題。任給一個報酬函數，遵循平均問題的方法， (請參照 Alili and Kyprianou [1], Kyprianou and Surya [16], Novikov and Shiryaev [23], and Surya [28])， 對於所對應之平均問題的解，我們提供了明確的公式。 透過此明確的公式， 對於美式最佳停止問題，我們得到最佳的執行邊界和最佳的報酬。

此外，在矩陣指數型跳躍擴散隨機過程之下，我們也考慮永久美式複合選擇權定價問題。遵循 Gapeev and Rodosthenous [12]，對於跳躍擴散的隨機過程而言，原先兩步的最佳停止問題可分解成，單步最佳停止問題的序列。在雙重指數型跳躍擴散模型之下，對於永久美式複合選擇權而言，我們得到了明確的定價公式。 藉由我們的方法，我們也涵蓋了 Gapeev and Rodosthenous [12] 的結果。

Optimal stopping problems for matrix-exponential jump-diffusion processes

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Abstract

In this dissertation, we consider the optimal stopping problems for a general class of reward functions under the matrix-exponential jump-diffusion processes. Given the reward function in this class, following the averaging problem approach(see, for example, Alili and Kyprianou [1], Kyprianou and Surya [16], Novikov and Shiryaev [22], and Surya [27]), we give an explicit formula for solutions of the corresponding averaging problem. Based on this explicit formula, we obtain the optimal boundary and the value function for the American optimal stopping problems. Also, we consider the pricing problems of perpetual American compound options under the matrix-exponential jump-diffusion processes. Following Gapeev and Rodosthenous [12], the initial two-step optimal stopping problems are decomposed into sequences of one-step problems for the underlying jump-diffusion process. In the double-exponential jump-diffusion model, we obtain the explicit pricing formula for the perpetual American compound option pricing problems. By our approach, we also recover results obtained in Gapeev and Rodosthenous [12]

誌　　謝

時間總是在不知不覺中飛逝，在鍵盤上敲下誌謝的同時，博士班的學業即將畫下句點，我也將展開人生另一段旅程。在充滿不確定的未來之中唯一能確定最令我懷念的，是單純地探索未知世界的那段時光。然而，我很慶幸有許許多多的貴人相助，使得我能堅持到底，達成了自己的目標。

首先想謝謝我的家人，謝謝我的爸爸、媽媽，養育、教育我；謝謝我的哥哥，在我生活各方面充分的支持我；同時也謝謝我姐姐以及她兩個可愛的女兒，讓我的博士班生活增添了許多樂趣。接著要感謝我的女朋友 Sora，謝謝妳的支持，使我能持續專注在研究工作上，感謝妳陪我走過人生中的困境，每當感到無助時，因為妳的鼓勵，讓我增添了許多的信心。接著要感謝我的指導老師許元春教授，在我進行博士班研究的期間，不管是在研究或生活上都給予我許多指導與充分的支援。透過和許老師一次次的深入討論，逐步調整研究方向，今日才有這本論文的誕生。此外，也感謝黃子偉教授，謝謝黃老師在我懵懂無知時，帶著我，指引我許多人生未來的方向，也因為您替我打下的數學基礎，我才有機會探索機率的世界，感謝您所做的一切。

Contents

1 Introduction

The optimal stopping problems we consider in this thesis will be of the form

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x(e^{-r\tau} g(X_\tau)) \tag{1.1}$$

where $X = \{X_t : t \geq 0\}$ under $\mathbb{P}_x$ is a Lévy process started from $X_0 = x$. Further, g is a measurable function, $r \geq 0$ and $\mathcal{T}$ is a family of stopping times with respect to the natural filtration generated by X, $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$. Here we assume that $\mathbb{E}_x(\sup_{0\leq t\leq\infty} e^{-rt} g(X_t)) < \infty$ for all x. The optimal stopping problem consists of finding the optimal stopping time τ^* such that

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x(e^{-r\tau} g(X_\tau)) = \mathbb{E}_x(e^{-r\tau^*} g(X_{\tau^*})).$$

Also we need to find the corresponding optimal reward (the value function)

$$V(x) = \mathbb{E}_x(e^{-r\tau^*} g(X_{\tau^*})).$$

In finance literature, we shall associate risk-free r, Markov stopping time τ, the payoff function g, and the log return of asset price X with the random variable

$$e^{-r\tau} g(X_\tau) = e^{-r\tau} g(X_{\tau(\omega)}(\omega)),$$

which can be interpreted as the discounted gain of the holder of the American option which is exercised at time τ. Then it will be natural to interpret the value $\mathbb{E}_x(e^{-r\tau} g(X_\tau))$ as the expected discounted payoff corresponding to a chosen time τ and an initial state x. Assume the buyer knows that all expected discounted payoff, then the value $V(x)$ is naturally seen as the rational price of the perpetual American option and the optimal stopping time τ^* is the optimal exercise strategy for the buyer. In credit risk modelling, for Leland's optimal capital structure model, we have to choose the optimal bankruptcy level so as to maximize the value of equality of the firm. Therefore, the optimal stopping boundary can be seen as the optimal bankruptcy level of the defaultable firm.

If the reward function g is nonnegative and continuous, then the general theory of optimal stopping rules for Markov processes say that the value function $V(x)$ is the smallest r-excessive majorant of $g(x)$, i.e., the smallest function $V(x)$ such that $V(x) \geq g(x)$ and $V(x) \geq e^{-rt}\mathbb{E}_x(V(X_t))$ for all $x \in \mathbb{R}$ and $t \geq 0$, and $\lim_{t\downarrow 0} e^{-rt}\mathbb{E}_x(V(X_t)) = V(x)$. (See, for example, Shiryayev [26] Lemma 3 p.118 and Theorem 1 p.124.) Let $C = \{x \in \mathbb{R}, V(x) > g(x)\}$ (the continuation region) and $D = \{x \in \mathbb{R}, V(x) = g(x)\}$ (the stopping region). If $\tau_D = \inf\{t > 0 : X_t \in D\} < \infty$ a.s., then τ_D is the optimal stopping time and $V(x) = \mathbb{E}_x(e^{-r\tau_D} g(X_{\tau_D}))$. Also, if there exists an optimal stopping time τ^* for the problem (1.1), then $\tau^* \geq \tau_D$ and $V(x) = \mathbb{E}_x(e^{-r\tau^*} g(X_{\tau^*})) = \mathbb{E}_x(e^{-r\tau_D} g(X_{\tau_D}))$. Hence finding the solution of the optimal stopping problem is reduce to finding the stopping region D and the function $\mathbb{E}_x(e^{-r\tau_D} g(X_{\tau_D}))$.

For diffusion processes, many authors obtained that the boundary of the stopping region D is determined by using the smooth pasting condition and solving the optimal stopping problem is reduced to solving a corresponding Stefan's free boundary problem. For details, see Section 3.8 in Shiryayev [26]. Unfortunately, once the problem (1.1) is driven by Lévy processes, the smooth pasting condition may break down. For example, Aili and Kyprianou [1] considered the problem of pricing the American put option under a Lévy process. They showed that the smooth pasting conditions hold if and only if 0 is regular for $(-\infty, 0)$ for the Lévy process X_t. Similar results were also obtained by Kyprianou and Surya [16] for the integer power function g. Recently, inspired by the works of Boyarchenko and Levendorskiǐ [7], Surya [27] proposed an averaging problem approach for solving the optimal stopping problem (1.1) in a general setting. The averaging problem approach does not appeal to a free boundary problem associated to the optimal stopping problem. Given a reward function g, in terms of solutions for the average problem, we have a fluctuation identity for overshoots for a Lévy process. With this identity, Surya [27] showed by martingale arguments that an optimal solution can be founded if the solutions to the averaging problem that has certain monotonicity properties. Use these approaches, he was able to recover many known results in literature(see, for example, [1], [16], [18] and [22]).

In this thesis, we consider a matrix-exponential jump-diffusion process X of the form given in (3.2). Under this model assumption, we give the explicit formulas in (4.23) and (4.66) for solutions of the averaging problem for a general class of reward function. (Our result depends on the recent work of Lewis and Mordecki in [17].) Moreover, we derive sufficient criterions on the reward function and the explicit solutions of corresponding averageing problem for the existence of optimal boundary. In particular, we can evaluate the optimal boundaries for the boundary value problem (1.1) by solving an equation and then obtain an explicit formula for the value functions. (For details, see Theorem 4.7 and 4.12.) To illustrate our results, we resolve the optimal stopping problem (1.1) in Section 5 for some specific reward functions. Our results are consistent with that of Kyprianou and Surya [16], Mordecki [18], Boyarchenko and Levendorskii [7], and many others. Also it is worth noting that, under our model assumption, our examples in Section 5 show that the sufficient conditions for optimality in Theorem 4.7 are easier to verify than that in the literature.

In addition, we also consider the pricing problem of perpetual American compound options under the matrix-exponential jump-diffusion processes of the form given in (3.2). A compound option is a standard option with another standard option being the underlying asset. There are four basic types of compound options: a call on a call, a put on a call, a call on a put, and a put on a put. Consider, for example, a call on a put of European type. On the first exercise date T1, the holder of the compound option is entitled to pay the first strike price, K1, and receive a put option. The put option gives the holder the right to sell the underlying asset for the second strike price, L2, on the second exercise date, T2. The compound option will be exercised on the first exercise date only the value of the option on that date is greater than the first strike price. In the Black-Scholes framework, European-style compound options can be valued analytically in terms of integrals of the bivariate normal distribution(see, for example, Geske [13]). For more general underlying dynamics, either explicit solutions do not exist or the integrals become difficult to evaluate. On the other hand, in the literature, many researchers considered the compound options of American type. In Chiarella and Kang [9], the authors formulated the pricing problem for American compound options as the solution to a two-step free boundary problem which is solved numerically via a sparse grid approach. In [12], Gapeev and Rodosthenous considered the pricing problem of perpetual American compound options when the underlying dynamics follow the geometric Brownian motion. By solving the associated sequence of one-sided free boundary problems and the martingale verification, they obtained explicit pricing formulas for all four types of perpetual American compound options. Following Gapeev and Rodosthenous [12], the initial two-step optimal stopping problems are decomposed into sequences of one-step problems for the underlying jump-diffusion process. In the double-exponential jump-diffusion model, we give explicit solutions for the associated optimal stopping problems and, hence obtain the explicit pricing formula for the perpetual American option pricing problems. By our approach, we also recover results obtained in Gapeev and Rodosthenous [12].

Finally, Mordecki [20] showed by the representation theory of excessive functions that if a Radon measure can be found such that the corresponding excessive function has very intuitive properties then the excessive function coincides with the value function of the optimal stopping problem. In section 9, we establish connections between the approaches of Surya [27] and Mordecki [20].

The outline of the thesis is as follows. In Section 2 we recall some main results of Surya [27] and [17]. In Section 3, we introduce the setting of matrix-exponential Lévy processes. When the reward function g is in a special class of functions, denoted by π_0 (resp., π_1), we present in Section 4 an explicit formula Q_g (resp., P_g) for the solutions of the corresponding averaging problem and obtain sufficient conditions on g and Q_g (resp., P_g) for the optimality. The special class of reward functions contains many known examples in literature. In particular we verify that if the reward function g is a sufficiently regular function, Q_g is consistent with that in Surya [27]. Section 5 shows some examples to illustrate our results. In Section 6, we consider the perpetual American compound options pricing problem under matrix-exponential jump-diffusion processes and acquire the sufficient conditions for the existence of optimal boundary of perpetual American compound options. Section 7 gives some numerical results. In Section 8, we find the optimal boundaries and explicit pricing formula for the perpetual American compound option pricing problem under the double-exponential jump-diffusion processes. In Section 9, we establish connections between the approaches of [27] and [20].

2 Preliminaries

We first quote the accounts on Markov process in [23]. For more details, refer to [23].

2.1 Markov processes

The modern definition of a time-homogeneous Markov process emphasizes both trajectories and the transition functions as well as their connection. To be more precise, we shall assume that the following objects are given:

(A) a state space $(E, \mathcal{E})$;

(B) a family of probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}; \mathbb{P}_x, x \in E)$ where each $\mathbb{P}_x$ is a probability measure on $(\Omega, \mathcal{F})$;

(C) a stochastic process $X = (X_t)_{t\geq 0}$ where each X_t is $\mathcal{F}_t/\mathcal{E}$-measurable.

Assume that the following conditions are satisfied:

(a) the function $P(t, x; B) = \mathbb{P}_x(X_t \in B)$ is $\mathcal{E}$-measurable in x;

(b) $P(0, x; E\backslash\{x\}) = 0, x \in E$;

(c) for all $s, t \geq 0$ and $B \in \mathcal{E}$, the following property is valid:

$$\mathbb{P}_x(X_{t+s} \in B|\mathcal{F}_t) = P(s, X_t; B) \quad \mathbb{P} - \text{a.s.};$$

(d) the space Ω is rich enough in the sense that for any $\omega \in \Omega$ and $h > 0$ there exists $\omega' \in \Omega$ such that $X_{t+h}(\omega) = X_t(\omega')$ for all $t \geq 0$.

Under the above assumptions the process $X = (X_t)_{t\geq 0}$ is said to be a (time homogeneous) Markov process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}; \mathbb{P}_x, x \in E)$ and the function $P(t, x; B)$ is called a transition function of this process. It is worth noting that the conditions (a) and (c) imply that $\mathbb{P}_x$-a.s.

$$\mathbb{P}_x(X_{t+s} \in B|\mathcal{F}_t) = \mathbb{P}_{X_t}(X_s \in B), \quad x \in E, \quad B \in \mathcal{E}.$$

The property indicates that Markov principle of the "future" is independent of the "past" for the fixed "present". Also, it is called the Markov property of a process $X = (X_t)_{t\geq 0}$ fulfilling the condition (a)-(d).

Also, in general theory of Markov processes an important role is played by those processes which, in addition to Markov property, have the following strong Markov property: for any Markov time $\tau = \tau(\omega)$ (w.r.t. $(\mathcal{F}_t)_{t\geq 0}$)

$$\mathbb{P}_x(X_{\tau+s} \in B|\mathcal{F}_\tau) = P(s, X_\tau; B) \quad (\mathbb{P}_x - \text{a.s. on}\{\tau < \infty\});$$

where $\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}$ is a σ-algebra of events observed on the time interval $[0, \tau]$.

Remark 2.1 *For $X_{\tau(\omega)}(\omega)$ to be $\mathcal{F}_\tau/\mathcal{E}$-measurable we have to impose an additional restriction-that of measurability-on the process X. For instance, it is sufficient to assume that for every $t \geq 0$ the function $X_s(\omega)$, $s \leq t$, defines a measurable mapping from $([0,t] \times \Omega, \mathcal{B}([0,t] \times \mathcal{F}_t)$ into the measurable space $(E, \mathcal{E})$.*

2.2 Lévy processes

We recall below the definition of a Lévy process.

Definition 2.2 *(Lévy Process) A process $X = \{X_t : t \geq 0\}$ is said to be a Lévy process if it possesses the following properties:*

(i) The paths of X are almost surely right continuous with left limits.

(ii) $X_0 = 0$ almost surely.

(iii) For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}.

(iv) For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_u : u \leq s\}$.

A Lévy process starts from $X_0 = x$ is simply defined as $x + X_t$ for $t \geq 0$.

Note that the properties of stationary and independent increments implies that a Lévy process is a Markov process. Thanks to almost sure right continuity of paths, one may show in addition that Lévy processes are also Strong Markov processes.

Henceforth we keep to the standard notation that $(X, \mathbb{P}_x)$ is a Lévy process issued from $x \in \mathbb{R}$. For convenience we shall write $\mathbb{P}$ in place of $\mathbb{P}_0$. The Levy-Khinchine formula states that

$$\mathbb{E}(e^{\mathrm{i}uX_t}) = e^{t\psi(u)}$$

where the characteristic exponent of X is

$$\psi(u) = \mathrm{i}au - \frac{1}{2}b^2u^2 + \int_{\mathbb{R}} (e^{\mathrm{i}ux} - 1 - \mathrm{i}ux \cdot 1_{\{|x|<1\}})\Pi(dx).$$

Here $a \in \mathbb{R}$, $b \geq 0$ and Π is a measure on $\mathbb{R}\backslash\{0\}$ such that $\int_{\mathbb{R}} 1 \wedge x^2 \Pi(dx) < \infty$.

Throughout this thesis, we denote by e_r an exponential random variable with parameter $r > 0$, independent of the process X. We shall work with the convention that when $r = 0$, the variable e_r is understood to be equal to ∞ with probability 1. In addition, we denote by

$$M_r = \sup_{0 \leq s \leq \mathrm{e}_r} X_s \quad \text{and} \quad I_r = \inf_{0 \leq s \leq \mathrm{e}_r} X_s,$$

the supremum and the infimum of the Lévy process X killed at the independent exponential random time e_r. It is well-known that

$$X_{\mathrm{e}_r} - I_r \quad \text{and} \quad I_r \quad \text{are independent,}$$

and

$$X_{e_r} - I_r \quad \text{has the same distribution as} \quad M_r.$$

From these, we obtain the following Wiener-Hopf factorization formula,

$$\mathbb{E}e^{iuX_{e_r}} = \frac{r}{r - \psi(u)} = \psi_r^+(u)\psi_r^-(u),$$

where the two factors $\psi_r^+(u)$ and $\psi_r^-(u)$ are defined, respectively by

$$\psi_r^+(u) = \mathbb{E}(e^{iu(X_{e_r} - I_r)}) = \mathbb{E}(e^{iuM_r})$$

and

$$\psi_r^-(u) = \mathbb{E}(e^{iuI_r}).$$

Next, we introduce the family of linear operators $G_r(r > 0)$ associated with the Lévy process, called the resolvent operators. They are given for every measurable function $f \geq 0$ by

$$G_r f(x) = \int_0^\infty e^{-rt} P_t f(x) dt = \mathbb{E}_x\Big(\int_0^\infty e^{-rt} f(X_t) dt\Big).$$

Moreover, the resolvent kernel $\{G_r : r \geq 0\}$ of X is defined by

$$G_r(x, A) := \int_0^\infty e^{-rt} \mathbb{P}_x(X_t \in A) dt, \tag{2.1}$$

where $x \in \mathbb{R}$ and A is a Borel subset of $\mathbb{R}$. From (2.1), it is clear that

$$rG_r(x, dy) = \mathbb{P}_x(X_{e_r} \in dy),$$

and by the equation $X_{e_r} = M_r + \widetilde{I}_r$ where $\widetilde{I}_r := X_{e_r} - M_r$ has the same distribution a I_r, we obtain that

$$rG_r(x, y) = \begin{cases} \displaystyle\int_{-\infty}^{y-x} f_I(t) f_M(y - x - t) dt, & y - x < 0 \\ \\ \displaystyle\int_{y-x}^{\infty} f_M(t) f_I(y - x - t) dt, & y - x > 0\,. \end{cases} \tag{2.2}$$

Here we assume that M_r and I_r have densities f_{M_r} and f_{I_r}, respectively.

2.3 Sufficient conditions for optimality

We quote below Lemma 9.1 in [15].

Lemma 2.3 *Consider the optimal stopping problem* (1.1) *for a nonnegative function g and $r \geq 0$ under the assumption that for all $x \in \mathbb{R}$,*

$$\mathbb{P}_x(\textit{there exists} \quad \lim_{t \uparrow \infty} e^{-rt} g(X_t) < \infty) = 1. \tag{2.3}$$

Suppose that $\tau^* \in \mathcal{T}$ is a candidate optimal strategy for the optimal stopping problem (1.1) *and let $V^*(x) = \mathbb{E}_x(e^{-r\tau^*} g(X_{\tau^*}))$. Then the pair (V^*, τ^*) is a solution if*

(i) *$V^*(x) \geq g(x)$ for all $x \in \mathbb{R}$,*

(ii) *the process $\{e^{-rt} V^*(X_t) : t \geq 0\}$ is a right continuous supermartingale.*

To solve the optimal stopping problem (1.1), we follow the approach of Surya [27]. For a given continuous function g and $r > 0$, the average problem for the American call-type optimal stopping problem consists of finding a function $\widetilde{Q}_g$ satisfying the condition

$$\mathbb{E}\left(\widetilde{Q}_g(x+M_r)\right) = g(x) \tag{2.4}$$

for every $x \in \mathbb{R}$. With this definition, Surya [27] gave the following sufficient conditions for optimality.

Theorem 2.4 *Given a continuous function g. Suppose that $\widetilde{Q}_g$ is a continuous function that solves the average problem (2.4) for every $x \in \mathbb{R}$ and there exists $\widehat{x} \in \mathbb{R}$ such that $\widetilde{Q}_g(\widehat{x}) = 0$, $\widetilde{Q}_g(x)$ is non-decreasing for $x > \widehat{x}$ and $\widetilde{Q}_g(x) \le 0$ for $x < \widehat{x}$. Denote by x^* the largest root of the equation*

$$\widetilde{Q}_g(x) = 0.$$

Then the solution to the optimal stopping problem (1.1) is given by

$$V(x) = \sup_{\tau\in M} \mathbb{E}_x(e^{-r\tau}g(X_\tau)) = \mathbb{E}_x(e^{-r\tau^*}g(X_{\tau^*})),$$

while the optimal stopping time is given by $\tau^* = \inf\{t > 0 : X_t > x^*\}$. Moreover, we have for every $x \in \mathbb{R}$,

$$V(x) = \mathbb{E}\left(\widetilde{Q}_g(x+M_r)1_{\{x+M_r>x^*\}}\right).$$

Proof. Please refer to [27].

By replacing X with its dual process $\widehat{X} = -X$ and x with $-x$, similar results were also obtained in Surya [27] for American put-type optimal stopping problems. For a given continuous function g and $r > 0$, the average problem for the American put-type optimal stopping problem contains finding a function $\widetilde{P}_g$ satisfying the condition

$$\mathbb{E}\left(\widetilde{P}_g(x+I_r)\right) = g(x) \tag{2.5}$$

for every $x \in \mathbb{R}$.

Theorem 2.5 *Given a continuous function g. Suppose that $\widetilde{P}_g$ is continuous function that solve the problem (2.5) and there exists $\widehat{x} \in \mathbb{R}$ such that $\widetilde{P}_g(\widehat{x}) = 0$, $\widetilde{P}_g(x)$ is non-increasing for $x < \widehat{x}$ and $\widetilde{P}_g(x) \le 0$ for $x \ge \widehat{x}$. Denote by x^* the smallest root of the equation*

$$\widetilde{P}_g(x) = 0.$$

Then the optimal solution to the optimal stopping problem with payoff g, is given for each $x \in \mathbb{R}$ by

$$V(x) = \mathbb{E}\left(\widetilde{P}_g(x+I_r)1_{\{x+I_r<x^*\}}\right),$$

while the optimal stopping time is given by $\tau^* = \inf\{t > 0 : X_t < x^*\}$. That is to say that

$$V(x) = \sup_{\tau\in M} \mathbb{E}_x(e^{-r\tau}g(X_\tau)1_{\{\tau<\infty\}}) = \mathbb{E}_x(e^{-r\tau^*}g(X_{\tau^*})1_{\{\tau^*<\infty\}}), x \in \mathbb{R}.$$

Proof. Please refer to [27].

Remark 2.6 *It is worth noting that if* $g(x) = \mathbb{E}\bigg[\int_0^\infty e^{-rt}h(x+X_t)dt\bigg]$ *for some bounded function* h, *then by the Wiener-Hopf fluctuation identity, the function* $\widetilde{Q}_g(x) = \mathbb{E}_x[h(I_r)]$ *is a solution of the average problem (2.4). From this observation and the above theorem, we recover parts of Theorem 2 of Deligiannidis et al. [10]. In fact, similar results were obtained earlier by Boyarchenko and Levendorskiĭ [8] when the reward function* g *is of call-like or put-like. For details, see Boyarchenko and Levendorskiĭ [8].*

Furthermore, Surya [27] also showed that given a continuous function g and $\overline{g} = \max\{g(x), 0\}$, if $V(x)$ and $\overline{V}$ are the optimal value function of the problem (1.1) with respect to g and $\overline{g}$, respectively then for every $x \in \mathbb{R}$, $V(x) = \overline{V}(x)$. For the nonnegative reward function g, following similar arguments (with some minor modifications) as in Surya [27], we have the following sufficient conditions of optimality .

Theorem 2.7 *Given a nonnegative function* g *and* $H := \{g > 0\} = (\widehat{a}, \infty)$ *for some* $\widehat{a} < \infty$. *Suppose that* $\widetilde{Q}_g$ *is a continuous function on* H *that solves the averaging problem (2.4) for every* $x \in H$. *We assume further that there exists* $\widehat{x} \in H$ *such that* $\widetilde{Q}_g(\widehat{x}) = 0$, $\widetilde{Q}_g(x)$ *is non-decreasing for* $x > \widehat{x}$ *and* $\widetilde{Q}_g(x) \le 0$ *for* $\widehat{a} < x < \widehat{x}$. *Then the solution to the optimal stopping problem (1.1) is given by*

$$V(x) = \sup_{\tau \in M} \mathbb{E}_x(e^{-r\tau} g(X_\tau)) = \mathbb{E}_x(e^{-r\tau^*} g(X_{\tau^*})),$$

where x^* *is the largest root of* $\widetilde{Q}_g(x) = 0$ *in* $(\widehat{a}, \infty)$ *and* $\tau^* = \inf\{t > 0 : X_t > x^*\}$. *Moreover, we have for every* $x \in \mathbb{R}$,

$$V(x) = \mathbb{E}\bigg(\widetilde{Q}_g(x + M_r)1_{\{x+M_r > x^*\}}\bigg).$$

To acquire the main result in Theorem 2.7, the following lemma is needed.

Lemma 2.8 *Assume that* $\widetilde{Q}$ *that solve the problem (2.4) for* $x \in H$ *is continuous on* H, *and satisfies the requirements in Theorem 2.7. For each* $r > 0$ *and* $y \in H$ *a candidate solution to the problem (1.1)*

$$V_y(x) \triangleq \mathbb{E}\bigg(\widetilde{Q}_g(x + M_r)1_{\{x+M_r > y\}}\bigg). \tag{2.6}$$

Let x^* *be the largest root of the equation* $\widetilde{Q}_g(x) = 0$ *and for fixed* $y \in H$ *denote the first strict passage time* τ_y^+ *above* y *by* $\tau_y^+ = \inf\{t > 0 : X_t > y\}$. *Then we have the following results.*

(a) For $x \in \mathbb{R}$ *and* $y \in H$, *we have* $V_y(x) = 1_{\{x>y\}}g(x) + 1_{\{x \le y\}}\mathbb{E}_x\Big(e^{-r\tau_y^+} g(X_{\tau_y^+})\Big)$.

(b) $\{e^{-rt}V_{x^*}(X_t)\}_{t \in \mathbb{R}^+}$ *is a right-continuous* $\mathbb{P}_x$*-supermartingale.*

(c) For any $x \in \mathbb{R}$, *we have* $V_{x^*}(x) \ge g(x)$.

Proof. (a) For $x > y \in H$. It follows from $M_r \geq 0$ almost surely and $\widetilde{Q}_g$ solves the average problem (2.4) for $x \in H$ that for $x > y$ and $y \in H$, $V_y(x) = g(x)$. For $x \leq y \in H$. Observe fact that conditionally on $\mathcal{F}_{\tau_y^+}$ and on the event $\{e_r > \tau_y^+\}$, $M_r - X_{\tau_y^+}$ is independent of $\mathcal{F}_{\tau_y^+}$ and has the same distribution as M_r, due to the lack of memory property of exponential random variable e_r and the stationary independent increment of X. Combining with the fact that the function $\widetilde{Q}_g(x)$ solves the averaging problem (2.4) for $x \in H$, we obtain using tower property of conditional expectation that for $y \in H$

$$
\begin{aligned}
\mathbb{E}_x\left(\widetilde{Q}_g(M_r)1_{\{M_r>y\}}\right) &= \mathbb{E}_x\left(\widetilde{Q}_g(M_r)1_{\{e_r>\tau_y^+\}}\right) \\
&= \mathbb{E}_x\left(\mathbb{E}\left(\widetilde{Q}_g(M_r)1_{\{e_r>\tau_y^+\}}\middle|\mathcal{F}_{\tau_y^+}\right)\right) \\
&= \mathbb{E}_x\left(1_{\{e_r>\tau_y^+\}}\mathbb{E}\left(\widetilde{Q}_g(M_r)\middle|\mathcal{F}_{\tau_y^+}\right)\right) \\
&= \mathbb{E}_x\left(1_{\{e_r>\tau_y^+\}}\mathbb{E}\left(\widetilde{Q}_g(M_r - X_{\tau_y^+} + X_{\tau_y^+})\middle|\mathcal{F}_{\tau_y^+}\right)\right) \\
&= \mathbb{E}_x\left(1_{\{e_r>\tau_y^+\}}\mathbb{E}_{X_{\tau_y^+}}\left(\widetilde{Q}_g(M_r - X_{\tau_y^+})\middle|\mathcal{F}_{\tau_y^+}\right)\right) \\
&= \mathbb{E}_x\left(1_{\{e_r>\tau_y^+\}}\mathbb{E}_{X_{\tau_y^+}}\left(\widetilde{Q}_g(M_r)\right)\right) \\
&= \mathbb{E}_x\left(1_{\{e_r>\tau_y^+\}}g(X_{\tau_y^+})\right) \\
&= \mathbb{E}_x\left(e^{-r\tau_y^+}g(X_{\tau_y^+})\right).
\end{aligned}
$$

(b) This proof is obtained by using the fact that conditionally on the event $\{e_r > t\}$, the identity $M_r = \overline{X}_t \vee (M + X_t)$ holds and conditionally on the filtration $\mathcal{F}_t$, the random variable M has the same distribution as M_r. Along with the fact that $x \mapsto \widetilde{Q}_g(x)$ is nondecreasing on H, we have that for every $r, t \geq 0$ and each $x \in \mathbb{R}$ that

$$
\begin{aligned}
V_{x^*}(x) &= \mathbb{E}_x\left(\widetilde{Q}_g(M_r)1_{\{M_r>x^*\}}\right) \\
&= \mathbb{E}_x\left(\mathbb{E}\left(\widetilde{Q}_g(M_r)1_{\{M_r>x^*\}}\middle|\mathcal{F}_t\right)\right) \\
&\geq \mathbb{E}_x\left(1_{\{e_r>t\}}\mathbb{E}\left(\widetilde{Q}_g(X_t + M)1_{\{X_t+M>x^*\}}\middle|\mathcal{F}_t\right)\right) \\
&= \mathbb{E}_x\left(1_{\{e_r>t\}}\mathbb{E}_{X_t}\left(\widetilde{Q}_g(M_r)1_{\{M_r>x^*\}}\right)\right) \\
&= \mathbb{E}_x(1_{\{e_r>t\}}V_{x^*}(X_t)) = \mathbb{E}_x(e^{-rt}V_{x^*}(X_t)).
\end{aligned}
$$

By this and the Markov property, $\{e^{-rt}V_{x^*}(X_t)\}_{t\in\mathbb{R}^+}$ is a $\mathbb{P}_x$-supermartingale. Combining this with the fact that $\widetilde{Q}_g$ is continuous on H gives (b).

(c) By (a), and the facts that V is nonnegative and $g(x) = 0$ for $x \in H^c$, we see that it is sufficient to verify $V_{x^*}(x) \geq g(x)$ on $x \in (\widehat{a}, x^*]$ for completing this proof. Since the function $\widetilde{Q}_g$ solves the averaging problem (2.4) for $x \in H$, we obtain for every $r > 0$ and every $x \in (\widehat{a}, x^*]$ that

$$\begin{aligned} V_{x^*}(x) &= \mathbb{E}\Big(\widetilde{Q}_g(x+M_r)1_{\{x+M_r>x^*\}}\Big) \\ &= \mathbb{E}\Big(\widetilde{Q}_g(x+M_r)\Big) - \mathbb{E}\Big(\widetilde{Q}_g(x+M_r)1_{\{x+M_r\leq x^*\}}\Big) \\ &\geq \mathbb{E}\Big(\widetilde{Q}_g(x+M_r)\Big) = g(x), \end{aligned}$$

where the inequality follows from that $\widetilde{Q}_g(x) \leq 0$ for all $x \in (\widehat{a}, x^*]$, while the last equality is due to the fact that $\widetilde{Q}_g$ solves the problem (2.4) for $x \in H$. Hence, we complete the proof. ■

Proof of Theorem 2.7. The proof is mainly based on the results of Lemma 2.8. According to (b) and (c) of Lemma 2.8, we obtain for every $x \in \mathbb{R}$ that

$$V_{x^*}(x) \geq \sup_{\tau\in\mathcal{T}} \mathbb{E}_x(e^{-r\tau}V_{x^*}(X_\tau)) \geq \sup_{\tau\in\mathcal{T}} \mathbb{E}_x(e^{-r\tau}g(X_\tau)).$$

On the other hand, using (a) of Lemma 2.8, we have for each $x \in \mathbb{R}$ that

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}_x(e^{-r\tau}g(X_\tau)) \geq \mathbb{E}_x(e^{-r\tau^*}g(X_{\tau^*})) = V_{x^*}(x).$$

Hence, all the inequalities become equalities and we have that

$$V_{x^*}(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x(e^{-r\tau}g(X_\tau)) = \mathbb{E}_x(e^{-r\tau^*}g(X_{\tau^*})).$$

Therefore, for every $x \in \mathbb{R}$ the optimal value function of the problem (1.1) coincides with the function V_{x^*} and the optimal stopping time is given by $\tau^+_{x^*}$. ■

Remark 2.9 *Given a continuous function g with $\{g > 0\} = (-\infty, \widehat{a})$. Set $\widehat{g}(y) = g(-y)$ and $\widehat{M}_r := \sup_{0\leq t\leq e_r} -X_t$ and assume that $\widetilde{Q}_{\widehat{g}}$ solves the averaging problem (2.4) for $\widehat{M}_r$ (i.e., $\widehat{g}(y) = \mathbb{E}\Big[\widetilde{Q}_{\widehat{g}}(y+\widehat{M}_r)\Big]$ for all $y > -\widehat{a}$). Write $\widetilde{P}_g(x) = \widetilde{Q}_{\widehat{g}}(-x)$ for $x < \widehat{a}$. Then we have*

$$g(x) = \widehat{g}(-x) = \mathbb{E}\Big[\widetilde{Q}_{\widehat{g}}(-x+\widehat{M}_r)\Big] = \mathbb{E}\Big[\widetilde{Q}_{\widehat{g}}(-x-I_r)\Big] = \mathbb{E}\Big[\widetilde{P}_g(x+I_r)\Big], \text{ for } x < \widehat{a}. \quad (2.7)$$

Therefore, $\widetilde{P}_g(x)$ solves the averaging problem (2.5) for $x < \widehat{a}$.

Using the similar approach in the proof of Theorem 2.7 and Lemma 2.8, we also obtain the following Theorem.

Theorem 2.10 *Given a nonnegative function g and $H := \{g > 0\} = (-\infty, \widehat{a})$ for some $\widehat{a} > -\infty$. Suppose that $\widetilde{P}_g$ is a continuous function on H that solves the averaging problem (2.5)* *for every $x \in H$. We assume further that there exists $\widehat{x} \in H$ such that $\widetilde{P}_g(\widehat{x}) = 0$, $\widetilde{P}_g(x)$ is non-increasing for $x < \widehat{x}$ and $\widetilde{P}_g(x) \leq 0$ for $\widehat{x} < x < \widehat{a}$. Then the solution to the optimal stopping problem (1.1) is given by*

$$V(x) = \sup_{\tau \in M} \mathbb{E}_x(e^{-r\tau} g(X_\tau)) = \mathbb{E}_x(e^{-r\tau^*} g(X_{\tau^*})),$$

where x^* is the smallest root of $\widetilde{P}_g(x) = 0$ in $(-\infty, \widehat{a})$ and $\tau^* = \inf\{t > 0 : X_t > x^*\}$. Moreover, we have for every $x \in \mathbb{R}$,

$$V(x) = \mathbb{E}\Big(\widetilde{P}_g(x + I_r) 1_{\{x+I_r<x^*\}}\Big).$$

Proof. We first write $\widehat{g}(x) = g(-x)$ and $\widehat{H} := \{\widehat{g} > 0\} = (-\widehat{a}, \infty)$. Set $\widetilde{Q}_{\widehat{g}}(x) = \widetilde{P}_g(-x)$ for $x > -\widehat{a}$. Observe that $\widetilde{Q}_{\widehat{g}}$ is a continuous function on $\widehat{H}$ that solves the averaging problem (2.4) for $\widehat{M_r}$ as shown in Remark 2.9. Also, by assumption, there exists $-\widehat{x} \in \widehat{H}$ such that $\widetilde{Q}_{\widehat{g}}(-\widehat{x}) = 0$, $\widetilde{Q}_{\widehat{g}}(x)$ is nondecreasing for $x > -\widehat{x}$ and $\widetilde{Q}_{\widehat{g}}(x) \leq 0$ for $-\widehat{a} < x < -\widehat{x}$. Set $Y_t = -X_t$ and $y = -x$ and then by Theorem 2.7, observe

$$\widehat{W}(y) := \sup_{\tau} \mathbb{E}_y[e^{-r\tau} \widehat{g}(Y_\tau)] = \mathbb{E}_y[e^{-r\tau^*} \widehat{g}(Y_{\tau^*})]$$

where $\tau^* = \inf\{t > 0 : -X_t > -x^*\}$ and $-x^*$ is the largest root of $\widetilde{Q}_{\widehat{g}}(x) = 0$ in $(-\widehat{a}, \infty)$. Also, note that

$$V(x) = \sup_{\tau} \mathbb{E}_x[e^{-r\tau} g(X_\tau)] = \sup_{\tau} \mathbb{E}[e^{-r\tau} g(x + X_\tau)] = \sup_{\tau} \mathbb{E}[e^{-r\tau} \widehat{g}(-x - X_\tau)] = \widehat{W}(-x).$$

Therefore, if we set $y = -x$ and $y^* = -x^*$ then we have

$$\begin{aligned} V(x) = \widehat{W}_{\widehat{g}}(y) =& \mathbb{E}_y\Big[e^{-r\tau^*} \widehat{g}(Y_{\tau^*})\Big] = \mathbb{E}\Big[e^{-r\tau^*} \widehat{g}(y + Y_{\tau^*})\Big] = \mathbb{E}\Big[e^{-r\tau^*} g(-y - Y_{\tau^*})\Big] \\ =& \mathbb{E}\Big[e^{-r\tau^*} g(x + X_{\tau^*})\Big] = \mathbb{E}_x\Big[e^{-r\tau^*} g(X_{\tau^*})\Big] \end{aligned}$$

and

$$\begin{aligned} V(x) = \widehat{W}_{\widehat{g}}(y) =& \mathbb{E}\Big[\widetilde{Q}_{\widehat{g}}(y + \widehat{M}_r) 1_{\{y+\widehat{M}_r>y^*\}}\Big] = \mathbb{E}\Big[\widetilde{P}_{\widehat{g}}(-y - \widehat{M}_r) 1_{\{-y-\widehat{M}_r<-y^*\}}\Big] \\ =& \mathbb{E}\Big[\widetilde{P}_{\widehat{g}}(-y + I_r) 1_{\{-y+I_r<-y^*\}}\Big] = \mathbb{E}\Big[\widetilde{P}_{\widehat{g}}(x + I_r) 1_{\{x+I_r<x^*\}}\Big]. \end{aligned}$$

■

On the other hand, the following results of Mordecki and Salmimen in [18] studied the optimal stopping problem for Hunt and Lévy processes via the representation theory of excessive functions.

Theorem 2.11 *Consider a Lévy process $\{X_t\}$, a non-negative continuous reward function g, and a discount rate $r \geq 0$ such that*

$$\mathbb{E}_x(\sup_{t \geq 0} e^{-rt} g(X_t)) < \infty \tag{2.8}$$

Assume that there exists a radon measure σ with support on the set $[x^*, \infty)$ such that the function

$$V(x) := \int_{[x^*,\infty)} G_r(x, y)\sigma(dy)$$

satisfies the following conditions:

(a) V *is continuous,*

(b) $V(x) \to 0 \quad x \to -\infty$.

(c) $V(x) = g(x)$ *when* $x \geq x^*$,

(d) $V(x) \geq g(x)$ *when* $x < x^*$. *Let*

$$\tau^* = \inf\{t \geq 0 : X_t \geq x^*\}.$$

Then τ^* *is an optimal stopping time and* V *is the value function of the optimal stopping problem for* X_t *with the reward function* g*, in other words,*

$$V(x) = \sup_{\tau \in M} \mathbb{E}_x(e^{-r\tau} g(X_\tau)) = \mathbb{E}_x(e^{-r\tau^*} g(X_{\tau^*})), x \in \mathbb{R}$$

3 Matrix-exponential Lévy processes

Definition 3.1 *A distribution F on $(0,\infty)$ with a probability density function f is called a* ***matrix-exponential*** *distribution if its Laplace transform is a rational function, or equivalently, if f takes the form*

$$dF(y) = \sum_{j=1}^{n} \sum_{k=1}^{n_j} A_{j,k} y^{k-1} e^{-\lambda_j y} dy, \quad y > 0. \tag{3.1}$$

(Note that the parameters $A_{j,k}$ and λ_j are not necessary real; see [17]).

Definition 3.2 *Let $N \in \mathbb{N}$. Assume $\boldsymbol{B}$ is an $N \times N$ nonsingular subintensity matrix, that is, $b_{ij} \geq 0$ for $i \neq j$, $b_{ii} \leq 0$ and $\boldsymbol{b}^T = -\boldsymbol{B}\mathbf{e}^T \in \mathbb{R}_+^N \backslash \{\mathbf{0}\}$. Here, $\mathbf{e} = [11 \cdots 1]$ and $\mathbf{0} = [00 \cdots 0]$. Let $\mathbf{a}$ be an N-dimensional probability function. The probability distribution function F with the density function*

$$f(x) = \mathbf{a} e^{x\boldsymbol{B}} \boldsymbol{b}^T 1_{\{x>0\}}$$

is called a ***phase-type*** *distribution. We denote this distribution by $PH(\mathbf{a}, \boldsymbol{B})$. We say that the representation $(\mathbf{a}, \boldsymbol{B})$ is minimal if there do not exist $N_0 < N$, $\mathbf{a}'$ of dimension N_0 and a nonsingular subintensity matrix $\boldsymbol{B}'$ of dimension N_0 such that $f(x) = \mathbf{a}' e^{x\boldsymbol{B}'} \boldsymbol{b}'^T 1_{\{x>0\}}$.*

It is worth noting that the two dense classes of distributions on $(0,\infty)$ -phase-type distributions and distributions which are linear combinations of exponential distributions-are both subclasses of matrix-exponential distributions, see [3] Theorem 4.2 and [6]. Special cases of phase-type distributions include exponential distributions, Gamma distributions with integer parameter, and mixtures of exponential distributions. For details, see [3] or [24].

We say that a Lévy process is a **matrix-exponential Lévy process** if its downward Lévy measure is a finite measure and has a rational Laplace transform. In other words, the upward Lévy measure of $-X$ is proportional to a matrix-exponential distribution dF given by (3.1). In particular, by Proposition 1 in [4], matrix-exponential Lévy process are dense in the class of Lévy processes.

From now on, we consider the jump-diffusion process X taking the following form

$$X_t = X_0 + at + bW_t + \sum_{n=1}^{N_t^\lambda} Y_n + \sum_{k=1}^{N_t^\mu} Z_k, \qquad t \geq 0. \tag{3.2}$$

Here, $a \in \mathbb{R}\backslash\{0\}$, $b \geq 0$, $W = (W_t, t \geq 0)$ is a standard Brownian motion, $N^\lambda = (N_t^\lambda; t \geq 0)$, as well as $N^\mu = (N_t^\mu; t \geq 0)$ are the Poisson process with rate $\lambda > 0$ and $\mu > 0$, respectively. Also, $Y = (Y_n, n \in \mathbb{N})$ and $Z = (Z_k, k \in \mathbb{N})$ are the sequence of independent random variable with identical matrix-exponential distribution given by,

$$dF^{(+)}(x) = p_1(x)dx = (1_{\{x>0\}} \sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \frac{c_{kj} \beta_k^j x^{j-1}}{(j-1)!} e^{-\beta_k x})dx$$

and

$$dF^{(-)}(x) = p_2(x)dx = (1_{\{x<0\}} \sum_{p=1}^{v_2} \sum_{m=1}^{\ell_p} \frac{\widetilde{c}_{pm} \alpha_p^m (-x)^{m-1}}{(m-1)!} e^{\alpha_p x})dx,$$

respectively. Here, the parameters $c_{kj}, \beta_k, \widetilde{c}_{pm}$, and α_p can in principle take complex values, but if we order α_p and β_k by their real parts then α_1 and β_1 must be real, while the others may be complex with

$$0 < \beta_1 < \mathcal{R}e\,(\beta_2) \leq \cdots \leq \mathcal{R}e\,(\beta_{v_1}) \text{ and } 0 < \alpha_1 < \mathcal{R}e\,(\alpha_2) \leq \cdots \leq \mathcal{R}e\,(\alpha_{v_2}). \tag{3.3}$$

The random variable W, N^λ, N^μ, Y and Z are assumed to be independent.

Note that the characteristic exponent of X is given by

$$\psi(z) = \mathrm{i}az - \frac{b^2z^2}{2} + \lambda\Bigg[\sum_{k=1}^{v_1}\sum_{j=1}^{n_k} c_{kj}\Big(\frac{\mathrm{i}\beta_k}{z+\mathrm{i}\beta_k}\Big)^j - 1\Bigg] + \mu\Bigg[\sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p} \widetilde{c}_{pm}\Big(\frac{-\mathrm{i}\alpha_p}{z-\mathrm{i}\alpha_p}\Big)^m - 1\Bigg]. \tag{3.4}$$

We quote the following results by Lewis and Mordecki in [17].

Theorem 3.3

(1) The equation $r - \psi(z) = 0$ has, in the half-plane $\mathcal{I}m\,(z) < 0$, μ_1 distinct roots

$$-\mathrm{i}\rho_1, \cdots, -\mathrm{i}\rho_{\mu_1}, \quad \textit{with respect multiplicities} \quad m_1, ..., m_{\mu_1},$$

ordered such that $0 < \mathcal{R}e\,(\rho_1) \leq \mathcal{R}e\,(\rho_2) \leq \cdots \leq \mathcal{R}e\,(\rho_{\mu_1})$. The root $-\mathrm{i}\rho_1$ is purely imaginary. Furthermore, the total root count $\sum_{j=1}^{\mu_1} m_j$ is equal to $\overline{n}$ when $-X^-$ is a subordinator and $\overline{n}+1$ when $-X^-$ is not a subordinator. Here, $\overline{n} = \sum_{k=1}^{v_1} n_k$. On the other hand, the equation $r - \psi(z) = 0$ has, in the half-plane $\mathcal{I}m\,(z) > 0$, μ_2 distinct roots

$$-\mathrm{i}\widetilde{\rho}_1, \cdots, -\mathrm{i}\widetilde{\rho}_{\mu_2}, \quad \textit{with respect multiplicities} \quad \widetilde{m}_1, ..., \widetilde{m}_{\mu_2},$$

ordered such that $\mathcal{R}e\,(\widetilde{\rho}_{\mu_2}) \leq \cdots \leq \mathcal{R}e\,(\widetilde{\rho}_1) < 0$. The root $-\mathrm{i}\widetilde{\rho}_1$ is purely imaginary. Furthermore, the total root count $\sum_{j=1}^{\mu_2} \widetilde{m}_j$ is equal to $\overline{\ell}$ when X^+ is a subordinator and $\overline{\ell}+1$ when X^+ is not a subordinator. Here, $\overline{\ell} = \sum_{p=1}^{v_2} \ell_p$.

(2) The minimum I_r has rational Laplace transform ψ_r^- given by

$$\psi_r^-(u) = \prod_{k=1}^{v_2}\Big(\frac{u-\mathrm{i}\alpha_k}{-\mathrm{i}\alpha_k}\Big)^{\ell_k}\prod_{j=1}^{\mu_2}\Big(\frac{\mathrm{i}\widetilde{\rho}_j}{u+\mathrm{i}\widetilde{\rho}_j}\Big)^{\widetilde{m}_j}$$

and the maximum M_r has rational Laplace transform ψ_r^+ given by

$$\psi_r^+(u) = \prod_{k=1}^{v_1}\Big(\frac{u+\mathrm{i}\beta_k}{\mathrm{i}\beta_k}\Big)^{n_k}\prod_{j=1}^{\mu_1}\Big(\frac{\mathrm{i}\rho_j}{u+\mathrm{i}\rho_j}\Big)^{m_j}.$$

Throughout the thesis, we follow the convention that $\prod_{k=1}^{0}(\dots) = 1$. Also, we denote by $\mathcal{Z}_r$ the collection of zeros of $r - \psi(\xi)$ and say that $\mathcal{Z}_r$ is **separable** if its members are distinct (i.e., $m_1 = m_2 = \cdots = m_{\mu_1} = \widetilde{m}_1 = \cdots = \widetilde{m}_{\mu_2} = 1$).

4 The main results

Recall that $\{X_t\}_{t\geq 0}$ is a jump-diffusion process of the form in (3.2) and $-\mathrm{i}\widetilde{\rho}_1, \cdots, -\mathrm{i}\widetilde{\rho}_{\mu_2}, -\mathrm{i}\rho_1, \cdots, -\mathrm{i}\rho_{\mu_1}$ are the roots of $r - \psi(z) = 0$ with $\mathcal{R}e\,(\widetilde{\rho}_{\mu_2}) \leq \cdots \leq \mathcal{R}e\,(\widetilde{\rho}_1) \leq 0 < \mathcal{R}e\,(\rho_1) \leq \mathcal{R}e\,(\rho_2) \leq \cdots \leq \mathcal{R}e\,(\rho_{\mu_1})$. We assume further that $\mathcal{Z}_r$ is separable (i.e., $m_1 = m_2 = \cdots = m_{\mu_1} = \widetilde{m}_1 = \cdots = \widetilde{m}_{\mu_2} = 1$). Under these assumptions, our goal is to solve the optimal stopping problem (1.1) for a given continuous function g and $r > 0$. To do this, we first give an explicit formula for solutions of the averaging problems in (2.4) and (2.5). Then we find sufficient conditions on g that guarantees the existence of the optimal stopping boundary for the problem (1.1).

We observe, by Theorem 3.3, that the distribution of $\inf_{0\leq s\leq e_r} X_s$ is given by

$$\mathbb{P}\left(\inf_{0\leq s\leq e_r} X_s \in dy\right) = 1_{\{a>0 \text{ and } b=0\}}\widetilde{d}_0\delta_0(dy) + 1_{\{\mu_2\geq 1\}}\left[(\sum_{\eta=1}^{\mu_2}\widetilde{d}_\eta\widetilde{\rho}_\eta e^{-\widetilde{\rho}_\eta y})1_{\{y<0\}}dy\right] \tag{4.1}$$

where

$$\widetilde{d}_0 = \prod_{j=1}^{\mu_2}(-\widetilde{\rho}_j)\prod_{k=1}^{v_2}\alpha_k^{-\ell_k} \tag{4.2}$$

and

$$\widetilde{d}_j = (-1)\prod_{k=1}^{v_2}\left(\frac{\widetilde{\rho}_j + \alpha_k}{\alpha_k}\right)^{\ell_k}\prod_{m=1, m\neq j}^{\mu_2}\left(\frac{\widetilde{\rho}_m}{-\widetilde{\rho}_j + \widetilde{\rho}_m}\right), \text{ for } 1 \leq j \leq \mu_2. \tag{4.3}$$

Also the distribution of $\sup_{0\leq s\leq e_r} X_s$ is given by

$$\mathbb{P}\left(\sup_{0\leq s\leq e_r} X_s \in dy\right) = 1_{\{a<0 \text{ and } b=0\}}d_0\delta_0(dy) + 1_{\{\mu_1\geq 1\}}\left[(\sum_{j=1}^{\mu_1}d_j\rho_j e^{-\rho_j y})1_{\{y>0\}}dy\right] \tag{4.4}$$

where

$$d_0 = \prod_{j=1}^{\mu_1}\rho_j\prod_{k=1}^{v_1}\beta_k^{-n_k} \tag{4.5}$$

and

$$d_k = \prod_{j=1}^{v_1}\left(\frac{\beta_j - \rho_k}{\beta_j}\right)^{n_j}\prod_{i=1, i\neq k}^{\mu_1}\left(\frac{\rho_i}{\rho_i - \rho_k}\right), \text{ for } 1 \leq k \leq \mu_1. \tag{4.6}$$

Throughout this thesis, we follow the convention that $\prod_{i=1,i\neq k}^{\mu_1}(\ldots) = 1$ (resp., $\prod_{m=1,m\neq j}^{\mu_2}(\ldots) = 1$) in the case $\mu_1 = 1$ (resp., $\mu_2 = 1$). Plugging (4.1) and (4.4) into (2.2) gives

$$rG_r(x,y) = \begin{cases} \displaystyle\sum_{\eta=1}^{\mu_2}\left(1_{\{a<0 \text{ and } b=0\}}d_0\widetilde{d}_\eta\widetilde{\rho}_\eta + 1_{\{\mu_1\geq 1 \text{ and } \mu_2\geq 1\}}\sum_{j=1}^{\mu_1}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta d_j\rho_j}{\rho_j - \widetilde{\rho}_\eta}\right)e^{-\widetilde{\rho}_\eta(y-x)}, & \text{if } y - x < 0 \\ \displaystyle\sum_{j=1}^{\mu_1}\left(1_{\{a>0 \text{ and } b=0\}}\widetilde{d}_0 d_j\rho_j + 1_{\{\mu_1\geq 1 \text{ and } \mu_2\geq 1\}}\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta d_j\rho_j}{\rho_j - \widetilde{\rho}_\eta}\right)e^{-\rho_j(y-x)}, & \text{if } y - x > 0\,. \end{cases} \tag{4.7}$$

We also need the following facts.

Lemma 4.1 *Suppose $\mu_1 \geq 1$ and $\mu_2 \geq 1$. Then*

(a) *For $i = 1, .., \mu_1$, we have*

$$-\lambda \sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \frac{(\beta_k)^j c_{kj}}{(\beta_k - \rho_i)^j} - \mu \sum_{p=1}^{v_2} \sum_{m=1}^{\ell_p} \frac{(\alpha_p)^m \widetilde{c}_{pm}}{(\alpha_p + \rho_i)^m} + (\mu + \lambda + r) - a\rho_i - \frac{b^2 \rho_i^2}{2} = 0, \tag{4.8}$$

and for $\xi = 1, ..., \mu_2$, we obtain

$$-\lambda \sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \frac{(\beta_k)^j c_{kj}}{(\beta_k - \widetilde{\rho}_\xi)^j} - \mu \sum_{p=1}^{v_2} \sum_{m=1}^{\ell_p} \frac{(\alpha_p)^m \widetilde{c}_{pm}}{(\alpha_p + \widetilde{\rho}_\xi)^m} + (\mu + \lambda + r) - a\widetilde{\rho}_\xi - \frac{b^2 \widetilde{\rho}_\xi{}^2}{2} = 0. \tag{4.9}$$

(b) *For $1 \le k \le v_1$ and $1 \le \xi \le n_k$, we get*

$$\sum_{j=1}^{\mu_1} \frac{d_j \rho_j}{(\beta_k - \rho_j)^\xi} = \begin{cases} 1_{\{\xi=1\}} \cdot d_0, & \text{if } a < 0 \text{ and } b = 0, \\ \\ 0, & \text{otherwise.} \end{cases} \tag{4.10}$$

(c) *For $1 \le p \le v_2$ and $1 \le m \le \ell_p$, we acquire*

$$\sum_{j=1}^{\mu_2} \frac{\widetilde{d}_j \widetilde{\rho}_j}{(\alpha_p + \widetilde{\rho}_j)^m} = \begin{cases} 1_{\{m=1\}} \widetilde{d}_0, & \text{if } a > 0 \text{ and } b = 0, \\ \\ 0, & \text{otherwise.} \end{cases} \tag{4.11}$$

(d) *For any complex numbers $A_{p,m}$ and ω, we have*

$$\sum_{\eta=1}^{\mu_2} \sum_{p=1}^{v_2} \sum_{m=1}^{\ell_p} \frac{A_{p,m} \widetilde{d}_\eta \widetilde{\rho}_\eta}{\omega - \widetilde{\rho}_\eta} \left(\frac{1}{(\alpha_p + \widetilde{\rho}_\eta)^m} - \frac{1}{(\alpha_p + \omega)^m} \right) = \sum_{p=1}^{v_2} \sum_{m=1}^{\ell_p} \frac{A_{p,m}}{(\alpha_p + \omega)^m} \left(\widetilde{d}_0 1_{\{a>0 \text{ and } b=0\}} \right). \tag{4.12}$$

(e) *For any complex numbers A_s, we obtain*

$$\sum_{s=1}^{\mu_1} \sum_{\eta=1}^{\mu_2} \frac{d_s \rho_s A_s \widetilde{d}_\eta \widetilde{\rho}_\eta}{(\rho_s - \widetilde{\rho}_\eta)} \left[\sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \frac{-\lambda (\beta_k)^j c_{kj}}{(\beta_k - \rho_s)^j} - a\rho_s - \frac{b^2 \rho_s^2}{2} - \left(\sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \frac{-\lambda (\beta_k)^j c_{kj}}{(\beta_k - \widetilde{\rho}_\eta)^j} - a\widetilde{\rho}_\eta - \frac{b^2 \widetilde{\rho}_\eta^2}{2} \right) \right]$$
$$= 1_{\{a>0 \text{ and } b=0\}} \sum_{s=1}^{\mu_1} \sum_{p=1}^{v_2} \sum_{m=1}^{\ell_p} \frac{(-\widetilde{d}_0) d_s \rho_s A_s \mu (\alpha_p)^m \widetilde{c}_{pm}}{(\alpha_p + \rho_s)^m}. \tag{4.13}$$

(f) *The following identities hold:*

$$\begin{cases} \dfrac{(-a)d_0}{r} \displaystyle\sum_{j=1}^{\mu_2} \widetilde{d}_j \widetilde{\rho}_j = 1, & \text{if } a < 0 \text{ and } b = 0 \\ \\ \dfrac{a\widetilde{d}_0}{r} \displaystyle\sum_{j=1}^{\mu_1} d_j \rho_j = 1, & \text{if } a > 0 \text{ and } b = 0 \\ \\ \dfrac{1}{r} \displaystyle\sum_{s=1}^{\mu_1} \sum_{\eta=1}^{\mu_2} d_s \rho_s \widetilde{d}_\eta \widetilde{\rho}_\eta (\frac{b^2}{2}) = 1, & \text{if } b \ne 0. \end{cases} \tag{4.14}$$

Proof. (a) The identities in part (a) follows from the facts that $-\mathrm{i}\widetilde{\rho}_1, \cdots, -\mathrm{i}\widetilde{\rho}_{\mu_2}$ and $-\mathrm{i}\rho_1, \cdots, -\mathrm{i}\rho_{\mu_1}$ are solutions of $r - \psi(z) = 0$ and (3.4).

(b) Since $m_j = 1$ for $j = 1, ..., \mu_1$, by Theorem 3.3, we obtain that

$$\psi_r^+(u) = \prod_{k=1}^{v_1} \left(\frac{u + \mathrm{i}\beta_k}{\mathrm{i}\beta_k} \right)^{n_k} \prod_{j=1}^{\mu_1} \left(\frac{\mathrm{i}\rho_j}{u + \mathrm{i}\rho_j} \right). \tag{4.15}$$

Note that if $a < 0$ and $b = 0$ then $\mu_1 = \sum_{k=1}^{v_1} n_k$; otherwise, $\mu_1 = \sum_{k=1}^{v_1} n_k + 1$. Applying fractional decomposition to the right hand side of (4.15) gives

$$\prod_{k=1}^{v_1} \left(\frac{u + \mathrm{i}\beta_k}{\mathrm{i}\beta_k} \right)^{n_k} \prod_{j=1}^{\mu_1} \left(\frac{\mathrm{i}\rho_j}{u + \mathrm{i}\rho_j} \right) = 1_{\{\mu_1 \geq 1\}} \sum_{j=1}^{\mu_1} \frac{d_j \rho_j \mathrm{i}}{u + \mathrm{i}\rho_j} + 1_{\{a<0 \text{ and } b=0\}} d_0 \tag{4.16}$$

where $\lim_{u \to -\mathrm{i}\rho_j} (u + \mathrm{i}\rho_j)\psi_r^+(u) = d_j \rho_j \mathrm{i}$ and d_j are given by (4.6). For $1 \leq k \leq v_1$, and $1 \leq \xi \leq n_k$, our results follow by differentiating both sides of (4.16) $(\xi - 1)$-times at $u = -\mathrm{i}\beta_k$.

(c) Note that we have

$$\psi_r^-(u) = \prod_{k=1}^{v_2} \left(\frac{u - \mathrm{i}\alpha_k}{-\mathrm{i}\alpha_k} \right)^{\ell_k} \prod_{j=1}^{\mu_2} \left(\frac{\mathrm{i}\widetilde{\rho}_j}{u + \mathrm{i}\widetilde{\rho}_j} \right). \tag{4.17}$$

Note that if $a > 0$ and $b = 0$ then $\mu_2 = \sum_{k=1}^{v_2} \ell_k$; otherwise, $\mu_2 = \sum_{k=1}^{v_2} \ell_k + 1$. Applying fractional decomposition to the right hand side of (4.17) gives

$$\prod_{k=1}^{v_2} \left(\frac{u - \mathrm{i}\alpha_k}{-\mathrm{i}\alpha_k} \right)^{\ell_k} \prod_{j=1}^{\mu_2} \left(\frac{\mathrm{i}\widetilde{\rho}_j}{u + \mathrm{i}\widetilde{\rho}_j} \right) = 1_{\{\mu_2 \geq 1\}} \sum_{j=1}^{\mu_2} \frac{\widetilde{d}_j \widetilde{\rho}_j (-\mathrm{i})}{u + \mathrm{i}\widetilde{\rho}_j} + 1_{\{a>0 \text{ and } b=0\}} \widetilde{d}_0 \tag{4.18}$$

where $\lim_{u \to -\mathrm{i}\widetilde{\rho}_j} (u + \mathrm{i}\widetilde{\rho}_j)\psi_r^-(u) = (-\mathrm{i})\widetilde{d}_j \widetilde{\rho}_j$ and $\widetilde{d}_j$ is given by (4.3). For $1 \leq p \leq v_2$ and $1 \leq m \leq \ell_p$, we get our result by differentiating both sides of (4.18) $(m-1)$-times at $u = \mathrm{i}\alpha_k$.

(d) Observe that

$$\sum_{q=1}^{m} \frac{1}{(\alpha_p + \widetilde{\rho}_\eta)^q (\alpha_p + \omega)^{m-q+1}} = \frac{1}{\omega - \widetilde{\rho}_\eta} \left(\frac{1}{(\alpha_p + \widetilde{\rho}_\eta)^m} - \frac{1}{(\alpha_p + \omega)^m} \right).$$

Given any complex numbers $A_{p,m}$ and w, we have

$$\begin{aligned}
& \sum_{\eta=1}^{\mu_2} \sum_{p=1}^{v_2} \sum_{m=1}^{\ell_p} \frac{A_{p,m} \widetilde{d}_\eta \widetilde{\rho}_\eta}{\omega - \widetilde{\rho}_\eta} \left(\frac{1}{(\alpha_p + \widetilde{\rho}_\eta)^m} - \frac{1}{(\alpha_p + \omega)^m} \right) \\
= & \sum_{\eta=1}^{\mu_2} \sum_{p=1}^{v_2} \sum_{m=1}^{\ell_p} \sum_{q=1}^{m} \frac{A_{p,m} \widetilde{d}_\eta \widetilde{\rho}_\eta}{(\alpha_p + \widetilde{\rho}_\eta)^q (\alpha + \omega)^{m-q+1}} \\
= & \sum_{p=1}^{v_2} \sum_{m=1}^{\ell_p} \sum_{q=1}^{m} \frac{A_{p,m}}{(\alpha_p + \omega)^{m-q+1}} \left(\sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta}{(\alpha_p + \widetilde{\rho}_\eta)^q} \right) \\
= & \sum_{p=1}^{v_2} \sum_{m=1}^{\ell_p} \frac{A_{p,m}}{(\alpha_p + \omega)^m} \left(\widetilde{d}_0 1_{\{a>0 \text{ and } b=0\}} \right),
\end{aligned}$$

where the last equality follows from (4.11). The proof is complete.

(e) It is clear from (4.8) and (4.9) that

$$\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}\frac{d_s\rho_s A_s\widetilde{d}_\eta\widetilde{\rho}_\eta}{(\rho_s-\widetilde{\rho}_\eta)}\times\Bigg[-\lambda\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{(\beta_k)^j c_{kj}}{(\beta_k-\rho_s)^j}-\mu\sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p}\frac{(\alpha_p)^m\widetilde{c}_{pm}}{(\alpha_p+\rho_s)^m}-a\rho_s-\frac{b^2\rho_s^2}{2}$$
$$-\left(-\lambda\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^j}-\mu\sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p}\frac{(\alpha_p)^m\widetilde{c}_{pm}}{(\alpha_p+\widetilde{\rho}_\eta)^m}-a\widetilde{\rho}_\eta-\frac{b^2\widetilde{\rho}_\eta^2}{2}\right)\Bigg]=0. \tag{4.19}$$

This together with (4.12) yields (4.13).

(f) From Theorem 3.3, we have the following observation.

$$\begin{cases}\mu_1=\sum_{p=1}^{v_1}n_p \text{ and } \mu_2=\sum_{m=1}^{v_2}\ell_m+1, & \text{if } a<0 \text{ and } b=0\\ \mu_1=\sum_{p=1}^{v_1}n_p+1 \text{ and } \mu_2=\sum_{m=1}^{v_2}\ell_m, & \text{if } a>0 \text{ and } b=0\\ \mu_1=\sum_{p=1}^{v_1}n_p+1 \text{ and } \mu_2=\sum_{m=1}^{v_2}\ell_m+1, & \text{if } b\neq 0.\end{cases} \tag{4.20}$$

By applying the Wiener-Hopf factorization formula and combining with (4.16) and (4.18), we see that for $b=0$,

$$\begin{aligned}\frac{r}{r-\psi(u)} &= \frac{r\prod_{p=1}^{v_1}(u+\mathrm{i}\beta_p)^{n_p}\prod_{m=1}^{v_2}(u-\mathrm{i}\alpha_m)^{\ell_m}}{(-\mathrm{i}a)\prod_{j=1}^{\mu_1}(u+\mathrm{i}\rho_j)\prod_{k=1}^{\mu_2}(u+\mathrm{i}\widetilde{\rho}_k)}\\ &= \left(1_{\{\mu_1\geq 1\}}\sum_{j=1}^{\mu_1}\frac{d_j\rho_j\mathrm{i}}{u+\mathrm{i}\rho_j}+1_{\{a<0 \text{ and } b=0\}}d_0\right)\left(1_{\{\mu_2\geq 1\}}\sum_{j=1}^{\mu_2}\frac{\widetilde{d}_j\widetilde{\rho}_j(-\mathrm{i})}{u+\mathrm{i}\widetilde{\rho}_j}+1_{\{a>0 \text{ and } b=0\}}\widetilde{d}_0\right),\end{aligned} \tag{4.21}$$

and for $b\neq 0$,

$$\begin{aligned}\frac{r}{r-\psi(u)} &= \frac{r\prod_{p=1}^{v_1}(u+\mathrm{i}\beta_p)^{n_p}\prod_{m=1}^{v_2}(u-\mathrm{i}\alpha_m)^{\ell_m}}{\frac{b^2}{2}\prod_{j=1}^{\mu_1}(u+\mathrm{i}\rho_j)\prod_{k=1}^{\mu_2}(u+\mathrm{i}\widetilde{\rho}_k)}\\ &= \left(\sum_{j=1}^{\mu_1}\frac{d_j\rho_j\mathrm{i}}{u+\mathrm{i}\rho_j}\right)\left(\sum_{j=1}^{\mu_2}\frac{\widetilde{d}_j\widetilde{\rho}_j(-\mathrm{i})}{u+\mathrm{i}\widetilde{\rho}_j}\right).\end{aligned} \tag{4.22}$$

For the case $b=0$, our results follow by multiplying both sides of (4.21) by u, letting $u\to\infty$ and using (4.20). For the case $b\neq 0$, we obtain our result by multiplying both sides of (4.22) by u^2, letting $u\to\infty$ and using (4.20). ■

Observe that if $v_1\geq 1$ (i.e., there are upside jumps for the process X), then the function $\psi(\mathrm{i}z)$ is a real analytic function in $(-\beta_1,0)$ with $\psi(0)=0$ and $\lim_{z\downarrow-\beta_1}\psi(\mathrm{i}z)=\infty$. Hence, we have that $0<\rho_1<\beta_1$.

Definition 4.2 *We write $g \in \pi_0$ if the function $g : \mathbb{R} \to \mathbb{R}$ is absolutely continuous on every compact interval and moreover, for $v_1 \geq 1$, there exist $A_1 > 0, A_2 > 0$ and $\theta \in (0, \rho_1)$ such that $|g(x)| \leq A_1 + A_2 e^{\theta x}, \forall x \in \mathbb{R}$.*

For any $g \in \pi_0$, we define $Q_g(x)$ by the formula

$$\begin{aligned} & Q_g(x) \\ = \; & 1_{\{\mu_2 \geq 1\}} \sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta}{r} \Bigg\{ \sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \sum_{\ell=1}^{j} \frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k - \widetilde{\rho}_\eta)^\ell (j-\ell)!} e^{\beta_k x} \int_x^\infty (y-x)^{j-\ell} g(y) e^{-\beta_k y} dy \\ & \qquad - \Big((a + \frac{b^2 \widetilde{\rho}_\eta}{2}) g(x) + \frac{b^2}{2} g'(x) \Big) \Bigg\} \\ & + 1_{\{a>0 \text{ and } b=0\}} \frac{\widetilde{d}_0}{r} \Bigg\{ \sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \frac{-\lambda(\beta_k)^j c_{kj}}{(j-1)!} e^{\beta_k x} \int_x^\infty (u-x)^{j-1} g(u) e^{-\beta_k u} du \\ & \qquad + (\lambda + \mu + r) g(x) - a g'(x) \Bigg\}. \end{aligned} \tag{4.23}$$

We shows below that Q_g is a solution of the average problem (2.4).

Theorem 4.3 *For any $g \in \pi_0$ and $r > 0$,*

$$E\Big[Q_g(M_r + x)\Big] = g(x), \text{ for any } x \in \mathbb{R}.$$

Proof. Observe that

$$\mathbb{E} Q_g(M_r + x) = \int_0^\infty Q_g(y + x) f_M(y) dy = \int_x^\infty Q_g(u) f_M(u - x) du$$

where

$$f_{M_r}(z) = 1_{\{a<0 \text{ and } b=0\}} d_0 \delta_0(dz) + 1_{\{\mu_1 \geq 1\}} \sum_{j=1}^{\mu_1} d_j \rho_j e^{-\rho_j z} 1_{\{z>0\}}. \tag{4.24}$$

We write

$$Q_g(x) = 1_{\{\mu_2 \geq 1\}} \Big(Q_g^{(1)}(x) + Q_g^{(2)}(x) + Q_g^{(3)}(x) \Big) + 1_{\{a>0 \text{ and } b=0\}} \Big(Q_g^{(4)}(x) + Q_g^{(5)}(x) + Q_g^{(6)}(x) \Big), \tag{4.25}$$

where

$$Q_g^{(1)}(x) = \sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta}{r} \Bigg\{ \sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \sum_{\ell=1}^{j} \frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k - \widetilde{\rho}_\eta)^\ell (j-\ell)!} e^{\beta_k x} \int_x^\infty (y-x)^{j-\ell} g(y) e^{-\beta_k y} dy \Bigg\}, \tag{4.26}$$

$$Q_g^{(2)}(x) = \Big(- \sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta}{r} \Big) \frac{b^2}{2} g'(x), \tag{4.27}$$

$$Q_g^{(3)}(x) = - \sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta}{r} (a + \frac{b^2 \widetilde{\rho}_\eta}{2}) g(x), \tag{4.28}$$

$$Q_g^{(4)} = \frac{\widetilde{d}_0}{r}\left\{\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda(\beta_k)^j c_{kj}}{(j-1)!}e^{\beta_k x}\int_x^\infty (u-x)^{j-1}g(u)e^{-\beta_k u}du\right\}, \tag{4.29}$$

$$Q_g^{(5)} = \frac{\widetilde{d}_0(-a)}{r}g'(x), \tag{4.30}$$

and

$$Q_g^{(6)} = \frac{\widetilde{d}_0(\lambda+\mu+r)}{r}g(x). \tag{4.31}$$

Taking account of $g \in \pi_0$ and using integration by parts, we obtain

$$\begin{aligned} & e^{\rho_s x}\int_x^\infty e^{(\beta_k-\rho_s)u}\int_u^\infty (y-u)^{j-\ell}g(y)e^{-\beta_k y}dydu \\ = & \sum_{\xi=1}^{j-\ell+1}\frac{-(j-\ell)!e^{\beta_k x}}{(j-\ell+1-\xi)!(\beta_k-\rho_s)^\xi}\int_x^\infty (y-x)^{j-\ell+1-\xi}g(y)e^{-\beta_k y}dy \\ & +\frac{(j-\ell)!e^{\rho_s x}}{(\beta_k-\rho_s)^{j-\ell+1}}\int_x^\infty g(y)e^{-\rho_s y}dy. \end{aligned} \tag{4.32}$$

(For details, see the Appendix). For simplicity, we write

$$I_{s,k,j-\ell}^{(1)} = e^{\rho_s x}\int_x^\infty e^{(\beta_k-\rho_s)u}\int_u^\infty (y-u)^{j-\ell}g(y)e^{-\beta_k y}dydu$$

and

$$I_{k,j-\ell+1-\xi}^{(2)} = e^{\beta_k x}\int_x^\infty (y-x)^{j-\ell+1-\xi}g(y)e^{-\beta_k y}dy.$$

Using (4.24), (4.26), and (4.32), we obtain for $\mu_2 \geq 1$

$$\begin{aligned} & \int_x^\infty Q_g^{(1)}(u)f_M(u-x)du \\ = & 1_{\{\mu_1\geq 1\}}\frac{1}{r}\left\{\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\sum_{\ell=1}^{j}\frac{d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta(-\lambda)(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^\ell(j-\ell)!}\left(I_{s,k,j-\ell}^{(1)}\right)\right\} + 1_{\{a<0 \text{ and } b=0\}}d_0Q_g^{(1)}(x) \\ = & 1_{\{\mu_1\geq 1\}}\frac{1}{r}\left\{\sum_{\eta=1}^{\mu_2}\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\sum_{\ell=1}^{j}\sum_{\xi=1}^{j-\ell+1}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^\ell(j-\ell+1-\xi)!}\left(\sum_{s=1}^{\mu_1}\frac{d_s\rho_s}{(\beta_k-\rho_s)^\xi}\right)I_{k,j-\ell+1-\xi}^{(2)}\right. \\ & \left.+1_{\{\mu_1\geq 1\}}\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\left(\sum_{\ell=1}^{j}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^\ell(\beta_k-\rho_s)^{j-\ell+1}}\right)e^{\rho_s x}\int_x^\infty g(y)e^{-\rho_s y}dy\right\} \\ & +1_{\{a<0 \text{ and } b=0\}}d_0Q_g^{(1)}(x). \end{aligned} \tag{4.33}$$

For $\mu_1 \geq 1$, it follows from (b) of Lemma 4.1 that for $k = 1, ..., v_1$, and $\xi = 1, ..., n_k$

$$\sum_{j=1}^{\mu_1}\frac{d_j\rho_j}{(\beta_k-\rho_j)^\xi} = \begin{cases} 1_{\{\xi=1\}}\cdot d_0, & \text{if } a<0 \text{ and } b=0 \\ 0, & \text{otherwise.} \end{cases}$$

This implies that for $\mu_2 \geq 1$

$$\begin{aligned} & 1_{\{\mu_1\geq 1\}}\frac{1}{r}\sum_{\eta=1}^{\mu_2}\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\sum_{\ell=1}^{j}\sum_{\xi=1}^{j-\ell+1}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^\ell(j-\ell+1-\xi)!}\left(\sum_{s=1}^{\mu_1}\frac{d_s\rho_s}{(\beta_k-\rho_s)^\xi}\right)I_{k,j-\ell+1-\xi}^{(2)} \\ = & -1_{\{a<0 \text{ and } b=0\}}d_0Q_g^{(1)}(x). \end{aligned}$$

By this and the identity

$$\sum_{\ell=1}^{j}\frac{1}{(\beta_k-\widetilde{\rho}_\eta)^\ell(\beta_k-\rho_s)^{j-\ell+1}}=\frac{1}{\rho_s-\widetilde{\rho}_\eta}\cdot\left(\frac{1}{(\beta_k-\rho_s)^j}-\frac{1}{(\beta_k-\widetilde{\rho}_\eta)^j}\right)$$

(4.33) becomes

$$\begin{aligned}&\int_x^\infty Q_g^{(1)}(u)f_M(u-x)du\\ =\;& 1_{\{\mu_1\ge1\}}\frac{1}{r}\left\{\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}\frac{d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta}{\rho_s-\widetilde{\rho}_\eta}\left(\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda(\beta_k)^jc_{kj}}{(\beta_k-\rho_s)^j}+\frac{\lambda(\beta_k)^jc_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^j}\right)e^{\rho_sx}\int_x^\infty g(y)e^{-\rho_sy}dy\right\}.\end{aligned} \tag{4.34}$$

Again, by using integration by parts together with $g\in\pi_0$, we get

$$e^{\rho_sx}\int_x^\infty g'(y)e^{-\rho_sy}dy=-g(x)+\rho_se^{\rho_sx}\int_x^\infty g(y)e^{-\rho_sy}dy. \tag{4.35}$$

Combining (4.35) with (4.24) and (4.27) gives for $\mu_2\ge1$

$$\begin{aligned}&\int_x^\infty Q_g^{(2)}(u)f_M(u-x)du\\ =\;& 1_{\{\mu_1\ge1\}}(-\frac{b^2}{2r})\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta\left(e^{\rho_sx}\int_x^\infty g'(y)e^{-\rho_sy}dy\right)\\ =\;& 1_{\{\mu_1\ge1\}}(-\frac{b^2}{2r})\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta\left(-g(x)+\rho_se^{\rho_sx}\int_x^\infty g(y)e^{-\rho_sy}dy\right)\\ =\;& 1_{\{\mu_1\ge1\}}(\frac{b^2}{2r})g(x)\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta-1_{\{\mu_1\ge1\}}\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}\frac{d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta}{\rho_s-\widetilde{\rho}_\eta}\left(\frac{\rho_sb^2}{2r}(\rho_s-\widetilde{\rho}_\eta)\right)e^{\rho_sx}\int_x^\infty g(y)e^{-\rho_sy}dy.\end{aligned} \tag{4.36}$$

Also, using (4.24) and (4.28), we have for $\mu_2\ge1$

$$\begin{aligned}&\int_x^\infty Q_g^{(3)}(u)f_M(u-x)du\\ =\;& 1_{\{\mu_1\ge1\}}\frac{(-1)}{r}\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta(a+\frac{b^2\widetilde{\rho}_\eta}{2})\left(e^{\rho_sx}\int_x^\infty g(y)e^{-\rho_sy}dy\right)+1_{\{a<0,b=0\}}d_0Q_g^{(3)}(x)\\ =\;& 1_{\{\mu_1\ge1\}}\frac{(-1)}{r}\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}\frac{d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta}{\rho_s-\widetilde{\rho}_\eta}\left(a\rho_s+\frac{b^2\widetilde{\rho}_\eta\rho_s}{2}-(a\widetilde{\rho}_\eta+\frac{b^2\widetilde{\rho}_\eta^2}{2})\right)e^{\rho_sx}\int_x^\infty g(y)e^{-\rho_sy}dy\\ &+1_{\{a<0\text{ and }b=0\}}d_0Q_g^{(3)}(x).\end{aligned} \tag{4.37}$$

Combining (4.34), (4.36) and (4.37) gives for $\mu_2\ge1$

$$\begin{aligned}
&E\Big[(Q_g^{(1)}+Q_g^{(2)}+Q_g^{(3)})(M_r+x)\Big] \\
= \;& 1_{\{\mu_1\ge 1\}}\frac{1}{r}\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}\frac{d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta}{(\rho_s-\widetilde{\rho}_\eta)}e^{\rho_s x}\int_x^\infty g(y)e^{-\rho_s y}dy \\
&\times\Bigg[-\lambda\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{(\beta_k)^j c_{kj}}{(\beta_k-\rho_s)^j}-a\rho_s-\frac{b^2\rho_s^2}{2}-\Bigg(-\lambda\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^j}-a\widetilde{\rho}_\eta-\frac{b^2\widetilde{\rho}_\eta^2}{2}\Bigg)\Bigg] \\
&+1_{\{a<0 \text{ and } b=0\}}d_0Q_g^{(3)}(x)+1_{\{\mu_1\ge 1\}}(\frac{b^2}{2r})g(x)\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta.
\end{aligned}$$

Using (e) of Lemma 4.1, we obtain that for $\mu_2\ge 1$

$$\begin{aligned}
&E\Big[(Q_g^{(1)}+Q_g^{(2)}+Q_g^{(3)})(M_r+x)\Big] \\
= \;& 1_{\{\mu_1\ge 1\}}1_{\{a>0 \text{ and } b=0\}}\sum_{s=1}^{\mu_1}\sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p}\frac{(-\widetilde{d}_0)d_s\rho_s\mu(\alpha_p)^m\widetilde{c}_{pm}}{r(\alpha_p+\rho_s)^m}e^{\rho_s x}\int_x^\infty g(y)e^{-\rho_s y}dy \\
&+1_{\{a<0 \text{ and } b=0\}}d_0Q_g^{(3)}(x)+1_{\{\mu_1\ge 1\}}(\frac{b^2}{2r})g(x)\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta. \qquad (4.38)
\end{aligned}$$

Consequently, using (4.20), (4.14) and (4.25), we see that for the case $b\ne 0$,

$$\begin{aligned}
E\Big[Q_g(M_r+x)\Big] &= E\Big[(Q_g^{(1)}+Q_g^{(2)}+Q_g^{(3)})(M_r+x)\Big] \\
&= g(x)\Bigg(\frac{b^2}{2r}\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta\Bigg)=g(x),
\end{aligned}$$

and that for the case $a<0$ and $b=0$,

$$\begin{aligned}
E\Big[Q_g(M_r+x)\Big] &= E\Big[(Q_g^{(1)}+Q_g^{(2)}+Q_g^{(3)})(M_r+x)\Big] \\
&= g(x)\Bigg(\frac{(-a)d_0}{r}\sum_{\eta=1}^{\mu_2}\widetilde{d}_\eta\widetilde{\rho}_\eta\Bigg)=g(x).
\end{aligned}$$

It remains to consider the case $a>0$ and $b=0$. By (4.20), we have $\mu_1\ge 1$. It follows from (4.32), (4.24) and (4.29) that

$$\begin{aligned}
&\int_x^\infty Q_g^{(4)}(u)f_M(u-x)du \\
= \;& \sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\sum_{\xi=1}^{j}\frac{\widetilde{d}_0\lambda(\beta_k)^j c_{kj}e^{\beta_k x}}{r(j-\xi)!}\Bigg(\sum_{s=1}^{\mu_1}\frac{d_s\rho_s}{(\beta_k-\rho_s)^\xi}\Bigg)\int_x^\infty (y-x)^{j-\xi}g(y)e^{-\beta_k y}dy \\
&+\sum_{s=1}^{\mu_1}\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{\widetilde{d}_0d_s\rho_s(-\lambda)(\beta_k)^j c_{kj}e^{\rho_s x}}{r(\beta_k-\rho_s)^j}\int_x^\infty g(y)e^{-\rho_s y}dy \\
= \;& \sum_{s=1}^{\mu_1}\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{\widetilde{d}_0d_s\rho_s(-\lambda)(\beta_k)^j c_{kj}e^{\rho_s x}}{r(\beta_k-\rho_s)^j}\int_x^\infty g(y)e^{-\rho_s y}dy. \qquad (4.39)
\end{aligned}$$

The last equality comes from (b) of Lemma 4.1. Using (4.35), (4.24), and (4.30), we have

$$\int_x^\infty Q_g^{(5)}(u) f_M(u-x)du = (\frac{-a\widetilde{d}_0}{r})\sum_{s=1}^{\mu_1} d_s\rho_s\Big(-g(x)+\rho_s e^{\rho_s x}\int_x^\infty g(y)e^{-\rho_s y}dy\Big). \tag{4.40}$$

By (4.24) and (4.31), we have

$$\int_x^\infty Q_g^{(6)}(u) f_M(u-x)du = \frac{\widetilde{d}_0(\lambda+\mu+r)}{r}\sum_{s=1}^{\mu_1} d_s\rho_s e^{\rho_s x}\int_x^\infty g(y)e^{-\rho_s y}dy. \tag{4.41}$$

It follows from (4.25) (4.38), (4.39), (4.40), and (4.41) that

$$\begin{aligned} & E\Big[Q_g(M_r+x)\Big] \\ = \ & 1_{\{a>0 \text{ and } b=0\}}\sum_{s=1}^{\mu_1}\frac{\widetilde{d}_0 d_s\rho_s e^{\rho_s x}}{r}\int_x^\infty g(y)e^{-\rho_s y}dy \\ & \times\Big[-\lambda\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{(\beta_k)^j c_{kj}}{(\beta_k-\rho_s)^j}-\mu\sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p}\frac{(\alpha_p)^m\widetilde{c}_{pm}}{(\alpha_p+\rho_s)^m}+(\mu+\lambda+r)-a\rho_s\Big] \\ & +1_{\{a>0 \text{ and } b=0\}}\frac{\widetilde{d}_0 a}{r}(\sum_{j=1}^{\mu_1} d_j\rho_j)g(x). \end{aligned}$$

By (a) and (f) of Lemma 4.1, we obtain $E\Big[Q_g(M_r+x)\Big]=g(x)$. The proof is complete. ■

We write $g\in\mathcal{R}$ if $g:\mathbb{R}\to\mathbb{R}$ is a L_1-integrable function such that the Fourier transform $\widehat{g}$, defind by

$$\widehat{g}(\omega)=\int_{-\infty}^\infty e^{-\mathrm{i}\omega x}g(x)dx \tag{4.42}$$

satisfies the integrability condition $\int_{-\infty}^\infty(1+|\omega|^3)|\widehat{g}(\omega)|d\omega<\infty$. As noted in Surya [27], the set $\mathcal{R}$ belongs to the class of $\mathcal{C}_b^3$. This implies that every element in $\mathcal{R}$ also is in π_0. In [27], Surya showed that if $g\in\mathcal{R}$, the function $\frac{1}{2\pi}\int_{-\infty}^\infty e^{\mathrm{i}\omega x}\frac{\widehat{g}(\omega)}{\psi_r^+(\omega)}d\omega$ solves the American put-type averaging problem. In the following, we show that the function $\frac{1}{2\pi}\int_{-\infty}^\infty e^{\mathrm{i}\omega x}\frac{\widehat{g}(\omega)}{\psi_r^+(\omega)}d\omega$ coincides with the $Q_g(x)$ given by (4.23).

Proposition 4.4 *Given $g\in\mathcal{R}$ and $r>0$. Then*

$$Q_g(x)=\frac{1}{2\pi}\int_{-\infty}^\infty e^{\mathrm{i}\omega x}\frac{\widehat{g}(\omega)}{\psi_r^+(\omega)}d\omega. \tag{4.43}$$

Proof. By the Fourier inversion formula, it is sufficient to prove that

$$\widehat{Q}_g(\omega):=\int_{-\infty}^\infty Q_g(y)e^{-\mathrm{i}\omega y}dy=\Big(\psi_r^+(\omega)\Big)^{-1}\widehat{g}(\omega). \tag{4.44}$$

As in (4.25), we write

$$Q_g(x)=1_{\{\mu_2\geq 1\}}\Big(Q_g^{(1)}(x)+Q_g^{(2)}(x)+Q_g^{(3)}(x)\Big)+1_{\{a>0 \text{ and } b=0\}}\Big(Q_g^{(4)}(x)+Q_g^{(4)}(x)+Q_g^{(6)}(x)\Big). \tag{4.45}$$

To compute $\int_{-\infty}^{\infty} Q_g^{(1)}(y)e^{-\mathrm{i}\omega y}dy$, by using integration by parts, we have

$$\begin{aligned}
&\int_x^z e^{(\beta_k-\mathrm{i}\omega)y}\int_y^\infty (u-y)^{j-\ell}e^{-\beta_k u}g(u)dudy\\
=\ &\sum_{\xi=1}^{j-\ell+1}\frac{(j-\ell)!e^{(\beta_k-\mathrm{i}\omega)t}}{(j-\ell+1-\xi)!(\beta_k-\mathrm{i}\omega)^\xi}\int_t^\infty (y-t)^{j-\ell+1-\xi}e^{-\beta_k y}g(y)dy\Big|_{t=x}^{t=z}\\
&+\frac{(j-\ell)!}{(\beta_k-\mathrm{i}\omega)^{j-\ell+1}}\int_x^z g(y)e^{-\mathrm{i}\omega y}dy.
\end{aligned}\tag{4.46}$$

Here, for $1\le\xi\le j-\ell+1$, we have

$$\lim_{x\to-\infty} e^{-\mathrm{i}\omega x}\int_x^\infty (y-x)^{j-\ell+1-\xi}e^{-\beta_k(y-x)}g(y)dy=0 \tag{4.47}$$

and

$$\lim_{z\to\infty} e^{-\mathrm{i}\omega z}\int_z^\infty (y-z)^{j-\ell+1-\xi}e^{-\beta_k(y-z)}g(y)dy=0. \tag{4.48}$$

To prove these, observe that

$$\left|e^{-\mathrm{i}\omega t}\int_t^\infty (y-t)^{j-\ell+1-\xi}e^{-\beta_k(y-t)}g(y)dy\right|\le\int_0^\infty \left|u^{j-\ell+1-\xi}e^{-\beta_k u}g(u+t)\right|du. \tag{4.49}$$

Notice that $g\in\mathcal{R}$ implies $g\in\mathcal{C}_b^3$. Therefore g is uniformly continuous on $\mathbb{R}$. Combining this with $g\in L_1$ yields

$$\lim_{|x|\to\infty} g(x)=0. \tag{4.50}$$

This together with the Dominated Convergence Theorem, implies that the right hand side of (4.49) converges to zero, as $t\to-\infty$ or $t\to\infty$. Therefore, we justify (4.47) and (4.48). From these, we observe for $\mu_2\ge1$

$$\begin{aligned}
&\int_{-\infty}^\infty Q_g^{(1)}(y)e^{-\mathrm{i}\omega y}dy\\
=\ &\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{r}\left\{\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\sum_{\ell=1}^{j}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^\ell(j-\ell)!}\int_{-\infty}^\infty e^{(\beta_k-\mathrm{i}\omega)y}\int_y^\infty (u-y)^j e^{-\beta_k u}g(u)dudy\right\}\\
=\ &\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{r}\left\{\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\left(\sum_{\ell=1}^{j}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^\ell(\beta_k-\mathrm{i}\omega)^{j-\ell+1}}\right)\int_{-\infty}^\infty g(y)e^{-\mathrm{i}\omega y}dy\right\}\\
=\ &\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{r}\left\{\frac{1}{\mathrm{i}\omega-\widetilde{\rho}_\eta}\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\left(\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\mathrm{i}\omega)^j}+\frac{\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^j}\right)\widehat{g}(\omega)\right\}.
\end{aligned}\tag{4.51}$$

Similarly, we have

$$\int_{-\infty}^\infty Q_g^{(4)}(y)e^{-\mathrm{i}\omega y}dy=\frac{\widetilde{d}_0}{r}\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\mathrm{i}\omega)^j}\widehat{g}(\omega). \tag{4.52}$$

By using integration by parts along with (4.50), we have for $\mu_2\ge1$

$$\int_{-\infty}^\infty Q_g^{(2)}(y)e^{-\mathrm{i}\omega y}dy=-\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta b^2}{2r}\int_{-\infty}^\infty e^{-\mathrm{i}\omega x}g'(x)dx=-\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta\mathrm{i}\omega b^2}{2r}\widehat{g}(\omega), \tag{4.53}$$

and

$$\int_{-\infty}^{\infty} Q_g^{(5)}(y)e^{-\mathrm{i}\omega y}dy = \frac{\widetilde{d}_0(-a)\mathrm{i}\omega}{r}\widehat{g}(\omega). \tag{4.54}$$

Furthermore, it is clear that for $\mu_2 \geq 1$

$$\int_{-\infty}^{\infty} Q_g^{(3)}(y)e^{-\mathrm{i}\omega y}dy = -\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{r}(a+\frac{b^2\widetilde{\rho}_\eta}{2})\widehat{g}(\omega), \tag{4.55}$$

and

$$\int_{-\infty}^{\infty} Q_g^{(6)}(y)e^{-\mathrm{i}\omega y}dy = \frac{\widetilde{d}_0(\lambda+\mu+r)}{r}\widehat{g}(\omega). \tag{4.56}$$

Combining (4.45), and (4.51)-(4.56), we see that

$$\begin{aligned}
\widehat{Q}_g(\omega) &= \int_{-\infty}^{\infty} Q_g(y)e^{-\mathrm{i}\omega y}dy \\
&= \widehat{g}(\omega)\Bigg\{1_{\{\mu_2\geq 1\}}\frac{1}{r}\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{\mathrm{i}\omega-\widetilde{\rho}_\eta}\Bigg[(\mathrm{i}\omega-\widetilde{\rho}_\eta)(\frac{-\mathrm{i}\omega b^2}{2}-a-\frac{b^2\widetilde{\rho}_\eta}{2}) \\
&\quad +\Bigg(\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\mathrm{i}\omega)^j}+\frac{\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^j}\Bigg)\Bigg] \\
&\quad +1_{\{a>0 \text{ and } b=0\}}\frac{\widetilde{d}_0}{r}\Bigg(\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda\beta_k^j c_{kj}}{(\beta_k-\mathrm{i}\omega)^j}-\mathrm{i}a\omega+(\lambda+\mu+r)\Bigg)\Bigg\}.
\end{aligned} \tag{4.57}$$

Using the facts $(\mathrm{i}\omega-\widetilde{\rho}_\eta)(\frac{-\mathrm{i}\omega b^2}{2}-a-\frac{b^2\widetilde{\rho}_\eta}{2}) = \frac{\omega^2 b^2}{2}-\mathrm{i}a\omega+a\widetilde{\rho}_\eta+\frac{b^2\widetilde{\rho}_\eta^2}{2}$ and (4.12), we obtain that

$$\begin{aligned}
\widehat{Q}_g(\omega) &= \int_{-\infty}^{\infty} Q_g(y)e^{-\mathrm{i}\omega y}dy \\
&= \widehat{g}(\omega)\Bigg\{1_{\{\mu_2\geq 1\}}\frac{1}{r}\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{\mathrm{i}\omega-\widetilde{\rho}_\eta}\Bigg[\frac{\omega^2 b^2}{2}-\mathrm{i}a\omega+a\widetilde{\rho}_\eta+\frac{b^2\widetilde{\rho}_\eta^2}{2} \\
&\quad +\Bigg(\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\mathrm{i}\omega)^j}+\frac{\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^j}\Bigg)+\Bigg(\sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p}\frac{\mu(\alpha_p)^m\widetilde{c}_{pm}}{(\alpha_p+\widetilde{\rho}_\eta)^m}-\frac{\mu(\alpha_p)^m\widetilde{c}_{pm}}{(\alpha_p+\mathrm{i}\omega)^m}\Bigg)\Bigg] \\
&\quad +1_{\{a>0 \text{ and } b=0\}}\frac{\widetilde{d}_0}{r}\Bigg[\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\mathrm{i}\omega)^j}-\sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p}\frac{\mu(\alpha_p)^m\widetilde{c}_{pm}}{(\alpha_p+\mathrm{i}\omega)^m}-\mathrm{i}a\omega+(\lambda+\mu+r)\Bigg]\Bigg\}.
\end{aligned}$$

This together with (3.4) and (a) of Lemma 4.1 implies that

$$\begin{aligned}
\widehat{Q}_g(\omega) &= \widehat{g}(\omega)\Bigg\{\frac{r-\psi_r(\omega)}{r}\Bigg(1_{\{\mu_2\geq 1\}}\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{\mathrm{i}\omega-\widetilde{\rho}_\eta}+1_{\{a>0 \text{ and } b=0\}}\widetilde{d}_0\Bigg)\Bigg\} \\
&= \widehat{g}(\omega)\Bigg\{\Bigg(\frac{r-\psi_r(\omega)}{r}\Bigg)\psi_r^-(\omega)\Bigg\} = \widehat{g}(\omega)\Big(\psi_r^+(\omega)\Big)^{-1}.
\end{aligned} \tag{4.58}$$

■

Remark 4.5 *It follows from (4.57) and (4.58) that*

$$
\begin{aligned}
\left(\psi_r^{(+)}(\omega)\right)^{-1} =& 1_{\{\mu_2\ge 1\}}\frac{1}{r}\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{\mathrm{i}\omega-\widetilde{\rho}_\eta} \\
&\times\left[(\mathrm{i}\omega-\widetilde{\rho}_\eta)(\frac{-\mathrm{i}\omega b^2}{2}-a-\frac{b^2\widetilde{\rho}_\eta}{2})+\left(\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\mathrm{i}\omega)^j}+\frac{\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^j}\right)\right] \\
&+1_{\{a>0 \text{ and } b=0\}}\frac{\widetilde{d}_0}{r}\left[\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\mathrm{i}\omega)^j}-\mathrm{i}a\omega+(\lambda+\mu+r)\right]. \qquad (4.59)
\end{aligned}
$$

In the following, we study some properties of $Q_g(x)$. We write $g\in\widehat{\pi}_0$ if $g\in\pi_0$ and g is nondecreasing and $g\in C^1(\widehat{a},+\infty)$, where $\{g>0\}=(\widehat{a},+\infty)$ for some $\widehat{a}<\infty$.

Proposition 4.6 *Assume $\{X_t\}_{t\ge 0}$ is a jump-diffusion process of the form (3.2) with $c_{kj}>0$, $\beta_k>0$, and $\alpha_p>0$ for $1\le k\le v_1, 1\le j\le n_k, 1\le p\le v_2$. Consider the reward function $g\in\widehat{\pi}_0$ with $\{g>0\}=(\widehat{a},\infty)$ and assume Q_g is given by the formula in (4.23). Then*

(a) If there exists $\alpha>0$ such that $\lim_{x\to\infty}Q_g(x)\ge\alpha$, then there exists $x^*>\widehat{a}$ such that $Q_g(x^*)=0$.

(b) If $(\frac{g(u+x)}{g(x)})'<0$ and $(\frac{g'(x)}{g(x)})'<0$ for any $x>\widehat{a}$ and $u>0$, then there exists at most one $x^*\in(\widehat{a},+\infty)$ such that $Q_g(x^*)=0$.

(c) If both conditions in (a) *and* (b) *hold, then there exists a unique $x^*\in(\widehat{a},+\infty)$ such that $Q_g(x^*)=0$. Moreover, $Q_g(x)$ is increasing for $x>x^*$ and $Q_g(x)\le 0$ for $\widehat{a}<x\le x^*$.*

Proof. Observe that

$$
\begin{aligned}
&Q_g(x) \\
=&\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\sum_{\ell=1}^{j}\frac{-\lambda(\beta_k)^j c_{kj}}{r(j-\ell)!} \\
&\times\left[\left(1_{\{\mu_2\ge 1\}}\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{(\beta_k-\widetilde{\rho}_\eta)^\ell}+1_{\{a>0\text{ and }b=0\}}1_{\{\ell=1\}}\widetilde{d}_0\right)\int_0^\infty u^{j-\ell}g(u+x)e^{-\beta_k u}du\right] \\
&-\left[1_{\{\mu_2\ge 1\}}\frac{b^2}{2r}\left(\sum_{\eta=1}^{\mu_2}\widetilde{d}_\eta\widetilde{\rho}_\eta\right)+1_{\{a>0\text{ and }b=0\}}\frac{a\widetilde{d}_0}{r}\right]g'(x) \\
&-\left[1_{\{\mu_2\ge 1\}}\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{r}(a+\frac{b^2\widetilde{\rho}_\eta}{2})-1_{\{a>0\text{ and }b=0\}}\frac{\widetilde{d}_0}{r}(\lambda+\mu+r)\right]g(x). \qquad (4.60)
\end{aligned}
$$

We first show that $\lim_{t\to\widehat{a}^+}Q_g(t)<0$. To do this, we first claim that for $\mu_2\ge 1$ and $b\ne 0$

$$\sum_{\eta=1}^{\mu_2}\widetilde{d}_\eta\widetilde{\rho}_\eta=\frac{\prod_{j=1}^{\mu_2}-\widetilde{\rho}_j}{\prod_{k=1}^{v_2}(\alpha_k)^{\ell_k}}>0, \qquad (4.61)$$

and for $a>0$ and $b=0$

$$\widetilde{d}_0=\frac{\prod_{j=1}^{\mu_2}-\widetilde{\rho}_j}{\prod_{k=1}^{v_2}(\alpha_k)^{\ell_k}}>0. \qquad (4.62)$$

Also, we will show that for $1 \le k \le v_1$ and $1 \le \ell \le n_k$,

$$1_{\{\mu_2\ge 1\}}\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{(\beta_k-\widetilde{\rho}_\eta)^\ell}+1_{\{a>0\text{ and }b=0\}}1_{\{\ell=1\}}\widetilde{d}_0>0. \tag{4.63}$$

From identities (4.17) and (4.18), we acquire

$$\mathbb{E}[e^{uI_r}]=\prod_{k=1}^{v_2}\left(\frac{u+\alpha_k}{\alpha_k}\right)^{\ell_k}\prod_{j=1}^{\mu_2}\left(\frac{-\widetilde{\rho}_j}{u-\widetilde{\rho}_j}\right)=1_{\{\mu_2\ge 1\}}\sum_{j=1}^{\mu_2}\frac{\widetilde{d}_j\widetilde{\rho}_j}{u-\widetilde{\rho}_j}+1_{\{a>0\text{ and }b=0\}}\widetilde{d}_0. \tag{4.64}$$

We obtain (4.61) by multiplying both sides by u, letting $u\to\infty$ in (4.64) and using the fact that $\mu_2=\sum_{k=1}^{v_2}\ell_k+1$. Similarly, (4.62) follows by letting $u\to\infty$ in (4.64) and using the fact that $\mu_2=\sum_{k=1}^{v_2}\ell_k$. To verify (4.63), we note that differentiating both sides of (4.64) at $u=\beta_k$ for ξ-times implies

$$\mathbb{E}[(I_r)^\xi e^{\beta_k I_r}]=1_{\{\mu_2\ge 1\}}(-1)^\xi\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{(\beta_k-\widetilde{\rho}_\eta)^{\xi+1}}+1_{\{a>0\text{ and }b=0\}}1_{\{\xi=0\}}\widetilde{d}_0.$$

This yields (4.63). Using (4.61)-(4.63) and (4.60), we obtain

$$\lim_{t\to\widehat{a}^+}Q_g(t)<0. \tag{4.65}$$

To prove (a), notice that by the assumption in (a), we have $\lim_{x\to+\infty}Q_g(x)>0$ and $Q_g(x)\in C(\widehat{a},+\infty)$. These together with (4.65) and the intermediate-value theorem, imply that there exists at least one x^* in $(\widehat{a},\infty)$.

To prove (b), we write $Q_g(x)=g(x)h(x)$ for $x\in(\widehat{a},\infty)$, where

$$\begin{aligned}
&h(x)\\
&=\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\sum_{\ell=1}^{j}\frac{-\lambda(\beta_k)^j c_{kj}}{r(j-\ell)!}\\
&\quad\times\left[\left(1_{\{\mu_2\ge 1\}}\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{(\beta_k-\widetilde{\rho}_\eta)^\ell}+1_{\{a>0\text{ and }b=0\}}1_{\{\ell=1\}}\widetilde{d}_0\right)\int_0^\infty u^{j-\ell}\frac{g(u+x)}{g(x)}e^{-\beta_k u}du\right]\\
&\quad-\left[1_{\{\mu_2\ge 1\}}\frac{b^2}{2r}\left(\sum_{\eta=1}^{\mu_2}\widetilde{d}_\eta\widetilde{\rho}_\eta\right)+1_{\{a>0\text{ and }b=0\}}\frac{a\widetilde{d}_0}{r}\right]\frac{g'(x)}{g(x)}\\
&\quad-\left[1_{\{\mu_2\ge 1\}}\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{r}(a+\frac{b^2\widetilde{\rho}_\eta}{2})-1_{\{a>0\text{ and }b=0\}}\frac{\widetilde{d}_0}{r}(\lambda+\mu+r)\right].
\end{aligned}$$

Taking account of the equations (4.61)-(4.63) and the conditions $(\frac{g(u+x)}{g(x)})'<0$, $(\frac{g'(x)}{g(x)})'<0$ for any $x>\widehat{a}$ and $u>0$, we see that $h'(x)>0$ for any $x>\widehat{a}$. This implies that there exists at most one $x^*\in(\widehat{a},\infty)$ such that $h(x^*)=0$. Hence, $Q_g(x)=0$ has at most one solution in $(\widehat{a},+\infty)$.

To prove (c), by (a) and (b) we see that there exists only one $x^*\in(\widehat{a},\infty)$ such that $Q_g(x^*)=0$. Furthermore, since $\lim_{t\to\widehat{a}^+}Q_g(t)<0$, Q_g is continuous on $(\widehat{a},\infty)$ and $Q_g(x)=0$

has an unique solution on $(\widehat{a},\infty)$, we have $Q_g(x)<0$, for $x\in(\widehat{a},x^*)$. For $x>x^*$, we have $Q_g(x)=g(x)h(x)$ and hence $Q'_g(x)=g'(x)h(x)+g(x)h'(x)$. Since each term of the right hand side is nonnegative, and $g(x)$ and $h'(x)$ are positive, we obtain $Q'_g(x)>0$ for $x>x^*$ and hence $Q_g(x)$ is increasing on (x^*,∞). ■

Combining Theorem 2.7, Theorem 4.3, and Proposition 4.6 gives the following main result.

Theorem 4.7 *Assume $\{X_t\}_{t\geq 0}$ is a jump-diffusion process of the form (3.2) with $c_{kj}>0$, $\beta_k>0$, and $\alpha_p>0$ for $1\leq k\leq v_1, 1\leq j\leq n_k, 1\leq p\leq v_2$. Consider a payoff function $g(x)\in\widehat{\pi}_0$ with $Q_g(x)$ given by (4.23). Assume that the following conditions hold:*

(a) There exists $\alpha>0$ such that $\lim_{x\to\infty}Q_g(x)\geq\alpha$.

(b) $(\frac{g(u+x)}{g(x)})'<0$ and $(\frac{g'(x)}{g(x)})'<0$ for any $x>\widehat{a}$ and $u>0$.

Then the optimal stopping time for the optimal stopping problem (1.1) is given by $\tau^*=\inf\{t>0:X_t>x^*\}$ and the value function is given by

$$V(x)=\mathbb{E}_x(e^{-r\tau^*}g(X_{\tau^*}))=\int_{x^*-x}^{\infty}Q_g(x+m)f_{M_r}(m)dm.$$

Here x^* is the unique solution of the equation $Q_g(x)=0$ in $(\widehat{a},+\infty)$ and f_{M_r} is given by (4.4).

Remark 4.8 *The inspiration of giving the explicit formula for solutions of the averaging problem (2.4) comes from the well-known result. If $g\in C_0^2(\mathbb{R}^n)$ then*

$$\mathbb{E}_x\Big[\int_0^\infty e^{-rt}(r-\mathcal{A})g(X_t)dt\Big]=g(x).$$

Furthermore, observe that

$$\begin{aligned}\mathbb{E}_x\Big[\int_0^\infty e^{-rt}(r-\mathcal{A})g(X_t)dt\Big]&=\mathbb{E}\Big[(\frac{r-\mathcal{A}}{r})g(x+X_{e_r})\Big]\\&=\mathbb{E}\Big[\mathbb{E}\Big((\frac{r-\mathcal{A}}{r})g(x+M_r+I_r)|M_r\Big)\Big]\\&=\mathbb{E}\Big[Q_g(x+M_r)\Big]\end{aligned}$$

where

$$Q_g(z)=\mathbb{E}\Big[(\frac{r-\mathcal{A}}{r})g(z+I_r)\Big].$$

In fact, the identity (4.23) follows from expanding the above expression of $Q_g(x)$. In addition, using similar argument, we also obtain

$$g(x)=\mathbb{E}_x\Big[\int_0^\infty e^{-rt}(r-\mathcal{A})g(X_t)dt\Big]=\mathbb{E}\Big[P_g(x+I_r)\Big]$$

where

$$P_g(z)=\mathbb{E}\Big[(\frac{r-\mathcal{A}}{r})g(z+M_r)\Big].$$

Observe that if $v_2 \geq 1$ (i.e., there are downside jumps for the process X), then the function $\psi(\mathrm{i}z)$ is a real analytic function in $(0, \alpha)$ with $\psi(0) = 0$ and $\lim_{z \uparrow \alpha_1} \psi(\mathrm{i}z) = \infty$. Hence, we have that $0 < -\widetilde{\rho}_1 < \alpha$.

Definition 4.9 *We write $g \in \pi_1$ if the function $g : \mathbb{R} \to \mathbb{R}$ is absolutely continuous on every compact interval and moreover, for $v_2 \geq 1$, there exist $A_1 > 0, A_2 > 0$ and $\theta \in (0, -\widetilde{\rho}_1)$ such that $|g(x)| \leq A_1 + A_2 e^{-\theta x}, \forall x \in \mathbb{R}$.*

For any $g \in \pi_1$, we define $P_g(x)$ by the formula

$$\begin{aligned} &P_g(x) \\ =&\sum_{p=1}^{v_2} \sum_{m=1}^{\ell_p} \sum_{k=1}^{m} \frac{-\mu(\alpha_p)^m \widetilde{c}_{pm}}{r(m-k)!} \\ &\times \left[\left(1_{\{\mu_1 \geq 1\}} \sum_{j=1}^{\mu_1} \frac{d_j \rho_j}{(\alpha_p + \rho_j)^k} + 1_{\{a<0 \text{ and } b=0\}} 1_{\{k=1\}} d_0 \right) \int_{-\infty}^{0} (-t)^{m-k} g(t+x) e^{\alpha_p t} dt \right] \\ &+ \left[1_{\{\mu_1 \geq 1\}} \frac{b^2}{2r} \left(\sum_{j=1}^{\mu_1} d_j \rho_j \right) - 1_{\{a<0 \text{ and } b=0\}} \frac{a d_0}{r} \right] g'(x) \\ &+ \left[1_{\{\mu_1 \geq 1\}} \sum_{j=1}^{\mu_1} \frac{d_j \rho_j}{r} (a + \frac{b^2 \rho_j}{2}) + 1_{\{a<0 \text{ and } b=0\}} \frac{d_0}{r} (\lambda + \mu + r) \right] g(x). \end{aligned} \tag{4.66}$$

Remark 4.10 $P_g(-x) = \widehat{Q}_{\widehat{g}}(x)$ *where $\widehat{Q}_{\widehat{g}}$ is given in (4.23) for $\widehat{g}(x) = g(-x)$ and the process $-X_t$.*

In the following, we study some properties of $P_g(x)$. We write $g \in \widehat{\pi}_1$ if $g \in \pi_1$ and g is non-increasing and $g \in C^1(-\infty, \widehat{a})$, where $\{g > 0\} = (-\infty, \widehat{a})$ for some $\widehat{a} > -\infty$.

Proposition 4.11 *Assume $\{X_t\}_{t \geq 0}$ is a jump-diffusion process of the form (3.2) with $\widetilde{c}_{pm} > 0$, $\beta_k > 0$, and $\alpha_p > 0$ for $1 \leq p \leq v_2, 1 \leq m \leq \ell_p, 1 \leq k \leq v_1$. Consider the reward function $g \in \widehat{\pi}_1$ with $\{g > 0\} = (-\infty, \widehat{a})$ and assume P_g is given by the formula in (4.66). Then*

(a) If there exists $\beta > 0$ such that $\lim_{x \to -\infty} P_g(x) \geq \beta$, then there exists $x^* < \widehat{a}$ such that $P_g(x^*) = 0$.

(b) If $(\frac{g(t+x)}{g(x)})' > 0$ and $(\frac{g'(x)}{g(x)})' < 0$ for any $x < \widehat{a}$ and $t < 0$, then there exists at most one $x^* \in (-\infty, \widehat{a})$ such that $P_g(x^*) = 0$.

(c) If both conditions in (a) *and* (b) *hold, then there exists a unique $x^* \in (-\infty, \widehat{a})$ such that $P_g(x^*) = 0$. Moreover, $P_g(x)$ is decreasing for $x < x^*$ and $P_g(x) \leq 0$ for $x^* \leq x < \widehat{a}$.*

Proof. We first show that $\lim_{t \to \widehat{a}^-} P_g(t) < 0$. To do this, we first verify that for $\mu_1 \geq 1$ and $b \neq 0$

$$\sum_{j=1}^{\mu_1} d_j \rho_j = \frac{\prod_{\eta=1}^{\mu_1} \rho_\eta}{\prod_{k=1}^{v_1} \beta_k} > 0 \tag{4.67}$$

and for $a < 0$ and $b = 0$

$$d_0 = \frac{\prod_{\eta=1}^{\mu_1} \rho_\eta}{\prod_{k=1}^{v_1} \beta_k} > 0. \tag{4.68}$$

Moreover, we will show that $1 \le p \le v_2$ and $1 \le k \le \ell_p$

$$1_{\{\mu_1 \ge 1\}} \sum_{j=1}^{\mu_1} \frac{d_j \rho_j}{(\alpha_p + \rho_j)^k} + 1_{\{a<0 \text{ and } b=0\}} 1_{\{k=1\}} d_0 > 0. \tag{4.69}$$

From identities (4.15) and (4.16), we have that

$$\mathbb{E}_x[e^{-uM_r}] = \prod_{k=1}^{v_1} \left(\frac{u+\beta_k}{\beta_k} \right)^{n_k} \prod_{j=1}^{\mu_1} \left(\frac{\rho_j}{u+\rho_j} \right) = 1_{\{\mu_1 \ge 1\}} \sum_{j=1}^{\mu_1} \frac{d_j \rho_j}{u+\rho_j} + 1_{\{a<0 \text{ and } b=0\}} d_0. \tag{4.70}$$

We obtain (4.67) by multiplying both sides by u, letting $u \to \infty$ in (4.70) and using the fact that $\mu_1 = \sum_{k=1}^{v_1} n_k + 1$. Similarly, (4.68) follows by letting $u \to \infty$ in (4.70) and using the fact that $\mu_1 = \sum_{k=1}^{v_1} n_k$. To verify (4.69), we note that differentiating both sides of (4.70) at $u = \alpha_p$ for ξ-times implies

$$\mathbb{E}_x[(-M_r)^\xi e^{(-M_r)\alpha_p}] = 1_{\{\mu_1 \ge 1\}} (-1)^\xi \left(\sum_{j=1}^{\mu_1} \frac{d_j \rho_j}{(\alpha_p + \rho_j)^{\xi+1}} + 1_{\{a<0 \text{ and } b=0\}} 1_{\{\xi=0\}} d_0 \right).$$

This yields (4.69). Using (4.67)-(4.69) and (4.66), we obtain

$$\lim_{t \to \widehat{a}^-} P_g(t) < 0. \tag{4.71}$$

To prove (a), notice that by the assumption in (a), we have $\lim_{x \to -\infty} P_g(x) > 0$ and $P_g(x) \in C(-\infty, \widehat{a})$. These together with (4.71) and the intermediate-value theorem, imply that there exists at least one x^* in $(-\infty, \widehat{a})$.

To prove (b), we write $P_g(x) = g(x)h(x)$ for $x \in (-\infty, \widehat{a})$, where

$$\begin{aligned} &h_g(x) \\ =&\sum_{p=1}^{v_2} \sum_{m=1}^{\ell_p} \sum_{k=1}^{m} \frac{-\mu(\alpha_p)^m \widetilde{c}_{pm}}{r(m-k)!} \\ &\times \left[\left(1_{\{\mu_1 \ge 1\}} \sum_{j=1}^{\mu_1} \frac{d_j \rho_j}{(\alpha_p + \rho_j)^k} + 1_{\{a<0 \text{ and } b=0\}} 1_{\{k=1\}} d_0 \right) \int_{-\infty}^0 (-t)^{m-k} \frac{g(t+x)}{g(x)} e^{\alpha_p t} dt \right] \\ &+ \left[1_{\{\mu_1 \ge 1\}} \frac{b^2}{2r} \left(\sum_{j=1}^{\mu_1} d_j \rho_j \right) - 1_{\{a<0 \text{ and } b=0\}} \frac{a d_0}{r} \right] \frac{g'(x)}{g(x)} \\ &+ \left[1_{\{\mu_1 \ge 1\}} \sum_{j=1}^{\mu_1} \frac{d_j \rho_j}{r} (a + \frac{b^2 \rho_j}{2}) + 1_{\{a<0 \text{ and } b=0\}} \frac{d_0}{r} (\lambda + \mu + r) \right]. \end{aligned}$$

Taking account of the equations (4.67)-(4.69) and the conditions $(\frac{g(t+x)}{g(x)})' > 0$, $(\frac{g'(x)}{g(x)})' < 0$ for any $x < \widehat{a}$ and $t < 0$, we see that $h'(x) < 0$ for any $x < \widehat{a}$. This implies that there exists at most one $x^* \in (-\infty, \widehat{a})$ such that $h(x^*) = 0$. Hence, $P_g(x) = 0$ has at most one solution in $(-\infty, \widehat{a})$.

To prove (c), by (a) and (b) we see that there exists only one $x^* \in (-\infty, \widehat{a})$ such that $P_g(x^*) = 0$. Furthermore, since $\lim_{t \to \widehat{a}^-} P_g(t) < 0$, P_g is continuous on $(-\infty, \widehat{a})$ and $P_g(x) = 0$

has an unique solution on $(-\infty, \widehat{a})$, we have $P_g(x) < 0$, for $x \in (x^*, \widehat{a})$. For $x < x^*$, we have $P_g(x) = g(x)h(x)$ and hence $P_g'(x) = g'(x)h(x) + g(x)h'(x)$. By means of the facts that $g' \leq 0$, $g > 0$, $h' < 0$ and $h > 0$, we have that $P_g'(x) < 0$ for $x < x^*$ and hence $P_g(x)$ is decreasing on $(-\infty, x^*)$. ■

Theorem 4.12 *Assume $\{X_t\}_{t\geq 0}$ is a jump-diffusion process of the form (3.2) with $\widetilde{c}_{pm} > 0$, $\beta_k > 0$, and $\alpha_p > 0$ for $1 \leq p \leq v_2, 1 \leq m \leq \ell_p, 1 \leq k \leq v_1$. Consider a payoff function $g(x) \in \widehat{\pi}_1$ with $P_g(x)$ given by (4.66). Assume that the following conditions hold:*

(a) There exists $\beta > 0$ such that $\lim_{x\to-\infty} P_g(x) \geq \beta$.

(b) $(\frac{g(t+x)}{g(x)})' > 0$ and $(\frac{g'(x)}{g(x)})' < 0$ for any $x < \widehat{a}$ and $t < 0$.

Then the optimal stopping time for the optimal stopping problem (1.1) is given by $\tau^* = \inf\{t > 0 : X_t < x^*\}$ and the value function is given by

$$V(x) = \mathbb{E}_x(e^{-r\tau^*} g(X_{\tau^*})) = \int_{-\infty}^{x^*-x} P_g(x+z) f_{I_r}(z) dz.$$

Here x^* is the unique solution of the equation $P_g(x) = 0$ in $(-\infty, \widehat{a})$ and f_{I_r} is given by (4.1).

5 Some examples

In the following, we apply our results to some concrete examples. In particular we reproduce the special results of those discussed, among others, in Kyprianou and Surya [16], Novikov and Shiryaev [22], and Deligiannidis et al. [10]. In all examples below, we always assume that $\{X_t\}_{t\geq 0}$ is a jump-diffusion process of the form (3.2) with $c_{kj} > 0$, $\beta_k > 0$, and $\alpha_p > 0$ for $1 \leq k \leq v_1, 1 \leq j \leq n_k, 1 \leq p \leq v_2$. We consider some special functions g and verify that it satisfies conditions (a)-(b) in Theorem 4.7.

Example 5.1 *(Option with power function).*
Consider the optimal stopping problem (1.1) with $g(x) = (x^+)^\gamma$, $\gamma > 1$. *According to* (4.23), $Q_g(x)$ *is given by the formula*

$$\begin{aligned}
& Q_g(x) \\
= \ & 1_{\{\mu_2\geq 1\}} \sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta}{r} \Bigg[\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\sum_{\ell=1}^{j} \frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^\ell (j-\ell)!} \int_0^\infty u^{j-\ell} e^{-\beta_k u}(u+x)^\gamma du \\
& \qquad -\Big((a+\frac{b^2\widetilde{\rho}_\eta}{2})x^\gamma + \frac{b^2}{2}\gamma x^{\gamma-1}\Big)\Bigg] \\
& +1_{\{a>0 \ and \ b=0\}} \frac{\widetilde{d}_0}{r} \Bigg\{ \sum_{k=1}^{v_1}\sum_{j=1}^{n_k} \frac{-\lambda(\beta_k)^j c_{kj}}{(j-1)!} \int_0^\infty u^{j-1} e^{-\beta_k u}(u+x)^\gamma du \\
& \qquad +(\lambda+\mu+r)x^\gamma - a\gamma x^{\gamma-1}\Bigg\}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
& \lim_{x\to\infty} Q_g(x)x^{-\gamma} \\
= \ & 1_{\{\mu_2\geq 1\}} \sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta}{r} \Bigg\{ \sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\sum_{\ell=1}^{j} \frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^\ell \beta_k^{j-\ell+1}} - (a+\frac{b^2\widetilde{\rho}_\eta}{2}) \Bigg\} \\
& +1_{\{a>0 \ and \ b=0\}} \frac{\widetilde{d}_0}{r}(\mu+r).
\end{aligned}$$

By using the identity that

$$\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\Big(\sum_{\ell=1}^{j} \frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^\ell(\beta_k-\theta)^{j-\ell+1}}\Big) = \frac{1}{\theta-\widetilde{\rho}_\eta}\Big(\sum_{k=1}^{v_1}\sum_{j=1}^{n_k} \frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\theta)^j} + \frac{\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^j}\Big) \quad (5.1)$$

and Remark 4.5, we see that $\lim_{x\to\infty} Q_g(x)x^{-\gamma} = \Big(\psi_r^+(0)\Big)^{-1} = 1$, *which implies that* $\lim_{x\to\infty} Q_g(x) = \infty$. *Also, observe that for* $x > 0$ *and* $u > 0$

$$\Big(\frac{g(u+x)}{g(x)}\Big)' = \Big((1+\frac{u}{x})^\gamma\Big)' = \gamma(1+\frac{u}{x})^{\gamma-1}(\frac{-u}{x^2}) < 0,$$

and

$$\Big(\frac{g'(x)}{g(x)}\Big)' = (\gamma x^{-1})' = -\gamma x^{-2} < 0.$$

By Theorem 4.7, there exists a unique x^* such that $Q_g(x^*) = 0$ and $\tau^* := \inf\{t \ge 0 : X_t \ge x^*\}$ is the optimal stopping time for the optimal stopping problem (1.1) with $g(x) = (x^+)^\gamma$, $\gamma > 1$.

■

Remark 5.2 *Assume that $g(x) = (x^+)^n$, where $n \in \mathbb{N} \cup \{0\}$. Write $Q_n(x) = Q_g(x)$. Direct calculations show that $Q_n(x)$ satisfies $Q_0(x) = \Big(\psi_r^+(0)\Big)^{-1} = 1$, $\frac{d}{dx}Q_n(x) = nQ_{n-1}(x)$ and $\mathbb{E}\big[Q_n(M_r)\big] = 0$. Hence the functions $Q_n(x)$ are just the Appell polynomials for the random variable M_r in [16]. For Appell functions of any order $\gamma \ne 0$ and related works, see Novikov and Shiryaev [22] and Deligiannidis et al. [10].*

■

In the following example, we consider a special jump-diffusion model so that we can obtain a simple form for the value function.

Example 5.3 *Consider the case that $g(x) = (x^+)^\gamma$ with $\gamma > 1$, and $X_t = at + \sum_{i=1}^{N_t^\lambda} Y_i^\beta$ where $a < 0$ and $\{Y_i^\beta : i = 1, 2, ...\}$ is a sequence of independent exponentially-distributed random variables with parameter β. Under these model assumptions, we have $\psi(z) = \mathrm{i}az - \frac{\lambda z}{z + \mathrm{i}\beta}$ and $f_{M_r}(y) = d_0\delta_0(dy) + d_1\rho_1 e^{-\rho_1 y}dy$, where $d_1 = \frac{\beta - \rho_1}{\beta}$, $d_0 = \frac{\rho_1}{\beta}$, and $\{-\mathrm{i}\rho_1, -\mathrm{i}\widetilde{\rho}_1\}$ are the solutions of $r - \psi(z) = 0$. Also,we have*

$$Q_g(x) = \frac{-\widetilde{\rho}_1}{r}\left(-\frac{\lambda\beta}{\beta - \widetilde{\rho}_1}\int_x^\infty y^\gamma e^{-\beta(y-x)}dy - ax^\gamma\right),$$

for every $x > 0$. Hence, for each $x < x^*$, the value function is given by the formula

$$\begin{aligned} V(x) &= \mathbb{E}\Big(Q_g(x + M_r)1_{\{x+M_r > x^*\}}\Big) = d_1\rho_1 \int_{x^*}^\infty e^{-\rho_1(y-x)}Q_g(y)dy \\ &= \frac{-(\beta - \rho_1)\rho_1\widetilde{\rho}_1}{\beta r}e^{\rho_1 x}\int_{x^*}^\infty e^{-\rho_1 y}\left(-ay^\gamma - \frac{\lambda\beta}{\beta - \widetilde{\rho}_1}e^{\beta y}\int_y^\infty z^\gamma e^{-\beta z}dz\right)dy. \end{aligned}$$

Since

$$\int_{x^*}^\infty e^{(\beta-\rho_1)y}\int_y^\infty z^\gamma e^{-\beta z}dzdy = \frac{1}{\beta - \rho_1}\left(\int_{x^*}^\infty e^{-\rho_1 z}z^\gamma - e^{(\beta-\rho_1)x^*}\int_{x^*}^\infty e^{-\beta z}z^\gamma dz\right),$$

we see that

$$\begin{aligned} V(x) &= \frac{-(\beta - \rho_1)\rho_1\widetilde{\rho}_1}{\beta r}e^{\rho_1 x}\Bigg[(-a - \frac{\lambda\beta}{(\beta - \rho_1)(\beta - \widetilde{\rho}_1)})\int_{x^*}^\infty e^{-\rho_1 z}z^\gamma dz \\ &\qquad + \frac{\lambda\beta}{(\beta - \rho_1)(\beta - \widetilde{\rho}_1)})e^{(\beta-\rho_1)x^*}\int_{x^*}^\infty e^{-\beta z}z^\gamma dz\Bigg]. \end{aligned}$$

Since $-\mathrm{i}\rho_1$ and $-\mathrm{i}\widetilde{\rho}_1$ are the solutions of $r - \psi(z) = 0$, we obtain $-a - \frac{\lambda\beta}{(\beta-\rho_1)(\beta-\widetilde{\rho}_1)} = 0$. This together with $Q_g(x^*) = 0$ yields $V(x) = \frac{a\widetilde{\rho}_1\rho_1}{\beta r}e^{\rho_1(x-x^*)}(x^*)^\gamma$. Furthermore, since $\psi(0) = 0$ and $r - \psi(z) = \frac{(-\mathrm{i}a)(z+\mathrm{i}\rho_1)(z+\mathrm{i}\widetilde{\rho}_1)}{z+\mathrm{i}\beta}$, we have $\frac{a\widetilde{\rho}_1\rho_1}{\beta r} = 1$. This implies that $V(x) = e^{\rho_1(x-x^*)}(x^*)^\gamma$.

Clearly, V is continuous at the optimal boundary x^*. Since $V'(x^{*-}) = \rho_1(x^*)^\gamma$ and $g'(x^*) = \gamma(x^*)^{\gamma-1}$, there is no smooth fit at x^* as $x^* \neq \frac{\gamma}{\rho_1}$. To show $x^* \neq \frac{\gamma}{\rho_1}$, we set

$$F(x) = \frac{-\lambda\beta}{a(\beta - \widetilde{\rho}_1)} \int_0^\infty (1 + \frac{z}{x})^\gamma e^{-\beta z} dz.$$

Then using the inequality

$$(1 + \frac{z}{x})^\gamma \leq e^{\frac{\gamma z}{x}},$$

we observe that

$$F(x) < \frac{\lambda\beta}{(-a)(\beta - \widetilde{\rho}_1)} \cdot \frac{1}{\beta - \frac{\gamma}{x}}.$$

This implies that $F(\frac{\gamma}{\rho_1}) < \frac{\lambda\beta}{(-a)(\beta-\widetilde{\rho}_1)(\beta-\rho_1)} = 1$ and hence $Q_g(\frac{\gamma}{\rho_1}) > 0$. Consequently, $x^* < \frac{\gamma}{\rho_1}$. Note that $\{0\}$ is not regular for the half-line $(0, \infty)$ for the process $\{X_t\}$. Our results show no contradiction with the general results of Theorem 5.1 in Surya [27]. Similar results were obtained for the case $r = 0$ by Mordecki and Salminen [20].

■

Example 5.4 *(Perpetual American call option).*
We consider the optimal stopping problem (1.1) with $g(x) = (e^x - K)^+$ and assume $\rho_1 > 1$. By (4.23), *we have for $x > \ln K$*

$$\begin{aligned}
& Q_g(x) \\
= \; & 1_{\{\mu_2 \geq 1\}} \sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta}{r} \left[\sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \sum_{\ell=1}^{j} \frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k - \widetilde{\rho}_\eta)^\ell (j-\ell)!} \int_0^\infty u^{j-\ell} e^{-\beta_k u} (e^{u+x} - K) du \right. \\
& \left. - \left((a + \frac{b^2 \widetilde{\rho}_\eta}{2})(e^x - K) + \frac{b^2}{2} e^x \right) \right] \\
& + 1_{\{a>0 \text{ and } b=0\}} \frac{\widetilde{d}_0}{r} \left\{ \sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \frac{-\lambda(\beta_k)^j c_{kj}}{(j-1)!} \int_0^\infty u^{j-1} e^{-\beta_k u} (e^{u+x} - K) du \right. \\
& \left. + (\lambda + \mu + r)(e^x - K) - a e^x \right\}.
\end{aligned}$$

Moreover, observe that

$$\begin{aligned}
& \lim_{x\to\infty} Q_g(x) e^{-x} \\
= \; & 1_{\{\mu_2 \geq 1\}} \sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta}{r} \left\{ \sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \sum_{\ell=1}^{j} \frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k - \widetilde{\rho}_\eta)^\ell (\beta_k - 1)^{j-\ell+1}} - (a + \frac{b^2 \widetilde{\rho}_\eta}{2}) - \frac{b^2}{2} \right\} \\
& + 1_{\{a>0 \text{ and } b=0\}} \frac{\widetilde{d}_0}{r} \left(\sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k - 1)^j} + (\lambda + \mu + r) - a \right).
\end{aligned}$$

By using (5.1) and Remark 4.5, we see that $\lim_{x\to\infty} Q_g(x) e^{-x} = \left(\psi_r^+(-\mathrm{i}) \right)^{-1}$. This along with $1 < \rho_1$ implies that $\lim_{x\to\infty} Q_g(x) = \infty$. Also, we have that for $x > \ln K$ and $u > 0$

$$\left(\frac{g(u+x)}{g(x)} \right)' = \frac{K e^x (e^x - e^{u+x})}{(e^x - K)^2} < 0, \text{ and } \left(\frac{g'(x)}{g(x)} \right)' = \frac{-K e^{2x}}{(e^x - K)^2} < 0.$$

Hence, by Theorem 4.7, we obtain the optimal stopping boundary x^* and the pricing formula in terms of Q_g and f_{M_r}. The solution was obtained earlier by Mordecki [18] for general Lévy process. ■

Remark 5.5 *(Perpetual American Call Option). If $g(x) = \sum_{m=1}^{M} h_m e^{\theta_m x}$ with $0 \le \max\{\theta_m : 1 \le m \le M\} < \rho_1$ then*

$$Q_g(x) = \sum_{m=1}^{M} h_m e^{\theta_m x} \Big(\psi_r^+(-\mathrm{i}\theta_m) \Big)^{-1}. \tag{5.2}$$

(For a proof, see the Appendix). The result is consistent with [18], [7], and [27]. In particular, if $g(x) = (e^x - K)^+$ and $\rho_1 > 1$, then $Q_g(x) = e^x \Big(\psi_r^+(-\mathrm{i}) \Big)^{-1} - K$. Denote by x^* the unique root of $Q_g(x) = 0$. Then x^* is the optimal stopping boundary and the value function for $x < x^*$ is given by the formula $V(x) = \int_{x^*-x}^{\infty} Q_g(x+m) f_{M_r}(m) dm$.

■

Remark 5.6 *(Perpetual American Put Option). If $g(x) = \sum_{m=1}^{M} h_m e^{\theta_m x}$ where $\theta_m \ge 0$ for $1 \le m \le M$, then*

$$P_g(x) = \sum_{m=1}^{M} h_m e^{\theta_m x} \Big(\psi_r^-(-\mathrm{i}\theta_m) \Big)^{-1} \tag{5.3}$$

solves the averaging problem (2.5). In particular, if $g(x) = (K - e^x)^+$. then $P_g(x) = K - e^x \Big(\psi_r^-(-\mathrm{i}) \Big)^{-1}$. Denote by x^* the unique root of $P_g(x) = 0$. Then x^* is the optimal stopping boundary and for $x > x^*$, $V(x) = \int_{-\infty}^{x^*} P_g(u) f_{I_r}(u-x) du$.

Example 5.7 *Consider the optimal stopping problem (1.1) with $g(x) = \ln(x+1)1_{\{x \ge 0\}}$. To check conditions (a)-(b) in Theorem 4.7, we first substitute $g(x) = \ln(x+1)1_{\{x\ge 0\}}$ into (4.23). Multiplying both sides of (4.23) by $(\ln(x+1))^{-1}$ and using Remark 4.5, we see that $\lim_{x\to\infty} Q_g(x)(\ln(x+1))^{-1} = \Big(\psi_r^+(0) \Big)^{-1} = 1$, which implies that $\lim_{x\to\infty} Q_g(x) = \infty$. Next, observe that for $x > 0$ and $u > 0$*

$$\left(\frac{g(u+x)}{g(x)} \right)' = \left(\frac{\ln(u+x+1)}{\ln(x+1)} \right)' = \frac{\frac{\ln(x+1)}{u+x+1} - \frac{\ln(u+x+1)}{x+1}}{(\ln(x+1))^2} < 0,$$

and

$$\left(\frac{g'(x)}{g(x)} \right)' = \frac{-(x+1)^{-2} \cdot \ln(x+1) - (x+1)^{-2}}{(\ln(x+1))^{-2}} < 0.$$

By Theorem 4.7, there exists a unique $x^* > 0$ such that $Q_g(x^*) = 0$ and $\tau^* := \inf\{t \ge 0 : X_t \ge x^*\}$ is the optimal stopping time for the optimal stopping problem (1.1) with $g(x) = \ln(x+1)1_{\{x>0\}}$.

■

6 American compound options

In this section, we consider the pricing problem of the perpetual American compound options. Compound options are option on options. There are four main type of compound options. The perpetual American compound option have two strikes prices. For example, the call-on-call option gives its holder the right to buy at an random time τ for the strike price K_1 a call option with the strike price K_2 and the exercise time ζ, where $\zeta \geq \tau$. Also, the put-on-call option gives its holder the right to sell at an exercise time τ for the strike price L_1 a call option with the strike price K_2 and the random time ζ, where $\zeta \geq \tau$. The rational prices of such perpetual American options can be formulated by the values of the optimal stopping problems

$$\text{(call-on-call)} \qquad V_1(x) = \sup_{\tau} \mathbb{E}_x \Big[e^{-r\tau} H_1^+(X_\tau) \Big].$$

$$\text{(call-on-put)} \qquad V_2(x) = \sup_{\tau} \mathbb{E}_x \Big[e^{-r\tau} H_2^+(X_\tau) \Big].$$

$$\text{(put-on-call)} \qquad V_3(x) = \sup_{\tau} \mathbb{E}_x \Big[e^{-r\tau} H_3^+(X_\tau) \Big].$$

$$\text{(put-on-put)} \qquad V_4(x) = \sup_{\tau} \mathbb{E}_x \Big[e^{-r\tau} H_4^+(X_\tau) \Big].$$

Here the reward function $H_j(x)$, $j = 1, ..., 4$, are given by

$$H_1(x) = W(x) - K_1, \quad H_2(x) = U(x) - K_1, \quad H_3(x) = L_1 - W(x), \quad H_4(x) = L_1 - U(x)$$

for all $x \in \mathbb{R}$. Also, $W(x)$ and $U(x)$ denote the rational prices of the perpetual American call and put options with the strike prices K_2 and L_2, respectively and are given by

$$W(x) = \sup_{\eta} \mathbb{E}_x \Big[e^{-r\eta}(e^{X_\eta} - K_2)^+ \Big] \text{ and } U(x) = \sup_{\eta} \mathbb{E}_x \Big[e^{-r\eta}(L_2 - e^{X_\eta})^+ \Big] \tag{6.1}$$

where the suprema are taken over the stopping times η of the process X.

From now on, we assume that $\{X_t\}_{t\geq 0}$ takes the forms (3.2) with $n_k = 1$, $\ell_p = 1$, $c_{k1} > 0$, $\beta_k > 0$, $\widetilde{c}_{p1} > 0$ and $\alpha_p > 0$, for $1 \leq k \leq v_1$ and $1 \leq p \leq v_2$. For simplicity, we assume that $b \neq 0$. (Our approach also works for the case $b = 0$.) In this case, $\mu_1 = v_1 + 1$, $\mu_2 = v_2 + 1$ and all roots are simple and purely imaginary. Also, they satisfy the conditions

$$1 < \rho_1 < \beta_1 < \rho_2 < \cdots < \beta_{\mu_1 - 1} < \rho_{\mu_1} \tag{6.2}$$

and

$$0 < -\widetilde{\rho}_1 < \alpha_1 < -\widetilde{\rho}_2 < \cdots < \alpha_{\mu_2 - 1} < -\widetilde{\rho}_{\mu_2}. \tag{6.3}$$

We assume further that $\rho_1 > 1$ and $-\widetilde{\rho}_1 > 1$. Recall that

$$f_{M_r}(y) = \sum_{j=1}^{\mu_1} d_j \rho_j e^{-\rho_j y} 1_{\{y>0\}}$$

and

$$f_{I_r}(y) = \sum_{\eta=1}^{\mu_2} \widetilde{d}_\eta \widetilde{\rho}_\eta e^{-\widetilde{\rho}_\eta y} 1_{\{y<0\}}.$$

Hence, we also have

$$\psi_r^+(u) = \sum_{k=1}^{\mu_1} \frac{d_k \rho_k \mathrm{i}}{u + \mathrm{i}\rho_k}$$

and

$$\psi_r^-(u) = -\sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta \mathrm{i}}{u + \mathrm{i}\widetilde{\rho}_\eta}.$$

Here,

$$d_k = \prod_{j=1}^{v_1} \frac{\beta_j - \rho_k}{\beta_j} \prod_{i=1, i\neq k}^{\mu_1} \frac{\rho_i}{\rho_i - \rho_k}, \text{ for } 1 \leq k \leq \mu_1 \tag{6.4}$$

and

$$\widetilde{d}_\eta = -\prod_{k=1}^{v_2} \frac{\widetilde{\rho}_\eta + \alpha_k}{\alpha_k} \prod_{m=1, m\neq\eta}^{\mu_2} \frac{\widetilde{\rho}_m}{-\widetilde{\rho}_\eta + \widetilde{\rho}_m}, \text{ for } 1 \leq \eta \leq \mu_2. \tag{6.5}$$

Set $g_1(x) = e^x - K_2$. Then

$$Q_{g_1}(x) = e^x \left(\psi_r^+(-\mathrm{i})\right)^{-1} - K_2. \tag{6.6}$$

Denote by x_c^* the unique solution of $Q_{g_1}(x) = 0$ in $(\ln K_2, \infty)$. Then the value function $W(x)$ in (6.1) is given by the formula

$$W(x) = 1_{\{x \geq x_c^*\}}(e^x - K_2) + 1_{\{x < x_c^*\}} \sum_{j=1}^{\mu_1} \frac{d_j K_2 e^{\rho_j (x - x_c^*)}}{\rho_j - 1}. \tag{6.7}$$

By (6.2) and (6.4), we obtain that $d_k > 0$ for all k. Hence, $W(x)$ is a strictly increasing function with $\lim_{x\to-\infty} W(x) = 0$ and $\lim_{x\to\infty} W(x) = \infty$.

On the other hand, set $g_2(x) = L_2 - e^x$. Then

$$P_{g_2}(x) = L_2 - \left(\psi_r^-(-\mathrm{i})\right)^{-1} e^x. \tag{6.8}$$

Denote by x_p^* the unique solution of $P_{g_2}(x) = 0$ in $(-\infty, \ln L_2)$. The value function $U(x)$ in (6.1) is given by the formula

$$U(x) = 1_{\{x \leq x_p^*\}}(L_2 - e^x) + 1_{\{x > x_p^*\}} \sum_{\eta=1}^{\mu_2} \frac{-\widetilde{d}_\eta L_2 e^{\widetilde{\rho}_\eta (x - x_p^*)}}{1 - \widetilde{\rho}_\eta}. \tag{6.9}$$

By (6.3) and (6.5), $\widetilde{d}_\eta < 0$ for all η and so $U(x)$ is a strictly decreasing function with $\lim_{x\to-\infty} U(x) = L_2$ and $\lim_{x\to\infty} U(x) = 0$.

Also, if H is in π_0, we have

$$Q_H(x) = -\sum_{k=1}^{v_1}\sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta \lambda \beta_k c_{k1}}{r(\beta_k - \widetilde{\rho}_\eta)} e^{\beta_k x} \int_x^\infty H(y) e^{-\beta_k y} dy - \sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta}{r}(a + \frac{b^2 \widetilde{\rho}_\eta}{2}) H(x) - \sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta b^2 H'(x)}{2r}. \tag{6.10}$$

If H is in π_1, we have

$$P_H(x) = -\sum_{p=1}^{v_2}\sum_{j=1}^{\mu_1} \frac{d_j \rho_j \mu \alpha_p \widetilde{c}_{p1}}{r(\alpha + \rho_j)} \int_{-\infty}^0 H(t+x) e^{\alpha_p t} dt + \sum_{j=1}^{\mu_1} \frac{d_j \rho_j}{r}(a + \frac{b^2 \rho_j}{2}) H(x) + \sum_{j=1}^{\mu_1} \frac{d_j \rho_j b^2 H'(x)}{2r}. \tag{6.11}$$

(**Call-on-Call Option**). We consider the call-on-call option. The payoff function $H_1(x)$ is given by

$$H_1(x) = W(x) - K_1 = 1_{\{x \geq x_c^*\}}(e^x - K_1 - K_2) + 1_{\{x < x_c^*\}}\left(\sum_{j=1}^{\mu_1} \frac{d_j K_2 e^{\rho_j (x - x_c^*)}}{\rho_j - 1} - K_1\right). \tag{6.12}$$

Clearly, $H_1(x)$ is a strictly increasing function with $\lim_{x\to-\infty} H_1(x) = -K_1$ and $\lim_{x\to\infty} H_1(x) = \infty$. Note that $H_1 \in \pi_0$ and $Q_{H_1}(x) = e^x\left(\psi_r^+(-\mathrm{i})\right)^{-1} - K_1 - K_2$ for $x \geq x_c^*$. Furthermore, if there exists x_2^* such that $Q_{H_1}(x_2^*) = 0$, $Q_{H_1}(x) \leq 0$ for $\widehat{a} < x < x_2^*$ and $Q_{H_1}(x)$ is non-decreasing on (x_2^*, ∞) then by Theorem 2.7, we deduce that x_2^* is the optimal boundary and the value function (the rational price) is given by the formula

$$V_1(x) = \int_{x_2^* - x}^\infty Q_{H_1}(x+m) f_{M_r}(m) dm. \tag{6.13}$$

(**Call-on-Put Option**). We consider the call-on-put option. Then we have

$$H_2(x) = U(x) - K_1 = 1_{\{x \leq x_p^*\}}(L_2 - K_1 - e^x) + 1_{\{x > x_p^*\}}\left(\sum_{\eta=1}^{\mu_2} \frac{-\widetilde{d}_\eta L_2 e^{\widetilde{\rho}_\eta (x - x_p^*)}}{1 - \widetilde{\rho}_\eta} - K_1\right). \tag{6.14}$$

Clearly, $H_2(x)$ is a strictly decreasing function with $\lim_{x\to-\infty} H_2(x) = L_2 - K_1$ and $\lim_{x\to\infty} H_2(x) = -K_1$. Note that $H_2 \in \pi_1$ and $P_{H_2}(x) = L_2 - K_1 - e^x\left(\psi_r^-(-\mathrm{i})\right)^{-1}$ for $x \leq x_p^*$. Moreover, if there exists x_2^* such that $P_{H_2}(x_2^*) = 0$, $P_{H_2}(x) \leq 0$ for $x_2^* < x < \widehat{a}$ and $P_{H_2}(x)$ is non-increasing on $(-\infty, x_2^*)$ then by Theorem 2.10, we conclude that x_2^* is the optimal boundary and the value function (the rational price) is given by the formula

$$V_2(x) = \int_{-\infty}^{x_2^* - x} P_{H_2}(x+z) f_{I_r}(z) dz. \tag{6.15}$$

(**Put-on-Call Option**). We consider the put-on-call option. The payoff function is given by

$$H_3(x) = L_1 - W(x) = 1_{\{x \geq x_c^*\}}(L_1 + K_2 - e^x) + 1_{\{x < x_c^*\}}\left(L_1 - \sum_{j=1}^{\mu_1} \frac{d_j K_2 e^{\rho_j (x - x_c^*)}}{\rho_j - 1}\right). \tag{6.16}$$

Clearly, $H_3(x)$ is a strictly decreasing function with $\lim_{x\to-\infty} H_3(x) = L_1$ and $\lim_{x\to\infty} H_3(x) = -\infty$. Note that $H_3 \in \pi_1$ and $P_{H_3}(x) = L_1 - \sum_{j=1}^{\mu_1} \frac{d_j K_2(\psi_r^-(-\mathrm{i}\rho_j))^{-1} e^{\rho_j(x-x_c^*)}}{\rho_j - 1}$ for $x \le x_c^*$. Furthermore, if there exists x_2^* such that $P_{H_3}(x_2^*) = 0$, $P_{H_3}(x) \le 0$ for $x_2^* < x < \widehat{a}$ and $P_{H_3}(x)$ is non-increasing on $(-\infty, x_2^*)$ then by Theorem 2.10, we conclude that x_2^* is the optimal boundary and the value function (the rational price) is given by the formula

$$V_3(x) = \int_{-\infty}^{x_2^*-x} P_{H_3}(x+z) f_{I_r}(z) dz. \tag{6.17}$$

(**Put-on-Put Option**). We consider the put-on-put compound option. Then we have

$$H_4(x) = L_1 - U(x) = 1_{\{x \le x_p^*\}}(e^x + L_1 - L_2) + 1_{\{x > x_p^*\}}\left(L_1 - \sum_{\eta=1}^{\mu_2} \frac{-\widetilde{d}_\eta L_2 e^{\widetilde{\rho}_\eta(x-x_p^*)}}{1-\widetilde{\rho}_\eta}\right). \tag{6.18}$$

Clearly, $H_4(x)$ is a strictly increasing function with $\lim_{x\to-\infty} H_4(x) = L_1 - L_2$ and $\lim_{x\to\infty} H_4(x) = L_1$. Note that $H_4 \in \pi_0$ and $Q_{H_4}(x) = L_1 - \sum_{\eta=1}^{\mu_2} \frac{-\widetilde{d}_\eta L_2(\psi_r^+(-\mathrm{i}\widetilde{\rho}_\eta))^{-1} e^{\widetilde{\rho}_\eta(x-x_p^*)}}{1-\widetilde{\rho}_\eta}$ for $x \ge x_p^*$. Moreover, if there exists x_2^* such that $Q_{H_4}(x_2^*) = 0$, $Q_{H_4}(x) \le 0$ for $\widehat{a} < x < x_2^*$ and $Q_{H_4}(x)$ is non-decreasing on (x_2^*, ∞) then by Theorem 2.7, we deduce that x_2^* is the optimal boundary and the value function (the rational price) is given by the formula

$$V_4(x) = \int_{x_2^*-x}^{\infty} Q_{H_4}(x+m) f_{M_r}(m) dm. \tag{6.19}$$

7 Numerical results

Example 7.1 *(call-on-call option). We consider the strike prices* $K_1 = 10$ *and* $K_2 = 50$. *For the diffusion process,* $x_c^* = 4.2120$, $x_2^* = 4.3943$, $(d_1, \widetilde{d}_1) = (1, -1)$ *and* $(\rho_1, \widetilde{\rho}_1) = (3.8577, -0.4977)$. *For the exponential jump-diffusion process, we acquire that* $x_c^* = 5.3363$, $x_2^* = 5.5186$, $(d_1, d_2, \widetilde{d}_1, \widetilde{d}_2) = (0.6275, 0.3724, -0.8955, -0.1044)$ *and* $(\rho_1, \rho_2, \widetilde{\rho}_1, \widetilde{\rho}_2) = (1.2029, 6.9435, -0.2359, -3.4791)$. *For the mixture-exponential jump-diffusion process, we have that* $x_c^* = 4.7666$, $x_2^* = 4.9490$, $(d_1, d_2, d_3, \widetilde{d}_1, \widetilde{d}_2, \widetilde{d}_3)$ $= (0.4405, 0.1445, 0.4149, -0.7975, -0.05012, -0.1523)$ *and* $(\rho_1, \rho_2, \rho_3, \widetilde{\rho}_1, \widetilde{\rho}_2, \widetilde{\rho}_3)$ $= (1.3605, 3.3113, 7.2730, -0.03213, -0.2879, -2.8148)$.

Example 7.2 *(call-on-put option). We consider the strike prices* $K_1 = 20$ *and* $L_2 = 50$. *For the diffusion process,* $x_p^* = 2.8103$, $x_2^* = 2.2995$, $(d_1, \widetilde{d}_1) = (1, -1)$ *and* $(\rho_1, \widetilde{\rho}_1) = (3.8577, -0.4977)$. *For the exponential jump-diffusion process, we acquire that* $x_p^* = 2.5340$, $x_2^* = 2.0232$, $(d_1, d_2, \widetilde{d}_1, \widetilde{d}_2) = (0.6275, 0.3724, -0.8955, -0.1044)$ *and* $(\rho_1, \rho_2, \widetilde{\rho}_1, \widetilde{\rho}_2) = (1.2029, 6.9435, -0.2359, -3.4791)$. *For the mixture-exponential jump-diffusion process, we have that* $x_p^* = 2.0042$, $x_2^* = 1.4934$, $(d_1, d_2, d_3, \widetilde{d}_1, \widetilde{d}_2, \widetilde{d}_3)$ $= (0.4405, 0.1445, 0.4149, -0.7975, -0.05012, -0.1523)$ *and* $(\rho_1, \rho_2, \rho_3, \widetilde{\rho}_1, \widetilde{\rho}_2, \widetilde{\rho}_3)$ $= (1.3605, 3.3113, 7.2730, -0.03213, -0.2879, -2.8148)$.

Example 7.3 *(put-on-call option). We consider the strike prices* $L_1 = 300$ *and* $K_2 = 50$. *For the diffusion process, we have that* $x_c^* = 4.2120$, $x_2^* = 4.7562$, $(d_1, \widetilde{d}_1) = (1, -1)$ *and* $(\rho_1, \widetilde{\rho}_1) = (3.8577, -0.4977)$. *For the exponential jump-diffusion process, we acquire that* $x_c^* = 5.3363$, $x_2^* = 4.6442$, $(d_1, d_2, \widetilde{d}_1, \widetilde{d}_2) = (0.6275, 0.3724, -0.8955, -0.1044)$ *and* $(\rho_1, \rho_2, \widetilde{\rho}_1, \widetilde{\rho}_2) = (1.2029, 6.9435, -0.2359, -3.4791)$. *For the mixture-exponential jump-diffusion process, we have that* $x_c^* = 4.7666$, $x_2^* = 4.3958$, $(d_1, d_2, d_3, \widetilde{d}_1, \widetilde{d}_2, \widetilde{d}_3) = (0.4405, 0.1445, 0.4149, -0.7975, -0.0501, -0.1523)$ *and* $(\rho_1, \rho_2, \rho_3, \widetilde{\rho}_1, \widetilde{\rho}_2, \widetilde{\rho}_3) = (1.3605, 3.3113, 7.2730, -0.0321, -0.2879, -2.8148)$.

Example 7.4 *(put-on-put option). We consider the strike prices* $L_1 = 45$ *and* $L_2 = 50$. *For the diffusion process,* $x_p^* = 2.8103$, $x_2^* = 1.9094$, $(d_1, \widetilde{d}_1) = (1, -1)$ *and* $(\rho_1, \widetilde{\rho}_1) = (3.8577, -0.4977)$. *For the exponential jump-diffusion process, we acquire that* $x_p^* = 2.5340$, $x_2^* = 2.4091$, $(d_1, d_2, \widetilde{d}_1, \widetilde{d}_2) = (0.6275, 0.3724, -0.8955, -0.1044)$ *and* $(\rho_1, \rho_2, \widetilde{\rho}_1, \widetilde{\rho}_2) = (1.2029, 6.9435, -0.2359, -3.4791)$. *For the mixture-exponential jump-diffusion process, we acquire that* $x_p^* = 2.0042$, $x_2^* = 2.0176$, $(d_1, d_2, d_3, \widetilde{d}_1, \widetilde{d}_2, \widetilde{d}_3) = (0.4405, 0.1445, 0.4149, -0.7975, -0.0501, -0.1523)$ *and* $(\rho_1, \rho_2, \rho_3, \widetilde{\rho}_1, \widetilde{\rho}_2, \widetilde{\rho}_3) = (1.3605, 3.3113, 7.2730, -0.0321, -0.2879, -2.8148)$.

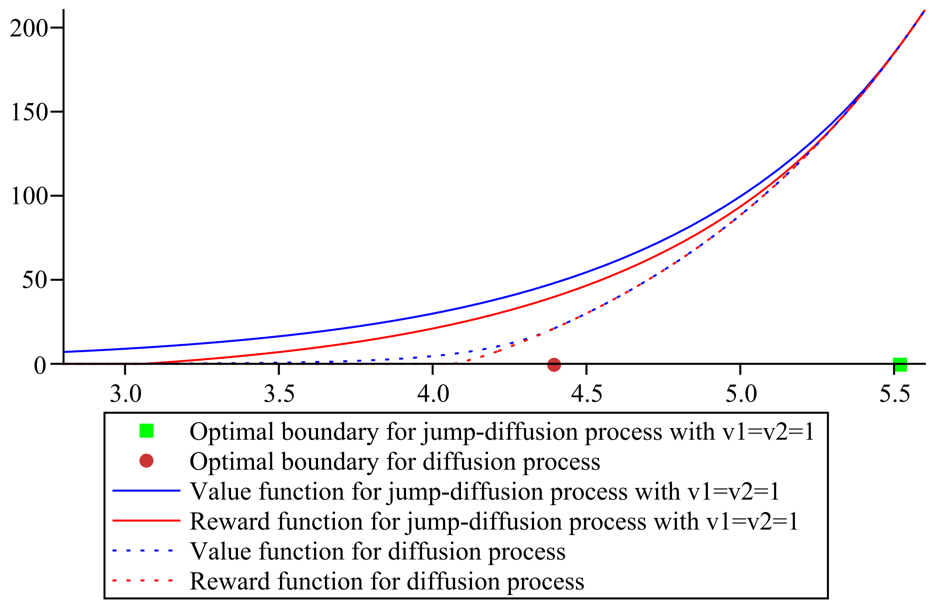

Figure 1: call-on-call options for jump-diffusion process with v1=v2=1 and diffusion process.

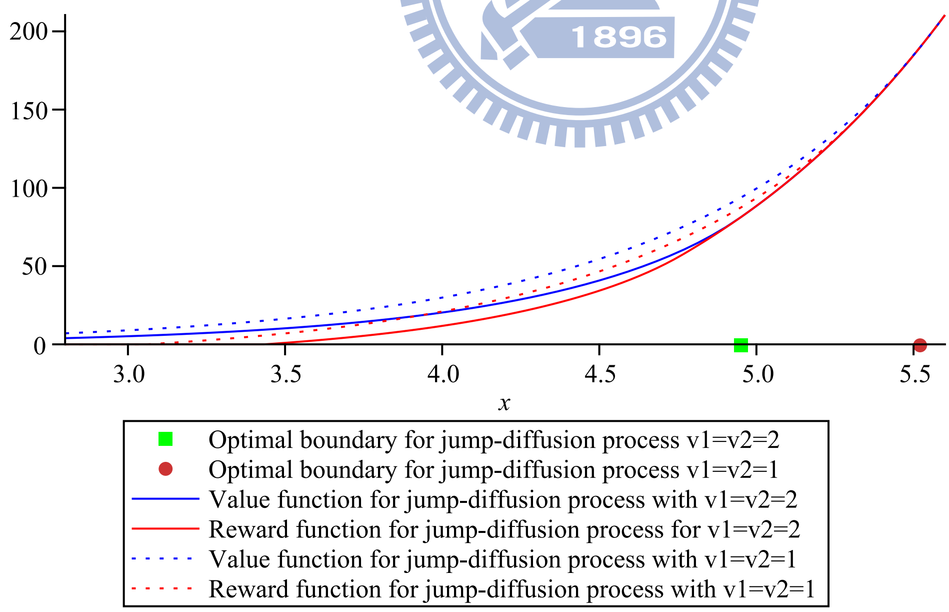

Figure 2: call-on-call options for jump-diffusion process with v1=v2=1 and v1=v2=2.

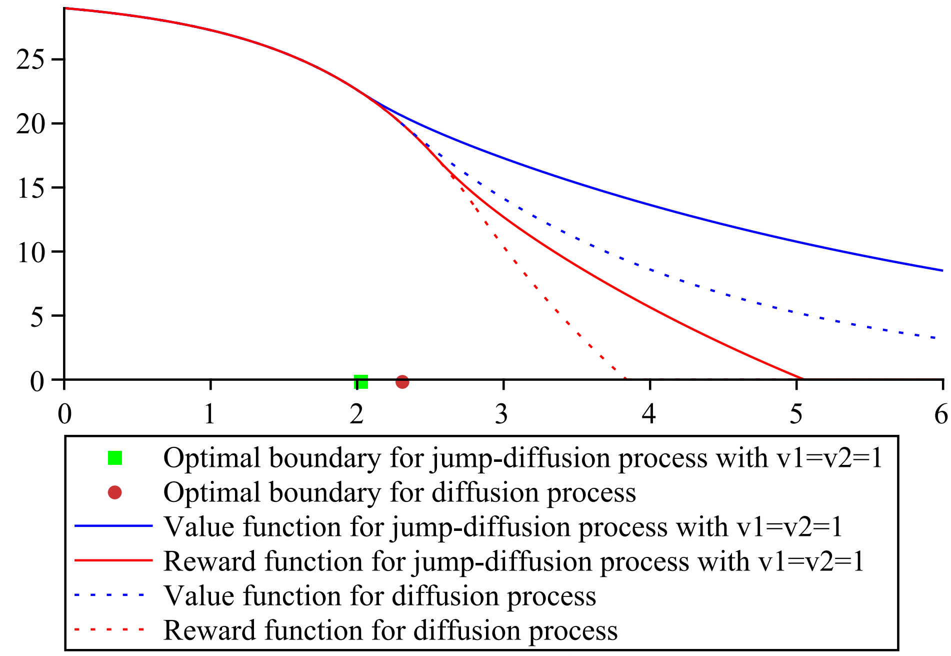

Figure 3: call-on-put options for jump-diffusion process with v1=v2=1 and diffusion process.

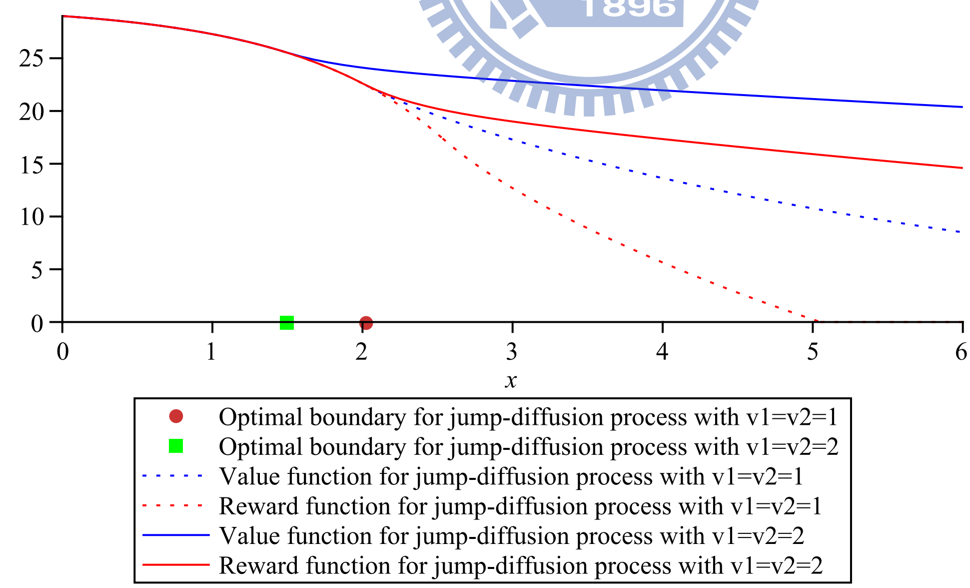

Figure 4: call-on-put options for jump-diffusion process with v1=v2=1 and v1=v2=2.

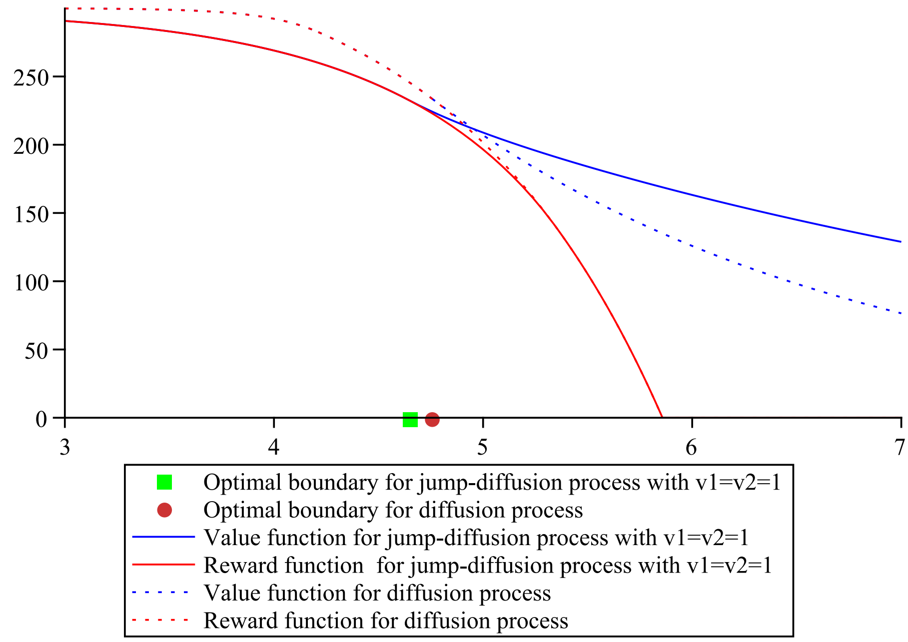

Figure 5: put-on-call options for jump-diffusion process with v1=v2=1 and diffusion process.

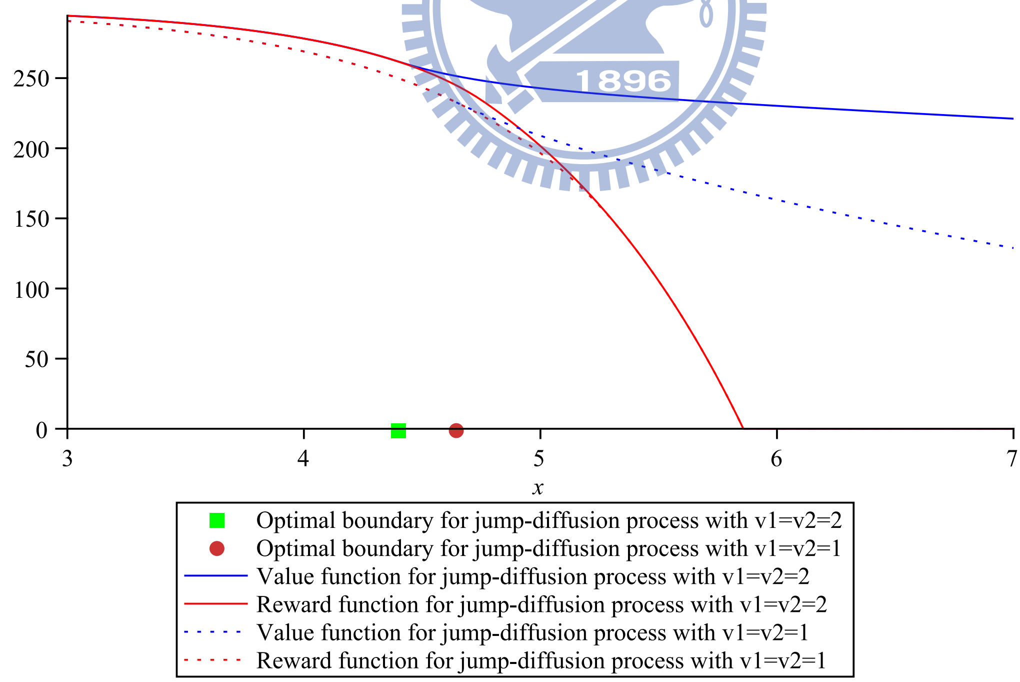

Figure 6: put-on-call options for jump-diffusion process with v1=v2=1 and v1=v2=2.

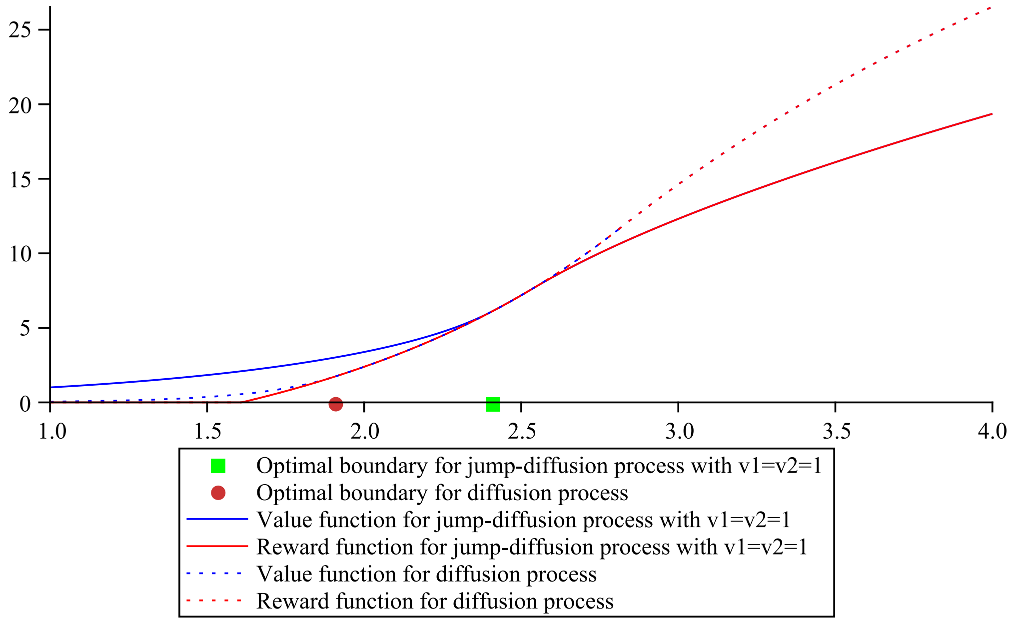

Figure 7: put-on-put options for jump-diffusion process with v1=v2=1 and diffusion process.

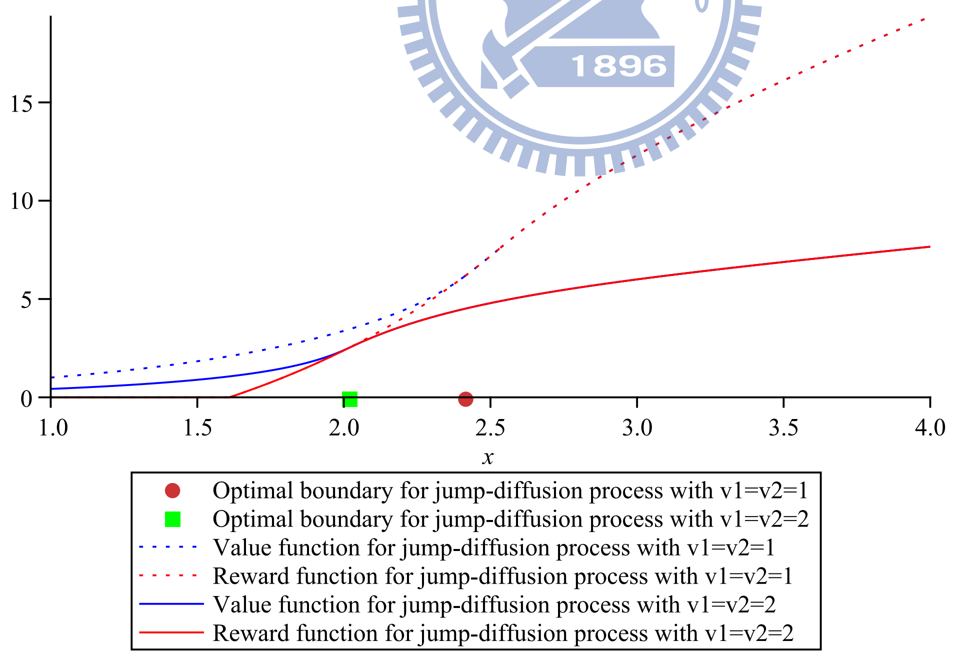

Figure 8: put-on-put options for jump-diffusion process with v1=v2=1 and v1=v2=2.

Table 1: Parameters for Jump-Diffusion Processes

	Diffusion Process	Exponent Jump Diffusion Process	Mixture Exponent Jump Diffusion Process
a	-0.105	-0.105	-0.105
b	0.25	0.25	0.25
λ	-	0.3	0.3
μ	-	0.3	0.3
β_1	-	2.5	2
α_1	-	1.428571429	0.1333333333
β_2	-	-	4
α_2	-	-	0.4166666667
c_{11}	-	1	0.5
c_{21}	-	-	0.5
$\widetilde{c}_{11}$	-	1	0.5
$\widetilde{c}_{21}$	-	-	0.5

Table 2: Compound options

	Diffusion Process	Exponent Jump Diffusion Process	Mixture Exponent Jump Diffusion Process
call-on-call			
K_1	10	10	10
K_2	50	50	50
x_c^*	4.212076578	5.336364625	4.766693426
x_2^*	4.394398135	5.518686182	4.949014981
call-on-put			
K_1	20	20	20
L_2	50	50	50
x_p^*	2.810341610	2.534084694	2.004288173
x_2^*	2.299515986	2.023259069	1.493462547
put-on-call			
L_1	300	300	300
K_2	50	50	50
x_c^*	4.212076578	5.336364625	4.766693426
x_2^*	4.756251761	4.644257544	4.395873979
put-on-put			
L_1	45	45	45
L_2	50	50	50
x_p^*	2.810341610	2.534084694	2.004288173
x_2^*	1.909491485	2.409138962	2.017624207

8 Verification of optimality

From now on, we consider the process

$$X_t = at + bW_t + \sum_{i=1}^{N_t^\lambda} Y_i - \sum_{j=1}^{N_t^\mu} Z_j.$$

Here $\{Y_i^\beta : i = 1, 2, ...\}$ and $\{Z_j^\alpha : j = 1, 2, ...\}$ are sequences of independent exponentially distributed random variables with parameters β and α, respectively. First, recall that $g_1(x) = e^x - K_2$ and $W(x) = 1_{\{x \geq x_c^*\}}(e^x - K_2) + 1_{\{x < x_c^*\}}(\frac{d_1K_2}{\rho_1-1}e^{\rho_1(x-x_c^*)} + \frac{d_2K_2}{\rho_2-1}e^{\rho_2(x-x_c^*)})$. Here $e^{x_c^*} = \psi_r^+(-\mathrm{i})K_2$. In addition, $g_2(x) = L_2 - e^x$ and $U(x) = 1_{\{x \leq x_p^*\}}(L_2 - e^x) + 1_{\{x > x_p^*\}}\left(\sum_{\eta=1}^2 \frac{-\widetilde{d}_\eta L_2 e^{\widetilde{\rho}_\eta(x-x_p^*)}}{1-\widetilde{\rho}_\eta}\right)$. Here $e^{x_p^*} = \psi_r^-(-\mathrm{i})L_2$.

(**Call-on-Call Option**). Note that $H_1(x) = W(x) - K_1 = 1_{\{x \geq x_c^*\}}(e^x - K_1 - K_2) + 1_{\{x \leq x_c^*\}}\left(\frac{d_1K_2}{\rho_1-1}e^{\rho_1(x-x_c^*)} + \frac{d_2K_2}{\rho_2-1}e^{\rho_2(x-x_c^*)} - K_1\right)$ and

$$\begin{aligned} Q_{H_1}(x) =& \left(\frac{\widetilde{d}_1\widetilde{\rho}_1}{r} \cdot \frac{-\lambda\beta}{\beta - \widetilde{\rho}_1} + \frac{\widetilde{d}_2\widetilde{\rho}_2}{r} \cdot \frac{-\lambda\beta}{\beta - \widetilde{\rho}_2}\right) e^{\beta x} \int_x^\infty H_1(y)e^{-\beta y}dy \\ & - \left[\frac{\widetilde{d}_1\widetilde{\rho}_1}{r} \cdot (a + \frac{b^2\widetilde{\rho}_1}{2}) + \frac{\widetilde{d}_2\widetilde{\rho}_2}{r} \cdot (a + \frac{b^2\widetilde{\rho}_2}{2})\right] H_1(x) - \left[\frac{b^2}{2r}(\widetilde{d}_1\widetilde{\rho}_1 + \widetilde{d}_2\widetilde{\rho}_2)\right] H_1'(x). \quad (8.1) \end{aligned}$$

We show that the rational price of the call-on-call option is the rational price of the perpetual American call option with the strike price $K_1 + K_2$. That is $V_1(x) = 1_{\{x \geq x_2^*\}}(e^x - K_1 - K_2) + 1_{\{x \leq x_2^*\}}\left(\frac{d_1(K_1+K_2)}{\rho_1-1}e^{\rho_1(x-x_2^*)} + \frac{d_2(K_1+K_2)}{\rho_2-1}e^{\rho_2(x-x_2^*)}\right)$. Here, $e^{x_2^*} = \psi_r^+(-\mathrm{i})(K_1 + K_2)$. (Note that $x_2^* > x_c^*$.)

Case 1: $e^{x_c^*} - K_1 - K_2 \leq 0$. It follows from $H_1^+(x) = (e^x - K_1 - K_2)^+$ that the result above holds.

Case 2: $e^{x_c^*} - K_1 - K_2 > 0$. Since $H_1(x)$ is increasing, we get $\{H_1 > 0\} = (\widehat{a}, \infty)$, for some $\widehat{a} < x_c^*$. For $\widehat{a} < x < x_c^*$, we have

$$\begin{aligned} e^{\beta x} \int_x^\infty H_1(y)e^{-\beta y}dy =& e^{\beta(x-x_c^*)}\left(\frac{d_1K_2}{(\rho_1-1)(\rho_1-\beta)} + \frac{d_2K_2}{(\rho_2-1)(\rho_2-\beta)} + \frac{\psi_r^+(-\mathrm{i})K_2}{\beta-1} - \frac{K_2}{\beta}\right) \\ & - \frac{d_1K_2e^{\rho_1(x-x_c^*)}}{(\rho_1-1)(\rho_1-\beta)} - \frac{d_2K_2e^{\rho_2(x-x_c^*)}}{(\rho_2-1)(\rho_2-\beta)} - \frac{K_1}{\beta}. \end{aligned}$$

Plugging this into (8.1) gives

$$\begin{aligned} Q_{H_1}(x) =& \left(\frac{\widetilde{d}_1\widetilde{\rho}_1}{r} \cdot \frac{-\lambda\beta}{\beta - \widetilde{\rho}_1} + \frac{\widetilde{d}_2\widetilde{\rho}_2}{r} \cdot \frac{-\lambda\beta}{\beta - \widetilde{\rho}_2}\right) \\ & \times \left[e^{\beta(x-x_c^*)}\left(\frac{d_1K_2}{(\rho_1-1)(\rho_1-\beta)} + \frac{d_2K_2}{(\rho_2-1)(\rho_2-\beta)} + \frac{\psi_r^+(-\mathrm{i})K_2}{\beta-1} - \frac{K_2}{\beta}\right)\right. \\ & \left. - \frac{d_1K_2e^{\rho_1(x-x_c^*)}}{(\rho_1-1)(\rho_1-\beta)} - \frac{d_2K_2e^{\rho_2(x-x_c^*)}}{(\rho_2-1)(\rho_2-\beta)} - \frac{K_1}{\beta}\right] \\ & - \left[\frac{\widetilde{d}_1\widetilde{\rho}_1}{r} \cdot (a + \frac{b^2\widetilde{\rho}_1}{2}) + \frac{\widetilde{d}_2\widetilde{\rho}_2}{r} \cdot (a + \frac{b^2\widetilde{\rho}_2}{2})\right]\left((\frac{d_1K_2}{\rho_1-1}e^{\rho_1(x-x_c^*)} + \frac{d_2K_2}{\rho_2-1}e^{\rho_2(x-x_c^*)}) - K_1\right) \\ & - \left[\frac{b^2}{2r}(\widetilde{d}_1\widetilde{\rho}_1 + \widetilde{d}_2\widetilde{\rho}_2)\right]\left(\frac{d_1\rho_1K_2}{\rho_1-1}e^{\rho_1(x-x_c^*)} + \frac{d_2\rho_2K_2}{\rho_2-1}e^{\rho_2(x-x_c^*)}\right). \quad (8.2) \end{aligned}$$

Notice that

$$\begin{aligned}
&\frac{d_1K_2}{(\rho_1-1)(\rho_1-\beta)}+\frac{d_2K_2}{(\rho_2-1)(\rho_2-\beta)}+\frac{\psi_r^+(-\mathrm{i})K_2}{\beta-1}-\frac{K_2}{\beta}\\
=&\frac{d_1K_2}{(\rho_1-1)(\rho_1-\beta)}+\frac{d_2K_2}{(\rho_2-1)(\rho_2-\beta)}+(\frac{K_2}{\beta-1})(\frac{d_1\rho_1}{\rho_1-1}+\frac{d_2\rho_2}{\rho_2-1})-\frac{K_2}{\beta}\\
=&\frac{K_2}{\beta(\beta-1)}\left(\frac{d_1\rho_1}{\rho_1-\beta}+\frac{d_2\rho_2}{\rho_2-\beta}\right)=0. \qquad (8.3)
\end{aligned}$$

Also, using the Wiener-Hopf factorization, we have that

$$\begin{aligned}
\frac{r}{r-\psi(-\mathrm{i}z)}=&\frac{r(\beta-z)(\alpha+z)}{-\left[\frac{b^2z^2}{2}(\beta-z)(\alpha+z)+az(\beta-z)(\alpha+z)+\lambda z(\alpha+z)-\mu z(\beta-z)\right]+r(\beta-z)(\alpha+z)}\\
=&\frac{\rho_1\rho_2(\beta-z)}{\beta(\rho_1-z)(\rho_2-z)}\sum_{k=1}^{2}\frac{\widetilde{d}_k\widetilde{\rho}_k}{z-\widetilde{\rho}_k}. \qquad (8.4)
\end{aligned}$$

Observe that evaluating both sides of (8.4) at $z=\beta$ implies that

$$\sum_{k=1}^{2}\frac{\widetilde{d}_k\widetilde{\rho}_k}{\beta-\widetilde{\rho}_k}=\frac{r(\rho_1-\beta)(\rho_2-\beta)}{-\lambda\rho_1\rho_2}. \qquad (8.5)$$

Also, by multiplying both sides of (8.4) by z^2 and letting $z\to\infty$, we see that

$$\sum_{k=1}^{2}\widetilde{d}_k\widetilde{\rho}_k=\frac{r\beta}{\frac{b^2}{2}\rho_1\rho_2}. \qquad (8.6)$$

Moreover, it follows from (a) and (c) in Lemma 4.1 that

$$\frac{\widetilde{d}_1\widetilde{\rho}_1}{r}\cdot(a+\frac{b^2\widetilde{\rho}_1}{2})+\frac{\widetilde{d}_2\widetilde{\rho}_2}{r}\cdot(a+\frac{b^2\widetilde{\rho}_2}{2})=\frac{1}{r}\left(-r-\frac{\lambda\widetilde{d}_1\widetilde{\rho}_1}{\beta-\widetilde{\rho}_1}-\frac{\lambda\widetilde{d}_2\widetilde{\rho}_2}{\beta-\widetilde{\rho}_2}\right). \qquad (8.7)$$

Taking account (8.2)-(8.7) we have that for $\widehat{a}<x<x_c^*$,

$$\begin{aligned}
Q_{H_1}(x)=&e^{\rho_1(x-x_c^*)}\frac{d_1K_2}{\rho_1-1}\left[-\frac{\beta(\rho_2-\beta)}{\rho_1\rho_2}+1-\frac{(\beta-\rho_1)(\beta-\rho_2)}{\rho_1\rho_2}-\frac{\beta}{\rho_2}\right]\\
&+e^{\rho_2(x-x_c^*)}\frac{d_2K_2}{\rho_2-1}\left[-\frac{\beta(\rho_1-\beta)}{\rho_1\rho_2}+1-\frac{(\beta-\rho_1)(\beta-\rho_2)}{\rho_1\rho_2}-\frac{\beta}{\rho_1}\right]-K_1\\
=&-K_1.
\end{aligned}$$

Because $H_1(x)=e^x-K_1-K_2$ for $x>x_c^*$, we have $Q_{H_1}(x)=Q_{\widetilde{H}_1}(x)$ for $x>x_c^*>\ln(K_1+K_2)$, where $\widetilde{H}_1(x)=e^x-K_1-K_2$. By (5.2), $Q_{\widetilde{H}_1}(x)=e^x(\psi_r^+(-\mathrm{i}))^{-1}-K_1-K_2$. Hence, there is an unique $x_2^*>x_c^*$ such that $Q_{H_1}(x_2^*)=Q_{\widetilde{H}_1}(x_2^*)=0$, $Q_{H_1}(x)=Q_{\widetilde{H}_1}(x)<0$ on (x_c^*,x_2^*) and $Q_{H_1}(x)=Q_{\widetilde{H}_1}(x)$ is increasing on (x_2^*,∞). By Theorem 2.7, x_2^* is the optimal boundary and

$$\begin{aligned}
V_1(x)=&\int_{x_2^*-x}^{\infty}Q_{H_1}(x+m)f_{M_r}(m)dm\\
=&1_{\{x\geq x_2^*\}}(e^x-K_1-K_2)+1_{\{x\leq x_2^*\}}\left(\frac{d_1(K_1+K_2)}{\rho_1-1}e^{\rho_1(x-x_2^*)}+\frac{d_2(K_1+K_2)}{\rho_2-1}e^{\rho_2(x-x_2^*)}\right).
\end{aligned}$$

This complete the proof. ■

(**Call-on-Put option**). Note that $H_2(x) = U(x)-K_1 = 1_{\{x\le x_p^*\}}(L_2-K_1-e^x)+1_{\{x>x_p^*\}}\Big(\sum_{\eta=1}^{2} \frac{-\widetilde{d}_\eta L_2 e^{\widetilde{\rho}_\eta (x-x_p^*)}}{1-\widetilde{\rho}_\eta} - K_1\Big)$ and

$$\begin{aligned} P_{H_2}(x) =& \Big(\frac{d_1\rho_1}{r}\cdot\frac{-\mu\alpha}{\alpha+\rho_1}+\frac{d_2\rho_2}{r}\cdot\frac{-\mu\alpha}{\alpha+\rho_2}\Big)e^{-\alpha x}\int_{-\infty}^{x} H_2(y)e^{\alpha y}dy \\ &+\Big[\frac{d_1\rho_1}{r}\cdot(a+\frac{b^2\rho_1}{2})+\frac{d_2\rho_2}{r}\cdot(a+\frac{b^2\rho_2}{2})\Big]H_2(x)+\Big[\frac{b^2}{2r}(d_1\rho_1+d_2\rho_2)\Big]H_2^{'}(x). \quad (8.8)\end{aligned}$$

We show that the rational price of the call-on-put option is the rational price of the perpetual American put option with the strike price L_2-K_1. That is $V_2(x) = 1_{\{x\le x_2^*\}}(L_2-K_1-e^x) + 1_{\{x>x_2^*\}}\Big(\sum_{\eta=1}^{2}\frac{-\widetilde{d}_\eta(L_2-K_1)e^{\widetilde{\rho}_\eta(x-x_2^*)}}{1-\widetilde{\rho}_\eta}\Big)$. Here $e^{x_2^*} = \psi_r^-(-\mathrm{i})(L_2-K_1)$. (Note that $x_2^*<x_p^*$.)

Case 1: $L_2-K_1\le e^{x_p^*}$. It follows from $H_2^+(x) = (L_2-K_1-e^x)^+$ that the result above holds.

Case 2: $L_2-K_1 > e^{x_p^*}$. Since H_2 is decreasing, we observe $\{H_2>0\} = (-\infty,\widehat{a})$, for some $\widehat{a}>x_p^*$. For $x_p^*<x<\widehat{a}$, we first observe that

$$\begin{aligned} e^{-\alpha x}\int_{-\infty}^{x} H_2(u)e^{\alpha u}du =& e^{-\alpha(x-x_p^*)}\Big[\frac{\widetilde{d}_1L_2}{(\alpha+\widetilde{\rho}_1)(1-\widetilde{\rho}_1)}+\frac{\widetilde{d}_2L_2}{(\alpha+\widetilde{\rho}_2)(1-\widetilde{\rho}_2)}+\frac{L_2}{\alpha}-\frac{e^{x_p^*}}{\alpha+1}\Big] \\ &+e^{-\widetilde{\rho}_1(x_p^*-x)}\frac{-\widetilde{d}_1L_2}{(\alpha+\widetilde{\rho}_1)(1-\widetilde{\rho}_1)}+e^{-\widetilde{\rho}_2(x_p^*-x)}\frac{-\widetilde{d}_2L_2}{(\alpha+\widetilde{\rho}_2)(1-\widetilde{\rho}_2)}-\frac{K_1}{\alpha}. \end{aligned} \quad (8.9)$$

Also, since $L_2-\Big(\psi_r^-(-\mathrm{i})\Big)^{-1}e^{x_p^*}=0$ and $\psi_r^-(-\mathrm{i}) = \frac{\widetilde{d}_1\widetilde{\rho}_1}{1-\widetilde{\rho}_1}+\frac{\widetilde{d}_2\widetilde{\rho}_2}{1-\widetilde{\rho}_2}$, we have

$$\begin{aligned} &\frac{\widetilde{d}_1L_2}{(\alpha+\widetilde{\rho}_1)(1-\widetilde{\rho}_1)}+\frac{\widetilde{d}_2L_2}{(\alpha+\widetilde{\rho}_2)(1-\widetilde{\rho}_2)}+\frac{L_2}{\alpha}-\frac{e^{x_p^*}}{\alpha+1} \\ =&\frac{-L_2}{\alpha(\alpha+1)}\Big(\frac{\widetilde{d}_1\widetilde{\rho}_1}{\alpha+\widetilde{\rho}_1}+\frac{\widetilde{d}_2\widetilde{\rho}_2}{\alpha+\widetilde{\rho}_2}\Big)=0. \end{aligned}$$

This together with (8.8) and (8.9) yields that for $x_p^*<x<\widehat{a}$,

$$\begin{aligned} P_{H_2}(x) =& \Big(\frac{d_1\rho_1}{r}\cdot\frac{-\mu\alpha}{\alpha+\rho_1}+\frac{d_2\rho_2}{r}\cdot\frac{-\mu\alpha}{\alpha+\rho_2}\Big)\Big[\frac{-\widetilde{d}_1L_2e^{-\widetilde{\rho}_1(x_p^*-x)}}{(\alpha+\widetilde{\rho}_1)(1-\widetilde{\rho}_1)}+\frac{-\widetilde{d}_2L_2e^{-\widetilde{\rho}_2(x_p^*-x)}}{(\alpha+\widetilde{\rho}_2)(1-\widetilde{\rho}_2)}+\frac{-K_1}{\alpha}\Big] \\ &+\Big[\frac{d_1\rho_1}{r}\cdot(a+\frac{b^2\rho_1}{2})+\frac{d_2\rho_2}{r}\cdot(a+\frac{b^2\rho_2}{2})\Big]\Big[\frac{-\widetilde{d}_1L_2e^{-\widetilde{\rho}_1(x_p^*-x)}}{1-\widetilde{\rho}_1}+\frac{-\widetilde{d}_2L_2e^{-\widetilde{\rho}_2(x_p^*-x)}}{1-\widetilde{\rho}_2}-K_1\Big] \\ &+\Big[\frac{b^2}{2r}(d_1\rho_1+d_2\rho_2)\Big]\Big[\frac{-\widetilde{d}_1\widetilde{\rho}_1L_2e^{-\widetilde{\rho}_1(x_p^*-x)}}{1-\widetilde{\rho}_1}+\frac{-\widetilde{d}_2\widetilde{\rho}_2L_2e^{-\widetilde{\rho}_2(x_p^*-x)}}{1-\widetilde{\rho}_2}\Big]. \end{aligned} \quad (8.10)$$

In addition, applying similar arguments as in (8.5)-(8.7), we obtain that

$$\frac{d_1\rho_1}{\alpha+\rho_1}+\frac{d_2\rho_2}{\alpha+\rho_2}=\frac{r(\widetilde{\rho}_1+\alpha)(\widetilde{\rho}_2+\alpha)}{-\mu\widetilde{\rho}_1\widetilde{\rho}_2} \quad (8.11)$$

$$\sum_{j=1}^{2} d_j\rho_j = \frac{r\alpha}{\frac{b^2}{2}\widetilde{\rho}_1\widetilde{\rho}_2} \quad (8.12)$$

and

$$d_1(a\rho_1 + \frac{b^2\rho_1^2}{2}) + d_2(a\rho_2 + \frac{b^2\rho_2^2}{2}) = r + \mu\left(\frac{d_1\rho_1}{\alpha+\rho_1} + \frac{d_2\rho_2}{\alpha+\rho_2}\right). \tag{8.13}$$

Plugging (8.11)-(8.13) into (8.10), we obtain that for $x_p^* < x < \widehat{a}$

$$\begin{aligned} P_{H_2}(x) =& \frac{-\widetilde{d}_1 L_2 e^{\widetilde{\rho}_1(x-x_p^*)}}{1-\widetilde{\rho}_1}\left[\frac{\alpha(\widetilde{\rho}_2+\alpha)}{\widetilde{\rho}_1\widetilde{\rho}_2} + 1 - \frac{(\widetilde{\rho}_1+\alpha)(\widetilde{\rho}_2+\alpha)}{\widetilde{\rho}_1\widetilde{\rho}_2} + \frac{\alpha}{\widetilde{\rho}_2}\right] \\ &+ \frac{-\widetilde{d}_2 L_2 e^{\widetilde{\rho}_2(x-x_p^*)}}{1-\widetilde{\rho}_2}\left[\frac{\alpha(\widetilde{\rho}_1+\alpha)}{\widetilde{\rho}_1\widetilde{\rho}_2} + 1 - \frac{(\widetilde{\rho}_2+\alpha)(\widetilde{\rho}_1+\alpha)}{\widetilde{\rho}_1\widetilde{\rho}_2} + \frac{\alpha}{\widetilde{\rho}_1}\right] - K_1 \\ =& -K_1. \end{aligned}$$

Because $H_2(x) = L_2 - K_1 - e^x$ for $x < x_p^*$, we have $P_{H_2}(x) = P_{\widetilde{H}_2}(x)$ for $x < x_p^* < \ln(L_2 - K_1)$, where $\widetilde{H}_2 = L_2 - K_1 - e^x$. By (5.3), $P_{\widetilde{H}_2}(x) = L_2 - K_1 - e^x(\psi_r^-(-\mathrm{i}))^{-1}$. Hence, there is an unique $x_2^* < x_p^*$ such that $P_{H_2}(x_2^*) = P_{\widetilde{H}_2}(x_2^*) = 0$, $P_{H_2}(x) = P_{\widetilde{H}_2}(x) < 0$ on (x_2^*, x_p^*) and $P_{H_2}(x) = P_{\widetilde{H}_2}(x)$ is decreasing on $(-\infty, x_2^*)$. By Theorem 2.10, x_2^* is the optimal boundary and

$$\begin{aligned} V_2(x) =& \int_{-\infty}^{x_2^*-x} P_{H_2}(x+z) f_{I_r}(z)dz \\ =& 1_{\{x\le x_2^*\}}(L_2 - K_1 - e^x) + 1_{\{x > x_2^*\}}\left(\sum_{\eta=1}^{2} \frac{-\widetilde{d}_\eta (L_2-K_1)e^{\widetilde{\rho}_\eta(x-x_2^*)}}{1-\widetilde{\rho}_\eta}\right). \end{aligned}$$

We complete the proof. ■

(**Put-on-Call option**). Note that $H_3(x) = L_1 - W(x) = 1_{\{x\ge x_c^*\}}(L_1+K_2-e^x)^+ + 1_{\{x\le x_c^*\}}(L_1 - K_2(\frac{d_1}{\rho_1-1}e^{\rho_1(x-x_c^*)} + \frac{d_2}{\rho_2-1}e^{\rho_2(x-x_c^*)}))^+$ and

$$\begin{aligned} P_{H_3}(x) =& \left(\frac{d_1\rho_1}{r}\cdot\frac{-\mu\alpha}{\alpha+\rho_1} + \frac{d_2\rho_2}{r}\cdot\frac{-\mu\alpha}{\alpha+\rho_2}\right)e^{-\alpha x}\int_{-\infty}^{x} H_3(y)e^{\alpha y}dy \\ &+ \left[\frac{d_1\rho_1}{r}\cdot(a + \frac{b^2\rho_1}{2}) + \frac{d_2\rho_2}{r}\cdot(a + \frac{b^2\rho_2}{2})\right]H_3(x) + \left[\frac{b^2}{2r}(d_1\rho_1 + d_2\rho_2)\right]H_3^{'}(x). \end{aligned} \tag{8.14}$$

Case 1: $L_1 + K_2 - e^{x_c^*} \le 0$. On $\{H_3 > 0\}$, we have $H_3(x) = L_1 - K_2(\frac{d_1}{\rho_1-1}e^{\rho_1(x-x_c^*)} + \frac{d_2}{\rho_2-1}e^{\rho_2(x-x_c^*)})$. It follows from (5.3) that $P_{H_3}(x) = L_1 - \left(\sum_{j=1}^{2}\frac{d_j K_2(\psi_r^-(-\mathrm{i}\rho_j))^{-1}e^{\rho_j(x-x_c^*)}}{\rho_j-1}\right)$. Hence, there exists unique $x_2^* < x_c^*$ such that $P_{H_3}(x_2^*) = 0$. By Theorem 2.10, we deduce that x_2^* is the optimal boundary and

$$V_3(x) = \int_{-\infty}^{x_2^*-x} P_{H_3}(x+z) f_{I_r}(z)dz, \tag{8.15}$$

where $f_{I_r}(z) = \sum_{j=1}^{2}\widetilde{d}_j\widetilde{\rho}_j e^{-\widetilde{\rho}_j z}1_{\{z<0\}}$.

Case 2: $L_1 + K_2 - e^{x_c^*} > 0$. Since H_3 is decreasing, we have $\{H_3 > 0\} = (-\infty, \widehat{a})$, for some $x_c^* < \widehat{a}$. For $x_c^* < x < \widehat{a}$, direct calculation gives

$$\begin{aligned} & e^{-\alpha x}\int_{-\infty}^{x} H_3(u)e^{\alpha u}du \\ =& e^{-\alpha(x-x_c^*)}\left(-\frac{d_1K_2}{(\rho_1-1)(\rho_1+\alpha)} - \frac{d_2K_2}{(\rho_2-1)(\rho_2+\alpha)} + \left(\frac{d_1\rho_1}{\rho_1-1} + \frac{d_2\rho_2}{\rho_2-1}\right)\frac{K_2}{\alpha+1} - \frac{K_2}{\alpha}\right) - \frac{e^x}{\alpha+1} + \frac{L_1+K_2}{\alpha} \\ =& \frac{K_2e^{-\alpha(x-x_c^*)}}{\alpha(\alpha+1)}\left(-\frac{d_1\rho_1}{\rho_1+\alpha} - \frac{d_2\rho_2}{\rho_2+\alpha}\right) - \frac{e^x}{\alpha+1} + \frac{L_1+K_2}{\alpha}. \end{aligned} \tag{8.16}$$

In addition, by the similar approach as in (8.5)-(8.7), we have that

$$\frac{d_1\rho_1}{\alpha+\rho_1}+\frac{d_2\rho_2}{\alpha+\rho_2}=\frac{r(\widetilde{\rho}_1+\alpha)(\widetilde{\rho}_2+\alpha)}{-\mu\widetilde{\rho}_1\widetilde{\rho}_2}, \tag{8.17}$$

$$\sum_{j=1}^{2} d_j\rho_j=\frac{r\alpha}{\frac{b^2}{2}\widetilde{\rho}_1\widetilde{\rho}_2}, \tag{8.18}$$

and

$$d_1(a\rho_1+\frac{b^2\rho_1^2}{2})+d_2(a\rho_2+\frac{b^2\rho_2^2}{2})=r+\mu\Big(\frac{d_1\rho_1}{\alpha+\rho_1}+\frac{d_2\rho_2}{\alpha+\rho_2}\Big). \tag{8.19}$$

Therefore, taking account of (8.14) and (8.16)-(8.19), we have that for $x_c^*<x<\widehat{a}$,

$$\begin{aligned}P_{H_3}(x)=&\Big(\frac{d_1\rho_1}{r}\cdot\frac{-\mu\alpha}{\alpha+\rho_1}+\frac{d_2\rho_2}{r}\cdot\frac{-\mu\alpha}{\alpha+\rho_2}\Big)\Big[\frac{K_2e^{-\alpha(x-x_c^*)}}{\alpha(\alpha+1)}\Big(-\frac{d_1\rho_1}{\rho_1+\alpha}-\frac{d_2\rho_2}{\rho_2+\alpha}\Big)-\frac{e^x}{\alpha+1}+\frac{L_1+K_2}{\alpha}\Big]\\&+\Big[1+\frac{\mu}{r}\Big(\frac{d_1\rho_1}{\alpha+\rho_1}+\frac{d_2\rho_2}{\alpha+\rho_2}\Big)\Big](L_1+K_2-e^x)-\Big[\frac{b^2}{2r}(d_1\rho_1+d_2\rho_2)\Big]e^x\\=&\frac{\mu K_2}{r(\alpha+1)}\Big(\frac{d_1\rho_1}{\alpha+\rho_1}+\frac{d_2\rho_2}{\alpha+\rho_2}\Big)^2e^{-\alpha(x-x_c^*)}\\&+\Big[-1-\frac{\mu}{r(\alpha+1)}\Big(\frac{d_1\rho_1}{\alpha+\rho_1}+\frac{d_2\rho_2}{\alpha+\rho_2}\Big)-\frac{b^2}{2r}(d_1\rho_1+d_2\rho_2)\Big]e^x+L_1+K_2.\end{aligned} \tag{8.20}$$

From the identity above, we see that $P_{H_3}(x)$ is decreasing on $(x_c^*,\widehat{a})$ and

$$\begin{aligned}\lim_{x\to(x_c^*)^+}P_{H_3}(x)=&L_1+K_2+\frac{\mu K_2}{r(\alpha+1)}\Big(\frac{d_1\rho_1}{\alpha+\rho_1}+\frac{d_2\rho_2}{\alpha+\rho_2}\Big)^2\\&-\Big[1+\frac{\mu}{r(\alpha+1)}\Big(\frac{d_1\rho_1}{\alpha+\rho_1}+\frac{d_2\rho_2}{\alpha+\rho_2}\Big)+\frac{b^2}{2r}(d_1\rho_1+d_2\rho_2)\Big]e^{x_c^*}.\end{aligned}$$

Next, we verify that $P_{H_3}(x)$ is decreasing on $(-\infty,x_c^*)$. By Remark 5.6, we have that for $x<x_c^*$,

$$P_{H_3}(x)=-d_1K_2e^{\rho_1(x-x_c^*)}\Big[\frac{\alpha(\rho_1-\widetilde{\rho}_1)(\rho_1-\widetilde{\rho}_2)}{\widetilde{\rho}_1\widetilde{\rho}_2(\rho_1-1)(\rho_1+\alpha)}\Big]-d_2K_2e^{\rho_2(x-x_c^*)}\Big[\frac{\alpha(\rho_2-\widetilde{\rho}_1)(\rho_2-\widetilde{\rho}_2)}{\widetilde{\rho}_1\widetilde{\rho}_2(\rho_2-1)(\rho_2+\alpha)}\Big]+L_1. \tag{8.21}$$

Because the coefficients of $P_{H_3}(x)$ in (8.21) are both negative, we conclude that $P_{H_3}(x)$ is decreasing on $(-\infty,x_c^*)$. Notice that $\lim_{x\to-\infty}P_{H_3}(x)=L_1$ and $\lim_{x\to\widehat{a}}P_{H_3}(x)<0$. Hence, there is an unique $x_2^*<\widehat{a}$ such that $P_{H_3}(x_2^*)=0$. (Note that if $\lim_{x\to(x_c^*)^+}P_{H_3}(x)<0$, then $x_2^*<x_c^*$; otherwise, $x_2^*\geq x_c^*$.) By Theorem 2.10, we deduce that x_2^* is the optimal boundary and the value function is given as in (8.15). ■

(**Put-on-Put option**). Note that $H_4(x)=L_1-U(x)=1_{\{x\leq x_p^*\}}(e^x+L_1-L_2)+1_{\{x>x_p^*\}}\Big(L_1-\sum_{\eta=1}^{\mu_2}\frac{-\widetilde{d}_\eta L_2e^{\widetilde{\rho}_\eta(x-x_p^*)}}{1-\widetilde{\rho}_\eta}\Big)$ and

$$\begin{aligned}Q_{H_4}(x)=&\Big(\frac{\widetilde{d}_1\widetilde{\rho}_1}{r}\cdot\frac{-\lambda\beta}{\beta-\widetilde{\rho}_1}+\frac{\widetilde{d}_2\widetilde{\rho}_2}{r}\cdot\frac{-\lambda\beta}{\beta-\widetilde{\rho}_2}\Big)e^{\beta x}\int_x^\infty H_4(y)e^{-\beta y}dy\\&-\Big[\frac{\widetilde{d}_1\widetilde{\rho}_1}{r}\cdot(a+\frac{b^2\widetilde{\rho}_1}{2})+\frac{\widetilde{d}_2\widetilde{\rho}_2}{r}\cdot(a+\frac{b^2\widetilde{\rho}_2}{2})\Big]H_4(x)-\Big[\frac{b^2}{2r}(\widetilde{d}_1\widetilde{\rho}_1+\widetilde{d}_2\widetilde{\rho}_2)\Big]H_4'(x).\end{aligned} \tag{8.22}$$

Case 1: $e^{x_p^*}+L_1-L_2\le 0$. On $\{H_4>0\}$, we have $H_4(x)=L_1-\sum_{\eta=1}^{2}\frac{-\widetilde{d}_\eta L_2 e^{\widetilde{\rho}_\eta(x-x_p^*)}}{1-\widetilde{\rho}_\eta}$. It follows from (5.2) that $Q_{H_4}(x)=L_1-\sum_{\eta=1}^{2}\frac{-\widetilde{d}_\eta L_2(\psi_r^+(-\mathrm{i}\widetilde{\rho}_\eta))^{-1}e^{\widetilde{\rho}_\eta(x-x_p^*)}}{1-\widetilde{\rho}_\eta}$. Therefore, there exists unique $x_2^*>x_p^*$ such that $Q_{H_4}(x_2^*)=0$. By Theorem 2.7, we deduce that x_2^* is the optimal boundary and

$$V_4(x)=\int_{x_2^*-x}^{\infty}Q_{H_4}(x+m)f_{M_r}(m)dm, \tag{8.23}$$

where $f_{M_r}(m)=\sum_{j=1}^{2}d_j\rho_j e^{-\rho_j m}1_{\{m>0\}}$.

Case 2: $e^{x_p^*}+L_1-L_2>0$. Since H_4 is increasing, we get $\{H_4>0\}=(\widehat{a},\infty)$, for some $\widehat{a}<x_p^*$. For $\widehat{a}<x<x_p^*$, we have

$$\begin{aligned}
&e^{\beta x}\int_x^\infty H_4(y)e^{-\beta y}dy\\
=&e^{\beta(x-x_p^*)}\left(\frac{e^{x_p^*}}{1-\beta}+\frac{L_2}{\beta}+\frac{-\widetilde{d}_1L_2}{(1-\widetilde{\rho}_1)(\widetilde{\rho}_1-\beta)}+\frac{-\widetilde{d}_2L_2}{(1-\widetilde{\rho}_2)(\widetilde{\rho}_2-\beta)}\right)-\frac{e^x}{1-\beta}+\frac{L_1-L_2}{\beta}\\
=&\frac{L_2e^{\beta(x-x_p^*)}}{\beta(\beta-1)}\left(\frac{-\widetilde{d}_1\widetilde{\rho}_1}{\beta-\widetilde{\rho}_1}+\frac{-\widetilde{d}_2\widetilde{\rho}_2}{\beta-\widetilde{\rho}_2}\right)-\frac{e^x}{1-\beta}+\frac{L_1-L_2}{\beta}.
\end{aligned}\tag{8.24}$$

Plugging (8.24) into (8.22)and using (8.7) gives for $\widehat{a}<x\le x_p^*$,

$$\begin{aligned}
Q_{H_4}(x)=&\left(\frac{\widetilde{d}_1\widetilde{\rho}_1}{r}\cdot\frac{-\lambda\beta}{\beta-\widetilde{\rho}_1}+\frac{\widetilde{d}_2\widetilde{\rho}_2}{r}\cdot\frac{-\lambda\beta}{\beta-\widetilde{\rho}_2}\right)\left[\frac{L_2e^{\beta(x-x_p^*)}}{\beta(\beta-1)}\left(\frac{-\widetilde{d}_1\widetilde{\rho}_1}{\beta-\widetilde{\rho}_1}+\frac{-\widetilde{d}_2\widetilde{\rho}_2}{\beta-\widetilde{\rho}_2}\right)-\frac{e^x}{1-\beta}+\frac{L_1-L_2}{\beta}\right]\\
&+\left[1+\frac{\lambda}{r}\left(\frac{\widetilde{d}_1\widetilde{\rho}_1}{\beta-\widetilde{\rho}_1}+\frac{\widetilde{d}_2\widetilde{\rho}_2}{\beta-\widetilde{\rho}_2}\right)\right]\left(e^x+L_1-L_2\right)-\left[\frac{b^2}{2r}(\widetilde{d}_1\widetilde{\rho}_1+\widetilde{d}_2\widetilde{\rho}_2)\right]e^x\\
=&\frac{\lambda L_2}{r(\beta-1)}\left(\frac{-\widetilde{d}_1\widetilde{\rho}_1}{\beta-\widetilde{\rho}_1}+\frac{-\widetilde{d}_2\widetilde{\rho}_2}{\beta-\widetilde{\rho}_2}\right)^2e^{\beta(x-x_p^*)}\\
&+\left[1+\frac{\lambda}{r(1-\beta)}\left(\frac{\widetilde{d}_1\widetilde{\rho}_1}{\beta-\widetilde{\rho}_1}+\frac{\widetilde{d}_2\widetilde{\rho}_2}{\beta-\widetilde{\rho}_2}\right)-\frac{b^2}{2r}(\widetilde{d}_1\widetilde{\rho}_1+\widetilde{d}_2\widetilde{\rho}_2)\right]e^x+L_1-L_2.
\end{aligned}\tag{8.25}$$

By using (8.5), (8.6) and the fact that $\beta\rho_1\rho_2>\beta\rho_1+\beta\rho_2-\beta$, we obtain that

$$1+\frac{\lambda}{r(1-\beta)}\left(\frac{\widetilde{d}_1\widetilde{\rho}_1}{\beta-\widetilde{\rho}_1}+\frac{\widetilde{d}_2\widetilde{\rho}_2}{\beta-\widetilde{\rho}_2}\right)-\frac{b^2}{2r}(\widetilde{d}_1\widetilde{\rho}_1+\widetilde{d}_2\widetilde{\rho}_2)=1-\frac{\beta\rho_1+\beta\rho_2-\beta-\rho_1\rho_2}{\rho_1\rho_2(\beta-1)}>0.$$

This together with (8.25) leads to the facts that $Q_{H_4}(x)$ is increasing on $(\widehat{a},x_p^*)$ and

$$\begin{aligned}
\lim_{x\to(x_p^*)^-}Q_{H_4}(x)=&\frac{\lambda L_2}{r(\beta-1)}\left(\frac{-\widetilde{d}_1\widetilde{\rho}_1}{\beta-\widetilde{\rho}_1}+\frac{-\widetilde{d}_2\widetilde{\rho}_2}{\beta-\widetilde{\rho}_2}\right)^2\\
&+\left[1+\frac{\lambda}{r(1-\beta)}\left(\frac{\widetilde{d}_1\widetilde{\rho}_1}{\beta-\widetilde{\rho}_1}+\frac{\widetilde{d}_2\widetilde{\rho}_2}{\beta-\widetilde{\rho}_2}\right)-\frac{b^2}{2r}(\widetilde{d}_1\widetilde{\rho}_1+\widetilde{d}_2\widetilde{\rho}_2)\right]e^{x_p^*}+L_1-L_2.
\end{aligned}\tag{8.26}$$

It remains to verify that $Q_{H_4}(x)$ is increasing on (x_p^*,∞). By Remark 5.5, we have that for $x>x_p^*$

$$Q_{H_4}(x)=\frac{(-\widetilde{d}_1)L_2\beta(\rho_1-\widetilde{\rho}_1)(\rho_2-\widetilde{\rho}_1)e^{\widetilde{\rho}_1(x-x_p^*)}}{\rho_1\rho_2(1-\widetilde{\rho}_1)(\widetilde{\rho}_1-\beta)}+\frac{(-\widetilde{d}_2)L_2\beta(\rho_1-\widetilde{\rho}_2)(\rho_2-\widetilde{\rho}_2)e^{\widetilde{\rho}_2(x-x_p^*)}}{\rho_1\rho_2(1-\widetilde{\rho}_2)(\widetilde{\rho}_2-\beta)}+L_1. \tag{8.27}$$

Moreover, because the coefficients of $Q_{H_4}(x)$ in (8.27) are both negative, we conclude that $Q_{H_4}(x)$ is increasing on (x_p^*, ∞). Also, notice that $\lim_{x\to\infty} Q_{H_4}(x) = L_1$ and $\lim_{x\to\widehat{a}} Q_{H_4}(x) < 0$. Hence, there exists an unique $x_2^* > \widehat{a}$ such that $Q_{H_4}(x_2^*) = 0$. (Note that if $\lim_{x\to(x_p^*)} Q_{H_4}(x) < 0$, $x_2^* > x_p^*$; otherwise, $x_2^* \leq x_p^*$.) By Theorem 2.7, we see that x_2^* is the optimal boundary and the value function is given in (8.23). ■

Next, we consider the compound options for diffusion processes and assume that $X_t = at + bW_t$. In this case, $d_1 = 1$, $\widetilde{d}_1 = -1$ and $\widetilde{\rho}_1 < 0 < \rho_1$ are the solutions of $r - az - \frac{b^2z^2}{2} = 0$. Recall that $g_1(x) = e^x - K_2$ and $W(x) = 1_{\{x\geq x_c^*\}}(e^x - K_2) + 1_{\{x<x_c^*\}}\frac{d_1K_2}{\rho_1-1}e^{\rho_1(x-x_c^*)}$. Here $Q_{g_1}(x) = e^x(\psi_r^+(-\mathrm{i}))^{-1} - K_2$ and x_c^* is the unique solution of $Q_{g_1}(x) = 0$, i.e.,

$$e^{x_c^*} = \frac{\rho_1 K_2}{\rho_1 - 1}. \tag{8.28}$$

In addition, $g_2(x) = L_2 - e^x$ and $U(x) = 1_{\{x\leq x_p^*\}}(L_2 - e^x) + 1_{\{x>x_p^*\}}\frac{-\widetilde{d}_1 L_2 e^{\widetilde{\rho}_1(x-x_p^*)}}{1-\widetilde{\rho}_1}$. Here $P_{g_2}(x) = L_2 - e^x(\psi_r^-(-\mathrm{i}))^{-1}$ and x_p^* is the unique solution of $P_{g_2}(x) = 0$, i.e.,

$$e^{x_p^*} = \frac{-\widetilde{\rho}_1 L_2}{1-\widetilde{\rho}_1}. \tag{8.29}$$

(**Call-on-Call option**). Notice that $H_1(x) = 1_{\{x\geq x_c^*\}}(e^x - K_1 - K_2) + 1_{\{x\leq x_c^*\}}\left(\frac{d_1K_2}{\rho_1-1}e^{\rho_1(x-x_c^*)} - K_1\right)$ and

$$Q_{H_1}(x) = -\frac{\widetilde{d}_1\widetilde{\rho}_1}{r}\cdot(a + \frac{b^2\widetilde{\rho}_1}{2})H_1(x) - \frac{b^2\widetilde{d}_1\widetilde{\rho}_1}{2r}H_1^{'}(x). \tag{8.30}$$

First, notice that if $e^{x_c^*} - K_1 - K_2 \leq 0$, then the rational price of the call-on-call option is the rational price of the perpetual American call option with the strike price $K_1 + K_2$. Next, consider the case $e^{x_c^*} - K_1 - K_2 > 0$ and write $\{H_1 > 0\} = (\widehat{a}, \infty)$, for some $\widehat{a} < x_c^*$. By using the facts that $d_1 = 1$, $\widetilde{d}_1 = -1$, $\frac{\widetilde{d}_1\widetilde{\rho}_1}{r}(a + \frac{b^2\widetilde{\rho}_1}{2}) = -1$ and $-\frac{b^2}{2r}\widetilde{\rho}_1\rho_1 = 1$, , we see that for $\widehat{a} < x < x_c^*$

$$\begin{aligned} Q_{H_1}(x) =& -\frac{\widetilde{d}_1\widetilde{\rho}_1}{r}\cdot(a + \frac{b^2\widetilde{\rho}_1}{2})\left(\frac{d_1K_2}{\rho_1-1}e^{\rho_1(x-x_c^*)} - K_1\right) - \frac{b^2}{2r}(\widetilde{d}_1\widetilde{\rho}_1)\left(\frac{d_1\rho_1K_2}{\rho_1-1}e^{\rho_1(x-x_c^*)}\right) \\ =& -K_1. \end{aligned}$$

For $x \geq x_c^*$, $Q_{H_1}(x) = e^x - K_1 - K_2 - \frac{e^x}{\rho_1}$. Therefore, it is easy to see that there is an unique $x_2^* > x_c^*$ such that $Q_{H_1}(x_2^*) = 0$ and $Q_{H_1}(x)$ is increasing on (x_2^*, ∞). By Theorem 2.7, x_2^* is the optimal boundary and

$$V_1(x) = \int_{x_2^*-x}^{\infty} Q_{H_1}(x+m)f_{M_r}(m)dm. \tag{8.31}$$

Observe that $e^{x_c^*} = \frac{\rho_1 K_2}{\rho_1 - 1}$ and $e^{x_2^*}(\frac{\rho_1-1}{\rho_1}) = K_1 + K_2$. Therefore, we have

$$e^{x_2^*} = \frac{\rho_1 K_1}{\rho_1 - 1} + e^{x_c^*}. \tag{8.32}$$

Plugging the identities $Q_{H_1}(x) = (\frac{\rho_1-1}{\rho_1})e^x - K_1 - K_2$ and $f_{M_r}(x) = \rho_1 e^{-\rho_1 x}$ into (8.31), we get that for $x < x_2^*$

$$\begin{aligned} V(x) =& \int_{x_2^*-x}^{\infty} \left(\frac{\rho_1 - 1}{\rho_1} e^{x+m} - K_1 - K_2 \right) \rho_1 e^{-\rho_1 m} dm = \left(e^{x_2^*} - K_1 - K_2 \right) e^{-\rho_1 (x_2^* - x)} \\ =& \frac{e^{x_2^*}}{\rho_1} e^{-\rho_1 (x_2^* - x)} \end{aligned} \tag{8.33}$$

(The last identity follows from $Q_{H_1}(x_2^*) = 0$). Moreover, for $x > x_2^*$, $V(x) = e^x - K_1 - K_2$. ■

(**Call-on-Put option**). Notice that $H_2(x) = 1_{\{x \le x_p^*\}}(L_2 - K_1 - e^x) + 1_{\{x > x_p^*\}}\left(\frac{-\widetilde{d}_1 L_2 e^{\widetilde{\rho}_1 (x - x_p^*)}}{1 - \widetilde{\rho}_1} - K_1 \right)$ and

$$P_{H_2}(x) = \frac{d_1 \rho_1}{r}(a + \frac{b^2 \rho_1}{2}) H_2(x) + \frac{b^2 d_1 \rho_1}{2r} H_2'(x). \tag{8.34}$$

First, notice that if $L_2 - K_1 - e^{x_p^*} \le 0$ then the rational price of the call-on-put option is the rational price of the perpetual American put option with the strike price $L_2 - K_1$. Next, consider the case in which $L_2 - K_1 - e^{x_p^*} > 0$ and write $\{H_2 > 0\} = (-\infty, \widehat{a})$, for some $\widehat{a} > x_p^*$. Taking account of the facts that $d_1 = 1$, $\frac{d_1 \rho_1}{r}(a + \frac{b^2 \rho_1}{2}) = 1$ and $-\frac{d_1 \rho_1 \widetilde{\rho}_1}{2r} = 1$, we have that for $x_p^* < x < \widehat{a}$,

$$P_{H_2}(x) = \frac{L_2 e^{\widetilde{\rho}_1 (x - x_p^*)}}{1 - \widetilde{\rho}_1} - K_1 + \frac{1}{(-\widetilde{\rho}_1)} \frac{\widetilde{\rho}_1 L_2 e^{\widetilde{\rho}_1 (x - x_p^*)}}{1 - \widetilde{\rho}_1} = -K_1.$$

Also, for $x \le x_p^*$, we have $P_{H_2}(x) = (-1 + \frac{1}{\widetilde{\rho}_1})e^x + L_2 - K_1$. Therefore, it is easy to see that there exists an unique $x_2^* < x_p^*$ such that $P_{H_2}(x_2^*) = 0$ and $P_{H_2}(x)$ is decreasing on $(-\infty, x_2^*)$. By Theorem 2.10, we conclude that x_2^* is the optimal boundary and

$$V_2(x) = \int_{-\infty}^{x_2^* - x} P_{H_2}(x + z) f_{I_r}(z) dz. \tag{8.35}$$

By using the facts that $e^{x_p^*} = \frac{(-\widetilde{\rho}_1)L_2}{1 - \widetilde{\rho}_1}$ and $(\frac{1 - \widetilde{\rho}_1}{\widetilde{\rho}_1})e^x + L_2 - K_1 = 0$, we obtain

$$e^{x_2^*} = \frac{\widetilde{\rho}_1 K_1}{1 - \widetilde{\rho}_1} + e^{x_p^*}. \tag{8.36}$$

Plugging the identities $P_{H_2}(x) = (\frac{1 - \widetilde{\rho}_1}{\widetilde{\rho}_1})e^x - K_1 + L_2$ and $f_{I_r}(x) = -\widetilde{\rho}_1 e^{-\widetilde{\rho}_1 x}$ into (8.35), we have that for $x \ge x_2^*$,

$$\begin{aligned} V_2(x) =& \int_{-\infty}^{x_2^* - x} \left((\frac{1 - \widetilde{\rho}_1}{\widetilde{\rho}_1}) e^{x+z} - K_1 + L_2 \right) (-\widetilde{\rho}_1) e^{-\widetilde{\rho}_1 z} dz \\ =& \frac{e^{x_2^*}}{-\widetilde{\rho}_1} e^{\widetilde{\rho}_1 (x - x_2^*)}. \end{aligned} \tag{8.37}$$

Furthermore, for $x < x_2^*$, $V_2(x) = L_2 - K_1 - e^x$. ■

(**Put-on-Call option**). Note that $H_3(x) = 1_{\{x \geq x_c^*\}}(L_1 + K_2 - e^x) + 1_{\{x \leq x_c^*\}}(L_1 - \frac{d_1 K_2}{\rho_1 - 1} e^{\rho_1(x - x_c^*)})$ and

$$P_{H_3}(x) = \frac{d_1\rho_1}{r} \cdot (a + \frac{b^2\rho_1}{2}) H_3(x) + \frac{b^2}{2r}(d_1\rho_1) H_3^{'}(x). \tag{8.38}$$

First, notice that if $L_1 + K_2 \leq e^{x_c^*}$, then on $\{H_3 > 0\}$, we have $H_3(x) = L_1 - \frac{d_1 K_2}{\rho_1 - 1} e^{\rho_1(x - x_c^*)}$. It follows from (5.3) that $P_{H_3}(x) = L_1 - \frac{d_1 K_2 (\psi_r^-(-\mathrm{i}\rho_1))^{-1} e^{\rho_1(x - x_c^*)}}{\rho_1 - 1}$. Therefore, there exists an unique $x_2^* < x_c^*$ such that $P_{H_3}(x_2^*) = 0$. By Theorem 2.10, we deduce that x_2^* is the optimal boundary and

$$V_3(x) = \int_{-\infty}^{x_2^* - x} P_{H_3}(x + z) f_{I_r}(z) dz, \tag{8.39}$$

where $f_{I_r}(z) = \widetilde{d}_1 \widetilde{\rho}_1 e^{-\widetilde{\rho}_1 z} 1_{\{z < 0\}}$.
Next, consider the case $L_1 + K_2 > e^{x_c^*}$. Since H_3 is decreasing, we have $\{H_3 > 0\} = (-\infty, \widehat{a})$ for some $\widehat{a} > x_c^*$. Plugging $H_3(x) = L_1 + K_2 - e^x$ into (8.38) and using the facts $d_1 = 1$, $\frac{d_1\rho_1}{r}(a + \frac{b^2\rho_1}{2}) = 1$ and $-\frac{d_1\rho_1\widetilde{\rho}_1 b^2}{2r} = 1$, we get that for $x_c^* \leq x < \widehat{a}$,

$$P_{H_3}(x) = \frac{d_1\rho_1}{r}(a + \frac{b^2\rho_1}{2})\Big[L_1 + K_2 - e^x\Big] - \frac{d_1\rho_1 b^2}{2r} e^x = L_1 + K_2 + (\frac{1}{\widetilde{\rho}_1} - 1)e^x. \tag{8.40}$$

On the other hand, for $x < x_c^*$,

$$\begin{aligned} P_{H_3}(x) =& L_1 - \frac{K_2 e^{\rho_1(x - x_c^*)}}{\rho_1 - 1} + \frac{1}{(-\widetilde{\rho}_1)}\left(-\frac{\rho_1 K_2 e^{\rho_1(x - x_c^*)}}{\rho_1 - 1}\right) \\ =& \frac{K_2 e^{\rho_1(x - x_c^*)}}{\rho_1 - 1}\left(-1 + \frac{\rho_1}{\widetilde{\rho}_1}\right) + L_1. \end{aligned} \tag{8.41}$$

It follows from (8.40) and (8.41) that $P_{H_3}(x)$ is a decreasing function on $(-\infty, \widehat{a})$. By (8.28) , we obtain that

$$\lim_{x \to (x_c^*)^+} P_{H_3}(x) = L_1 + K_2 + (\frac{1 - \widetilde{\rho}_1}{\widetilde{\rho}_1}) e^{x_c^*} = L_1 - \left(\frac{\widetilde{\rho}_1 - \rho_1}{\widetilde{\rho}_1(\rho_1 - 1)}\right) K_2. \tag{8.42}$$

If $L_1 < \left(\frac{\widetilde{\rho}_1 - \rho_1}{\widetilde{\rho}_1(\rho_1 - 1)}\right) K_2$, then there is only one $x_2^* < x_c^*$ such that $P_{H_3}(x_2^*) = 0$. Also, by (8.28), we have

$$e^{\rho_1 x_2^*} = \frac{L_1 \widetilde{\rho}_1 \rho_1 e^{(\rho_1 - 1)x_c^*}}{\widetilde{\rho}_1 - \rho_1}. \tag{8.43}$$

Therefore, by Theorem 2.10, we deduce that x_2^* is the optimal boundary and for $x \geq x_2^*$,

$$\begin{aligned} V_3(x) =& \int_{-\infty}^{x_2^*} P_{H_1}(u) f_{I_r}(u - x) du = \int_{-\infty}^{x_2^*} \left[\frac{K_2 e^{\rho_1(u - x_c^*)}}{\rho_1 - 1}\left(-1 + \frac{\rho_1}{\widetilde{\rho}_1}\right) + L_1\right](-\widetilde{\rho}_1) e^{-\widetilde{\rho}_1(u - x)} du \\ =& e^{\widetilde{\rho}_1(x - x_2^*)}\left(\frac{K_2 e^{-\rho_1(x_c^* - x_2^*)}}{1 - \rho_1} + L_1\right) = (-\frac{1}{\widetilde{\rho}_1}) e^{x_c^*} \cdot e^{\widetilde{\rho}_1(x - x_2^*)} \cdot e^{\rho_1(x_2^* - x_c^*)} \end{aligned}$$

(The last identity comes form the relation $\frac{K_2 e^{\rho_1(x_2^*-x_c^*)}}{\rho_1-1}\Big(-1+\frac{\rho_1}{\widetilde{\rho}_1}\Big)+L_1=0$ and (8.28)). Also, for $x<x_2^*$,

$$V_3(x)=L_1-\frac{d_1K_2e^{\rho_1(x-x_c^*)}}{\rho_1-1}. \tag{8.44}$$

On the other hand, if $L_1\geq\frac{\widetilde{\rho}_1-\rho_1}{\widetilde{\rho}_1(\rho_1-1)}K_2$, then there is only one $x_2^*\geq x_c^*$ such that $P_{H_3}(x_2^*)=0$, which leads to the fact that

$$e^{x_2^*}=\frac{\widetilde{\rho}_1(L_1+K_2)}{\widetilde{\rho}_1-1}. \tag{8.45}$$

By Theorem 2.10, we deduce that x_2^* is the optimal boundary and the value function is given as follows.

$$\begin{aligned}V_3(x)=&\int_{-\infty}^{x_c^*}P_H(u)f_{I_r}(u-x)du+\int_{x_c^*}^{x_2^*}P_H(u)f_{I_r}(u-x)du\\=&e^{\widetilde{\rho}_1(x-x_c^*)}\Big(e^{x_c^*}+\frac{\rho_1K_2}{1-\rho_1}\Big)+e^{\widetilde{\rho}_1(x-x_2^*)}\Big(L_1+K_2-e^{x_2^*}\Big)\\=&e^{\widetilde{\rho}_1(x-x_2^*)}\frac{e^{x_2^*}}{-\widetilde{\rho}_1}.\end{aligned} \tag{8.46}$$

(The last identity comes from (8.28) and (8.45).) Also, for $x<x_c^*$,

$$V_3(x)=L_1-\frac{d_1K_2e^{\rho_1(x-x_c^*)}}{\rho_1-1} \tag{8.47}$$

and for $x_c^*\leq x\leq x_2^*$, $V_3(x)=L_1+K_2-e^x$. ■

(**Put-on-Put option**). Notice that $H_4(x)=1_{\{x\leq x_p^*\}}(e^x+L_1-L_2)+1_{\{x>x_p^*\}}\Big(L_1-\frac{-\widetilde{d}_1L_2e^{\widetilde{\rho}_1(x-x_p^*)}}{1-\widetilde{\rho}_1}\Big)$ and

$$Q_{H_4}(x)=-\frac{\widetilde{d}_1\widetilde{\rho}_1}{r}\cdot(a+\frac{b^2\widetilde{\rho}_1}{2})H_4(x)-\frac{b^2\widetilde{d}_1\widetilde{\rho}_1}{2r}H_4^{'}(x). \tag{8.48}$$

First, notice that if $e^{x_p^*}+L_1-L_2\leq 0$, then on $\{H_4>0\}$, we have $H_4(x)=1_{\{x\geq x_p^*\}}(L_1-\frac{(-\widetilde{d}_1)L_2e^{\widetilde{\rho}_1(x-x_p^*)}}{1-\widetilde{\rho}_1})$. It follows from (5.2) that there exists an unique $x_2^*>x_p^*$ such that $Q_{H_4}(x_2^*)=0$. By Theorem 2.7, x_2^* is the optimal boundary and

$$V_4=\int_{x_2^*-x}^{\infty}Q_{H_4}(x+m)f_{M_r}(m)dm, \tag{8.49}$$

where $f_{M_r}(m)=d_1\rho_1e^{-\rho_1m}1_{\{m>0\}}$. Next, consider the case in which $e^{x_p^*}+L_1-L_2>0$. Since H_4 is increasing, we get $\{H_4>0\}=(\widehat{a},\infty)$, for some $\widehat{a}<x_p^*$. By using the facts that $d_1=1$, $\widetilde{d}_1=-1$, $\frac{\widetilde{d}_1\widetilde{\rho}_1}{r}\cdot(a+\frac{b^2\widetilde{\rho}_1}{2})=-1$ and $-\frac{b^2}{2r}\widetilde{\rho}_1\rho_1=1$, we see that for $\widehat{a}<x\leq x_p^*$

$$Q_{H_4}(x)=e^x+L_1-L_2+\frac{e^x}{(-\rho_1)}=(1-\frac{1}{\rho_1})e^x+L_1-L_2 \tag{8.50}$$

and for $x>x_p^*$

$$Q_{H_4}(x)=L_1-\frac{L_2e^{\widetilde{\rho}_1(x-x_p^*)}}{1-\widetilde{\rho}_1}+\frac{\widetilde{\rho}_1L_2e^{\widetilde{\rho}_1(x-x_p^*)}}{\rho_1(1-\widetilde{\rho}_1)}=\frac{L_2e^{\widetilde{\rho}_1(x-x_p^*)}}{1-\widetilde{\rho}_1}(-1+\frac{\widetilde{\rho}_1}{\rho_1})+L_1. \tag{8.51}$$

It follows from (8.50) and (8.51) that $Q_{H_4}(x)$ is an increasing function on $(\widehat{a}, \infty)$. By (8.29), we see that

$$\lim_{x \to (x_p^*)^-} Q_{H_4}(x) = (\frac{\rho_1 - 1}{\rho_1})e^{x_p^*} + L_1 - L_2 = \frac{(\widetilde{\rho}_1 - \rho_1)L_2}{\rho_1(1-\widetilde{\rho}_1)} + L_1. \tag{8.52}$$

If $\frac{(\widetilde{\rho}_1-\rho_1)L_2}{\rho_1(1-\widetilde{\rho}_1)} + L_1 < 0$ then there exists only one $x_2^* > x_p^*$ such that $Q_{H_4}(x_2^*) = 0$. By (8.29), we obtain

$$e^{\widetilde{\rho}_1 x_2^*} = \frac{L_1 \rho_1 \widetilde{\rho}_1 e^{\widetilde{\rho}_1 x_p^*}}{(\widetilde{\rho}_1 - \rho_1)e^{x_p^*}}. \tag{8.53}$$

By Theorem 2.7, we see that x_2^* is the optimal boundary and for $x < x_2^*$,

$$\begin{aligned} V_4(x) &= \int_{x_2^*}^{\infty} Q_{H_4}(u) f_{M_r}(u-x)du = \int_{x_2^*}^{\infty} \left[\frac{L_2(\widetilde{\rho}_1 - \rho_1)}{(1-\widetilde{\rho}_1)\rho_1} e^{\widetilde{\rho}_1(u-x_p^*)} + L_1\right] \rho_1 e^{-\rho_1(u-x)} du \\ &= e^{\rho_1(x-x_2^*)} e^{\widetilde{\rho}_1(x_2^*-x_p^*)} \frac{-\widetilde{\rho}_1 L_2}{\rho_1(1-\widetilde{\rho}_1)} = e^{\rho_1(x-x_2^*)} e^{\widetilde{\rho}_1(x_2^*-x_p^*)} \frac{e^{x_p^*}}{\rho_1} \end{aligned} \tag{8.54}$$

(The last equality follows from (8.29)). Also, for $x \geq x_2^*$, $V_4(x) = L_1 - \frac{(-\widetilde{d}_1)L_2 e^{\widetilde{\rho}_1(x-x_p^*)}}{1-\widetilde{\rho}_1}$. On the other hand, if $\frac{(\widetilde{\rho}_1-\rho_1)L_2}{\rho_1(1-\widetilde{\rho}_1)} + L_1 > 0$ then there is only one $x_2^* < x_p^*$ such that $Q_{H_4}(x_2^*) = 0$. In this case, we have

$$e^{x_2^*} = \frac{(L_2 - L_1)\rho_1}{\rho_1 - 1}. \tag{8.55}$$

By Theorem 2.7, x_2^* is the optimal boundary and for $x < x_2^*$,

$$\begin{aligned} V_4(x) &= \int_{x_2^*}^{x_p^*} Q_{H_4}(u) f_{M_r}(u-x)du + \int_{x_p^*}^{\infty} Q_{H_4}(u) f_{M_r}(u-x)du \\ &= e^{\rho_1(x-x_p^*)}\left(e^{x_p^*} + L_2 - \frac{L_2}{1-\widetilde{\rho}_1}\right) + e^{\rho_1(x-x_2^*)}\left(e^{x_2^*} + L_1 - L_2\right) \\ &= e^{\rho_1(x-x_2^*)} \frac{e^{x_2^*}}{\rho_1}. \end{aligned} \tag{8.56}$$

The last identity comes from (8.29) and (8.55). Also, we have for $x > x_2^*$,

$$V_4(x) = 1_{\{x_2^* \leq x \leq x_p^*\}} e^x - L_1 - L_2 + 1_{\{x > x_p^*\}} L_1 - \frac{(-\widetilde{d}_1)L_2 e^{\widetilde{\rho}_1(x-x_p^*)}}{1-\widetilde{\rho}_1}. \tag{8.57}$$

9 Connections between $Q_g(x)$ and $\sigma_g(x)$

In this section, we establish a relation between Q_g and σ_g. Here, we assume that the process $\{X_t\}_{t\geq 0}$ is a jump-diffusion process with positive matrix-exponential jumps (there is no downside jumps). Also, it follows from (4.20) and (4.7) that if X^+ is not a subordinator (i.e., $\mu_2 \geq 1$),

$$G_r(x,y) = \begin{cases} \frac{1}{r}\Big(1_{\{a<0 \text{ and } b=0\}} d_0(-\widetilde{\rho}_1) + 1_{\{\mu_1\geq 1\}} \sum_{j=1}^{\mu_1} \frac{(-\widetilde{\rho}_1) d_j \rho_j}{\rho_j - \widetilde{\rho}_1}\Big) e^{-\widetilde{\rho}_1(y-x)}, & y-x<0, \\ \\ \frac{1}{r}\Big(1_{\{\mu_1\geq 1\}} \sum_{j=1}^{\mu_1} \frac{(-\widetilde{\rho}_1) d_j \rho_j}{\rho_j - \widetilde{\rho}_1}\Big) e^{-\rho_j(y-x)}, & y-x>0; \end{cases} \tag{9.1}$$

otherwise,

$$G_r(x,y) = 1_{\{y-x>0\}} \frac{1}{r} f_M(y-x). \tag{9.2}$$

We write $g \in \pi_2$ if g and g' are in π_0. Note that if $g \in \pi_2$, then $Q_g'(x)$ exists almost everywhere. For $g \in \pi_2$, we define

$$\sigma_g'(x) = \begin{cases} \frac{-r}{\widetilde{\rho}_1}\Big(- \widetilde{\rho}_1 Q_g(x) + Q_g'(x)\Big), & \text{if } X^+ \text{ is not a subordinator,} \\ \\ rQ_g(x), & \text{if } X^+ \text{ is a subordinator.} \end{cases} \tag{9.3}$$

Theorem 9.1 *Assume that $g \in \pi_2$ and $Q_g(x^*) = 0$ for some $x^* \in \mathbb{R}$. Set $\widetilde{V}(x) = \int_{x^*}^{\infty} G_r(x,y)\sigma_g'(y)dy$ for all x. Then we have*

(a) For every $x \geq x^*, \widetilde{V}(x) = g(x)$.

(b) For any $x < x^*, \widetilde{V}(x) = \mathbb{E}\Big[Q_g(M_r + x); M_r + x \geq x^*\Big]$ and $\lim_{x\to-\infty} \widetilde{V}(x) = 0$.

(c) If $Q_g(x) \leq 0$ for $x \leq x^*$, then we obtain $\widetilde{V}(x) \geq g(x)$ and in addition, if $Q_g(x)$ is non-decreasing for $x > x^*$ then $\widetilde{V}(x)$ is the value function for the optimal stopping problem (1.1).

Proof. (a) We consider first that X^+ is not a subordinator. Using integration by parts, we have for $x \geq x^*$

$$\int_{x^*}^{x} e^{-\widetilde{\rho}_1(y-x)} \sigma_g'(y)dy = \frac{-r}{\widetilde{\rho}_1}\Big(Q_g(x) - e^{\widetilde{\rho}_1(x-x^*)} Q_g(x^*)\Big). \tag{9.4}$$

On the other hand, notice that for $\mu_1 \geq 1$, $g \in \pi_2$ implies that

$$\lim_{z\to\infty} Q_g(z) e^{-\rho_i z} = 0. \tag{9.5}$$

This along with integration by parts yields that for $1 \leq i \leq \mu_1$,

$$\int_{x}^{\infty} e^{-\rho_i(y-x)} \sigma_g'(y)dy = \frac{-r}{\widetilde{\rho}_1}\Big(- Q_g(x) + (\rho_i - \widetilde{\rho}_1) \int_{x}^{\infty} e^{-\rho_i(y-x)} Q_g(y)dy\Big). \tag{9.6}$$

It follows from (9.4), (9.6), and (9.1) that

$$\int_{x^*}^{x} G_r(x,y)\sigma_g'(y)dy$$
$$= \Big(1_{\{a<0 \text{ and } b=0\}}d_0 + 1_{\{\mu_1\ge 1\}}\sum_{j=1}^{\mu_1}\frac{d_j\rho_j}{\rho_j-\widetilde{\rho}_1}\Big)\Big(Q_g(x) - e^{\widetilde{\rho}_1(x-x^*)}Q_g(x^*)\Big) \tag{9.7}$$

and

$$\int_{x}^{\infty} G_r(x,y)\sigma_g'(y)dy$$
$$= 1_{\{\mu_1\ge 1\}}\sum_{j=1}^{\mu_1}\frac{d_j\rho_j}{\rho_j-\widetilde{\rho}_1}\Big(-Q_g(x) + (\rho_j-\widetilde{\rho}_1)\int_x^\infty e^{-\rho_j(y-x)}Q_g(y)dy\Big). \tag{9.8}$$

Therefore, for $x \ge x^*$,

$$\begin{aligned}
\widetilde{V}(x) &:= \int_{x^*}^{\infty} G_r(x,y)\sigma_g'(y)dy = \int_{x^*}^{x} G_r(x,y)\sigma_g'(y)dy + \int_{x}^{\infty} G_r(x,y)\sigma_g'(y)dy \\
&= 1_{\{a<0 \text{ and } b=0\}}d_0Q_g(x) + 1_{\{\mu_1\ge 1\}}\sum_{j=1}^{\mu_1}\int_x^\infty d_j\rho_j e^{-\rho_j(y-x)}Q_g(y)dy \\
&= \int_x^\infty Q_g(y)f_M(y-x)dy = \mathbb{E}\Big[Q_g(M_r+x)\Big] = g(x).
\end{aligned} \tag{9.9}$$

Now we consider the case that X^+ is a subordinator. It follows from (9.3) and (9.2) that for $x \ge x^*$,

$$\begin{aligned}
\widetilde{V}(x) &:= \int_{x^*}^{\infty} G_r(x,y)\sigma_g'(y)dy = \int_{x^*}^{x} G_r(x,y)\sigma_g'(y)dy + \int_{x}^{\infty} G_r(x,y)\sigma_g'(y)dy \\
&= \int_x^\infty Q_g(y)f_M(y-x)dy = \mathbb{E}\Big[Q_g(M_r+x)\Big] = g(x).
\end{aligned} \tag{9.10}$$

(b) Assume X^+ is not a subordinator. For $\mu_1 \ge 1$, we see by (9.5) and integration by parts that

$$\int_{x^*}^\infty e^{-\rho_i(y-x)}Q_g'(y)dy = \rho_i\int_{x^*}^\infty e^{-\rho_i(y-x)}Q_g(y)dy.$$

This together with $\sigma_g'(x) = \frac{-r}{\widetilde{\rho}_1}\Big(-\widetilde{\rho}_1Q_g(x) + Q_g'(x)\Big)$ implies

$$\int_{x^*}^\infty e^{-\rho_i(y-x)}\sigma_g'(y)dy = (\frac{-r}{\widetilde{\rho}_1})(\rho_i-\widetilde{\rho}_1)\int_{x^*}^\infty e^{-\rho_i(y-x)}Q_g(y)dy. \tag{9.11}$$

Taking account of (9.11) and (9.1), we obtain that for $x < x^*$

$$\begin{aligned}
\widetilde{V}(x) &:= \int_{x^*}^{\infty} G_r(x,y)\sigma_g'(y)dy \\
&= 1_{\{\mu_1\ge 1\}}\sum_{j=1}^{\mu_1}d_j\rho_j\int_{x^*}^\infty e^{-\rho_j(y-x)}Q_g(y)dy \\
&= \int_{x^*}^\infty Q_g(y)f_M(y-x)dy = \mathbb{E}\Big[Q_g(M_r+x); M_r+x\ge x^*\Big].
\end{aligned}$$

Next we assume X^+ is a subordinator. It follows from (9.3) and (9.2) that for $x < x^*$

$$\begin{aligned} \widetilde{V}(x) &:= \int_{x^*}^{\infty} G_r(x,y)\sigma_g'(y)dy \\ &= \int_{x^*}^{\infty} Q_g(y) f_M(y-x)dy = \mathbb{E}\Big[Q_g(M_r+x); M_r+x \ge x^*\Big]. \end{aligned}$$

In both cases, since $\mathcal{R}e\,(\rho_j) > 0$ for $1 \le j \le \mu_1$, we see that

$$\lim_{x\to-\infty} \int_{x^*}^{\infty} e^{-\rho_j(y-x)} Q_g(y)dy = 0. \tag{9.12}$$

This implies $\lim_{x\to-\infty} \widetilde{V}(x) = 0$.

(c) It follows from (b) and Q_g is a solution of the averaging problem that

$$\begin{aligned} \widetilde{V}(x) &= \mathbb{E}\Big[Q_g(M_r+x); M_r+x \ge x^*\Big] \\ &= g(x) - \mathbb{E}\Big[Q_g(M_r+x); M_r+x < x^*\Big]. \end{aligned} \tag{9.13}$$

This along with $Q_g(x) \le 0$, for any $x < x^*$ yields $\widetilde{V}(x) \ge g(x)$. Additionally, if $Q_g(x)$ is non-decreasing for $x > x^*$, then using Theorem 2.7, we obtain that $\widetilde{V}(x)$ is the value function for the optimal stopping problem (1.1). ∎

Example 9.2 *(Example 5.3 continued). We have*

$$Q_g(x) = \frac{-\widetilde{\rho}_1}{r}\bigg(-\frac{\lambda\beta}{\beta-\widetilde{\rho}_1}\int_x^{\infty} y^\gamma e^{-\beta(y-x)}dy - ax^\gamma\bigg)$$

and

$$Q_g'(x) = \frac{-\widetilde{\rho}_1}{r}\bigg(-\frac{\lambda\beta^2}{\beta-\widetilde{\rho}_1}\int_x^{\infty} y^\gamma e^{-\beta(y-x)}dy + \frac{\lambda\beta}{\beta-\widetilde{\rho}_1}x^\gamma - a\gamma x^{\gamma-1}\bigg).$$

Therefore, we have

$$\begin{aligned} \sigma_g'(x) &= \frac{-r}{\widetilde{\rho}_1}\Big(-\widetilde{\rho}_1 Q_g(x) + Q_g'(x)\Big) \\ &= -\lambda\beta e^{\beta x}\int_x^{\infty} y^\gamma e^{-\beta y}dy + \bigg(\frac{\lambda\beta}{\beta-\widetilde{\rho}_1} + a\widetilde{\rho}_1\bigg)x^\gamma - a\gamma x^{\gamma-1}. \end{aligned}$$

Note that $-\mathrm{i}\widetilde{\rho}_1$ *is a solution of the equation* $r - \psi(z) = 0$, *where* $\psi(z) = \mathrm{i}az + \lambda\Big(\frac{\mathrm{i}\beta}{z+\mathrm{i}\beta} - 1\Big)$. *This implies that* $r + \lambda = a\widetilde{\rho}_1 + \frac{\lambda\beta}{\beta-\widetilde{\rho}_1}$ *and* $\sigma_g'(x) = -\lambda\beta e^{\beta x}\int_x^{\infty} y^\gamma e^{-\beta y}dy + (r+\lambda)x^\gamma - a\gamma x^{\gamma-1}$.

10 Appendix

Proof of Lemma 2.3. We quote below the proof of Lemma 9.1 in [15]. The definition of V^* implies that

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}_x(e^{-r\tau}g(X_\tau)) \geq V^*(x)$$

for all $x \in \mathbb{R}$. On the other hand, property (ii) together with Doob's Optional Stopping Theorem imply that for all $t \geq 0$, $x \in \mathbb{R}$ and $\sigma \in \mathcal{T}$,

$$V^*(x) \geq \mathbb{E}_x(e^{-r(t\wedge\sigma)}V^*(X_{t\wedge\sigma}))$$

and hence by property (i), Fatou's Lemma, the non-negativity of g and assumption (2.3)

$$\begin{aligned} V^*(x) &\geq \liminf_{t\uparrow\infty} \mathbb{E}_x(e^{-r(t\wedge\sigma)}g(X_{t\wedge\sigma})) \\ &\geq \mathbb{E}_x(\liminf_{t\uparrow\infty} e^{-r(t\wedge\sigma)}g(X_{t\wedge\sigma})) \\ &= \mathbb{E}_x(e^{-r\sigma}g(X_\sigma)). \end{aligned}$$

As $\sigma \in \mathcal{T}$ is arbitrary, it follows that for all $x \in \mathbb{R}$

$$V^*(x) \geq \sup_{\tau\in\mathcal{T}} \mathbb{E}_x(e^{-r\tau}g(X_\tau)).$$

In conclusion it must hold that

$$V^*(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x(e^{-r\tau}g(X_\tau))$$

for all $x \in \mathbb{R}$. ■

Proof of (4.32).

$$\begin{aligned} & e^{\rho_s x}\int_x^z e^{(\beta_k-\rho_s)u}\int_u^\infty (y-u)^{j-\ell}g(y)e^{-\beta_k y}dydu \\ = & \sum_{\xi=1}^{j-\ell+1} \frac{(j-\ell)!}{(j-\ell+1-\xi)!(\beta_k-\rho_s)^\xi} \times \\ & \left[e^{\rho_s(x-z)}e^{\beta_k z}\int_z^\infty (y-z)^{j-\ell+1-\xi}g(y)e^{-\beta_k y}dy - e^{\beta_k x}\left(\int_x^\infty (y-x)^{j-\ell+1-\xi}g(y)e^{-\beta_k y}dy\right)\right] \\ & + \frac{(j-\ell)!e^{\rho_s x}}{(\beta_k-\rho_s)^{j-\ell+1}}\int_x^z g(y)e^{-\rho_s y}dy. \end{aligned} \tag{10.1}$$

First, observe that the condition(2) of Definition 4.2, implies that the last term in (10.1) will converge, as $z \to \infty$. So, it suffices to verify the first term converges to zero. Indeed, we have

$$
\begin{aligned}
&\left| e^{\rho_s(x-z)} e^{\beta_k z} \int_z^\infty (y-z)^{j-\ell+1-\xi} g(y) e^{-\beta_k y} dy \right| \\
= &\left| e^{\rho_s(x-z)} \int_0^\infty u^{j-\ell+1-\xi} g(u+z) e^{-\beta_k u} du \right| \\
\leq &\left| e^{\rho_s(x-z)} \right| \int_0^\infty u^{j-\ell+1-\xi} |g(u+z)| |e^{-\beta_k u}| du \\
\leq &\left| e^{\rho_s(x-z)} \right| \int_0^\infty u^{j-\ell+1-\xi} (A_1 + A_2 e^{\theta(u+z)}) |e^{-\beta_k u}| du.
\end{aligned}
$$

Taking account of the fact $0 < \theta < \rho_1 < \beta_k$, we see that the last term above converges to zero, as $z \to \infty$. This yields

$$
\lim_{z\to\infty} \left| e^{\rho_s(x-z)} e^{\beta_k z} \int_z^\infty (y-z)^{j-\ell+1-\xi} g(y) e^{-\beta_k y} dy \right| = 0.
$$

The proof is complete. ■

Proof of Theorem 2.11. We quote below the proof of Theorem 3.1 in [20]. Since V is an r-excessive function (for details, see [14] Proposition 7.6 p.501) and, from condition (c) and (d), a major of g, it follows by Dynkin's characterization of the value function as the least excessive majorant, that

$$
V(x) \geq \sup_{\tau \in \mathcal{T}} \mathbb{E}(e^{-r\tau} g(X_\tau)). \tag{10.2}
$$

In order to conclude the proof, we establish the equality in equation (10.2). Consider for each $n \geq 1$ the stopping time

$$
\tau_n = \inf\{t \geq 0 : X_t \notin (-n, x^* - \frac{1}{n})\}.
$$

for $\omega \in \{\tau^* < \infty\}$ define $\overline{\tau} = \lim_{n\to\infty} \tau_n$. We have

$$
\tau_1 \leq \tau_2 \leq \cdots \leq \overline{\tau} \leq \tau^*.
$$

For n large enough, $X_{\tau_n} \geq x^* - \frac{1}{n}$, and, as the process is quasi-left continuous, $\lim_{n\to\infty} X_{\tau_n} = X_{\overline{\tau}}$, and, hence $X_{\overline{\tau}} \geq x^*$. This gives us that $\overline{\tau} = \tau^*$ a.s.

As V is r-excessive, the sequence $\{e^{-r\tau_n} V(X_{\tau_n})\}$ is a non-negative supermartingale, and consequently, it converges a.s. to a random variable. Because $X_{\tau_n} \to X_{\tau^*}$ a.s., and V is continuous, we identify the limit as $e^{-r\tau^*} V(X_{\tau^*})$. From assumption (a) and (b) it follows that

$$
C_V := \sup_{x \leq x^*} V(x) < \infty.
$$

Furthermore, as

$$
e^{-r\tau_n} V(X_{\tau_n}) = e^{-r\tau_n} V(X_n) 1_{\{\tau_n < \tau^*\}} + e^{-r\tau^*} g(X_{\tau^*}) 1_{\{\tau_n = \tau^*\}} \leq C_V + \sup_{t \geq 0} e^{-rt} g(X_t)
$$

we obtain, in view of condition (2.8), using the Lebesgue dominated convergence theorem

$$\mathbb{E}(e^{-r\tau_n}V(X_{\tau_n})) \downarrow \mathbb{E}(e^{-r\tau^*}V(X_{\tau^*})) \quad \text{as} \quad n \to \infty.$$

Furthermore, as the representing measure σ does not change the open set $(-\infty, x^*)$, the function V is harmonic on the set, and, as τ_n are exit times from the open sets $(-n, x^* - \frac{1}{n})$, we conclude that

$$V(x) = \mathbb{E}(e^{-r\tau_n}V(X_{\tau_n})) \downarrow \mathbb{E}(e^{-r\tau^*}V(X_{\tau^*})),$$

and the proof is complete. ■

Proof of (5.2). By (4.23), we have

$$\begin{aligned} & Q_g(x) \\ = & \sum_{m=1}^{M} h_m \Bigg\{ 1_{\{\mu_2 \geq 1\}} \sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta}{r} \Bigg[\sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \sum_{\ell=1}^{j} \frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k - \widetilde{\rho}_\eta)^\ell (j-\ell)!} \int_0^\infty u^{j-\ell} e^{-\beta_k u} e^{\theta_m(u+x)} du \\ & \qquad - \Big((a + \frac{b^2 \widetilde{\rho}_\eta}{2}) e^{\theta_m x} + \frac{\theta_m b^2}{2} e^{\theta_m x} \Big) \Bigg] \\ & + 1_{\{a>0 \text{ and } b=0\}} \frac{\widetilde{d}_0}{r} \Bigg\{ \sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \frac{-\lambda(\beta_k)^j c_{kj}}{(j-1)!} \int_0^\infty u^{j-1} e^{-\beta_k u} e^{\theta_m(u+x)} du \\ & \qquad + (\lambda + \mu + r) e^{\theta_m x} - a\theta_m e^{\theta_m x} \Bigg\} \Bigg\}. \end{aligned}$$

Also, by using the fact that $0 \leq \max\{\theta_m : 1 \leq m \leq M\} < \rho_1 < \min\{\mathcal{R}e\,(\beta_k) : 1 \leq k \leq v_1\}$ and applying integration by parts, we have

$$\int_0^\infty u^{j-\ell} e^{(\theta_m - \beta_k)u} du = \frac{(j-\ell)!}{(\beta_k - \theta_m)^{j-\ell+1}}.$$

This together with the fact

$$\sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \Bigg(\sum_{\ell=1}^{j} \frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k - \widetilde{\rho}_\eta)^\ell (\beta_k - \theta_m)^{j-\ell+1}} \Bigg) = \frac{1}{\theta_m - \widetilde{\rho}_\eta} \Bigg(\sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k - \theta_m)^j} + \frac{\lambda(\beta_k)^j c_{kj}}{(\beta_k - \widetilde{\rho}_\eta)^j} \Bigg)$$

yields

$$\begin{aligned} Q_g(x) = & \sum_{m=1}^{M} h_m e^{\theta_m x} \Bigg\{ 1_{\{\mu_2 \geq 1\}} \frac{1}{r} \sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta}{\theta_m - \widetilde{\rho}_\eta} \Bigg[\Bigg(\sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k - \theta_m)^j} + \frac{\lambda(\beta_k)^j c_{kj}}{(\beta_k - \widetilde{\rho}_\eta)^j} \Bigg) \\ & \qquad - \Big(\frac{\theta_m b^2}{2} + a + \frac{b^2 \widetilde{\rho}_\eta}{2} \Big) \Big(\theta_m - \widetilde{\rho}_\eta \Big) \Bigg] \\ & + 1_{\{a>0 \text{ and } b=0\}} \frac{\widetilde{d}_0}{r} \Bigg(\sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k - \theta_m)^j} - a\theta_m + (\lambda + \mu + r) \Bigg) \Bigg\}. \end{aligned}$$

It follows from Remark 4.5 that

$$Q_g(x) = \sum_{m=1}^{M} h_m e^{\theta_m x} \Big(\psi^+(-\mathrm{i}\theta_m) \Big)^{-1}.$$

■

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