Computers Math. Applic. Vol. 20, No. 2, pp. 35–40, 1990
Printed in Great Britain.

0097-4943/90 $3.00 + 0.00

EXISTENCE OF SOLUTIONS FOR BERMAN'S EQUATION FROM LAMINAR FLOWS IN A POROUS CHANNEL WITH SUCTION

C.-A. Wang,[1]† T.-W. Hwang[2] and Y.-Y. Chen[3]

[1]Institute of Applied Mathematics, National Chung Cheng University, Minshone, Taiwan 62117, R.O.C.
[2]Department of Mathematics, National Kaoshiung Normal University, Kaoshiung, Taiwan 80264, R.O.C.
[3]Institute of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan 30050, R.O.C.

(*Received* 10 *February* 1989; *in revised form* 21 *August* 1989)

Abstract—We study the Berman problem

$$f''' + \mathrm{Re}(f'^2 - ff'') = \mathrm{K}, \qquad \mathrm{Re} > 0, \ (' = \mathrm{d}/\mathrm{d}\eta),$$

subject to the conditions $f(0) = f''(0) = f'(1) = f(1) - 1 = 0$, which arises from laminar flows in a porous channel with suction. The existence of nonnegative concave solutions for each Re is verified by applying a topological method. With an *a priori* estimate, the uniqueness for small Re is also shown.

1. INTRODUCTION

The study of the flow of a viscous fluid confined by a porous wall is important, such as the separation of ^{235}U from ^{238}U by gaseous diffusion and the transpiration cooling of a heated surface. The separation process is performed by first converting uranium to the gas UF_6 and then forcing the converted gas through a porous wall via a pressure gradient. Consequently, a concentration of the desired component is obtained due to the difference in the rates of diffusion through the porous wall, which is caused by the difference in the molecular weights. Moreover, the transpiration cooling of heated surfaces such as a rocket wall or a wing surface in high-speed flight is carried out by injecting a cooler fluid through the porous-metal combustion-chamber lining to form a protective layer of cooler fluid near the wall for cooling the surface.

Suppose that the porous wall is uniform and the rectangular channel is long enough such that the end effects are negligible. Berman [1] first showed that the corresponding two-dimensional Navier–Stokes system can be reduced to a similarity two-point boundary value problem:

$$f'''(\eta) + \mathrm{Re}[(f'(\eta))^2 - f(\eta)f''(\eta)] = \mathrm{K}, \tag{1}$$

with

$$f(0) = f''(0) = f'(1) = f(1) - 1 = 0. \tag{2}$$

It was found that the solutions of problem (1,2) are governed by the crossflow Reynold number $\mathrm{Re} = Vd/v$, where V represents the normal velocity at the wall, v is the kinematic viscosity of the fluid and d denotes the half-width of the rectangular channel. It should be pointed out that a positive V represents the suction for a separation problem, while a negative V corresponds to the injection for cooling. Moreover, the similarity function f is related to the stream function, η is the normalized coordinate, with $\eta = \pm 1$ at the wall, and K is an integration constant from the derivation of problem (1,2). For convenience, we call problem (1,2) the Berman problem.

Preliminary studies [1–5] of the Berman problem concentrated on the numerical reports for the one, and only, solution for each Re. However, a second similarity solution was first found by Raithby [6] for large suction, i.e. Re is large. Robinson [7] further presented numerically the second and third solutions for some moderate positive Re. With a transformation and the shooting method, Skalak and Wang [8] then studied an equivalent initial value problem and classified all possible solutions for both suction and injection. For injection, the Berman problem can only

†To whom all correspondence should be addressed.

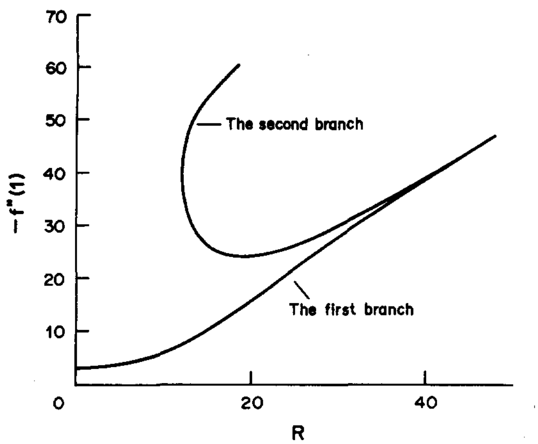

Fig. 1

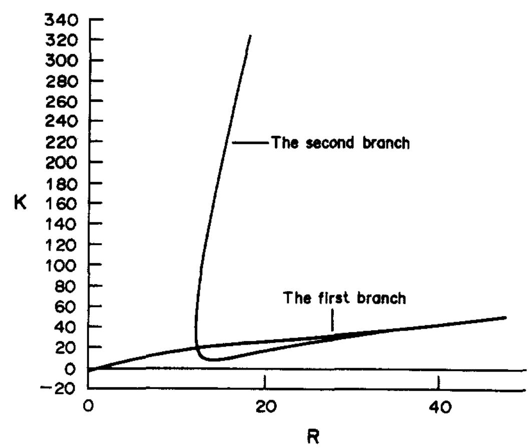

Fig. 2

possess nonnegative solutions. Shih [9] then gave a mathematical verification for the existence of solutions with each injection $\mathrm{Re} < 0$.

For suction, Skalak and Wang classified three different types of solutions. Especially, if the parameters Re, $-f''(1)$ and K are chosen on the first branch of the bifurcation diagrams, as shown in Figs 1 and 2, the Berman problem has only nonnegative concave solutions. In this paper, the existence of such solutions for each positive Re and the uniqueness of the solution with small Re are verified.

2. THE EXISTENCE THEOREM

To study the existence property, some qualitative properties of the solutions are given in the following lemma:

Lemma 1. For $\mathrm{Re} \neq 0$, if f is a solution of the Berman problem, then
(i) $\mathrm{Re} f^{(\mathrm{iv})} < 0$ and $f'f'' - ff''' > 0$ on $(0, 1]$;
(ii) $0 \leqslant f'^2 - ff'' \leqslant -f''(1)$ on $(0, 1]$.

The proof of assertion (i) was given by Skalak and Wang [8]. Let $F(f) = f'^2 - ff''$. Then, from assertion (i), $(F(f))(\eta)$ is increasing. But, $(F(f))(0) = f'(0)^2 \geqslant 0$ and $(F(f))(1) = -f''(1)$. Hence, assertion (ii) follows immediately and $f(\eta)$ can not be convex near $\eta = 1$. Now we state the main result:

Theorem 1. For each $\mathrm{Re} > 0$, there exists a real number K such that the Berman problem has at least one nonnegative concave solution.

The theorem will be proved by a topological method proposed originally by Hastings [10]. That is, we shall fix a positive Re and find all nonempty open sets of shooting parameter pairs (α, K), $\alpha = f'(0)$, in $\mathbb{R}^2$ on which the Berman problem does not possess any desired solution. If the union of these sets is a proper subset of $\mathbb{R}^2$, then the Berman problem must have a solution. To support the desired property, we first quote two important topological lemmas:

Lemma 2 [11]. Let p and q be two points of $\mathbb{R}^2$ which are separated by a closed set K (i.e. p and q lie in distinct components of $\mathbb{R}^2 - K$). Then p and q are separated by some component of K.

Lemma 3 [11]. Let S be a connected subset of $\mathbb{R}^2$ which intersects both U and $\mathbb{R}^2 - U$, where U is a subset of $\mathbb{R}^2$. Then S intersects ∂U, the boundary of U.

Let $f(\eta; \alpha, K)$ be the solution of equation (1) which satisfies the initial conditions

$$f(0; \alpha, K) = f''(0; \alpha, K) = f'(0; \alpha, K) - \alpha = 0. \tag{3}$$

Now we define the following sets:

$$S_+ = \{(\alpha, K) \in \mathbb{R}^2 : f(1; \alpha, K) > 1\},$$
$$S_- = \{(\alpha, K) \in \mathbb{R}^2 : f(1; \alpha, K) < 1\},$$
$$S'_+ = \{(\alpha, K) \in \mathbb{R}^2 : f'(1; \alpha, K) > 0\}$$

and

$$S'_- = \{(\alpha, K) \in \mathbb{R}^2 : f'(1; \alpha, K) < 0\}.$$

It is clear that for each (α, K) in either set, the Berman problem possesses no solution. Moreover, the sets S_+, S_-, S'_+ and S'_- are open, and both $S_+ \cap S_-$, $S'_+ \cap S'_-$ are empty. Now, the following lemma verifies the property that S_+ and S'_+ are nonempty:

Lemma 4. (i) S'_+ contains a subset $\{(\alpha, \mathrm{Re}\,\alpha^2): \alpha > 0\}$;
(ii) S_+ contains a subset $\{(\alpha, \mathrm{Re}\,\alpha^2): \alpha > 1\}$.

Proof. It is clear that $\alpha\eta$ is a solution of equation (1), subject to the initial condition (3). Then, by the uniqueness of the solution of the initial value problem, $f(\eta; \alpha\ \mathrm{Re}\,\alpha^2) = \alpha\eta$. This yields that

$$f(1; \alpha, \mathrm{Re}\,\alpha^2) = f'(1; \alpha, \mathrm{Re}\,\alpha^2) = \alpha. \tag{4}$$

Thus, the lemma follows immediately. Q.E.D.

Furthermore, the nonemptyness of S_- and S'_- can be obtained by the following lemma:

Lemma 5. (i) S_- contains a subset $\{(\alpha, K): K - \mathrm{Re}\,\alpha^2 + 6\alpha - 6 \leqslant 0\} - \{(1, \mathrm{Re})\}$,
(ii) S'_- contains a subset $\{(\alpha, K): K - \mathrm{Re}\,\alpha^2 + 2\alpha \leqslant 0\} - \{(0, 0)\}$.

Proof. We consider the following two cases.

Case 1. $K - \mathrm{Re}\,\alpha^2 \neq 0$. From equation (1), we have $f'''(0; \alpha, K) = K - \mathrm{Re}\,\alpha^2$. Then by Lemma 1, $f'''(\eta; \alpha, K)$ is decreasing. Hence, from condition (3), we have

$$f'''(\eta; \alpha, K) < K - \mathrm{Re}\,\alpha^2$$
$$f''(\eta; \alpha, K) < (K - \mathrm{Re}\,\alpha^2)\eta,$$
$$f'(\eta; \alpha, K) < (K - \mathrm{Re}\,\alpha^2)\eta^2/2 + \alpha$$

and

$$f(\eta; \alpha, K) < (K - \mathrm{Re}\,\alpha^2)\eta^3/6 + \alpha\eta,$$

$\forall \eta > 0$. Thus, we obtain that

$$f'(1; \alpha, K) < 0, \quad \text{if } K - \mathrm{Re}\,\alpha^2 + 2\alpha \leqslant 0,$$

and

$$f(1; \alpha, K) < 1, \quad \text{if } K - \mathrm{Re}\,\alpha^2 + 6\alpha - 6 \leqslant 0.$$

Case 2. $K - \mathrm{Re}\,\alpha^2 = 0$. From condition (4), we obtain that

$$f(1; \alpha, \mathrm{Re}\,\alpha^2) < 1, \quad \text{if } \alpha < 1$$

and

$$f'(1; \alpha, \mathrm{Re}\,\alpha^2) < 0, \quad \text{if } \alpha < 0.$$

Now combine the results in both cases, and the assertions of Lemma 5 are obtained. Q.E.D.

The correlations between the curves $K - \mathrm{Re}\,\alpha^2 = 0$, $K - \mathrm{Re}\,\alpha^2 + 2\alpha = 0$ and $K - \mathrm{Re}\,\alpha^2 + 6\alpha - 6 = 0$, when $\mathrm{Re} = 2$ and $\mathrm{Re} = 3/2$, are shown in Fig. 3. Moreover, it is now necessary to study a property concerning the set $S_+ \cup S'_-$.

Lemma 6. $S_+ \cup S'_-$ contains a subset $W = \{(\alpha, K): \alpha > 2, K - \mathrm{Re}\,\alpha^2 < 0\}$.

Proof. For (α, K) in W, both $f''(\eta; \alpha, K)$ and $f'''(\eta; \alpha, K)$ are nonpositive. Hence $f'(\eta; \alpha, K)$ is a concave, decreasing function. Now if $(\alpha, K) \in \mathbb{R}^2 - S'_-$, then $f'(1; \alpha, K) \geqslant 0$ and

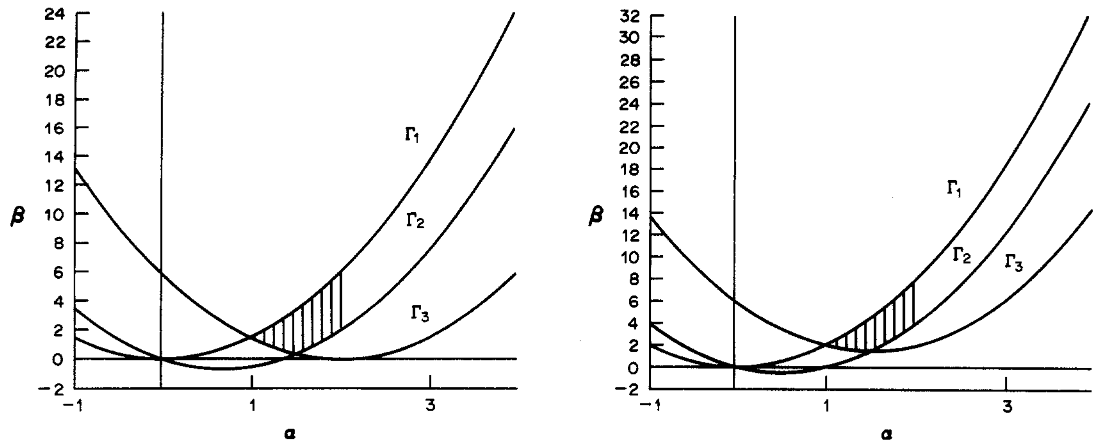

Fig. 3(a) Fig. 3(b)

$f'(\eta;\alpha,K) \geqslant \alpha - \alpha\eta\ \forall\eta \in [0,1]$. This shows that $f(1;\alpha,K) \geqslant \alpha/2 > 1$ and completes the proof. Q.E.D.

To show that $S_+ \cup S_- \cup S'_+ \cup S'_- \neq \mathbb{R}^2$, some properties concerning the components of these sets must be studied. Now let T'_+ be the component of S'_+ containing the set $\{(\alpha, \operatorname{Re}\alpha^2): \alpha > 0\}$ and T'_- be the component of S'_- containing the set $\{(\alpha, K): K - \operatorname{Re}\alpha^2 + 2\alpha \leqslant 0\} - \{(0,0)\}$. Since T'_+ and T'_- are disjoint, then any point p in T'_- must be separated from any point q in T'_+ by $\partial T'_-$, the boundary of T'_-. By Lemma 2, there exists a component Γ of $\partial T'_-$ which separates p from q. In fact, the choice of Γ is independent of p and q, due to the fact that all points in the connected open set T'_- must lie in the same component of the complement of Γ, and the corresponding property holds for all points in T'_+.

Let $\Gamma_+ = \{(\alpha, K) \in \Gamma: \alpha \geqslant 0, \operatorname{Re}\alpha^2 - 2\alpha \leqslant K \leqslant \operatorname{Re}\alpha^2\}$. Suppose one can verify that Γ_+ is a connected and unbounded set containing the origin $(0,0)$. Then Γ_+ must intersect S_- and W. Letting $\tilde{p}$ be a point in $\Gamma_+ \cap W$, then Lemma 6 implies that the point $\tilde{p}$ must be in S_+. Then, by Lemma 3, Γ_+ intersect ∂S_+. That is, there is a point $\bar{p}$ in $\partial S'_- \partial S_+$. Moreover, $S_+ \cap S_-$ and $S'_+ \cap S'_-$ are empty. Hence, the point $\bar{p}$ is not in the union of S_+, S_-, S'_+ and S'_-. Note that the point $\bar{p}$ must lie in the shaded region in Fig. 3. Thus, there exists at least a point (α, K) in $\mathbb{R}^2$ such that $f(1;\alpha,K) = 1$ and $f'(1;\alpha,K) = 0$, and the result of Theorem 1 is obtained. In fact, we have the desired lemma as follows:

Lemma 7. The set Γ_+ is a nonempty, connected and unbounded set containing the origin $(0,0)$.

Proof. Suppose that the origin $(0,0)$ is not in Γ_+, then there exists a neighborhood U of the origin such that U and Γ_+ are disjoint. Hence we may choose α small enough that both $(\alpha, \operatorname{Re}\alpha^2)$ and $(\alpha, \operatorname{Re}\alpha^2 - 2\alpha)$ are contained in U. This leads to a contradiction, since $(\alpha, \operatorname{Re}\alpha^2)$ and $(\alpha, \operatorname{Re}\alpha^2 - 2\alpha)$ are separated by Γ_+.

Let $V = \{(\alpha, K): \alpha > 0, \operatorname{Re}\alpha^2 - 2\alpha < K < \operatorname{Re}\alpha^2\}$. If Γ_+ is disconnected, then there are two open sets A and B such that $A \cap B = \emptyset$, $(0,0) \in B$ and $\Gamma_+ \subset A \cup B$. Thus, $\Gamma \subset \tilde{A} \cup \tilde{B}$, where $\tilde{A} = A \cap V$ and $\tilde{B} = B \cup (\mathbb{R}^2 - \bar{V})$. This violates the connectness of Γ.

If Γ_+ is bounded, then for sufficiently large α the points $(\alpha, \operatorname{Re}\alpha^2)$ and $(\alpha, \operatorname{Re}\alpha^2 - 2\alpha)$ lie in the same component of $\mathbb{R}^2 - \Gamma_+$. This leads to a contradiction again. Q.E.D.

3. THE UNIQUENESS OF THE SOLUTION WITH SMALL Re

To show the uniqueness, it is necessary to obtain an *a priori* estimate of solutions for the Berman problem. Now integrating equation (1) and applying condition (2), the Berman problem is equivalent to

$$f(\eta) = h(\eta) + \operatorname{Re}\int_0^1 [J(\eta,t) - h(\eta)J(1,t)][f'^2 - ff'']\,\mathrm{d}t \tag{5}$$

and

$$K = -3 + 3\,\mathrm{Re}\int_0^1 J(1,t)[f'^2 - ff'']\,dt, \tag{6}$$

where $J(\eta, t)$ is given by

$$J(\eta,t) = \begin{cases} (1-t)\eta, & 0 \leqslant \eta \leqslant t \leqslant 1, \\ -\frac{1}{2}(\eta^2 + t^2) + \eta, & 0 \leqslant t \leqslant \eta \leqslant 1, \end{cases}$$

and

$$h(\eta) = 3\int_0^1 J(\eta, s)\,ds = \tfrac{1}{2}\eta(3-\eta^2) = f_0(\eta).$$

Note that $f_0(\eta)$ is the unique solution of the Berman problem when Re = 0. It is clear that $J(\eta, t)$ is Green's function of $-v''' = 0$, satisfying $v(0) = v''(0) = v'(1) = 0$. From equation (5), $f''(\eta)$ can be written as

$$f''(\eta) = -3\eta - \mathrm{Re}\int_0^\eta (f'^2 - ff'')\,dt + 3\,\mathrm{Re}\,\eta\int_0^1 J(1,t)(f'^2 - ff'')\,dt. \tag{7}$$

Note that $\int_0^1 J(1,t)\,dt = 1/3$. Then, by Theorem 1, we conclude that

$$-3\eta + \mathrm{Re} f''(1)\eta \leqslant f''(\eta) \leqslant -3\eta - \mathrm{Re} f''(1)\eta, \tag{8}$$

since Re is positive. To establish an *a priori* bound of f'', it is essential to estimate $f''(1)$. However, $3J(1,t) - 1$ is nonnegative on $[0, 1/\sqrt{3}]$. Then, from equation (7), we get

$$\begin{aligned} f''(1) &= -3 + \mathrm{Re}\int_0^1 [3J(1,t) - 1][f'^2 - ff'']\,dt \\ &\geqslant -3 + \mathrm{Re}\int_{1/\sqrt{3}}^1 [3J(1,t) - 1][f'^2 - ff'']\,dt \\ &\geqslant -3 - \mathrm{Re} f''(1)\int_{1/\sqrt{3}}^1 [3J(1,t) - 1]\,dt = -3 + \mathrm{Re}/(3\sqrt{3}) f''(1). \end{aligned}$$

Then, for each $\mathrm{Re} \in (0, 3\sqrt{3})$,

$$0 \geqslant f''(1) \geqslant \frac{-3}{1 - \mathrm{Re}/(3\sqrt{3})}.$$

Hence, we have the following lemma:

Lemma 8. For every $\mathrm{Re} \in (0, 3\sqrt{3})$, if f is a solution of the Berman problem, then

$$\|f''\|_\infty \leqslant 3 + \frac{3\,\mathrm{Re}}{1 - \mathrm{Re}/(3\sqrt{3})}.$$

Now the uniqueness of the solution with small Re can be given in the following theorem:

Theorem 2. The Berman problem has a unique solution, if $\mathrm{Re} \in (0, \mathrm{Re}_0)$, where

$$\mathrm{Re}_0 = \frac{-(72\sqrt{3}+1) + \sqrt{(72\sqrt{3}+1)^2 + 12\sqrt{3}(72\sqrt{3} - 24)}}{48(3\sqrt{3} - 1)} \approx 4.005014 \times 10^{-2}.$$

Proof. Assume that g_1 and g_2 are two solutions of the Berman problem. Recall that $F(f) = (f')^2 - ff''$. Then, we have

$$\begin{aligned} |F(g_1) - F(g_2)| &\leqslant |g_1' - g_2'|[|g_1'| + |g_2'|] + |g_1'||g_1'' - g_2''| + |g_2''||g_1 - g_2| \\ &\leqslant 12\left[1 + \frac{\mathrm{Re}}{1 - \mathrm{Re}/(3\sqrt{3})}\right]\|g_1'' - g_2''\|_\infty. \end{aligned}$$

Hence, from equation (7), we get that

$$\|g_1'' - g_2''\|_\infty \leqslant 24\,\mathrm{Re}\left[\frac{\mathrm{Re}}{1-\mathrm{Re}/(3\sqrt{3})}\right]\|g_1'' - g_2''\|_\infty .$$

To reach contradiction, one requires that

$$24\,\mathrm{Re}\left[1+\frac{\mathrm{Re}}{1-\mathrm{Re}/(3\sqrt{3})}\right] < 1.$$

Hence, the condition $0 < \mathrm{Re} < \mathrm{Re}_0$ is sufficient. Q.E.D.

In fact, let

$$B = \{u \in C^2[0, 1]: u(0) = u'(1) = u''(0) = 0\}$$

and

$$\|u\|_B = \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty ,$$

one can show that B is a Banach space. But $\|u\|_\infty \leqslant \|u'\|_\infty \leqslant \|u''\|_\infty$. Then we can choose

$$\delta(\mathrm{Re}) = 10 + \frac{9\,\mathrm{Re}}{1-\mathrm{Re}/(3\sqrt{3})}$$

and define an open set $D = \{u \in B\colon \|u\|_B < \delta(\mathrm{Re})\}$. Now for each f in D, let the operator $(T(f))(\eta)$ be the r.h.s. of equation (5). Then, by routine arguments, one can apply the Leray–Schauder fixed-point theorem and verify the existence of solutions for every fixed $\mathrm{Re} \in (0, 3\sqrt{3})$, although it is a weak result.

As shown in Figs 1 and 2, we have verified only a small portion of the observed result from the bifurcation diagrams. However, Skalak and Wang [8] did classify all possible solutions and indicated that the Berman problem possesses three different types of solutions when $\mathrm{Re} > \mathrm{Re}^*$, for some $\mathrm{Re}^* > 0$. Therefore, further mathematical investigation is required to verify the existence of two other types of solutions, and the multiplicity of solutions also.

REFERENCES

1. A. S. Berman, Linear flow in channels with porous walls. *J. appl. Phys.* **24,** 1232–1235 (1953).
2. A. S. Berman, Effects of porous boundaries on the flow of fluids in systems of various geometries. *Proc. 2nd U.N. Int. Conf. Uses atom. Energy* **4,** 351–358 (1958).
3. E. R. G. Echert, P. L. Donoughe and B. J. Moore, Velocity and friction characteristics of laminar viscous boundary-layer and channel flow over surfaces with ejection of suction. Report TN4102, National Advisory Committee on Aeronautics, Washington, D.C. (1957).
4. M. Morduchow, On laminar flow through a channel or tube with injection: applications of method of averages. *Q. appl. Math.* **XIV,** 361–368 (1957).
5. J. P. Quaile and E. K. Levy, Laminar flow in a porous tube with suction. *J. Heat Mass Transfer* **97,** 223–243 (1975).
6. G. Raithby, Laminar heat transfer in the thermal entrance region of circular tubes and two-dimensional rectangular ducts with wall suction and injection. *Heat Mass Transfer* **14,** 223–243 (1971).
7. W. A. Robinson, The existence of multiple solutions for the laminar flow in a uniformly porous channel with suction at both walls. *J. Engng Math.* **10,** 23–40 (1976).
8. F. M. Skalak and C. Y. Wang, On the nonunique solutions of laminar flow through a porous tube or channel. *SIAM Jl appl. Math.* **34,** 535–544 (1978).
9. K. E. Shih, On the existence of solutions of an equation arising in the theory of laminar flow in a uniformly porous channel with injection. *SIAM Jl appl. Math.* **47,** 526–533 (1987).
10. S. P. Hastings, An existence theorem for a problem from boundary layer theory. *Archs Ration. Mech. Analysis* **33,** 103–109 (1969).
11. M. H. A. Newman, *Element of the Topology of Plane Sets of Points*, 2nd edn. Cambridge Univ. Press, Cambs. (1961).