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碩士論文

有旋轉項的 Gross-Pitaevskii 方程之半古典極限

Semiclassical Limit of the Gross-Pitaevskii Equation with Rotation

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摘　　要

在本論文中，我們用兩種不同的做法研究有旋轉項的 Gross-Pitaevskii 方程之半古典極限。首先，我們使用修改過的 Madelung 變換以著重在與量子流體動力學方程（quantum hydrodynamical equations）等價的擬線性雙曲對稱系統（quasilinear symmetric hyperbolic system）。我們建立在極限系統奇點形成之前，當普朗克常數趨近於零時，量子密度與量子動量收斂到可壓縮的旋轉歐拉方程(compressible rotational Euler equation)之唯一解。此外，我們證明在維度 2 之可壓縮的旋轉歐拉方程之局部解的存在性與唯一性。其次，我們考慮量子密度與量子動量在恆定狀態(1,0)附近的情形。我們建立有旋轉項的 Gross-Pitaevskii 方程弱收斂到等價於線性波動方程(linear wave equation)的波映射方程(wave map equation)。這方法的結果引領聲波（acoustic wave）的討論。

關鍵詞: 旋轉，Gross-Pitaevskii 方程，半古典極限，歐拉方程。

Semiclassical Limit of the Gross-Pitaevskii Equation with Rotation

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ABSTRACT

In this paper, we perform the semiclassical limit of the Gross-Pitaevskii equation with rotation by two different approaches. First, we use the modified Madelung transformation to focus on the quasilinear symmetric hyperbolic system, which is equivalent to the quantum hydrodynamical equations. We establish that before the formation of singularities in the limiting system, the quantum density and quantum momentum converge to the unique solution of the compressible rotational Euler equation as the Planck constant $\hbar$ tends to zero. In addition, we prove the existence and uniqueness of local solutions of the compressible rotational Euler equation in dimension 2. Second, we consider the case when the quantum density and quantum momentum are near the constant state (1,0). We establish that the Gross-Pitaevskii equation with rotation converges weakly to the wave map equation, equivalently the linear wave equation. The result of this approach leads the discussion of the acoustic wave.

Keywords：rotation, Gross-Pitaevskii equation , semiclassical limit, Euler equation.

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You are the stars, the moon, and the sunshine in my life.

「養成隨手的計算紙都能寫得整整齊齊的好習慣。」大二升大三的那年暑假，林琦焜老師深深烙印在我心靈的話，使我開始養成做數學有條不紊的好習慣。我深深謝謝老師開啟我清晰學習的思緒。

深深謝謝我的指導教授－林琦焜老師啟發我找到雪亮的勇氣。用批判存疑的眼睛看世界、看問題、看眼前，用欣賞的心態看別人，用希望的心態看未來。深深謝謝老師引領我成為獨立的人。直觀的見識，精確的態度，自信的靈魂，省思的自我。我一步步隨著論文的完成，找到更清澈的自己。論文是我的學習結果，對數學的熱情堅持與認真嚴謹是老師深刻在我心中的榜樣。每當我出奇不意地至老師辦公室問問題時，老師總是無時不刻正在做數學與閱讀。我感動不停止的前進。老師的教學渲染我體會數學生命的美，胸懷科學家的氣度。這份論文的完成是具體的，然而心靈氣度的成長是生命這段旅程最珍貴的豐收。深深謝謝我的指導教授讓我從辨別、反省、改變中更加進步。

深深謝謝口試委員：中央研究院研究員李志豪老師及交通大學王夏聲老師。李志豪老師是我大二升大三那年參與暑期研討班的指導教授。從我就讀大學至研究所，老師總是給我細微的教導及滿滿的關懷和協助，真誠的溫暖蔓延在每一地方每一成長階段。口試後，老師幫我整理了需弄懂及更正的要點。我深深感謝老師的用心與提攜。王夏聲老師是我碩一時的實變老師。從我學習當一名研究生開始，他總是看見我學習過程中遭遇的猶疑與問題，叮嚀、提點、建議我學習的方式，分享、提示、鼓勵我精進的方向，真切的妙囊出現在每一適時每一關卡。口試後，老師再次提醒我改善的問題點。我深深感謝老師的教導與開導。深深謝謝老師們的導引，讓我在學習的過程及生活的態度豁然開朗。

深深謝謝交通大學師長們。謝謝教導我常微分方程的石至文老師與量子力學的孟心飛老師，奠基我學問的知識。謝謝教導我學術英文寫作的吳思葦老師，精進我英文寫作能力。謝謝吳恭儉學長在我遇見瓶頸、迷惘、遲疑、困頓時，給予我建議，分享我經驗，提供我可能的突破方向。每次 seminar 結束，學長總會針對我們遭遇的問題，提出精進探尋的意見。在學長幫助下，我逐步克服疑惑，成長邁進。另外，謝謝陳冠羽學長和梁育豪學長在我碩士生涯裡盡其可能為我解惑及給予我協助。舉凡課業和生活上的疑問雜事，他們總是充滿耐心地教導我，盡力地幫助我解決。特別謝謝梁育豪學長在我口試當天協助我準備前置作業，細心地提供我完善的器材，讓我保持穩定的心情完成口試。

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畢業是新旅程的開始。我會成為自信、和煦、分享的人，讓生命光耀且溫暖。我期許自己懷有寧靜、智慧與勇氣，面對人生挑戰自在、浩然、豪邁。

最後，謹此論文獻給呵護我長大，支持我夢想的母親－黃麗英女士。

蔡佳穎
謹誌于　國立交通大學
2011年六月

Contents

CHAPTER 1

Introduction

Bose-Einstein condensation (BEC) is a phenomenon that a macroscopic fraction of the atoms occupy the same quantum level and behave as a coherent matter wave at very low temperature. An important issue is the relationship between BEC and superfluidity. The properties of rotating condensates in traps have increasingly been the object of study in recent years. The readers are referred to [**1, 2, 3, 7, 9**] for more experimental and theoretical work, and these developments have been reviewed in [**6, 17, 18**].

The primary research questions to be addressed in this paper are as follows. In the case of a dilute Bose gas at temperature much smaller than the critical condensation temperature, the time-dependent Gross-Pitaevskii equation with rotation is the equation of motion in the frame rotating with the trap. It takes the form

$$\begin{aligned} i\hbar\partial_t\psi^\hbar &= -\frac{\hbar^2}{2m}\Delta\psi^\hbar + g|\psi^\hbar|^2\psi^\hbar + V(x)\psi^\hbar + i\hbar\omega x \times \nabla\psi^\hbar \\ &= -\frac{\hbar^2}{2m}\Delta\psi^\hbar + g|\psi^\hbar|^2\psi^\hbar + V(x)\psi^\hbar + i\hbar\omega x^\perp \cdot \nabla\psi^\hbar, \end{aligned} \tag{1.1}$$

where $x^\perp = (-x_2, x_1)$. The macroscopic wave function $\psi^\hbar(t,x)$ is inherently a complex function. m is the atomic mass. In the non-linear potential term, g characterizes the strength of the short-range interparticle potential. The external potential $V(x)$ does not depend on the time. The rotating term is composed of the angular velocity ω of the rotating trap and the x_3 component of the angular momentum operator

$$L_{x_3} = -i\hbar x \times \nabla = -i\hbar\left(x_1\partial_{x_2} - x_2\partial_{x_1}\right) = -i\hbar x^\perp \cdot \nabla, \tag{1.2}$$

where $x = (x_1, x_2)$ is the coordinate in the rotating frame. The operator L_{x_3} is only a scalar in two dimensional case and represents the rotation. The Gross-Pitaevskii equation with rotation (1.1) is also called the rotating nonlinear Schrödinger equation. In this paper, we focus on a simple case in which g and ω are constants.

A singular limit is an interesting problem. It makes a connection between different fields and contributes to the understanding of the nature of the problem. The study of the semiclassical limit has a great importance for determining the limiting behaviour of any function of the field $\psi^\hbar$. It may also describe superfluids and provide rich dynamical phenomena in rotating BEC gases. In this work, we discuss the semiclassical limit by two different approaches : the modified Madelung transformation and the density fluctuation.

First of all, we use the Madelung transformation

$$\psi^\hbar(t,x) = A^\hbar(t,x)e^{\frac{i}{\hbar}S^\hbar(t,x)}, \tag{1.3}$$

where both $A^\hbar(t,x)$ and $S^\hbar(t,x)$ are real-valued functions. As density $\rho^\hbar$ and momentum $\mu_\omega^\hbar$ are given by

$$\rho^\hbar = |A^\hbar|^2 = \left|\psi^\hbar\right|^2, \qquad \mu_\omega^\hbar = \rho^\hbar\left(\frac{1}{m}\nabla S^\hbar - \omega x^\perp\right), \tag{1.4}$$

the quantum hydrodynamic equations of the rotating nonlinear Schrödinger equation (1.1) then are

$$\begin{aligned}
&\partial_t\rho^\hbar + \nabla\cdot\mu_\omega^\hbar = 0,\\
&\partial_t\mu_\omega^\hbar + \nabla\cdot\left(\frac{\mu_\omega^\hbar\otimes\mu_\omega^\hbar}{\rho^\hbar}\right) + \nabla\left(\frac{g}{2m}(\rho^\hbar)^2\right)\\
&\qquad +\rho^\hbar\nabla\left(\frac{1}{m}V - \frac{1}{2}\omega^2|x|^2\right) = \frac{\hbar^2}{4m^2}\nabla\cdot\left(\rho^\hbar\nabla^2\log\rho^\hbar\right) - 2\omega(\mu_\omega^\hbar)^\perp,
\end{aligned} \tag{1.5}$$

with initial data

$$\rho^\hbar(0,x) = \rho_0^\hbar(x), \quad \mu_\omega^\hbar(0,x) = \mu_{\omega,0}^\hbar(x). \tag{1.6}$$

To overcome the difficulty caused by the nonlinear term, we introduce the modified Madelung transformation suggested by Grenier [**8**]. There has been a change of emphasis from the real-valued function $A^\hbar(t,x)$ to the complex-valued one. The main method to carry out this study is based on transforming the rotating nonlinear Schrödinger equation (1.1) into a dispersive perturbation of a quasilinear symmetric hyperbolic system, to which the Lax-Friedrich-Kato's theory can be applied, by the modified Madelung transformation. The readers are referred to good tutorials in [**5, 8, 11, 12, 13**] for an introduction to the above approaches and its applications. When $\hbar$ tends to zero, we formally have the compressible rotational Euler equation

$$\begin{aligned} &\partial_t \rho + \nabla \cdot \mu_\omega = 0, \\ &\partial_t \mu_\omega + \nabla \cdot \left(\frac{\mu_\omega \otimes \mu_\omega}{\rho} \right) + \nabla \left(\frac{g}{2m} \rho^2 \right) + \rho \nabla \left(\frac{1}{m} V - \frac{1}{2} \omega^2 |x|^2 \right) = -2\omega \mu_\omega^\perp, \end{aligned} \tag{1.7}$$

with initial data

$$\rho(0,x) = \rho_0(x), \quad \mu_\omega(0,x) = \mu_{\omega,0}(x). \tag{1.8}$$

It follows that the system of quantum mechanics converges to the system obeying Newton mechanics.

It is obviously required that the existence and uniqueness of solutions of the compressible rotational Euler equation are determined. Much research has been devoted on the existence and uniqueness of solutions of the Euler equation. For the incompressible Euler equation, the existence of a global solution in the two dimensional case and a local solution in the three dimensional case has been established [**20**]. Roger Temam also has given another new short proof for the three dimensional case by representing the unknown pressure as Poisson's equation and applying the Galerkin method with a special basis in his paper [**20**]. For the compressible Euler equation, P. L. Lions has discussed global entropy solutions in the one dimensional case [**15**]. However, little research has been done on the existence of solutions of the Euler equation with rotation. We propose a clear proof of the existence of a local solution of the compressible rotational Euler equation in dimension 2 by the equivalent relation, resulting from the conclusion of the above semiclassical limit.

Let us then discuss the semiclassical limit in the view of $\dfrac{|\psi^\hbar|^2 - 1}{\hbar}$. It means that we consider the case when $(\rho^\hbar, \mu^\hbar)$ is near the constant state $(1,0)$. The semiclassical (dispersion) limit concludes that the rotating nonlinear Schrödinger equation (1.1) converges weakly to the wave map equation (6.38). Moreover, the wave map equation is equivalent to the linear wave equation (6.39). This is just the beginning of studying the acoustic wave.

The remainder of this paper is organized into five chapters. In Chapter 2, we start by deriving the hydrodynamical structure of the rotating nonlinear Schrödinger equation

(1.1). The readers are also referred to [**10**] for the deriving process. To obtain the local existence of smooth solutions and perform the semiclassical limit, the procedure is displayed in Chapter 3 and consists of three main parts. First, we transform the rotating nonlinear Schrödinger equation into a quasilinear hyperbolic system by the modified Madelung transformation in Section 3.1. Second, a priori estimate, which allows to pass to the limit $\hbar \to 0$ and justfy the WKB hierarchy, is employed in Subsection 3.2.1. For a discussion of a priori estimate, also see [**5, 8, 11, 12, 13, 16**]. Third, some compactness arguments are the tools of attaining our desired results. For a full account of this part, also see [**14**]. Chapter 4 contains a description of the WKB expansion. The readers are also referred to [**5, 8, 11**]. The local existence and uniqueness proof of solutions of the compressible rotational Euler equation is outlined in Chapter 5. We see a connection between the rotating nonlinear Schrödinger equation and the acoustic wave in Chapter 6. We perform the semiclassical (dispersion) limit in the view of the density fluctuation in Section 6.1. The readers are also referred to [**4, 14**] for further details of the density fluctuation.

CHAPTER 2

Hydrodynamical Structure

The physical content of the rotating nonlinear Schrödinger equation (1.1) may be revealed by reformulating it as a pair of hydrodynamic equations (1.5), which we will use two methods to derive. Initially, we use Noether's theorem to determine the conservation laws. The Lagrangian density for the rotating nonlinear Schrödinger equation (1.1) is

$$\begin{aligned}\mathfrak{L} = \frac{i\hbar}{2}\left[\left(\psi^\hbar\right)^* \partial_t \psi^\hbar - \psi^\hbar \partial_t \left(\psi^\hbar\right)^*\right] - \frac{\hbar^2}{2m}\nabla\left(\psi^\hbar\right)^* \cdot \nabla\psi^\hbar - g|\psi^\hbar|^2 \left(\psi^\hbar\right)^* \psi^\hbar \\ -V(x)\left(\psi^\hbar\right)^*\psi^\hbar - \frac{i\hbar\omega}{2}\left[\left(\psi^\hbar\right)^* x^\perp \cdot \nabla\psi^\hbar - \psi^\hbar x^\perp \cdot \nabla\left(\psi^\hbar\right)^*\right].\end{aligned} \tag{2.1}$$

The action $\mathfrak{S} = \iint \mathfrak{L} dxdt$ is invariant under the following transformations

$$\psi^\hbar_\epsilon(t,x) = e^{i\epsilon}\psi^\hbar(t,x) \quad \text{with generator} \quad \delta\psi^\hbar = i\psi^\hbar, \tag{2.2}$$

$$\psi^\hbar_\epsilon(t_\epsilon,x) = \psi^\hbar(t_\epsilon-\epsilon,x) = \psi^\hbar(t,x) \quad \text{with generator} \quad \delta\psi^\hbar = \partial_t\psi^\hbar. \tag{2.3}$$

However, the action $\mathfrak{S}$ is not invariant under the transformation

$$\psi^\hbar_\epsilon(t,x_\epsilon) = \psi^\hbar(t,x_\epsilon-\epsilon) = \psi^\hbar(t,x) \quad \text{with generator} \quad \delta\psi^\hbar = \nabla\psi^\hbar, \tag{2.4}$$

and about x_3 axis under the transformation

$$\psi^\hbar_\epsilon(t,x_\epsilon) = \psi^\hbar\left(t,R^T_\epsilon x\right) = \psi^\hbar(t,x) \quad \text{with generator} \quad \delta\psi^\hbar = L_{x_3}\psi^\hbar, \tag{2.5}$$

where for all $\epsilon \in \mathbb{R}$,

$$R_\epsilon = \begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix}, \quad R^T_\epsilon = \begin{pmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{pmatrix}. \tag{2.6}$$

Therefore, by Noether's theorem, the invariances generate the conservation laws, and the generators $i\psi^\hbar$ and $\partial_t\psi^\hbar$ correspond to the conservation of charge and energy, respectively. We have not the conservation of momentum due to the rotational term $-\omega L_{x_3}\psi^\hbar$ appearing in (1.1) and the conservation of angular momentum due to the linear potential term $V(x)\psi^\hbar$. We have the main result :

Theorem 2.1. *If $\psi^\hbar$ is a smooth function, then the hydrodynamical formulation of the rotating nonlinear Schrödinger equation* (1.1) *is*

(1) *the conservation of charge* :

$$\partial_t \rho^\hbar + \nabla \cdot (\rho^\hbar (u^\hbar - \omega x^\perp)) = 0,$$

(2) *the equation of momentum* :

$$\partial_t \mu_\omega^\hbar + \nabla \cdot \left(\frac{\mu_\omega^\hbar \otimes \mu_\omega^\hbar}{\rho^\hbar} \right) + \nabla \left(\frac{g}{2m} (\rho^\hbar)^2 \right)$$

$$+ \rho^\hbar \nabla \left(\frac{1}{m} V - \frac{1}{2} \omega^2 |x|^2 \right) = \frac{\hbar^2}{4m^2} \nabla \cdot \left(\rho^\hbar \nabla^2 \log \rho^\hbar \right) - 2\omega (\mu_\omega^\hbar)^\perp,$$

(3) *the conservation of energy* :

$$\partial_t \theta^\hbar + \nabla \cdot \left[\frac{\mu_\omega^\hbar}{\rho^\hbar} \left(\theta^\hbar + \frac{g}{2m} (\rho^\hbar)^2 \right) \right] = \frac{\hbar^2}{4m^2} \nabla \cdot \left[\frac{\mu_\omega^\hbar \Delta \rho^\hbar}{\rho^\hbar} - \frac{\nabla \cdot \mu_\omega^\hbar \nabla \rho^\hbar}{\rho^\hbar} \right],$$

(4) *the equation of angular momentum* :

$$\partial_t \left(x^\perp \cdot \mu^\hbar \right) + \nabla \cdot \left[\mu^\hbar \left(x^\perp \cdot \frac{\mu^\hbar}{\rho^\hbar} \right) + x^\perp \left(\frac{g}{2m} (\rho^\hbar)^2 - \omega x^\perp \cdot \mu^\hbar \right) \right]$$

$$+ x^\perp \cdot \frac{\rho^\hbar}{m} \nabla V = \frac{\hbar^2}{4m^2} \nabla \cdot \left[x^\perp \Delta \rho^\hbar - \frac{\nabla \rho^\hbar}{\rho^\hbar} \left(x^\perp \cdot \nabla \rho^\hbar \right) \right],$$

where

$$u_\omega^\hbar = u^\hbar - \omega x^\perp,$$
$$\mu_\omega^\hbar = \rho^\hbar u_\omega^\hbar = \mu^\hbar - \rho^\hbar \omega x^\perp,$$

and

$$\theta^\hbar = \frac{\left|\mu_\omega^\hbar\right|^2}{2\rho^\hbar} + \frac{\hbar^2}{8m^2} \frac{\left|\nabla \rho^\hbar\right|^2}{\rho^\hbar} + \frac{g}{2m} (\rho^\hbar)^2 + \rho^\hbar \left(\frac{1}{m} V - \frac{1}{2} \omega^2 \left|x\right|^2 \right).$$

2.1. Euler Equation (First Method)

We introduce the Madelung transformation $\psi^\hbar(t,x) = A^\hbar(t,x) e^{\frac{i}{\hbar} S^\hbar(t,x)}$, where both the amplitude $A^\hbar(t,x)$ and the phase $S^\hbar(t,x)$ are real-valued functions, and insert it into the rotating nonlinear Schrödinger equation (1.1). Separating the real and imaginary parts leads to

$$\partial_t A^\hbar + \frac{1}{m} \nabla S^\hbar \cdot \nabla A^\hbar + \frac{1}{2m} A^\hbar \Delta S^\hbar - \omega x^\perp \cdot \nabla A^\hbar = 0, \tag{2.7}$$

$$\partial_t S^\hbar + \frac{1}{2m} \left|\nabla S^\hbar\right|^2 + g \left|A^\hbar\right|^2 + V - \omega x^\perp \cdot \nabla S^\hbar = \frac{\hbar^2}{2m} \frac{\Delta A^\hbar}{A^\hbar}. \tag{2.8}$$

To understand the nature of the velocity of the fluid, we multiply (2.7) by $2A^\hbar$ and define the density, velocity, and momentum as

$$\rho^\hbar = |A^\hbar|^2 = \left|\psi^\hbar\right|^2, \qquad u^\hbar = \frac{1}{m}\nabla S^\hbar = (u_1^\hbar, u_2^\hbar), \qquad \mu^\hbar = \rho^\hbar u^\hbar, \tag{2.9}$$

respectively. It follows that

$$\partial_t \rho^\hbar + \nabla\cdot\mu^\hbar - \omega x^\perp \cdot \nabla\rho^\hbar = 0, \tag{2.10}$$

where $\rho^\hbar$ is a probability density. We also write (2.10) as the total differential form

$$\partial_t \rho^\hbar + \nabla\cdot\mu^\hbar - \nabla\cdot\left(\omega x^\perp \rho^\hbar\right) = 0. \tag{2.11}$$

Hence, we have the conservation of charge

$$\partial_t \rho^\hbar + \nabla\cdot(\rho^\hbar u_\omega^\hbar) = 0, \qquad u_\omega^\hbar = u^\hbar - \omega x^\perp, \tag{2.12}$$

where $u_\omega^\hbar$ is the modified velocity. Taking the gradient of (2.8) and then multiplying it by $\frac{1}{m}$, we obtain the equation of motion for the veloctiy

$$\begin{aligned}&\partial_t u^\hbar + \left(u^\hbar\cdot\nabla\right)u^\hbar + \frac{g}{m}\nabla\rho^\hbar + \frac{1}{m}\nabla V\\ &= \frac{\hbar^2}{2m^2}\nabla\left(\frac{\Delta\sqrt{\rho^\hbar}}{\sqrt{\rho^\hbar}}\right) - \omega(u^\hbar)^\perp + \omega(x^\perp\cdot\nabla)u^\hbar\end{aligned} \tag{2.13}$$

or

$$\begin{aligned}&\partial_t u_\omega^\hbar + \left(u_\omega^\hbar\cdot\nabla\right)u_\omega^\hbar + \frac{g}{m}\nabla\rho^\hbar\\ &+\nabla\left(\frac{1}{m}V - \frac{1}{2}\omega^2|x|^2\right) = \frac{\hbar^2}{2m^2}\nabla\left(\frac{\Delta\sqrt{\rho^\hbar}}{\sqrt{\rho^\hbar}}\right) - 2\omega(u_\omega^\hbar)^\perp,\end{aligned} \tag{2.14}$$

where $(u^\hbar)^\perp = (-u_2^\hbar, u_1^\hbar)$ and $(u_\omega^\hbar)^\perp = (u^\hbar)^\perp + \omega x$. To obtain the momentum, we multiply (2.13) by $\rho^\hbar$ and (2.10) by $u^\hbar$ and then add up to have

$$\begin{aligned}&\partial_t\mu^\hbar + \nabla\cdot\left(\frac{\mu^\hbar\otimes\mu^\hbar}{\rho^\hbar}\right) + \nabla\left(\frac{g}{2m}(\rho^\hbar)^2 - \omega x^\perp\cdot\mu^\hbar\right) + \frac{\rho^\hbar}{m}\nabla V\\ &= \frac{\hbar^2}{4m^2}\nabla\cdot\left(\rho^\hbar\nabla^2\log\rho^\hbar\right) + \omega x\left(\frac{(\mu^\hbar)^\perp}{\rho^\hbar}\cdot\nabla\rho^\hbar\right),\end{aligned} \tag{2.15}$$

where $(\mu^\hbar)^\perp = \rho^\hbar (u^\hbar)^\perp$. Therefore, we can derive the equation of the modified momentum

$$\begin{aligned}\partial_t \mu^\hbar_\omega + \nabla \cdot \left(\frac{\mu^\hbar_\omega \otimes \mu^\hbar_\omega}{\rho^\hbar} \right) + \nabla \left(\frac{g}{2m} (\rho^\hbar)^2 \right) \\ + \rho^\hbar \left(\frac{1}{m} \nabla V - \frac{1}{2} \omega^2 |x|^2 \right) = \frac{\hbar^2}{4m^2} \nabla \cdot \left(\rho^\hbar \nabla^2 \log \rho^\hbar \right) - 2\omega (\mu^\hbar)^\perp_\omega .\end{aligned} \tag{2.16}$$

Equations (2.12) and (2.16) are analogous to the continuity equation and the momentum equation in fluid mechanics, respectively. The above equations show that the rotation affects both the continuity equation and the momentum equation. Provided that $S^\hbar$ is not singular, we can conclude that the velocity field is irrotational; that is, the potential flow $\nabla \times u^\hbar = 0$. However, $\nabla \times u^\hbar_\omega = -2\omega$. Moreover, by (2.13), we have

$$\partial_t \left(x^\perp \cdot u^\hbar \right) + \nabla \cdot \left\{ x^\perp \left[\frac{|u^\hbar|^2}{2} + \frac{g}{m} \rho^\hbar + \frac{1}{m} V - \omega x^\perp \cdot u^\hbar \right] \right\} = \frac{\hbar^2}{2m^2} \nabla \cdot \left(\frac{\Delta \sqrt{\rho^\hbar}}{\sqrt{\rho^\hbar}} \right), \tag{2.17}$$

and by (2.15), we also have the equation of angular momentum

$$\begin{aligned}\partial_t \left(x^\perp \cdot \mu^\hbar \right) + \nabla \cdot \left[\mu^\hbar \left(x^\perp \cdot \frac{\mu^\hbar}{\rho} \right) + x^\perp \left(\frac{g}{2m} (\rho^\hbar)^2 - \omega x^\perp \cdot \mu^\hbar \right) \right] \\ + x^\perp \cdot \frac{\rho^\hbar}{m} \nabla V = \frac{\hbar^2}{4m^2} \nabla \cdot \left[x^\perp \Delta \rho^\hbar - \frac{\nabla \rho^\hbar}{\rho^\hbar} \left(x^\perp \cdot \nabla \rho^\hbar \right) \right].\end{aligned} \tag{2.18}$$

Since angular momentum about the axis of rotation is not conserved, the trap $V(x)$ has no axis of symmetry.

2.2. Euler Equation (Second Method)

We consider $\psi^\hbar$ and $\left(\psi^\hbar\right)^*$ to be solutions of the rotating nonlinear Schrödinger equation

$$i\hbar \partial_t \psi^\hbar = \left(-\frac{\hbar^2}{2m} \Delta + g \left| \psi^\hbar \right|^2 + V + i\hbar \omega x^\perp \cdot \nabla \right) \psi^\hbar, \tag{2.19}$$

$$-i\hbar \partial_t \left(\psi^\hbar\right)^* = \left(-\frac{\hbar^2}{2m} \Delta + g \left| \psi^\hbar \right|^2 + V - i\hbar \omega x^\perp \cdot \nabla \right) \left(\psi^\hbar\right)^*, \tag{2.20}$$

respectively. Multiplying (2.19) by $\left(\psi^\hbar\right)^*$ and (2.20) by $\psi^\hbar$, we can write

$$i\hbar \left(\psi^\hbar\right)^* \partial_t \psi^\hbar = -\frac{\hbar^2}{2m} \left(\psi^\hbar\right)^* \Delta \psi^\hbar + g \left|\psi^\hbar\right|^4 + V \left|\psi^\hbar\right|^2 + i\hbar\omega \left(\psi^\hbar\right)^* x^\perp \cdot \nabla \psi^\hbar, \tag{2.21}$$

$$-i\hbar \psi^\hbar \partial_t \left(\psi^\hbar\right)^* = -\frac{\hbar^2}{2m} \psi^\hbar \Delta \left(\psi^\hbar\right)^* + g \left|\psi^\hbar\right|^4 + V \left|\psi^\hbar\right|^2 - i\hbar\omega \psi^\hbar x^\perp \cdot \nabla \left(\psi^\hbar\right)^*. \tag{2.22}$$

Subtracting (2.22) from (2.21) and using the equality

$$\nabla \cdot \left[\left(\psi^{\hbar}\right)^* \nabla \psi^{\hbar} \right] = \left(\psi^{\hbar}\right)^* \Delta \psi^{\hbar} + \nabla \left(\psi^{\hbar}\right)^* \cdot \nabla \psi^{\hbar}, \tag{2.23}$$

there results

$$\partial_t |\psi^{\hbar}|^2 = i\frac{\hbar}{2m} \nabla \cdot \left[\left(\psi^{\hbar}\right)^* \nabla \psi^{\hbar} - \psi^{\hbar} \nabla \left(\psi^{\hbar}\right)^* \right] + \omega x^{\perp} \cdot \nabla \left|\psi^{\hbar}\right|^2. \tag{2.24}$$

Hence, we obtain equation (2.10) by setting

$$\rho^{\hbar} = \left|\psi^{\hbar}\right|^2, \qquad \mu^{\hbar} = -i\frac{\hbar}{2m} \left[\left(\psi^{\hbar}\right)^* \nabla \psi^{\hbar} - \psi^{\hbar} \nabla \left(\psi^{\hbar}\right)^* \right]. \tag{2.25}$$

Next, we do the similar steps to seek for (2.15). We multiply (2.19) by $\nabla \left(\psi^{\hbar}\right)^*$ and (2.20) by $\nabla \psi^{\hbar}$ and then add up to yield

$$\begin{aligned} &-i\hbar \left[\partial_t \left(\psi^{\hbar}\right)^* \nabla \psi^{\hbar} - \partial_t \psi^{\hbar} \nabla \left(\psi^{\hbar}\right)^* \right] = \\ &-\frac{\hbar^2}{2m} \left[\Delta \psi^{\hbar} \nabla \left(\psi^{\hbar}\right)^* + \Delta \left(\psi^{\hbar}\right)^* \nabla \psi^{\hbar} \right] + g \left|\psi^{\hbar}\right|^2 \nabla \left|\psi^{\hbar}\right|^2 + V \nabla \left|\psi^{\hbar}\right|^2 \\ &+ i\hbar\omega \left\{ \left(x^{\perp} \cdot \nabla \psi^{\hbar}\right) \nabla \left(\psi^{\hbar}\right)^* - \left[x^{\perp} \cdot \nabla \left(\psi^{\hbar}\right)^* \right] \nabla \psi^{\hbar} \right\}. \end{aligned} \tag{2.26}$$

On the other hand, we take the gradient of (2.19) and (2.20), multiply them by $\left(\psi^{\hbar}\right)^*$ and $\psi^{\hbar}$, respectively, and then add up to yield

$$\begin{aligned} &-i\hbar \left\{ \psi^{\hbar} \nabla \left[\partial_t \left(\psi^{\hbar}\right)^* \right] - \left(\psi^{\hbar}\right)^* \nabla \left(\partial_t \psi^{\hbar}\right) \right\} = \\ &-\frac{\hbar^2}{2m} \left\{ \left(\psi^{\hbar}\right)^* \nabla \left(\Delta \psi^{\hbar}\right) + \psi^{\hbar} \nabla \left[\Delta \left(\psi^{\hbar}\right)^* \right] \right\} \\ &+ 3g \left|\psi^{\hbar}\right|^2 \nabla \left|\psi^{\hbar}\right|^2 + 2 \left|\psi^{\hbar}\right|^2 \nabla V + V \nabla \left|\psi^{\hbar}\right|^2 \\ &+ i\hbar\omega \left\{ \left(\psi^{\hbar}\right)^* \nabla \left(x^{\perp} \cdot \nabla \psi^{\hbar}\right) - \psi^{\hbar} \nabla \left[x^{\perp} \cdot \nabla \left(\psi^{\hbar}\right)^* \right] \right\}. \end{aligned} \tag{2.27}$$

Now subtracting (2.27) from (2.26), we derive

$$\begin{aligned}
&-i\hbar\partial_t\left[\left(\psi^\hbar\right)^*\nabla\psi^\hbar-\psi^\hbar\nabla\left(\psi^\hbar\right)^*\right]= \\
&-\frac{\hbar^2}{2m}\left\{\Delta\psi^\hbar\nabla\left(\psi^\hbar\right)^*+\Delta\left(\psi^\hbar\right)^*\nabla\psi^\hbar-\left(\psi^\hbar\right)^*\nabla\left(\Delta\psi^\hbar\right)-\psi^\hbar\nabla\left[\Delta\left(\psi^\hbar\right)^*\right]\right\} \\
&-2g\left|\psi^\hbar\right|^2\nabla\left|\psi^\hbar\right|^2-2\left|\psi^\hbar\right|^2\nabla V-i\hbar\omega\nabla\left\{x^\perp\cdot\left[\left(\psi^\hbar\right)^*\nabla\psi^\hbar-\psi^\hbar\nabla\left(\psi^\hbar\right)^*\right]\right\} \\
&+2i\hbar\omega x\left\{\left[\nabla\left(\psi^\hbar\right)^*\right]^\perp\cdot\nabla\psi^\hbar\right\},
\end{aligned}\tag{2.28}$$

where $\left[\nabla\left(\psi^\hbar\right)^*\right]^\perp=\left(-\partial_{x_2}\left(\psi^\hbar\right)^*,\partial_{x_1}\left(\psi^\hbar\right)^*\right)$. After multiplying (2.28) by $\frac{1}{2m}$, we have

$$\begin{aligned}
\partial_t\mu^\hbar=-\frac{\hbar^2}{4m^2}&\Big\{\Delta\psi^\hbar\nabla\left(\psi^\hbar\right)^*+\Delta\left(\psi^\hbar\right)^*\nabla\psi^\hbar \\
&-\left(\psi^\hbar\right)^*\nabla\left(\Delta\psi^\hbar\right)-\psi^\hbar\nabla\left[\Delta\left(\psi^\hbar\right)^*\right]\Big\}-\frac{g}{m}\rho^\hbar\nabla\rho^\hbar \\
&-\frac{\rho^\hbar}{m}\nabla V+\omega\nabla\left(x^\perp\cdot\mu^\hbar\right)+\omega x\left(\frac{\left(\mu^\hbar\right)^\perp}{\rho^\hbar}\cdot\nabla\rho^\hbar\right).
\end{aligned}\tag{2.29}$$

Since

$$\begin{aligned}
\nabla\Delta\rho^\hbar&=\left[\left(\psi^\hbar\right)^*\nabla\Delta\psi^\hbar+\psi^\hbar\nabla\Delta\left(\psi^\hbar\right)^*\right] \\
&+2\nabla\left[\nabla\psi^\hbar\cdot\nabla\left(\psi^\hbar\right)^*\right]+\Delta\psi^\hbar\nabla\left(\psi^\hbar\right)^*+\Delta\left(\psi^\hbar\right)^*\nabla\psi^\hbar,
\end{aligned}\tag{2.30}$$

we can rewrite (2.29) as

$$\begin{aligned}
&\partial_t\mu^\hbar+\frac{\hbar^2}{2m^2}\nabla\left|\nabla\psi^\hbar\right|^2+\frac{g}{m}\rho^\hbar\nabla\rho^\hbar+\frac{\rho^\hbar}{m}\nabla V+\frac{\hbar^2}{2m^2}\Big[\Delta\psi^\hbar\nabla\left(\psi^\hbar\right)^* \\
&+\Delta\left(\psi^\hbar\right)^*\nabla\psi^\hbar\Big]=\frac{\hbar^2}{4m^2}\nabla\Delta\rho^\hbar+\omega\nabla\left(x^\perp\cdot\mu^\hbar\right)+\omega x\left(\frac{\left(\mu^\hbar\right)^\perp}{\rho^\hbar}\cdot\nabla\rho^\hbar\right).
\end{aligned}\tag{2.31}$$

Using the equality

$$\frac{\hbar^2}{2m^2}\left|\nabla\psi^\hbar\right|^2=\frac{\left|\mu^\hbar\right|^2}{2\rho^\hbar}+\frac{\hbar^2}{8m^2}\frac{\left|\nabla\rho^\hbar\right|^2}{\rho^\hbar},\tag{2.32}$$

we derive

$$\partial_t \mu^\hbar + \nabla\left(\frac{|\mu^\hbar|^2}{2\rho^\hbar}\right) + \nabla\left(\frac{g}{2m}(\rho^\hbar)^2\right) + \frac{\rho^\hbar}{m}\nabla V$$

$$+\frac{\hbar^2}{2m^2}\left[\Delta\psi^\hbar\nabla\left(\psi^\hbar\right)^* + \Delta\left(\psi^\hbar\right)^*\nabla\psi^\hbar\right] \tag{2.33}$$

$$= \frac{\hbar^2}{4m^2}\nabla\left(\Delta\rho^\hbar - \frac{|\nabla\rho^\hbar|^2}{2\rho^\hbar}\right) + \omega\nabla\left(x^\perp\cdot\mu^\hbar\right) + \omega x\left(\frac{(\mu^\hbar)^\perp}{\rho^\hbar}\cdot\nabla\rho^\hbar\right).$$

Since

$$\frac{\hbar^2}{2m^2}\left[\Delta\psi^\hbar\nabla\left(\psi^\hbar\right)^* + \Delta\left(\psi^\hbar\right)^*\nabla\psi^\hbar\right] = \nabla\cdot\left(\frac{\mu^\hbar\otimes\mu^\hbar}{\rho^\hbar}\right) - \nabla\left(\frac{|\mu^\hbar|^2}{2\rho^\hbar}\right)$$

$$+\frac{\hbar^2}{4m^2}\left[\frac{1}{\rho^\hbar}\nabla\cdot\left(\nabla\rho^\hbar\otimes\nabla\rho^\hbar\right) - \frac{|\nabla\rho^\hbar|^2}{2(\rho^\hbar)^2}\nabla\rho^\hbar - \frac{\nabla|\nabla\rho^\hbar|^2}{2\rho^\hbar}\right] \tag{2.34}$$

and

$$\nabla\Delta\rho^\hbar + \frac{|\nabla\rho^\hbar|^2}{(\rho^\hbar)^2}\nabla\rho^\hbar - \frac{1}{\rho^\hbar}\nabla\cdot\left(\nabla\rho^\hbar\otimes\nabla\rho^\hbar\right) = \nabla\cdot\left(\rho^\hbar\nabla^2\log\rho^\hbar\right), \tag{2.35}$$

we obtain equation (2.15). In addition, by observing the dimension of (2.32), we can conjecture that the kinetic energy is of the form $\frac{\hbar^2}{2m^2}\left|\nabla\psi^\hbar\right|^2$.

2.3. Energy Equation

We can derive the conservation of energy from (2.19) and (2.20) as follows. Multiplying (2.19) by $\partial_t\left(\psi^\hbar\right)^*$ and (2.20) by $\partial_t\psi^\hbar$ and then adding up, we can write

$$-\frac{\hbar^2}{2m}\left[\Delta\psi^\hbar\partial_t\left(\psi^\hbar\right)^* + \Delta\left(\psi^\hbar\right)^*\partial_t\psi^\hbar\right] + g\left|\psi^\hbar\right|^2\partial_t\left|\psi^\hbar\right|^2 + V\partial_t\left|\psi^\hbar\right|^2$$

$$+i\hbar\omega\left[x^\perp\cdot\nabla\psi^\hbar\partial_t\left(\psi^\hbar\right)^* - x^\perp\cdot\nabla\left(\psi^\hbar\right)^*\partial_t\psi^\hbar\right] = 0. \tag{2.36}$$

We use the equalities

$$\Delta\psi^\hbar\partial_t\left(\psi^\hbar\right)^* + \Delta\left(\psi^\hbar\right)^*\partial_t\psi^\hbar = \nabla\cdot\left[\partial_t\left(\psi^\hbar\right)^*\nabla\psi^\hbar + \partial_t\psi^\hbar\nabla\left(\psi^\hbar\right)^*\right] - \partial_t\left|\nabla\psi^\hbar\right|^2 \tag{2.37}$$

and

$$x^\perp\cdot\nabla\psi^\hbar\partial_t\left(\psi^\hbar\right)^* - x^\perp\cdot\nabla\left(\psi^\hbar\right)^*\partial_t\psi^\hbar$$

$$= \partial_t\left[\left(\psi^\hbar\right)^* x^\perp\cdot\nabla\psi^\hbar\right] - x^\perp\cdot\nabla\left[\left(\psi^\hbar\right)^*\partial_t\psi^\hbar\right] \tag{2.38}$$

and then multiply (2.36) by $\frac{1}{m}$ to have

$$\begin{aligned}&\partial_t\left\{\frac{\hbar^2}{2m^2}\left|\nabla\psi^\hbar\right|^2+\frac{g}{2m}\left|\psi^\hbar\right|^4+\frac{1}{m}V\left|\psi^\hbar\right|^2+\frac{i\hbar\omega}{m}\left[\left(\psi^\hbar\right)^*x^\perp\cdot\nabla\psi^\hbar\right]\right\}\\&-\nabla\cdot\left\{\frac{\hbar^2}{2m^2}\left[\partial_t\left(\psi^\hbar\right)^*\nabla\psi^\hbar+\partial_t\psi^\hbar\nabla\left(\psi^\hbar\right)^*\right]\right\}\\&-x^\perp\cdot\nabla\left\{\frac{i\hbar\omega}{m}\left[\left(\psi^\hbar\right)^*\partial_t\psi^\hbar\right]\right\}=0.\end{aligned}\tag{2.39}$$

Similarly, we use the equality

$$\begin{aligned}&x^\perp\cdot\nabla\psi^\hbar\partial_t\left(\psi^\hbar\right)^*-x^\perp\cdot\nabla\left(\psi^\hbar\right)^*\partial_t\psi^\hbar\\&=x^\perp\cdot\nabla\left[\psi^\hbar\partial_t\left(\psi^\hbar\right)^*\right]-\partial_t\left[\psi^\hbar x^\perp\cdot\nabla\left(\psi^\hbar\right)^*\right]\end{aligned}\tag{2.40}$$

to have

$$\begin{aligned}&\partial_t\left\{\frac{\hbar^2}{2m^2}\left|\nabla\psi^\hbar\right|^2+\frac{g}{2m}\left|\psi^\hbar\right|^4+\frac{1}{m}V\left|\psi^\hbar\right|^2-\frac{i\hbar\omega}{m}\left[\psi^\hbar x^\perp\cdot\nabla\left(\psi^\hbar\right)^*\right]\right\}\\&-\nabla\cdot\left\{\frac{\hbar^2}{2m^2}\left[\partial_t\left(\psi^\hbar\right)^*\nabla\psi^\hbar+\partial_t\psi^\hbar\nabla\left(\psi^\hbar\right)^*\right]\right\}+x^\perp\cdot\nabla\left\{\frac{i\hbar\omega}{m}\left[\psi^\hbar\partial_t\left(\psi^\hbar\right)^*\right]\right\}=0.\end{aligned}\tag{2.41}$$

Now adding up (2.39) and (2.41) and multiplying it by $\frac{1}{2}$, we obtain the energy equation expressed in terms of $\psi^\hbar$ as

$$\begin{aligned}&\partial_t\left\{\frac{\hbar^2}{2m^2}\left|\nabla\psi^\hbar\right|^2+\frac{g}{2m}\left|\psi^\hbar\right|^4+\frac{1}{m}V\left|\psi^\hbar\right|^2+\frac{i\hbar\omega}{2m}\left[\left(\psi^\hbar\right)^*x^\perp\cdot\nabla\psi^\hbar-\psi^\hbar x^\perp\cdot\nabla\left(\psi^\hbar\right)^*\right]\right\}\\&-\nabla\cdot\left\{\frac{\hbar^2}{2m^2}\left[\partial_t\left(\psi^\hbar\right)^*\nabla\psi^\hbar+\partial_t\psi^\hbar\nabla\left(\psi^\hbar\right)^*\right]\right\}\\&-\nabla\cdot\left\{x^\perp\frac{i\hbar\omega}{2m}\left[\left(\psi^\hbar\right)^*\partial_t\psi^\hbar-\psi^\hbar\partial_t\left(\psi^\hbar\right)^*\right]\right\}=0.\end{aligned}\tag{2.42}$$

Therefore, we derive the conservation of energy

$$\partial_t\theta^\hbar+\nabla\cdot\left[\frac{\mu_\omega^\hbar}{\rho^\hbar}\left(\theta^\hbar+\frac{g}{2m}(\rho^\hbar)^2\right)\right]=\frac{\hbar^2}{4m^2}\nabla\cdot\left[\frac{\mu_\omega^\hbar\Delta\rho^\hbar}{\rho^\hbar}-\frac{\nabla\cdot\mu_\omega^\hbar\nabla\rho^\hbar}{\rho^\hbar}\right],\tag{2.43}$$

where energy

$$\theta^\hbar = \frac{\hbar^2}{2m^2}\left|\nabla\psi^\hbar\right|^2 + \frac{g}{2m}\left|\psi^\hbar\right|^4 + \frac{1}{m}V\left|\psi^\hbar\right|^2 + \frac{i\hbar\omega}{2m}\left[\left(\psi^\hbar\right)^* x^\perp\cdot\nabla\psi^\hbar - \psi^\hbar x^\perp\cdot\nabla\left(\psi^\hbar\right)^*\right]$$

$$= \frac{\left|\mu^\hbar\right|^2}{2\rho^\hbar} + \frac{\hbar^2}{8m^2}\frac{\left|\nabla\rho^\hbar\right|^2}{\rho^\hbar} + \frac{g}{2m}(\rho^\hbar)^2 + \frac{1}{m}V\rho^\hbar - \omega\left(x^\perp\cdot\mu^\hbar\right)$$

$$= \frac{\left|\mu^\hbar_\omega\right|^2}{2\rho^\hbar} + \frac{\hbar^2}{8m^2}\frac{\left|\nabla\rho^\hbar\right|^2}{\rho^\hbar} + \frac{g}{2m}(\rho^\hbar)^2 + \rho^\hbar\left(\frac{1}{m}V - \frac{1}{2}\omega^2\left|x\right|^2\right)$$

$$= \frac{1}{2}\left(\frac{\left|\mu^\hbar_\omega\right|}{\sqrt{\rho^\hbar}}\right)^2 + \frac{\hbar^2}{2m^2}\left(\nabla\sqrt{\rho^\hbar}\right)^2 + \frac{g}{2m}\left[\rho^\hbar + \frac{m}{g}\left(\frac{1}{m}V - \frac{1}{2}\omega^2|x|^2\right)\right]^2$$

$$-\frac{m}{2g}\left(\frac{1}{m}V - \frac{1}{2}\omega^2|x|^2\right)^2. \tag{2.44}$$

If we confine $V(x)$ to satisfying $\frac{1}{m}V(x) - \frac{1}{2}\omega^2|x|^2 > 0$, energy $\theta^\hbar$ is positive definite.

CHAPTER 3

Semiclassical Limit of the Local Smooth Solutions

Let us consider the family, parameterized by $\hbar$, of solutions

$$\psi^{\hbar}(t,x) = A^{\hbar}(t,x)\exp\left(\frac{i}{\hbar}S^{\hbar}(t,x)\right), \quad t \in \mathbb{R}^{+}, \quad x \in \mathbb{R}^{2}, \tag{3.1}$$

of the rotating nonlinear Schrödinger equation (1.1) with rapidly oscillating initial condition

$$\psi^{\hbar}(0,x) = \psi_0^{\hbar}(x) = A_0^{\hbar}(x)\exp\left(\frac{i}{\hbar}S_0^{\hbar}(x)\right), \tag{3.2}$$

where the complex-valued function $A^{\hbar}(t,x)$ denotes the amplitude, and the real-valued function $S^{\hbar}(t,x)$ denotes the phase. Unlike the usual WKB method to look for the solution of the form

$$\psi^{\hbar}(t,x) = A^{\hbar}(t,x)\exp\left(\frac{i}{\hbar}S(t,x)\right), \tag{3.3}$$

where S is independent of $\hbar$, we allow $S^{\hbar}$ to depend on $\hbar$. The initial density and momentum satisfying the Euler equation (2.12) and (2.16) are then

$$\begin{aligned} \rho^{\hbar}(0,x) &= \left|A_0^{\hbar}(x)\right|^2, \\ \mu_{\omega}^{\hbar}(0,x) &= \left|A_0^{\hbar}(x)\right|^2\left(\frac{1}{m}\nabla S_0^{\hbar}(x) - \omega x^{\perp}\right). \end{aligned} \tag{3.4}$$

We will use the hydrodynamical structure derived in the preceding chapter to study the asymptotic behaviour of solutions $\psi^{\hbar}(t,x)$ of the rotating nonlinear Schrödinger equation (1.1) with initial condition (3.2) as $\hbar$ tends to zero. If we argue formally, it is natural to conjecture that the $O(\hbar^2)$ dispersive term appearing in (2.16) is negligible as $\hbar$ tends to zero, and the limiting density ρ and momentum μ_{ω} satisfy the compressible rotational Euler equation (1.7) with initial condition inferred from (3.4) given by

$$\begin{aligned} \rho(0,x) &= \left|A_0(x)\right|^2, \\ \mu_{\omega}(0,x) &= \left|A_0(x)\right|^2\left(\frac{1}{m}\nabla S_0(x) - \omega x^{\perp}\right). \end{aligned} \tag{3.5}$$

Because the $O(\hbar^2)$ dispersive term is nonlinear, we still have difficulty treating the problem directly from the hydrodynamical structure. According to Grenier [**8**], the modified Madelung transformation can be employed in the study of the semiclassical limit. The procedure for expounding and proving are divided into three sections.

3.1. Quasilinear Symmetric Hyperbolic System

The first step in studying the semiclassical limit is to show the existence of a smooth solution $\psi^\hbar$ of the rotating nonlinear Schrödinger equation (1.1) on a finite time interval $[0,T]$, independent of $\hbar$, for initial data $A_0^\hbar(x)$ and $S_0^\hbar(x)$ with Sobolev regularity. We will transform the rotating nonlinear Schrödinger equation into a dispersive perturbation of a quasilinear symmetric hyperbolic system. We will look for solutions (3.1) where $A^\hbar = a^\hbar + ib^\hbar$. After inserting (3.1) into (1.1), we obtain

$$\begin{aligned} i\hbar\partial_t A^\hbar - A^\hbar\partial_t S^\hbar = &-\frac{i\hbar}{m}\nabla S^\hbar\cdot\nabla A^\hbar - \frac{\hbar^2}{2m}\Delta A^\hbar + \frac{1}{2m}A^\hbar\left|\nabla S^\hbar\right|^2 \\ &-\frac{i\hbar}{2m}A^\hbar\Delta S^\hbar + g\left|A^\hbar\right|^2 A^\hbar + VA^\hbar + i\hbar\omega\left(x^\perp\cdot\nabla A^\hbar\right) - \omega\left(x^\perp\cdot\nabla S^\hbar\right)A^\hbar. \end{aligned} \tag{3.6}$$

We split (3.6) into

$$\partial_t S^\hbar + \frac{1}{2m}\left|\nabla S^\hbar\right|^2 + g\left|A^\hbar\right|^2 + V - \omega\left(x^\perp\cdot\nabla S^\hbar\right) = 0, \tag{3.7a}$$

$$\partial_t A^\hbar + \frac{1}{m}\nabla S^\hbar\cdot\nabla A^\hbar + \frac{1}{2m}A^\hbar\Delta S^\hbar - \omega\left(x^\perp\cdot\nabla A^\hbar\right) = \frac{i\hbar}{2m}\Delta A^\hbar, \tag{3.7b}$$

based on whether the term is of order $O(1)$ or of order $O(\hbar)$ and $O(\hbar^2)$. The expression (3.7a)–(3.7b) differs from (2.7)–(2.8), which are split into the real and imaginary parts, by the criteria of separating (3.6) into two parts. Notice that the second derivative term in (2.8) is highly nonlinear, whereas that in (3.7b) is linear. Therefore, the classical quasilinear hyperbolic theory provides an approach to the semiclassical limit of the rotating nonlinear Schrödinger equation (1.1). The change of variable $u^\hbar = (u_1^\hbar, u_2^\hbar) = \dfrac{1}{m}\nabla S^\hbar$ leads to

$$\partial_t u_1^\hbar + \frac{1}{2}\partial_{x_1}\left[(u_1^\hbar)^2 + (u_2^\hbar)^2\right] + \frac{g}{m}\partial_{x_1}\left|A^\hbar\right|^2 + \frac{1}{m}\partial_{x_1}V - \omega\partial_{x_1}\left(x^\perp\cdot u^\hbar\right) = 0, \tag{3.8a}$$

$$\partial_t u_2^\hbar + \frac{1}{2}\partial_{x_2}\left[(u_1^\hbar)^2 + (u_2^\hbar)^2\right] + \frac{g}{m}\partial_{x_2}\left|A^\hbar\right|^2 + \frac{1}{m}\partial_{x_2}V - \omega\partial_{x_2}\left(x^\perp\cdot u^\hbar\right) = 0, \tag{3.8b}$$

$$\partial_t A^\hbar + u^\hbar\cdot\nabla A^\hbar + \frac{1}{2}A^\hbar\nabla\cdot u^\hbar - \omega\left(x^\perp\cdot\nabla A^\hbar\right) = \frac{i\hbar}{2m}\Delta A^\hbar. \tag{3.8c}$$

Let $A^\hbar = a^\hbar + ib^\hbar$; we have

$$\begin{aligned}&\partial_t\left(u_1^\hbar+\omega x_2\right)+\frac{2g}{m}a^\hbar\partial_{x_1}a^\hbar+\frac{2g}{m}b^\hbar\partial_{x_1}b^\hbar+\left(u_1^\hbar+\omega x_2\right)\partial_{x_1}\left(u_1^\hbar+\omega x_2\right)\\&\quad+\left(u_2^\hbar-\omega x_1\right)\partial_{x_2}\left(u_1^\hbar+\omega x_2\right)-\omega^2x_1+\frac{1}{m}\partial_{x_1}V=2\omega\left(u_2^\hbar-\omega x_1\right),\end{aligned}\tag{3.9a}$$

$$\begin{aligned}&\partial_t\left(u_2^\hbar-\omega x_1\right)+\frac{2g}{m}a^\hbar\partial_{x_2}a^\hbar+\frac{2g}{m}b^\hbar\partial_{x_2}b^\hbar+\left(u_1^\hbar+\omega x_2\right)\partial_{x_1}\left(u_2^\hbar-\omega x_1\right)\\&\quad+\left(u_2^\hbar-\omega x_1\right)\partial_{x_2}\left(u_2^\hbar-\omega x_1\right)-\omega^2x_2+\frac{1}{m}\partial_{x_2}V=-2\omega\left(u_1^\hbar+\omega x_2\right),\end{aligned}\tag{3.9b}$$

$$\begin{aligned}&\partial_t a^\hbar+\left(u_1^\hbar+\omega x_2\right)\partial_{x_1}a^\hbar+\left(u_2^\hbar-\omega x_1\right)\partial_{x_2}a^\hbar\\&\quad+\frac{1}{2}a^\hbar\partial_{x_1}\left(u_1^\hbar+\omega x_2\right)+\frac{1}{2}a^\hbar\partial_{x_2}\left(u_2^\hbar-\omega x_1\right)=-\frac{\hbar}{2m}\Delta b^\hbar,\end{aligned}\tag{3.9c}$$

$$\begin{aligned}&\partial_t b^\hbar+\left(u_1^\hbar+\omega x_2\right)\partial_{x_1}b^\hbar+\left(u_2^\hbar-\omega x_1\right)\partial_{x_2}b^\hbar\\&\quad+\frac{1}{2}b^\hbar\partial_{x_1}\left(u_1^\hbar+\omega x_2\right)+\frac{1}{2}b^\hbar\partial_{x_2}\left(u_2^\hbar-\omega x_1\right)=\frac{\hbar}{2m}\Delta a^\hbar,\end{aligned}\tag{3.9d}$$

with initial data

$$\begin{aligned}&a^\hbar(0,x)=a_0^\hbar(x),\quad b^\hbar(0,x)=b_0^\hbar(x),\\&u_\omega^\hbar(0,x)=\left(u_1^\hbar(0,x)+\omega x_2,u_2^\hbar(0,x)-\omega x_1\right)\\&\qquad=\left(u_{1,0}^\hbar(x)+\omega x_2,u_{2,0}^\hbar(x)-\omega x_1\right)=u_0^\hbar(x)-\omega x^\perp=u_{\omega,0}^\hbar(x),\end{aligned}\tag{3.10}$$

satisfying

$$\left[a_0^\hbar(x)\right]^2+\left[b_0^\hbar(x)\right]^2=\left|A_0^\hbar(x)\right|^2,\quad u_{\omega,0}^\hbar(x)=\frac{1}{m}\nabla S_0^\hbar(x)-\omega x^\perp.\tag{3.11}$$

Let $U_\omega^\hbar=\left(a^\hbar,b^\hbar,u_1^\hbar+\omega x_2,u_2^\hbar-\omega x_1\right)^T$; the system can be written in the form

$$\begin{aligned}&\partial_t U_\omega^\hbar+M_1(U_\omega^\hbar)\partial_{x_1}U_\omega^\hbar+M_2(U_\omega^\hbar)\partial_{x_2}U_\omega^\hbar+G=\mathcal{L}(U_\omega^\hbar),\\&U_\omega^\hbar(0,x)=U_{\omega,0}^\hbar(x)=\left(a_0^\hbar(x),b_0^\hbar(x),u_{1,0}^\hbar(x)+\omega x_2,u_{2,0}^\hbar(x)-\omega x_1\right)^T,\end{aligned}\tag{3.12}$$

where

$$M_1(U_\omega^\hbar) = \begin{pmatrix} u_1^\hbar + \omega x_2 & 0 & \frac{1}{2}a^\hbar & 0 \\ 0 & u_1^\hbar + \omega x_2 & \frac{1}{2}b^\hbar & 0 \\ \frac{2g}{m}a^\hbar & \frac{2g}{m}b^\hbar & u_1^\hbar + \omega x_2 & 0 \\ 0 & 0 & 0 & u_1^\hbar + \omega x_2 \end{pmatrix}, \tag{3.13a}$$

$$M_2(U_\omega^\hbar) = \begin{pmatrix} u_2^\hbar - \omega x_1 & 0 & 0 & \frac{1}{2}a^\hbar \\ 0 & u_2^\hbar - \omega x_1 & 0 & \frac{1}{2}b^\hbar \\ 0 & 0 & u_2^\hbar - \omega x_1 & 0 \\ \frac{2g}{m}a^\hbar & \frac{2g}{m}b^\hbar & 0 & u_2^\hbar - \omega x_1 \end{pmatrix}, \tag{3.13b}$$

$$G = \begin{pmatrix} 0 \\ 0 \\ -\omega^2 x_1 + \frac{1}{m}\partial_{x_1} V \\ -\omega^2 x_2 + \frac{1}{m}\partial_{x_2} V \end{pmatrix}, \tag{3.13c}$$

and

$$\mathcal{L}(U_\omega^\hbar) = \begin{pmatrix} 0 & -\frac{\hbar}{2m}\Delta & 0 & 0 \\ \frac{\hbar}{2m}\Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\omega \\ 0 & 0 & -2\omega & 0 \end{pmatrix} \begin{pmatrix} a^\hbar \\ b^\hbar \\ u_1^\hbar + \omega x_2 \\ u_2^\hbar - \omega x_1 \end{pmatrix}. \tag{3.13d}$$

The matrix $\mathcal{L}$ is antisymmetric and reflects the dispersive nature of (3.12). For all $(\xi, \eta)^T \in \mathbb{R}^2$,

$$\xi M_1(U_\omega^\hbar) + \eta M_2(U_\omega^\hbar) = \begin{pmatrix} \lambda & 0 & \frac{\xi}{2m}a^\hbar & \frac{\eta}{2m}a^\hbar \\ 0 & \lambda & \frac{\xi}{2m}b^\hbar & \frac{\eta}{2m}b^\hbar \\ 2\xi g a^\hbar & 2\xi g b^\hbar & \lambda & 0 \\ 2\eta g a^\hbar & 2\eta g b^\hbar & 0 & \lambda \end{pmatrix}, \tag{3.14}$$

where $\lambda = \xi(u_1^\hbar + \omega x_2) + \eta(u_2^\hbar - \omega x_1)$. The matrix (3.14) can be symmetrized by

$$M_0(U_\omega^\hbar) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/4mg & 0 \\ 0 & 0 & 0 & 1/4mg \end{pmatrix}, \tag{3.15}$$

which is symmetric and positive definite if $g > 0$ for all $U_\omega^\hbar$, and has only real eigenvalues λ, λ, $\lambda \pm \sqrt{\frac{g}{m}\left(\xi^2 + \eta^2\right)\left[(a^\hbar)^2 + (b^\hbar)^2\right]}$. Thus, we write (1.1) as the dispersive perturbation of the quasilinear symmetric hyperbolic system

$$\begin{aligned} & M_0(U_\omega^\hbar)\partial_t U_\omega^\hbar + \widetilde{M}_1(U_\omega^\hbar)\partial_{x_1} U_\omega^\hbar + \widetilde{M}_2(U_\omega^\hbar)\partial_{x_2} U_\omega^\hbar + \widetilde{G} = \widetilde{\mathcal{L}}(U_\omega^\hbar), \\ & U_\omega^\hbar(0, x) = U_{\omega,0}^\hbar(x), \end{aligned} \tag{3.16}$$

where $\widetilde{M}_1 = M_0 M_1$, $\widetilde{M}_2 = M_0 M_2$, $\widetilde{G} = M_0 G$, and $\widetilde{\mathcal{L}} = M_0 \mathcal{L}$. Here $\widetilde{M}_1$ and $\widetilde{M}_2$ are symmetric, and $\widetilde{\mathcal{L}}$ is antisymmetric.

3.2. Classical Solutions

In order to carry out the existence of classical solutions, we proceed along the lines of the existence proof concerning the initial value problem for the quasilinear symmetric hyperbolic system with modifications. We utilize the iteration scheme for establishing the local existence in time. As a first approximation to the solution of (3.16), we consider $U_\omega^0(t, x, \hbar)$ defined by $U_\omega^0(t, x, \hbar) = U_{\omega,0}^\hbar(x)$, where $U_{\omega,0}^\hbar(x)$ denotes the initial data. We define successively $U_\omega^{p+1}(t,x,\hbar)$ as the solution of the linear equation

$$M_0(U_\omega^p)\partial_t U_\omega^{p+1} + \widetilde{M}_1(U_\omega^p)\partial_{x_1}U_\omega^{p+1} + \widetilde{M}_2(U_\omega^p)\partial_{x_2}U_\omega^{p+1} + \widetilde{G} = \widetilde{\mathcal{L}}(U_\omega^{p+1}),$$
$$U_\omega^{p+1}(0,x,\hbar) = U_{\omega,0}^{\hbar}(x). \tag{3.17}$$

for $p = 1,2,3,\cdots$. We call $U_\omega^{p+1}(t,x,\hbar)$, $p = 0,1,2,3,\cdots$ successive approximations to a solution of (3.16). We might expect that U_ω^p tends to $U_\omega^\hbar$ as p tends to ∞. For further reference, we ignore the superscripts p and consider both $U_\omega \in C^\infty$ and $W_\omega \in C^\infty$ satisfying

$$M_0(W_\omega)\partial_t U_\omega + \widetilde{M}_1(W_\omega)\partial_{x_1}U_\omega + \widetilde{M}_2(W_\omega)\partial_{x_2}U_\omega + \widetilde{G} = \widetilde{\mathcal{L}}(U_\omega),$$
$$U_\omega(0,x,\hbar) = U_{\omega,0}^{\hbar}(x). \tag{3.18}$$

3.2.1. A Priori Estimate. The energy estimate is used to prove the existence of approximate solutions U_ω^p. Assume that the matrices $\widetilde{M}_1$ and $\widetilde{M}_2$ together with their derivatives of any desired order are continuous and bounded uniformly in $[0,T]\times\mathbb{R}^2$. We perform the process of the energy estimate in three stages.

Stage 1. *L^2-norm.* The canonical energy associated with (3.18) is defined by the scalar product

$$E(t) = (M_0U_\omega, U_\omega) = \iint M_0U_\omega \cdot U_\omega dx_1dx_2 = \iint U_\omega^T M_0 U_\omega dx_1 dx_2. \tag{3.19}$$

For a certain T, let the function $U_\omega(t,x,\hbar)$ be a solution of (3.18) of class $C^2([0,T]\times\mathbb{R}^2)$. We use the symmetry of M_0, $\widetilde{M}_1$, and $\widetilde{M}_2$ and integration by parts to have the basic energy equality of Friedrich

$$\frac{d}{dt}E(t) = (\Gamma U_\omega, U_\omega) + 2\left(\widetilde{\mathcal{L}}(U_\omega), U_\omega\right) - 2\left(\widetilde{G}, U_\omega\right), \tag{3.20}$$

where $\Gamma = \partial_t M_0 + \partial_{x_1}\widetilde{M}_1 + \partial_{x_2}\widetilde{M}_2$, so that the classical energy estimate can be obtained immediately. Since $\widetilde{\mathcal{L}}$ is an antisymmetric matrix, and we derive

$$\begin{aligned}\left(\widetilde{\mathcal{L}}(U_\omega), U_\omega\right) &= \iint U_\omega^T \widetilde{\mathcal{L}} U_\omega dx_1dx_2 = \iint \left(U_\omega^T\widetilde{\mathcal{L}}U_\omega\right)^T dx_1dx_2 \\ &= \iint \left(\widetilde{\mathcal{L}}U_\omega\right)^T U_\omega dx_1dx_2 = \iint U_\omega^T\widetilde{\mathcal{L}}^T U_\omega dx_1dx_2 = -\iint U_\omega^T \widetilde{\mathcal{L}}U_\omega dx_1dx_2,\end{aligned} \tag{3.21}$$

the term $\left(\widetilde{\mathcal{L}}(U_\omega), U_\omega\right) = 0$ contributes nothing to the energy estimate. This means that the singular perturbation does not create energy. If $G \in L^2(\mathbb{R}^2)$, then we apply Cauchy-Schwarz's inequality and Young's inequality to have

$$\left(\widetilde{G}, U_\omega\right) < C + C\|U_\omega\|_{L^2}^2. \tag{3.22}$$

Using the positive definite and symmetric matrix M_0, we obtain

$$\frac{d}{dt}E(t) \leq \left(\|\Gamma\|_{L^\infty} + C\right)\|U_\omega\|_{L^2}^2 + C \leq k\left(\|\Gamma\|_{L^\infty} + C\right)E(t) + C, \tag{3.23}$$

where an appropriate constant $k > 1$ is set to ensure that the last inequality holds. Because of the initial data

$$E(0) = \left(M_0 U_{\omega,0}^\hbar, U_{\omega,0}^\hbar\right) \leq \|M_0\|_{L^\infty}\|U_{\omega,0}^\hbar\|_{L^2}^2, \tag{3.24}$$

it follows from Gronwall's inequality that for $t \in [0, T]$,

$$E(t) \leq \exp\left[k\left(\|\Gamma\|_{L^\infty} + C\right)t\right]\left(\|M_0\|_{L^\infty}\|U_{\omega,0}^\hbar\|_{L^2}^2 + Ct\right). \tag{3.25}$$

Furthermore,

$$\max_{0 \leq t \leq T} \|U_\omega(t,\hbar)\|_{L^2}^2 \leq \exp\left[k\left(\|\Gamma\|_{L^\infty} + C\right)T\right]\left(\|U_{\omega,0}^\hbar\|_{L^2}^2 + \frac{CT}{\|M_0\|_{L^\infty}}\right). \tag{3.26}$$

Stage 2. *H^1-norm.* If we multiply (3.18) by $(M_0(W_\omega))^{-1}$, differentiate with respect to x_1, and multiply it by $M_0(W_\omega)$, then we have

$$M_0(W_\omega)\partial_t\partial_{x_1}U_\omega + \widetilde{M_1}(W_\omega)\partial_{x_1}\partial_{x_1}U_\omega + \widetilde{M_2}(W_\omega)\partial_{x_2}\partial_{x_1}U_\omega = \widetilde{\mathcal{L}}(\partial_{x_1}U_\omega) + F_{1x_1},$$

$$\partial_{x_1}U_\omega(0,x,\hbar) = \left(U_\omega^\hbar\right)_{x_1,0}(x), \tag{3.27a}$$

where

$$F_{1x_1} = -\partial_{x_1}\widetilde{M_1}\partial_{x_1}U_\omega - \partial_{x_1}\widetilde{M_2}\partial_{x_2}U_\omega - \partial_{x_1}\widetilde{G}. \tag{3.27b}$$

Similarly, we differentiate with respect to x_2 to have

$$M_0(W_\omega)\partial_t\partial_{x_2}U_\omega + \widetilde{M}_1(W_\omega)\partial_{x_1}\partial_{x_2}U_\omega + \widetilde{M}_2(W_\omega)\partial_{x_2}\partial_{x_2}U_\omega = \widetilde{\mathcal{L}}(\partial_{x_2}U_\omega) + F_{1x_2},$$

$$\partial_{x_2}U_\omega(0,x,\hbar) = \left(U_\omega^\hbar\right)_{x_2,0}(x), \tag{3.28a}$$

where

$$F_{1x_2} = -\partial_{x_2}\widetilde{M}_1\partial_{x_1}U_\omega - \partial_{x_2}\widetilde{M}_2\partial_{x_2}U_\omega - \partial_{x_2}\widetilde{G}. \tag{3.28b}$$

We expect to bound $(M_0\partial_{x_1}U_\omega, \partial_{x_1}U_\omega)$ and $(M_0\partial_{x_2}U_\omega, \partial_{x_2}U_\omega)$, where $(\cdot,\cdot)$ is the usual L^2 scalar product. Assume $U_\omega \in C^2([0,T];C^3(\mathbb{R}^2))$. Since M_0, $\widetilde{M}_1$, and $\widetilde{M}_2$ are symmetric, we have

$$\partial_t\left(M_0\partial_{x_1}U_\omega, \partial_{x_1}U_\omega\right) = \left(\Gamma\partial_{x_1}U_\omega, \partial_{x_1}U_\omega\right) + 2\left(\widetilde{\mathcal{L}}(\partial_{x_1}U_\omega), \partial_{x_1}U_\omega\right) + 2\left(F_{1x_1}, \partial_{x_1}U_\omega\right) \tag{3.29}$$

and

$$\partial_t\left(M_0\partial_{x_2}U_\omega, \partial_{x_2}U_\omega\right) = \left(\Gamma\partial_{x_2}U_\omega, \partial_{x_2}U_\omega\right) + 2\left(\widetilde{\mathcal{L}}(\partial_{x_2}U_\omega), \partial_{x_2}U_\omega\right) + 2\left(F_{1x_2}, \partial_{x_2}U_\omega\right), \tag{3.30}$$

where $\Gamma = \partial_t M_0 + \partial_{x_1}\widetilde{M}_1 + \partial_{x_2}\widetilde{M}_2$. The antisymmetry of $\widetilde{\mathcal{L}}$ yields the result

$$\left(\widetilde{\mathcal{L}}(\partial_{x_1}U_\omega), \partial_{x_1}U_\omega\right) = \left(\widetilde{\mathcal{L}}(\partial_{x_2}U_\omega), \partial_{x_2}U_\omega\right) = 0. \tag{3.31}$$

From the structure of M_0, $\widetilde{M}_1$, and $\widetilde{M}_2$ and the application of Cauchy-Schwarz's inequality and Young's inequality, we find the following estimates

$$2\left(-\partial_{x_i}\widetilde{M}_1\partial_{x_1}U_\omega - \partial_{x_i}\widetilde{M}_2\partial_{x_2}U_\omega, \partial_{x_i}U_\omega\right) \le C\left\|\partial_{x_1}U_\omega\right\|_{L^2}^2 + C\left\|\partial_{x_2}U_\omega\right\|_{L^2}^2 \tag{3.32}$$

for $i = 1,2$. If $\partial_{x_i}G \in L^2(\mathbb{R}^2)$ for $i = 1,2$, then we apply again Cauchy-Schwarz's inequality and Young's inequality to obtain

$$2\left(-\partial_{x_1}\widetilde{G}, \partial_{x_1}U_\omega\right) + 2\left(-\partial_{x_2}\widetilde{G}, \partial_{x_2}U_\omega\right) \le C + C\left\|\partial_{x_1}U_\omega\right\|_{L^2}^2 + C\left\|\partial_{x_2}U_\omega\right\|_{L^2}^2. \tag{3.33}$$

Since M_0 is a positive definite and symmetric matrix, we have

$$\begin{aligned}&\partial_t\left[(M_0\partial_{x_1}U_\omega, \partial_{x_1}U_\omega) + (M_0\partial_{x_2}U_\omega, \partial_{x_2}U_\omega)\right]\\ &\le k\left(\|\,\Gamma\,\|_{L^\infty} + C\right)\left[(M_0\partial_{x_1}U_\omega, \partial_{x_1}U_\omega) + (M_0\partial_{x_2}U_\omega, \partial_{x_2}U_\omega)\right] + C\end{aligned} \tag{3.34}$$

for some $k > 1$. Because of the initial data

$$\begin{aligned}&\left(M_0\left(U_\omega^\hbar\right)_{x_1,0},\left(U_\omega^\hbar\right)_{x_1,0}\right)+\left(M_0\left(U_\omega^\hbar\right)_{x_2,0},\left(U_\omega^\hbar\right)_{x_2,0}\right)\\&\le\|M_0\|_{L^\infty}\left(\left\|\left(U_\omega^\hbar\right)_{x_1,0}\right\|_{L^2}^2+\left\|\left(U_\omega^\hbar\right)_{x_2,0}\right\|_{L^2}^2\right),\end{aligned}\tag{3.35}$$

we deduce from Gronwall's inequality and the strict positivity of M_0 that

$$\begin{aligned}&\max_{0\le t\le T}\left(\|\partial_{x_1}U_\omega(t,\hbar)\|_{L^2}^2+\|\partial_{x_2}U_\omega(t,\hbar)\|_{L^2}^2\right)\\&\le\exp\left[k\left(\|\,\Gamma\,\|_{L^\infty}+C\right)T\right]\cdot\left[\left(\left\|\left(U_\omega^\hbar\right)_{x_1,0}\right\|_{L^2}^2+\left\|\left(U_\omega^\hbar\right)_{x_2,0}\right\|_{L^2}^2\right)+\frac{CT}{\|M_0\|_{L^\infty}}\right].\end{aligned}\tag{3.36}$$

Stage 3. *H^2-norm.* From equations (3.27a) and (3.28a), we use the method similar to that in Stage 2 to obtain

$$\begin{aligned}&M_0(W_\omega)\partial_t\left(\partial_{x_i}\partial_{x_j}U_\omega\right)+\widetilde{M}_1(W_\omega)\partial_{x_1}\left(\partial_{x_i}\partial_{x_j}U_\omega\right)\\&+\widetilde{M}_2(W_\omega)\partial_{x_2}\left(\partial_{x_i}\partial_{x_j}U_\omega\right)=\widetilde{\mathcal{L}}\left(\partial_{x_i}\partial_{x_j}U_\omega\right)+F_{2x_ix_j},\end{aligned}\tag{3.37a}$$

where

$$\begin{aligned}F_{2x_ix_j}&=-\partial_{x_i}\partial_{x_j}\widetilde{M}_1\partial_{x_1}U_\omega-\partial_{x_i}\partial_{x_j}\widetilde{M}_2\partial_{x_2}U_\omega-\partial_{x_i}\widetilde{M}_1\partial_{x_1}\partial_{x_j}U_\omega\\&-\partial_{x_i}\widetilde{M}_2\partial_{x_2}\partial_{x_j}U_\omega-\partial_{x_j}\widetilde{M}_1\partial_{x_1}\partial_{x_i}U_\omega-\partial_{x_j}\widetilde{M}_2\partial_{x_2}\partial_{x_i}U_\omega-\partial_{x_i}\partial_{x_j}\widetilde{G},\end{aligned}\tag{3.37b}$$

for $i,j=1,2$. Assume $U_\omega\in C^2([0,T];C^4(\mathbb{R}^2))$. Because of the symmetry of M_0, $\widetilde{M}_1$, and $\widetilde{M}_2$, we have

$$\begin{aligned}&\partial_t\left(M_0\partial_{x_i}\partial_{x_j}U_\omega,\partial_{x_i}\partial_{x_j}U_\omega\right)=\left(\Gamma\partial_{x_i}\partial_{x_j}U_\omega,\partial_{x_i}\partial_{x_j}U_\omega\right)\\&+2\left(\widetilde{\mathcal{L}}\left(\partial_{x_i}\partial_{x_j}U_\omega\right),\partial_{x_i}\partial_{x_j}U_\omega\right)+2\left(F_{2x_ix_j},\partial_{x_i}\partial_{x_j}U_\omega\right),\end{aligned}\tag{3.38}$$

where $\Gamma=\partial_tM_0+\partial_{x_1}\widetilde{M}_1+\partial_{x_2}\widetilde{M}_2$. The first term on the right side of (3.38) can be bounded by

$$\left(\Gamma\partial_{x_i}\partial_{x_j}U_\omega,\partial_{x_i}\partial_{x_j}U_\omega\right)\le\|\Gamma\|_{L^\infty}\left\|\partial_{x_i}\partial_{x_j}U_\omega\right\|_{L^2}^2.\tag{3.39}$$

The antisymmetry of $\widetilde{\mathcal{L}}$ leads to

$$\left(\widetilde{\mathcal{L}}\left(\partial_{x_i}\partial_{x_j}U_\omega\right), \partial_{x_i}\partial_{x_j}U_\omega\right) = 0, \tag{3.40}$$

and the usual estimates on commutators lead to

$$\begin{aligned}
&2\big(-\partial_{x_i}\partial_{x_j}\widetilde{M_1}\partial_{x_1}U_\omega - \partial_{x_i}\partial_{x_j}\widetilde{M_2}\partial_{x_2}U_\omega - \partial_{x_i}\widetilde{M_1}\partial_{x_1}\partial_{x_j}U_\omega \\
&\quad -\partial_{x_i}\widetilde{M_2}\partial_{x_2}\partial_{x_j}U_\omega - \partial_{x_j}\widetilde{M_1}\partial_{x_1}\partial_{x_i}U_\omega - \partial_{x_j}\widetilde{M_2}\partial_{x_2}\partial_{x_i}U_\omega, \partial_{x_i}\partial_{x_j}U_\omega\big) \\
&\quad \le C\left\|\partial_{x_1}U_\omega\right\|_{L^2}^2 + C\left\|\partial_{x_2}U_\omega\right\|_{L^2}^2 + C\sum_{|\alpha|=2}\left\|\partial_x^\alpha U_\omega\right\|_{L^2}^2,
\end{aligned} \tag{3.41}$$

where α is a multi-index of length $|\alpha|$; that is, $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$. If $\partial_{x_i}\partial_{x_j}G \in L^2(\mathbb{R}^2)$, meaning $\partial_{x_i}\partial_{x_j}\left(\frac{1}{m}\nabla V(x)\right) \in L^2(\mathbb{R}^2)$, then we apply Cauchy-Schwarz's inequality and Young's inequality to obtain

$$2\left(-\partial_{x_i}\partial_{x_j}\widetilde{G}, \partial_{x_i}\partial_{x_j}U_\omega\right) < C + C\left\|\partial_{x_i}\partial_{x_j}U_\omega\right\|_{L^2}^2. \tag{3.42}$$

Therefore,

$$\begin{aligned}
&\partial_t \sum_{|\alpha|=2}\left(M_0\partial_x^\alpha U_\omega, \partial_x^\alpha U_\omega\right) \\
&\le \left(\|\Gamma\|_{L^\infty} + C\right)\sum_{|\alpha|=2}\left\|\partial_x^\alpha U_\omega\right\|_{L^2}^2 + \left(C\left\|\partial_{x_1}U_\omega\right\|_{L^2}^2 + C\left\|\partial_{x_2}U_\omega\right\|_{L^2}^2 + C\right) \\
&\le k\left(\|\Gamma\|_{L^\infty} + C\right)\sum_{|\alpha|=2}\left(M_0\partial_x^\alpha U_\omega, \partial_x^\alpha U_\omega\right) + \left(C\left\|\partial_{x_1}U_\omega\right\|_{L^2}^2 + C\left\|\partial_{x_2}U_\omega\right\|_{L^2}^2 + C\right)
\end{aligned} \tag{3.43}$$

for some $k > 1$. It follows from Gronwall's inequality and the strict positivity of M_0 that

$$\begin{aligned}
&\max_{0 \le t \le T}\sum_{|\alpha|=2}\left\|\partial_x^\alpha U_\omega(t,\hbar)\right\|_{L^2}^2 \le \exp\left[k\left(\|\Gamma\|_{L^\infty} + C\right)T\right] \\
&\quad \cdot \left[\sum_{|\alpha|=2}\left\|\partial_x^\alpha U_{\omega,0}^\hbar\right\|_{L^2}^2 + \frac{\left(C\left\|\partial_{x_1}U_\omega\right\|_{L^2}^2 + C\left\|\partial_{x_2}U_\omega\right\|_{L^2}^2 + C\right)T}{\|M_0\|_{L^\infty}}\right].
\end{aligned} \tag{3.44}$$

Setting $U_\omega(t,x,\hbar) = U(t,x,\hbar) - \omega(0,0,-x_2,x_1)$ and $U^\hbar_{\omega,0}(x) = U^\hbar_0(x) - \omega(0,0,-x_2,x_1)$, we can rewrite (3.44) as

$$\max_{0\le t\le T} \sum_{|\alpha|=2} \|\partial_x^\alpha U(t,\hbar)\|^2_{L^2} \le \exp\left[k\left(\|\Gamma\|_{L^\infty} + C\right)T\right]$$
$$\cdot\left[\sum_{|\alpha|=2} \|\partial_x^\alpha U_0^\hbar\|^2_{L^2} + \frac{\left(C\|\partial_{x_1}U_\omega\|^2_{L^2} + C\|\partial_{x_2}U_\omega\|^2_{L^2} + C\right)T}{\|M_0\|_{L^\infty}}\right]. \tag{3.45}$$

to present a clear perspective.

As described in stage 3, the results of the higher energy estimate are obtained. Summarizing the above estimates, we conclude that

$$\max_{0\le t\le T} \| U_\omega(t,\hbar)\|^2_{H^s} \le \exp\left[k\left(\|\Gamma\|_{L^\infty} + C\right)T\right]\left[\|U^\hbar_{\omega,0}\|^2_{H^s} + \frac{CT}{\|M_0\|_{L^\infty}}\right], \tag{3.46}$$

and we find that for sufficiently small T, we can estimate all $\partial_x^\alpha U_\omega$ for $|\alpha| \le s$, $s > 3$. This shows that if $\frac{1}{m}\nabla V(x) - \omega^2 x \in H^s(\mathbb{R}^2)$ and $U^\hbar_{\omega,0} \in H^s(\mathbb{R}^2)$, then the iteration scheme defined by (3.17) is well-defined, and we also obtain a priori estimate on the space derivatives of the type

$$\| U^p_\omega(t,\hbar)\|_{H^s} \le C, \quad t\in[0,T], \tag{3.47}$$

which denotes

$$U^p_\omega \in L^\infty([0,T]; H^s(\mathbb{R}^2)). \tag{3.48}$$

In addition, it follows that every component of U^p_ω belongs to $L^\infty([0,T];H^s(\mathbb{R}^2))$ and then from (3.16) or (3.17) that for $t\in[0,T]$,

$$\|\partial_t a^p(t,\hbar)\|_{H^{s-2}} \le C, \quad \|\partial_t b^p(t,\hbar)\|_{H^{s-2}} \le C,$$
$$\|\partial_t u_1^p(t,\hbar)\|_{H^{s-1}} \le C, \quad \|\partial_t u_2^p(t,\hbar)\|_{H^{s-1}} \le C. \tag{3.49}$$

The inclusion relation $H^{s-1}(\mathbb{R}^2) \subset H^{s-2}(\mathbb{R}^2)$ leads to

$$\|\partial_t U^p_\omega(t,\hbar)\|_{H^{s-2}} \le C, \quad t\in[0,T], \tag{3.50}$$

which denotes

$$\partial_t U^p_\omega \in L^\infty([0,T]; H^{s-2}(\mathbb{R}^2)). \tag{3.51}$$

Remark 3.1. *Assume* $U^\hbar = \left(a^\hbar, b^\hbar, u_1^\hbar, u_2^\hbar\right)^T$. *It is convenient to rewrite* $U_\omega^\hbar = U^\hbar - \omega(0,0,-x_2,x_1)^T$ *and is helpful for the finer analysis. If* $\frac{1}{m}\nabla V(x) - \omega^2 x \in H^1(\mathbb{R}^2)$, $\frac{1}{m}\nabla V(x) \in H^s(\mathbb{R}^2)$, $U_{\omega,0}^\hbar \in H^1(\mathbb{R}^2)$, *and* $U_0^\hbar \in H^s(\mathbb{R}^2)$, *then we construct approximate solutions* $U_\omega^p(t,\hbar) = U^p(t,\hbar) - \omega(0,0,-x_2,x_1)^T$, $p = 0,1,2,\cdots$ *satisfying*

$$U_\omega^p \in L^\infty([0,T];H^1(\mathbb{R}^2)), \quad U^p \in L^\infty([0,T];H^s(\mathbb{R}^2)), \quad \partial_t U^p \in L^\infty([0,T];H^{s-2}(\mathbb{R}^2)).$$

3.2.2. The Existence and Uniqueness Results.

Proposition 3.2. *Let* $s > 3$ *and the potential* $V(x)$ *satisfy*

$$\frac{1}{m}\nabla V(x) - \omega^2 x \in H^s(\mathbb{R}^2).$$

Assume that the initial data $U_{\omega,0}^\hbar = \left(a_0^\hbar, b_0^\hbar, u_{1,0}^\hbar + \omega x_2, u_{2,0}^\hbar - \omega x_1\right)^T \in [H^s(\mathbb{R}^2)]^4$ *satisfies the uniform bound*

$$\| U_{\omega,0}^\hbar \|_{H^s} < C_1.$$

Then there is a time interval $[0,T]$ *with* $T > 0$, *so that the IVP for* (3.12) *has a unique classical solution*

$$U_\omega^\hbar = \left(a^\hbar, b^\hbar, u_1^\hbar + \omega x_2, u_2^\hbar - \omega x_1\right)^T \in C^1([0,T];C^2(\mathbb{R}^2)).$$

Furthermore,

$$U_\omega^\hbar \in C([0,T];H^s(\mathbb{R}^2)) \cap C^1([0,T];H^{s-2}(\mathbb{R}^2)),$$

and T *depends on the bound* C_1, *and in particular, not on* $\hbar$. *In addition, the solution* $U_\omega^\hbar$ *satisfies the estimate*

$$\|U_\omega^\hbar(t,\cdot)\|_{H^s} < C_2$$

for all $t \in [0,T]$. *The constant* C_2 *is also independent of* $\hbar$.

Proof. Following the results obtained in Subsection 3.2.1, for any fixed $\hbar$, we have constructed a sequence $\{U_\omega^p\}_{p=0}^\infty$ belonging to

$$C([0,T];H^s(\mathbb{R}^2)) \cap C^1([0,T];H^{s-2}(\mathbb{R}^2)), \tag{3.52}$$

and satisfying (3.17) as well as the uniform estimate

$$\max_{0 \le t \le T} (\| U_\omega^p(t,\hbar) \|_{H^s} + \| \partial_t U_\omega^p(t,\hbar) \|_{H^{s-2}}) \le C. \tag{3.53}$$

We use the mean value theorem to show that for every p, for $0 < t_1, t_2 < T$ and $\xi \in (t_1, t_2)$,

$$\begin{aligned} \|U_\omega^p(t_2,\hbar) - U_\omega^p(t_1,\hbar)\|_{H^{s-2}} &= \|\partial_t U_\omega^p(\xi,\hbar)(t_2-t_1)\|_{H^{s-2}} \\ &= |t_2-t_1|\,\|\partial_t U_\omega^p(\xi,\hbar)\|_{H^{s-2}} \le \max_{0\le t\le T} \|\partial_t U_\omega^p(t,\hbar)\|_{H^{s-2}}\,|t_2-t_1|. \end{aligned} \tag{3.54}$$

Thus, if $U_\omega^p : [0,T] \to H^s(\mathbb{R}^2)$ is continuous and differentiable on $[0,T]$, and $\partial_t U_\omega^p$ is bounded for every t, then U_ω^p is the Lipschitz continuous function on $[0,T]$ with values in the norm topology of $H^{s-2}(\mathbb{R}^2)$. This also explains that $\{U_\omega^p\}_{p=0}^\infty$ is equicontinuous. It follows from the Arzela-Ascoli theorem that there exists $U_\omega^\hbar \in L^\infty([0,T]; H^s(\mathbb{R}^2)) \cap \mathrm{Lip}([0,T]; H^{s-2}(\mathbb{R}^2))$ such that

$$\max_{0\le t\le T} \left\|U_\omega^p - U_\omega^\hbar\right\|_{H^{s-2}} \to 0, \tag{3.55}$$

as p tends to ∞. Hence,

$$U_\omega^p \to U_\omega^\hbar \quad \text{in} \quad C([0,T]; H^{s-2}(\mathbb{R}^2)). \tag{3.56}$$

Furthermore, we use the interpolation inequality to show that for $0 < \theta < 1$,

$$\begin{aligned} \left\|U_\omega^p - U_\omega^\hbar\right\|_{H^{s-\alpha}} &\le \left\|U_\omega^p - U_\omega^\hbar\right\|_{H^s}^\theta \left\|U_\omega^p - U_\omega^\hbar\right\|_{H^{s-2}}^{1-\theta} \\ &\le \left(\|U_\omega^p\|_{H^s} + \left\|U_\omega^\hbar\right\|_{H^s}\right)^\theta \left\|U_\omega^p - U_\omega^\hbar\right\|_{H^{s-2}}^{1-\theta} \le C\left\|U_\omega^p - U_\omega^\hbar\right\|_{H^{s-2}}^{1-\theta} \to 0 \end{aligned} \tag{3.57}$$

as p tends to ∞ so that we have

$$U_\omega^p \to U_\omega^\hbar \quad \text{in} \quad C([0,T]; H^{s-\alpha}(\mathbb{R}^2)) \tag{3.58}$$

for an appropriate constant α with $0 < \alpha = 2 - 2\theta < 2$. When we choose s such that $s - \alpha - 2 > \frac{2}{2} = 1$, the space $H^s(\mathbb{R}^2)$ becomes an algebra, in which we can perform multiplication and keep the product stay, to overcome the difficulty of the nonlinearity. Indeed, it can be shown that

$$U_\omega^\hbar \in C([0,T]; H^s(\mathbb{R}^2)) \cap C^1([0,T]; H^{s-2}(\mathbb{R}^2)), \tag{3.59}$$

and $U_\omega^\hbar$ is a solution of (3.12). The Sobolev embedding theorem tells us that

$$H^{s-2}(\mathbb{R}^2) \hookrightarrow C^2(\mathbb{R}^2) \tag{3.60}$$

if $s \geq 5$. Thus, we have

$$C([0,T];H^s(\mathbb{R}^2)) \cap C^1([0,T];H^{s-2}(\mathbb{R}^2)) \hookrightarrow C^1([0,T];C^2(\mathbb{R}^2)) \tag{3.61}$$

and

$$U_\omega^\hbar \in C^1([0,T];C^2(\mathbb{R}^2)) \tag{3.62}$$

which indicates that the constructed solution is classical.

In order to show that no extraction of subsequence is needed, we still prove the uniqueness of the classical solution of IVP for (3.12) by doing a straightforward energy estimate for the difference of two solutions. Let $(U_\omega^\hbar)_1$ and $(U_\omega^\hbar)_2$ be two solutions satisfying $(U_\omega^\hbar)_1(0,x) = U_{\omega,0}^\hbar(x)$ and $(U_\omega^\hbar)_2(0,x) = U_{\omega,0}^\hbar(x)$. Let $H^\hbar = (U_\omega^\hbar)_1 - (U_\omega^\hbar)_2$; we get

$$M_0 \partial_t H^\hbar + \widetilde{M}_1\left[(U_\omega^\hbar)_1\right]\partial_{x_1} H^\hbar + \widetilde{M}_2\left[(U_\omega^\hbar)_1\right]\partial_{x_2} H^\hbar = \widetilde{\mathcal{L}}(H^\hbar) + F, \tag{3.63a}$$

where

$$F = \left\{\widetilde{M}_1\left[(U_\omega^\hbar)_2\right] - \widetilde{M}_1\left[(U_\omega^\hbar)_1\right]\right\}\partial_{x_1}(U_\omega^\hbar)_2 + \left\{\widetilde{M}_2\left[(U_\omega^\hbar)_2\right] - \widetilde{M}_2\left[(U_\omega^\hbar)_1\right]\right\}\partial_{x_2}(U_\omega^\hbar)_2. \tag{3.63b}$$

We do the same procedure as before and expect to bound the canonical energy $E(t) = (M_0 H^\hbar, H^\hbar)$. We have the basic energy equality of Friedrich

$$\partial_t\left(M_0 H^\hbar, H^\hbar\right) = \left(\Gamma H^\hbar, H^\hbar\right) + 2\left(\widetilde{\mathcal{L}}(H^\hbar), H^\hbar\right) + 2\left(F, H^\hbar\right), \tag{3.64}$$

where $\Gamma = \partial_t M_0 + \partial_{x_2}\widetilde{M}_1 + \partial_{x_2}\widetilde{M}_2$. On the right side of (3.64), the first term is bounded by the assumption, and the second term vanishes due to the antisymmetry of $\widetilde{\mathcal{L}}$. Applying Cauchy-Schwarz's inequality leads to

$$2\left(F, H^\hbar\right) \leq C\|H^\hbar\|_{L^2}^2. \tag{3.65}$$

Therefore,

$$\partial_t\left(M_0 H^\hbar, H^\hbar\right) \leq \left(\|\Gamma\|_{L^\infty} + C\right)\|H^\hbar\|_{L^2}^2 \leq k\left(\|\Gamma\|_{L^\infty} + C\right)\left(M_0 H^\hbar, H^\hbar\right), \tag{3.66}$$

for some $k > 1$. By Gronwall's inequality, it follows that for $t \in [0,T]$,

$$\left(M_0 H^\hbar, H^\hbar\right) \leq \exp\left[k\left(\|\Gamma\|_{L^\infty} + C\right)t\right]\left(M_0 H_0^\hbar, H_0^\hbar\right) = 0. \tag{3.67}$$

Since this holds for all $t \in [0,T]$, $H^\hbar = 0$; that is, $(U_\omega^\hbar)_1 = (U_\omega^\hbar)_2$. Thus, the classical solution is unique.

□

Remark 3.3. *Under the assumptions of Remark* 3.1*, in fact, there exists* $U^\hbar$ *belonging to* $C([0,T];H^s(\mathbb{R}^2))\cap C^1([0,T];H^{s-2}(\mathbb{R}^2))$ *such that* U^p *tends to* $U^\hbar$ *as* p *tends to* ∞*. However,* $U^\hbar_\omega$ *belongs to* $C([0,T];H^1(\mathbb{R}^2))$*.*

We have proven the local existence and uniqueness of classical solutions of the dispersive perturbation of the quasilinear symmetric hyperbolic system. The result of Proposition 3.2 is pulled back to (1.1) which is equivalent to (3.12) for smooth solutions.

Theorem 3.4. *Assume the hypotheses of Proposition* 3.2*. Then the initial value problem of* (1.1) *and* (3.2) *has a unique classical solution in* $C^1([0,T];C^2(\mathbb{R}^2))$ *of the form*

$$\psi^\hbar(t,x) = A^\hbar(t,x)\exp\left(\frac{i}{\hbar}S^\hbar(t,x)\right)$$

on the time interval $[0,T]$*. Moreover,* $A^\hbar$ *and* $\nabla S^\hbar$ *are bounded in* $L^\infty([0,T];H^s(\mathbb{R}^2))$ *uniformly in* $\hbar$*, and* $\dfrac{1}{m}\nabla S^\hbar - \omega x^\perp$ *is bounded in* $L^\infty([0,T];H^1(\mathbb{R}^2))$ *uniformly in* $\hbar$*.*

Proof. The finer insight in Remark 3.1 and Remark 3.3 gives us more information at a more detailed level. Since $U^\hbar_0 \in H^s(\mathbb{R}^2)$ and $U^\hbar_{\omega,0} \in H^1(\mathbb{R}^2)$, we have $(A^\hbar_0, S^\hbar_0) \in H^s(\mathbb{R}^2)\times H^{s+1}(\mathbb{R}^2)$ and $\dfrac{1}{m}\nabla S^\hbar_0 - \omega x^\perp \in H^1(\mathbb{R}^2)$. Because of the expression (3.1) of $\psi^\hbar$ in the initial value problem for the rotating nonlinear Schrödinger equation, $\psi^\hbar$ has the same regularity as $A^\hbar$. Hence, we will observe the properties of $A^\hbar = a^\hbar + ib^\hbar$ and $S^\hbar$. It follows from (3.59) that

$$A^\hbar \in C([0,T];H^s(\mathbb{R}^2))\cap C^1([0,T];H^{s-2}(\mathbb{R}^2)). \tag{3.68}$$

Further, the Sobolev embedding theorem implies that

$$C([0,T];H^s(\mathbb{R}^2))\cap C^1([0,T];H^{s-2}(\mathbb{R}^2)) \hookrightarrow C^1([0,T];C^2(\mathbb{R}^2)) \tag{3.69}$$

if $s \geq 5$. Since $\dfrac{1}{m}\nabla S^\hbar = u^\hbar \in C([0,T];H^s(\mathbb{R}^2))\cap C^1([0,T];H^{s-1}(\mathbb{R}^2))$, we have $S^\hbar(t,\cdot) \in H^{s+1}(\mathbb{R}^2)$ and $\partial_t S^\hbar(t,\cdot) \in H^s(\mathbb{R}^2)$; that is,

$$S^\hbar \in C([0,T];H^{s+1}(\mathbb{R}^2))\cap C^1([0,T];H^s(\mathbb{R}^2)). \tag{3.70}$$

Once again, the Sobolev embedding theorem implies that $H^s(\mathbb{R}^2) \hookrightarrow C^2(\mathbb{R}^2)$ if $s \geq 3$. Hence, we obtain

$$C([0,T];H^{s+1}(\mathbb{R}^2)) \cap C^1([0,T];H^s(\mathbb{R}^2)) \hookrightarrow C^1([0,T];C^2(\mathbb{R}^2)). \tag{3.71}$$

Moreover,

$$\frac{1}{m}\nabla S^\hbar - \omega x^\perp = u_\omega^\hbar \in C([0,T];H^1(\mathbb{R}^2)). \tag{3.72}$$

The initial value problem of (1.1) and (3.2) for the rotating nonlinear Schrödinger equation is equivalent to the dispersive quasilinear hyperbolic system (3.12) due to the existence of classical solutions. Applying this equivalent relation, we complete it. □

3.2.3. The Properties of $\rho^\hbar$. When we expect that the Euler equation (2.12) and (2.16) tends to the limiting Euler equation (1.7) as $\hbar$ tends to zero, $\rho^\hbar$ must be restricted to ensure that there is not a singularity in the $O(\hbar^2)$ dispersive term appearing in (2.16). Before exploring the properties of $\rho^\hbar$, we obtain more information about the phase function from the modified Madelung transformation. We employ the polar coordinates :

$$A^\hbar = a^\hbar + ib^\hbar = \sqrt{\rho^\hbar}\exp\left(i\theta^\hbar\right) = \sqrt{\rho^\hbar}\left(\cos\theta^\hbar + i\sin\theta^\hbar\right). \tag{3.73}$$

We use the equality

$$a^\hbar \Delta b^\hbar - b^\hbar \Delta a^\hbar = \nabla \cdot \left(\rho^\hbar \nabla\theta^\hbar\right), \tag{3.74}$$

and then from (3.9a)–(3.9d), we derive the system

$$\partial_t \rho^\hbar + \nabla\cdot\left[\rho^\hbar\left(u_\omega^\hbar + \frac{\hbar}{m}\nabla\theta^\hbar\right)\right] = 0, \tag{3.75a}$$

$$\partial_t \theta^\hbar + u_\omega^\hbar \cdot \nabla\theta^\hbar + \frac{\hbar}{2m}|\nabla\theta^\hbar|^2 = \frac{\hbar}{2m}\frac{\nabla\sqrt{\rho^\hbar}}{\sqrt{\rho^\hbar}}, \tag{3.75b}$$

$$\partial_t u_\omega^\hbar + \left(u_\omega^\hbar\cdot\nabla\right)u_\omega^\hbar + \nabla\left(\frac{g}{m}\rho^\hbar + \frac{1}{m}V\right) = \omega^2 x - 2\omega\left(u_\omega^\hbar\right)^\perp. \tag{3.75c}$$

Equation (3.75a) has an extra term of order $O(\hbar)$ in comparison with the usual continuity equation. Moreover, this system (3.75a)–(3.75c) is of order $O(\hbar)$, but not of order $O(\hbar^2)$ in comparison with both (2.12) and (2.14). Consider the limiting equation of (3.75b)

$$\partial_t\theta + u_\omega\cdot\nabla\theta = 0 \tag{3.76}$$

with initial data $\theta(0,x) = 0$. It follows immediately that

$$\theta(t,x) = 0 \tag{3.77}$$

along the characteristic differential equation $\dfrac{dx}{dt} = u_\omega(t,x)$ subject to the initial condition $x(0) = x_0$. We conclude that the limiting system of (3.75a)–(3.75c) are the same as the limiting equations of (2.12) and (2.14) when $\hbar$ tends to zero.

Proposition 3.5. *Assume the hypotheses of Proposition* 3.2. *If* $\rho_0^\hbar(x) = (a_0^\hbar)^2 + (b_0^\hbar)^2 > 0$, *then* $\rho^\hbar(t,x) > 0$ *for all* $t \geq 0$; *if* $\rho_0^\hbar$ *has a compact support, then* $\rho^\hbar(t,\cdot)$ *does too for any* $t \in [0,T]$, *and*

$$R\left\{\rho^\hbar(t,\cdot)\right\} \leq R\left\{\rho_0^\hbar\right\} + (1+\hbar)CT$$

where $R\{u\} \equiv \sup\{|x| : u(x) \neq 0\}$.

Proof. Let (τ,ξ) be an arbitrary fixed time-space point in $[0,T] \times \mathbb{R}^2$. Since

$$u_\omega^\hbar(t,x) + \frac{\hbar}{m}\nabla\theta^\hbar(t,x) \in C^1([0.T]; H^{s-2}(\mathbb{R}^2)) \cap L^\infty([0.T]; H^s(\mathbb{R}^2)), \tag{3.78}$$

the Existence-Uniqueness theorem for ordinary differential equations guarantees that the problem

$$\frac{dx}{dt} = u_\omega^\hbar(t,x) + \frac{\hbar}{m}\nabla\theta^\hbar(t,x), \quad x(\tau) = \xi, \tag{3.79}$$

has a unique and continuous solution $x = \Psi(t) \in C^1([0,T];\mathbb{R}^2)$. Moreover, equation (3.75a) is equivalent to an ODE

$$\begin{aligned}\frac{d}{dt}\rho^\hbar(t,\Psi(t)) &= \partial_t\rho^\hbar(t,\Psi(t)) + \nabla\rho^\hbar(t,\Psi(t)) \cdot \left(u_\omega^\hbar(t,x) + \frac{\hbar}{m}\nabla\theta^\hbar(t,x)\right) \\ &= -\rho^\hbar(t,\Psi(t))\,\nabla\cdot\left(u_\omega^\hbar(t,x) + \frac{\hbar}{m}\nabla\theta^\hbar(t,x)\right).\end{aligned} \tag{3.80}$$

Integrating the above equality over a time interval $[0,\tau]$, we obtain

$$\rho^\hbar(\tau,\xi) = \rho^\hbar(0,\Psi(0)) \exp\left\{-\int_0^\tau \nabla\cdot\left(u_\omega^\hbar(t,x) + \frac{\hbar}{m}\nabla\theta^\hbar(t,x)\right)dt\right\}. \tag{3.81}$$

Thus, $\rho^\hbar(\tau,\xi) \geq 0$ if $\rho^\hbar(0,\Psi(0)) = \rho_0^\hbar(\Psi(0)) \geq 0$. Denote $R\{u\} \equiv \sup\{|x| : u(x) \neq 0\}$ for $u \in C(\mathbb{R}^2)$. When $\rho^\hbar(\tau,\xi) \neq 0$, $\rho_0^\hbar(\Psi(0)) \neq 0$, so $|\Psi(0)| \leq R\{\rho_0^\hbar\}$, and

$$\begin{aligned} |\xi| = |\Psi(\tau)| &= \left|\Psi(0) + \int_0^\tau \left(u_\omega^\hbar(t,x) + \frac{\hbar}{m}\nabla\theta^\hbar(t,x)\right) dt\right| \\ &\leq |\Psi(0)| + \int_0^\tau \left\|u_\omega^\hbar(t,x)\right\|_{L^\infty} + \frac{\hbar}{m}\left\|\nabla\theta^\hbar(t,x)\right\|_{L^\infty} dt \\ &\leq R\left\{\rho_0^\hbar\right\} + (1+\hbar)CT. \end{aligned} \tag{3.82}$$

Hence, we obtain

$$R\left\{\rho^\hbar(t,\cdot)\right\} \leq R\left\{\rho_0^\hbar\right\} + (1+\hbar)CT. \tag{3.83}$$

□

3.3. Semiclassical Limit

Let $U_\omega = (a, b, u_1 + \omega x_2, u_2 - \omega x_1)^T$. The limiting system of (3.12) is the quasilinear hyperbolic system

$$\begin{aligned} &\partial_t U_\omega + M_1(U_\omega)\partial_{x_1} U_\omega + M_2(U_\omega)\partial_{x_2} U_\omega + G = \mathcal{L}_2(U_\omega), \\ &U_\omega(0,x) = U_{\omega,0}(x) = \left(a_0(x), b_0(x), u_{1,0}(x) + \omega x_2, u_{2,0}(x) - \omega x_1\right)^T, \end{aligned} \tag{3.84}$$

where

$$\mathcal{L}_2(U_\omega) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\omega \\ 0 & 0 & -2\omega & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ u_1 + \omega x_2 \\ u_2 - \omega x_1 \end{pmatrix}, \tag{3.85}$$

and (3.84) is equivalent to the compressible rotational Euler equation (1.7) as long as solutions are smooth. We will show that it is possible to pass to the limit $\hbar \to 0$ in (3.12).

Proposition 3.6. *Let $U_{\omega,0}^\hbar, U_{\omega,0} \in H^s(\mathbb{R}^2)$, $s > 3$. Suppose that $U_{\omega,0}^\hbar(x)$ converges to $U_{\omega,0}(x)$ in $H^s(\mathbb{R}^2)$ as $\hbar$ tends to zero. Let $[0,T]$ be the fixed interval determined in Proposition* 3.2*. Then as $\hbar$ tends to zero, there exists $U_\omega(t,x) \in L^\infty([0,T]; H^s(\mathbb{R}^2))$ such that for $0 < \delta < 2$,*

$$U_\omega^\hbar(t,x) \to U_\omega(t,x) \quad in \quad C([0,T]; H^{s-\delta}(\mathbb{R}^2)).$$

The function $U_\omega(t,x)$ belongs to $C([0,T];H^s(\mathbb{R}^2))\cap C^1([0,T];H^{s-2}(\mathbb{R}^2))$ and is a classical solution of (3.84) *with initial data $U_\omega(0,x)=U_{\omega,0}(x)$.*

Proof. Since $\{U_\omega^\hbar\}_\hbar$ is bounded in $H^s(\mathbb{R}^2)$ for all $t\in[0,T]$, a weak compactness argument shows that for any fixed time $t\in[0,T]$, there exist a subsequence of $\{U_\omega^\hbar\}_\hbar$ (always denoted by $\{U_\omega^\hbar\}_\hbar$ due to the uniqueness) and a function $U_\omega\in H^s(\mathbb{R}^2)$ such that $U_\omega^\hbar \rightharpoonup U_\omega$ in $H^s(\mathbb{R}^2)$ as $\hbar\to 0$. Similarly, $\partial_t U_\omega^\hbar \rightharpoonup \partial_t U_\omega$ in $H^{s-2}(\mathbb{R}^2)$ as $\hbar\to 0$. We use the mean value theorem to show that for all $\hbar$, for $0<t<T$ and $\xi\in(t,t+h)$,

$$\begin{aligned}\|U_\omega^\hbar(t+h)-U_\omega^\hbar(t)\|_{H^{s-2}} &= \|\partial_t U_\omega^\hbar(\xi)h\|_{H^{s-2}}\\ &\le h \max_{0\le t\le T}\|\partial_t U_\omega^\hbar(t)\|_{H^{s-2}}\to 0 \quad \text{as} \quad h\to 0,\end{aligned}\tag{3.86}$$

which denotes the sequence $\{U_\omega^\hbar\}_\hbar$ is equicontinuous. The Arzela-Ascoli theorem implies that there exists $U_\omega\in L^\infty([0,T];H^s(\mathbb{R}^2))\cap \mathrm{Lip}([0,T];H^{s-2}(\mathbb{R}^2))$ such that

$$\max_{0\le t\le T}\|U_\omega^\hbar(t)-U_\omega(t)\|_{H^{s-2}}\to 0\tag{3.87}$$

as $\hbar$ tends to zero. Therefore,

$$\partial_t U_\omega\in L^\infty([0,T];H^{s-2}(\mathbb{R}^2)),\tag{3.88}$$

and

$$U_\omega\in C([0,T];H^s(\mathbb{R}^2))\cap C^1([0,T];H^{s-2}(\mathbb{R}^2)).\tag{3.89}$$

The Sobolev embedding theorem shows that

$$C([0,T];H^s(\mathbb{R}^2))\cap C^1([0,T];H^{s-2}(\mathbb{R}^2))\hookrightarrow C^1([0,T];C^2(\mathbb{R}^2))\tag{3.90}$$

if $s>5$. We deduce from the interpolation inequality that

$$U_\omega^\hbar\to U_\omega \quad \text{in} \quad C([0,T];H^{s-\delta}(\mathbb{R}^2)),\tag{3.91}$$

where $0<\delta<2$. Therefore,

$$U_\omega^\hbar \quad \text{converges strongly in} \quad C([0,T];H^{s-\delta}(\mathbb{R}^2)) \quad \text{to a function} \quad U_\omega.\tag{3.92}$$

Furthermore, from the equation itself, we also have

$$U_\omega^\hbar \quad \text{converges strongly in} \quad C^1([0,T];H^{s-2-\delta}(\mathbb{R}^2)) \quad \text{to a function} \quad U_\omega.\tag{3.93}$$

Since $U^{\hbar}_{\omega,0}(x)$ converges strongly to $U_{\omega,0}(x)$ in $H^s(\mathbb{R}^2)$, this limiting system has the initial data $U_\omega(0,x) = U_{\omega,0}(x)$. In particular, we note that $\mathcal{L}(U^{\hbar}_{\omega})$ is uniformly bounded in $H^{s-2}(\mathbb{R}^2)$, so the perturbation term $\mathcal{L}(U^{\hbar}_{\omega})$ tends to $\mathcal{L}_2(U_\omega)$ as $\hbar$ tends to zero. The uniqueness proof of this system is like that in Subsection 3.2.2. Hence, the whole sequence converges to U_ω. □

Remark 3.7. *The strong convergence of $U^{\hbar}_{\omega,0}$ to $U_{\omega,0}$ implies that $U^{\hbar}_0$ converges strongly to U_0. For the same reason, the result of Proposition 3.6 reveals the fact that $U^{\hbar}$ converges strongly to U.*

The relation of equivalence between (3.84) and (1.7) leads us to have the following convergent result, link T to the existence time of a smooth solution of (1.7), and ensure the strong convergence of $\psi^{\hbar}$ to a classical solution of the compressible rotational Euler equation (1.7).

Theorem 3.8. *Assume that (ρ, μ_ω) is a solution of the compressible rotational Euler equation* (1.7) *for $0 \le t \le T$ and belongs to $C([0,T]; H^s(\mathbb{R}^2))$, $s > 3$, with initial condition*

$$\begin{aligned} \rho_0(x) &= \rho(0,x) = |A_0(x)|^2 , \\ \mu_{\omega,0}(x) &= \mu_\omega(0,x) = |A_0(x)|^2 \left(\frac{1}{m}\nabla S_0(x) - \omega x^{\perp}\right). \end{aligned}$$

Then there exists a critical value of $\hbar$, $\hbar_c$ depending on T, such that under the hypotheses

(1) $\frac{1}{m}\nabla V(x) - \omega^2 x \in H^s(\mathbb{R}^2)$,

(2) *$A^{\hbar}_0(x)$ converges strongly to $A_0(x)$ in $H^s(\mathbb{R}^2)$ as $\hbar$ tends to zero,*

(3) $(\rho_0, \mu_{\omega,0}) \in L^\infty([0,T]; H^s(\mathbb{R}^2))$,

(4) $0 < \hbar < \hbar_c$,

the initial value problem of (1.1) *and* (3.2) *has a unique classical solution $\psi^{\hbar}$ of the form* (3.1), *where $A^{\hbar}$ and $\nabla S^{\hbar}$ are bounded in $L^\infty([0,T]; H^s(\mathbb{R}^2))$ uniformly in $\hbar$, and $\frac{1}{m}\nabla S^{\hbar} - \omega x^{\perp}$ is bounded in $L^\infty([0,T]; H^1(\mathbb{R}^2))$ uniformly in $\hbar$, on $[0,T]$. Moreover, $(\rho^{\hbar}, \mu^{\hbar}_\omega)$ converges strongly to the solution (ρ, μ_ω) of* (1.7) *in $C([0,T]; H^{s-2}(\mathbb{R}^2))$ as $\hbar$ tends to zero.*

PROOF. Assume that there exists a solution (ρ, μ_ω) in $L^\infty([0,T]; H^s(\mathbb{R}^2))$ of (1.7) on a time interval $[0,T]$ with $s > 3$ for the initial data

$$\rho_0(x) = |A_0(x)|^2 = \left|\lim_{\hbar\to 0} A_0^\hbar(x)\right|^2,$$

$$\mu_{\omega,0}(x) = |A_0(x)|^2 \left(\frac{1}{m}\nabla S_0(x) - \omega x^\perp\right) = \left|\lim_{\hbar\to 0} A_0^\hbar(x)\right|^2 \left(\lim_{\hbar\to 0}\frac{1}{m}\nabla S_0^\hbar(x) - \omega x^\perp\right), \tag{3.94}$$

satisfying $\|\rho_0(\cdot)\|_{H^s} < C$, $\|\mu_{\omega,0}(\cdot)\|_{H^s} < C$. It makes sense since $\|U_{\omega,0}^\hbar(\cdot)\|_{H^s} < C_1$, and $U_{\omega,0}^\hbar(x)$ converges strongly to $U_{\omega,0}(x)$ in $H^s(\mathbb{R}^2)$ as $\hbar$ tends to zero.

The existence time T of solutions of (1.7) coincides with that in Proposition 3.2. There will be no confliction. Assume that the limiting system (3.84) admits a solution on a maximal time interval $[0,T^*]$. Let us prove that $T^* > T$. If $T^* \le T$, then ρ and μ_ω are in $L^\infty([0,T^*]; H^s(\mathbb{R}^2))$, so $u_\omega \in L^\infty([0,T^*]; H^s(\mathbb{R}^2))$. By using (3.9c) and (3.9d), we get that a and b are in $L^\infty([0,T^*]; H^{s-1}(\mathbb{R}^2))$, which is impossible since T^* is set to be the maximal time of existence. Hence, $T^* > T$.

Along the lines of the proof of Proposition 3.6, we consider the difference of (3.12) and (3.84). Set $H^\hbar = U_\omega^\hbar - U_\omega$. Then

$$\partial_t H^\hbar + M_1\left(H^\hbar + U_\omega\right)\partial_{x_1} H^\hbar + M_2\left(H^\hbar + U_\omega\right)\partial_{x_2} H^\hbar = \mathcal{L}\left(H^\hbar\right) + F^\hbar, \tag{3.95a}$$

where

$$\begin{aligned} F^\hbar = {} & \left(\mathcal{L} - \mathcal{L}_2\right)(U_\omega) - \left[M_1\left(H^\hbar + U_\omega\right) - M_1(U_\omega)\right]\partial_{x_1} U_\omega \\ & - \left[M_2\left(H^\hbar + U_\omega\right) - M_2(U_\omega)\right]\partial_{x_2} U_\omega. \end{aligned} \tag{3.95b}$$

Since M_0 is positive definite for all $(H^\hbar + U_\omega)$, (3.95a) becomes

$$M_0\partial_t H^\hbar + \widetilde{M_1}\left(H^\hbar + U_\omega\right)\partial_{x_1} H^\hbar + \widetilde{M_2}\left(H^\hbar + U_\omega\right)\partial_{x_2} H^\hbar = \widetilde{\mathcal{L}}\left(H^\hbar\right) + M_0 F^\hbar, \tag{3.96}$$

where $\widetilde{M_1} = M_0 M_1$, $\widetilde{M_2} = M_0 M_2$, and $\widetilde{\mathcal{L}} = M_0\mathcal{L}$. Here the matrices $\widetilde{M_1}\left(H^\hbar + U_\omega\right)$ and $\widetilde{M_2}\left(H^\hbar + U_\omega\right)$ are symmetric. The energy associated with (3.96) is

$$E(t) = \left(M_0 H^\hbar, H^\hbar\right) = \iint \left(H^\hbar\right)^T M_0 H^\hbar dx_1 dx_2, \tag{3.97}$$

and the Friedrich energy equality is written as

$$\frac{d}{dt}E(t) = \left(\Gamma^{\hbar} H^{\hbar}, H^{\hbar}\right) + 2\left(\widetilde{\mathcal{L}}\left(H^{\hbar}\right) + M_0 F^{\hbar}, H^{\hbar}\right), \tag{3.98}$$

where $\Gamma^{\hbar} = \partial_t M_0 + \partial_{x_1}\widetilde{M_1} + \partial_{x_2}\widetilde{M_2}$. Since $\widetilde{\mathcal{L}}$ is antisymmetric, we have

$$(\widetilde{\mathcal{L}}(H^{\hbar}), H^{\hbar}) = 0. \tag{3.99}$$

Applying Cauchy-Schwarz's inequality and Young's inequality leads to

$$\left(\left(\widetilde{\mathcal{L}} - \widetilde{\mathcal{L}_2}\right)(U_\omega), H^{\hbar}\right) \le \hbar C + \hbar C \left\|H^{\hbar}\right\|_{L^2}^2, \tag{3.100}$$

and for $i = 1, 2$,

$$\left(-\left[\widetilde{M}_i(H^{\hbar} + U_\omega) - \widetilde{M}_i(U_\omega)\right]\partial_{x_i}U_\omega, H^{\hbar}\right) \le C\left\|H^{\hbar}\right\|_{L^2}^2. \tag{3.101}$$

Hence,

$$\begin{aligned}\frac{d}{dt}E(t) &\le (\|\Gamma\|_{L^\infty} + \hbar C + C)\left\|H^{\hbar}\right\|_{L^2}^2 + \hbar C \\ &\le k(\|\Gamma\|_{L^\infty} + \hbar C + C)\left(M_0 H^{\hbar}, H^{\hbar}\right) + \hbar C\end{aligned} \tag{3.102}$$

for some $k > 1$. By applying Gronwall's inequality and the strict positive of M_0, we deduce that for $t \in [0, T]$,

$$\begin{aligned}\left\|H^{\hbar}\right\|_{L^2}^2 &\le \exp[k(\|\Gamma\|_{L^\infty} + \hbar C + C)t]\left[\left\|U_{\omega,0}^{\hbar}(x) - U_{\omega,0}(x)\right\|_{L^2} + \frac{\hbar C t}{\|M_0\|_{L^\infty}}\right] \\ &= C(\hbar) \to \exp[k(\|\Gamma\|_{L^\infty} + C)t]\cdot(0+0) = 0\end{aligned} \tag{3.103}$$

as $\hbar$ tends to zero. We complete the proof. □

These results indicate that the regularity (3.59) of solutions of the quasilinear hyperbolic system (3.12) controls that of solutions of the quantum hydrodynamic equations of the rotating nonlinear Schrödinger equation (1.1).

CHAPTER 4

WKB Expansion

We must be content with approximate solutions of the system (3.12) obtained by perturbation expansion :

$$U_\omega^\hbar = U_\omega^{(0)} + \hbar U^{(1)} + \hbar^2 U^{(2)} + \cdots + \hbar^N U^{(N)} + \cdots, \tag{4.1}$$

where $U_\omega^{(0)} = U^{(0)} - \omega\,(0, 0, -x_2, x_1)^T$. We write $M_1(U_\omega^\hbar)$ as the Taylor series expansion around $U_\omega^{(0)}$

$$\begin{aligned} M_1(U_\omega^\hbar) &= M_1\left(U_\omega^{(0)} + \hbar U^{(1)} + \hbar^2 U^{(2)} + \cdots + \hbar^N U^{(N)} + \cdots\right) \\ &= M_1(U_\omega^{(0)}) + DM_1(U_\omega^{(0)})\left(\hbar U^{(1)} + \hbar^2 U^{(2)} + \cdots\right) \\ &\quad + \frac{D^2 M_1(U_\omega^{(0)})}{2!}\left(\hbar U^{(1)} + \hbar^2 U^{(2)} + \cdots\right)^2 + \frac{D^3 M_1(U_\omega^{(0)})}{3!}\left(\hbar U^{(1)} + \hbar^2 U^{(2)} + \cdots\right)^3 \\ &\quad + \cdots + \frac{D^N M_1(U_\omega^{(0)})}{N!}\left(\hbar U^{(1)} + \hbar^2 U^{(2)} + \cdots\right)^N + \cdots. \end{aligned} \tag{4.2}$$

Similarly, we do the same to $M_2(U_\omega^\hbar)$. We present the hierarchy by the order of $\hbar$ as follows :

$$\partial_t U_\omega^{(0)} + M_1(U_\omega^{(0)})\partial_{x_1} U_\omega^{(0)} + M_2(U_\omega^{(0)})\partial_{x_2} U_\omega^{(0)} + G = \mathcal{L}_2(U_\omega^{(0)}), \tag{4.3a}$$

$$\begin{aligned} &\partial_t U^{(1)} + M_1(U_\omega^{(0)})\partial_{x_1} U^{(1)} + M_2(U_\omega^{(0)})\partial_{x_2} U^{(1)} \\ &+ DM_1(U_\omega^{(0)})U^{(1)}\partial_{x_1} U_\omega^{(0)} + DM_2(U_\omega^{(0)})U^{(1)}\partial_{x_2} U_\omega^{(0)} = \mathcal{L}_1(U_\omega^{(0)}) + \mathcal{L}_2(U^{(1)}), \end{aligned} \tag{4.3b}$$

$$\partial_t U^{(2)} + M_1(U_\omega^{(0)})\partial_{x_1} U^{(2)} + M_2(U_\omega^{(0)})\partial_{x_2} U^{(2)}$$

$$+DM_1(U_\omega^{(0)})\left[U^{(1)}\partial_{x_1}U^{(1)} + U^{(2)}\partial_{x_1}U_\omega^{(0)}\right] + DM_2(U_\omega^{(0)})\left[U^{(1)}\partial_{x_2}U^{(1)} + U^{(2)}\partial_{x_2}U_\omega^{(0)}\right]$$

$$+\frac{D^2M_1(U_\omega^{(0)})}{2!}\left[U^{(1)}\right]^2\partial_{x_1}U_\omega^{(0)} + \frac{D^2M_2(U_\omega^{(0)})}{2!}\left[U^{(1)}\right]^2\partial_{x_2}U_\omega^{(0)} = \mathcal{L}_1(U^{(1)}) + \mathcal{L}_2(U^{(2)}), \tag{4.3c}$$

$$\vdots$$

We have the general formula

$$\begin{aligned}
\partial_t U^{(N)} &+ \sum_{j=1}^{2} \sum_{\substack{k=0 \\ 0\le n_i\le k,\quad n_i\in\mathbb{N},\quad i=1,\cdots,N \\ n_1+n_2+\cdots+n_N=k \\ n_1\cdot 1+n_2\cdot 2+\cdots+n_N\cdot N+n_{x_j}=N \\ 0<n_{x_j}\le N,\quad n_{x_j}\in\mathbb{N}}}^{N} \frac{D^k M_j(U_\omega^{(0)})}{k!} \\
&\cdot \left[U^{(1)}\right]^{n_1}\left[U^{(2)}\right]^{n_2}\cdot\cdots\cdot\left[U^{(N)}\right]^{n_N}\partial_{x_j}U^{(n_{x_j})} \\
&+ \sum_{j=1}^{2} \sum_{\substack{k=0 \\ 0\le n_i\le k,\quad n_i\in\mathbb{N},\quad i=1,\cdots,N \\ n_1+n_2+\cdots+n_N=k \\ n_1\cdot 1+n_2\cdot 2+\cdots+n_N\cdot N=N}}^{N} \frac{D^k M_j(U_\omega^{(0)})}{k!} \\
&\cdot \left[U^{(1)}\right]^{n_1}\left[U^{(2)}\right]^{n_2}\cdot\cdots\cdot\left[U^{(N)}\right]^{n_N}\partial_{x_j}U_\omega^{(0)} \\
&= \mathcal{L}_1(U^{(N-1)}) + \mathcal{L}_2(U^{(N)}), \quad N=2,3,\cdots
\end{aligned} \tag{4.4}$$

where $\mathcal{L} = \hbar\mathcal{L}_1 + \mathcal{L}_2$, $\mathcal{L}_2$ is as that in (3.85), and

$$\mathcal{L}_1 = \begin{pmatrix} 0 & -\frac{1}{2m}\Delta & 0 & 0 \\ \frac{1}{2m}\Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.5}$$

In the next discussion, we shall observe the nature of the solution represented by each approximation.

Zeroth order approximation. (4.3a) is the limiting system of (3.12). We have that $U_\omega^\hbar$ is bounded in $C([0,T];H^s(\mathbb{R}^2))\cap C^1([0,T];H^{s-2}(\mathbb{R}^2))$, and $U_\omega^\hbar$ converges uniformly in $L^\infty([0.T];H^{s-2}(\mathbb{R}^2))$ to $U_\omega^{(0)}$. This implies that both $A^\hbar$ and $u_\omega^\hbar$ are bounded in $C([0,T];H^s(\mathbb{R}^2))\cap C^1([0,T];H^{s-2}(\mathbb{R}^2))$. By the Arzela-Ascoli theorem, there exists a subsequence of $\left\{(A^\hbar,u_\omega^\hbar)\right\}_\hbar$ such that $(A^\hbar,u_\omega^\hbar)$ converges uniformly in $L^\infty([0,T];H^{s-2}(\mathbb{R}^2))$ to $(A^{(0)},u_\omega^{(0)})$ which is a solution of

$$\partial_t A^{(0)}+\left(u_\omega^{(0)}\cdot\nabla\right)A^{(0)}+\frac{1}{2}A^{(0)}\left(\nabla\cdot u_\omega^{(0)}\right)=0, \tag{4.6a}$$

$$\partial_t u_\omega^{(0)}+\left(u_\omega^{(0)}\cdot\nabla\right)u_\omega^{(0)}+2\omega u_\omega^{(0)\perp}-\omega^2 x+\frac{g}{m}\nabla|A^{(0)}|^2+\frac{1}{m}\nabla V=0, \tag{4.6b}$$

where $u_\omega^{(0)\perp}=(-(u_2^{(0)}-\omega x_1),u_1^{(0)}+\omega x_2)$, with initial data

$$A^{(0)}(0,x)=\lim_{\hbar\to 0}A_0^\hbar(x),\quad u_\omega^{(0)}(0,x)=\frac{1}{m}\nabla S_0(x)-\omega x^\perp. \tag{4.7}$$

This system admits a unique solution. Therefore, all sequence $(A^\hbar,u_\omega^\hbar)\to(A^{(0)},u_\omega^{(0)})$ in $C([0,T];H^{s-2}(\mathbb{R}^2))$. Moreover, it follows from the interpolation theory that $(A^\hbar,u_\omega^\hbar)\to(A^{(0)},u_\omega^{(0)})$ in $C([0,T];H^{s-\theta}(\mathbb{R}^2))$ for $0<\theta<2$.

First order approximation. Let

$$\widetilde{U_1^\hbar}=\frac{U_\omega^\hbar-U_\omega^{(0)}}{\hbar}. \tag{4.8}$$

Then $\widetilde{U_1^\hbar}$ satisfies

$$\begin{aligned}&\partial_t\widetilde{U_1^\hbar}+M_1(U_\omega^\hbar)\partial_{x_1}\widetilde{U_1^\hbar}+M_2(U_\omega^\hbar)\partial_{x_2}\widetilde{U_1^\hbar}\\&+M_1(\widetilde{U_1^\hbar})\partial_{x_1}U_\omega^{(0)}+M_2(\widetilde{U_1^\hbar})\partial_{x_2}U_\omega^{(0)}=\mathcal{L}_1(U_\omega^{(0)})+\mathcal{L}(\widetilde{U_1^\hbar}).\end{aligned} \tag{4.9}$$

We want to get the convergence of the sequence $\left\{\widetilde{U_1^\hbar}\right\}_\hbar$ to $U^{(1)}$ which satisfies (4.3b). In particular, $U_\omega^\hbar$ converges uniformly in $C([0,T];H^{s-2}(\mathbb{R}^2))$ to $U_\omega^{(0)}$, which hints that we will estimate the H^{s-2}-norm of $\widetilde{U_1^\hbar}$. Since $U_\omega^{(0)}$ is bounded in $C([0,T];H^s(\mathbb{R}^2))\cap$

$C^1([0,T];H^{s-2}(\mathbb{R}^2))$, if $\widetilde{U_1^\hbar}(0,x)$ is bounded in $H^{s-2}(\mathbb{R}^2)$, then the energy estimate implies that

$$\widetilde{U_1^\hbar} \quad \text{is bounded in} \quad C([0,T];H^{s-2}(\mathbb{R}^2)) \cap C^1([0,T];H^{s-4}(\mathbb{R}^2)). \tag{4.10}$$

The application of the Arzela-Ascoli theorem and the interpolation theory shows that for $0<\alpha<2$,

$$\widetilde{U_1^\hbar} \to U^{(1)} \quad \text{in} \quad C([0,T];H^{s-2-\alpha}(\mathbb{R}^2)) \tag{4.11}$$

by passing to a subsequence in $\hbar$. Taking the limit of (4.9) and noting that for $i=1,2$,

$$\lim_{\hbar\to 0}\frac{M_i(\widetilde{U_1^\hbar})}{\widetilde{U_1^\hbar}} = \lim_{\hbar\to 0}\frac{\dfrac{M_i(U_\omega^\hbar)-M_i(U_\omega^{(0)})}{\hbar}}{\dfrac{U_\omega^\hbar-U_\omega^{(0)}}{\hbar}} = DM_i(U_\omega^{(0)}), \tag{4.12}$$

we deduce that $U^{(1)}$ is the unique solution of the linear equation (4.3b) with initial condition

$$U^{(1)}(0,x) = \lim_{\hbar\to 0}\frac{U_\omega^\hbar(0,x)-U_\omega^{(0)}(0,x)}{\hbar}. \tag{4.13}$$

The uniqueness makes us pass to the limit without the extraction of subsequence, so the whole sequence converges to $U^{(1)}$.

Higher order approximation. Assume that we have already obtained an asymptotic expansion up to order N

$$U_\omega^\hbar = U_\omega^{(0)} + \hbar U^{(1)} + \cdots + \hbar^N U^{(N)} + O(\hbar^N), \tag{4.14}$$

where the function $U_\omega^{(0)}$ belongs to $C([0,T];H^s(\mathbb{R}^2))\cap C^1([0,T];H^{s-2}(\mathbb{R}^2))$, and the function $U^{(j)}$ belongs to $C([0,T];H^{s-2j}(\mathbb{R}^2))\cap C^1([0,T];H^{s-2j-2}(\mathbb{R}^2))$, $j=1,2,\cdots,N$. Let

$$U_N^\hbar = \sum_{j=0}^{N}\hbar^j U^{(j)} - \omega\,(0,0,-x_2,x_1) \quad \text{and} \quad \widetilde{U_{N+1}^\hbar} = \frac{U_\omega^\hbar - U_N^\hbar}{\hbar^{N+1}}. \tag{4.15}$$

We write the equation for $\widetilde{U_{N+1}^\hbar}$

$$\begin{aligned}&\partial_t\widetilde{U_{N+1}^\hbar} + M_1(U_\omega^\hbar)\partial_{x_1}\widetilde{U_{N+1}^\hbar} + M_2(U_\omega^\hbar)\partial_{x_2}\widetilde{U_{N+1}^\hbar}\\ &\quad + M_1(\widetilde{U_{N+1}^\hbar})\partial_{x_1}U_N^\hbar + M_2(\widetilde{U_{N+1}^\hbar})\partial_{x_2}U_N^\hbar = \mathcal{L}_1(U^{(N)}) + \mathcal{L}(\widetilde{U_{N+1}^\hbar}) + B_N^\hbar,\end{aligned} \tag{4.16}$$

where

$$B_N^\hbar = \sum_{j=1}^{2} \left\{ \frac{1}{\hbar^{N+1}} \left[-M_j(U_N^\hbar)\partial_{x_j} U_N^\hbar + M_j(U_\omega^{(0)})\partial_{x_j} U_N^\hbar \right] + \sum_{k=1}^{N} \frac{\hbar^k}{\hbar^{N+1}} J_{j,k,N} \right\},$$

$$J_{j,k,N} = \sum_{\substack{m=1 \\ n_i \in \mathbb{N}, \quad i=1\cdots k \\ n_1+n_2+\cdots+n_k=m \\ n_1\cdot 1+n_2\cdot 2+\cdots+n_k\cdot k=k}}^{k} \frac{D^m M_j(U_\omega^{(0)})}{m!} [U^{(1)}]^{n_1} \cdots [U^{(k)}]^{n_k} \partial_{x_j} U_{N-k}^\hbar, \tag{4.17}$$

and $B_N^\hbar$ is a function which depends on $U_N^\hbar$. Observing the solution $U_\omega^\hbar$ of (3.12) and assuming $O(\hbar^N) = 0$, we have

$$\begin{aligned} &\partial_t \widetilde{U_{N+1}^\hbar} + M_1(U_\omega^\hbar)\partial_{x_1} \widetilde{U_{N+1}^\hbar} + M_2(U_\omega^\hbar)\partial_{x_2} \widetilde{U_{N+1}^\hbar} \\ &\quad + M_1(\widetilde{U_{N+1}^\hbar})\partial_{x_1} U_N^\hbar + M_2(\widetilde{U_{N+1}^\hbar})\partial_{x_2} U_N^\hbar = \mathcal{L}_1(U^{(N)}) + \mathcal{L}(\widetilde{U_{N+1}^\hbar}). \end{aligned} \tag{4.18}$$

Comparing (4.18) with (4.16), we conjecture that $B_N^\hbar$ is small enough to ignore. It can be shown that $B_N^\hbar$ is bounded in $L^\infty\left([0,T]; H^{s-2(N+1)}(\mathbb{R}^2)\right)$ uniformly in $\hbar$ (see [**8**]). By using again the energy estimate, we obtain

$$\widetilde{U_{N+1}^\hbar} \quad \text{is bounded in} \quad C([0,T]; H^{s-2(N+1)}(\mathbb{R}^2)) \cap C^1([0,T]; H^{s-2(N+1)-2}(\mathbb{R}^2)) \tag{4.19}$$

as soon as $\widetilde{U_{N+1}^\hbar}(0,x)$ is bounded in $H^{s-2(N+1)}(\mathbb{R}^2)$. Therefore, there exists a function $U^{(N+1)}$ such that for $0 < \alpha < 2$,

$$\widetilde{U_{N+1}^\hbar} \to U^{(N+1)} \quad \text{in} \quad C([0,T]; H^{s-2(N+1)-\alpha}(\mathbb{R}^2)) \tag{4.20}$$

without passing to a subsequence in $\hbar$ due to the uniqueness, and $U^{(N+1)}$ satisfies the general formula (4.4) for $N+1$ with initial data

$$U^{(N+1)}(0,x) = \lim_{\hbar \to 0} \frac{U_\omega^\hbar(0,x) - \left(U_\omega^{(0)}(0,x) + \cdots + \hbar^N U^{(N)}(0,x) \right)}{\hbar^{N+1}}. \tag{4.21}$$

This result is connected with $\psi^\hbar$ so that we have approximate solutions of the rotating nonlinear Schrödinger equation (1.1).

Theorem 4.1 (WKB expansion). *Suppose that the assumptions of Theorem 3.4 hold, and the initial amplitude $A_0^\hbar(x)$ admits the following expansion :*

$$A_0^\hbar(x) = \sum_{k=0}^{N} \hbar^k A_0^{(k)}(x) + \hbar^N R_N^{in}(x,\hbar),$$

where

$$\lim_{\hbar\to 0} \left\| R_N^{in}(x,\hbar) \right\|_{H^\sigma} = 0$$

for $N \in \mathbb{N}$ and $\sigma > 2N + 2 + \frac{2}{2}$, then the solution of the rotating nonlinear Schrödinger equation can be represented as

$$\psi^\hbar(t,x) = A^\hbar(t,x) \exp\left(\frac{i}{\hbar} S^\hbar(t,x)\right)$$

$$= \sum_{k=0}^{N} \hbar^k A^{(k)}(t,x) \exp\left(\frac{i}{\hbar} S(t,x)\right) + \hbar^N R_N(t,x,\hbar),$$

where

$$\lim_{\hbar\to 0} \| R_N(t,x,\hbar) \|_{C([0,T];H^{\sigma-2N-\epsilon}(\mathbb{R}^2))} = 0, \quad \forall \epsilon > 0.$$

CHAPTER 5

The Local Existence and Uniqueness

When we expound and prove the semiclassical limit, we must assure the existence of the limiting system. In this chapter, we sketch the existence and uniqueness proof of solutions of the compressible rotational Euler equation (1.7) by being based on the equivalent quasilinear hyperbolic system (3.84). The proof follows the same method mentioned in Proposition 3.2. We derive a short time existence theorem by constructing a sequence of approximate solutions and using the compactness argument. The procedure is decomposed into seven steps.

Step 1. *Conversion to the quasilinear symmetric hyperbolic system.* Constructing a symmetrizer M_0 (3.15) which is symmetric and positive definite if $g > 0$ for all U_ω, we will transform (3.84) into the quasilinear symmetric hyperbolic system

$$\begin{aligned} &M_0(U_\omega)\partial_t U_\omega + \widetilde{M_1}(U_\omega)\partial_{x_1} U_\omega + \widetilde{M_2}(U_\omega)\partial_{x_2} U_\omega + \widetilde{G} = \widetilde{\mathcal{L}}_2(U_\omega), \\ &U_\omega(0, x) = U_{\omega,0}(x), \end{aligned} \tag{5.1}$$

where $\widetilde{M_1} = M_0 M_1$, $\widetilde{M_2} = M_0 M_2$, $\widetilde{G} = M_0 G$, and $\widetilde{\mathcal{L}}_2 = M_0 \mathcal{L}_2$. Here $\widetilde{M_1}$ and $\widetilde{M_2}$ are symmetric, and $\widetilde{\mathcal{L}}_2$ is antisymmetric.

Step 2. *Construction of approximate solutions U_ω^q.* Our strategy will be to obtain a solution of (5.1) as a limit of solutions $U_\omega^{q+1}(t, x)$ of the linear equation

$$\begin{aligned} &M_0(U_\omega^q)\partial_t U_\omega^{q+1} + \widetilde{M_1}(U_\omega^q)\partial_{x_1} U_\omega^{q+1} + \widetilde{M_2}(U_\omega^q)\partial_{x_2} U_\omega^{q+1} + \widetilde{G} = \widetilde{\mathcal{L}}_2(U_\omega^{q+1}), \\ &U_\omega^{q+1}(0, x) = U_{\omega,0}(x), \end{aligned} \tag{5.2}$$

where $q = 0, 1, 2, 3, \cdots$ and $U_\omega^0(t, x) = U_{\omega,0}(x)$ denotes the initial data.

Step 3. *A priori estimate.* Our task will be to show that approximate solutions U_ω^q

exist for t in an interval independent of $q > 0$ and have a limit solving (5.1) as q tends to ∞. To do this, we estimate the H^s-norm of solutions of (5.2). Let $s > 2 + \frac{2}{2}$. Assume $U_\omega^{q+1} \in C^2([0,T]; C^{s+1}(\mathbb{R}^2))$. Following the process in Subsection 3.2.1 and summarizing the energy estimate associated with (5.2), we conculde that if $\frac{1}{m}\nabla V(x) - \omega^2 x \in H^s(\mathbb{R}^2)$ and $U_{\omega,0} \in H^s(\mathbb{R}^2)$, then the iteration scheme defined by (5.2) is well-defined, and approximate solutions U_ω^q satisfy a priori estimate on the space derivatives of the type

$$\|U_\omega^q(t)\|_{H^s} \leq C, \quad t \in [0,T], \tag{5.3}$$

which denotes

$$U_\omega^q \in L^\infty([0,T]; H^s(\mathbb{R}^2)). \tag{5.4}$$

In addition, it follows that every component, namely, a^q, b^q, $(u_1^q + \omega x_2)$, and $(u_2^q - \omega x_1)$, belongs to $L^\infty([0,T]; H^s(\mathbb{R}^2))$ and then from (5.2) that for $t \in [0,T]$,

$$\| \partial_t U_\omega^q(t) \|_{H^{s-1}} \leq C, \quad t \in [0,T], \tag{5.5}$$

which denotes

$$\partial_t U_\omega^q \in L^\infty([0,T]; H^{s-1}(\mathbb{R}^2)). \tag{5.6}$$

Note that if we write $U_\omega = U - \omega(0,0,-x_2,x_1)^T$, then the constructed approximate solutions $U_\omega^q = U^q - \omega(0,0,-x_2,x_1)^T$ satisfy

$$\begin{aligned} &U_\omega^q \in L^\infty([0,T]; H^1(\mathbb{R}^2)), \quad U^q \in L^\infty([0,T]; H^s(\mathbb{R}^2)), \\ &\partial_t U^q \in L^\infty([0,T]; H^{s-1}(\mathbb{R}^2)), \end{aligned} \tag{5.7}$$

as long as $\frac{1}{m}\nabla V(x) - \omega^2 x \in H^1(\mathbb{R}^2)$, $\frac{1}{m}\nabla V(x) \in H^s(\mathbb{R}^2)$, $U_{\omega,0} \in H^1(\mathbb{R}^2)$, and $U_0 \in H^s(\mathbb{R}^2)$.

Step 4. *A compactness result.* Show that the sequence $\{U_\omega^q\}_{q=0}^\infty$ is a relatively compact set in $C([0,T]; H^{s-1}(\mathbb{R}^2))$. We deduce from Step 3 that

$$\{U_\omega^q\}_{q=0}^\infty \quad \text{is bounded in} \quad L^\infty([0,T]; H^s(\mathbb{R}^2)), \tag{5.8}$$

$$\{\partial_t U_\omega^q\}_{q=0}^\infty \quad \text{is bounded in} \quad L^\infty([0,T]; H^{s-1}(\mathbb{R}^2)). \tag{5.9}$$

It follows from the classical compactness argument that there exist a subsequence of $\{U_\omega^q\}_{q=0}^\infty$ and a function $U_\omega \in L^\infty([0,T];H^s(\mathbb{R}^2))$ such that

$$U_\omega^q \rightharpoonup U_\omega \quad \text{weakly} \quad * \quad \text{in} \quad L^\infty([0,T];H^s(\mathbb{R}^2)), \tag{5.10}$$

$$\partial_t U_\omega^q \rightharpoonup \partial_t U_\omega \quad \text{weakly} \quad * \quad \text{in} \quad L^\infty([0,T];H^{s-1}(\mathbb{R}^2)). \tag{5.11}$$

As discussed in Subsection 3.2.2, the same technique is displayed. We appeal to Rellich's lemma which states that $H^s(\mathbb{R}^2) \hookrightarrow H^{s-1}(\mathbb{R}^2)$ is a compact embedding and the generalized Arzela-Ascoli theorem which states that $\{U_\omega^q\}_{q=0}^\infty$ is a relatively compact set in $C([0,T];H^{s-1}(\mathbb{R}^2))$ if and only if

(1) $\{U_\omega^q(t)\}_{q=0}^\infty$ is a relatively compact set in $H^{s-1}(\mathbb{R}^2)$ for all $t > 0$,

(2) $\{U_\omega^q\}_{q=0}^\infty$ is equicontinuous in $C([0,T];H^{s-1}(\mathbb{R}^2))$.

In addition, the foregoing result also implies that $\{U^q\}_{q=0}^\infty$ is a relatively compact set in $C([0,T];H^{s-1}(\mathbb{R}^2))$. Moreover, by the interpolation theory, we have that for $0 < \sigma < 1$,

$$U_\omega^q \to U_\omega \quad \text{in} \quad C([0,T];H^{s-\sigma}(\mathbb{R}^2)). \tag{5.12}$$

The uniqueness of U_ω shows that the whole sequence U_ω^q converges to U_ω.

Step 5. *Passage to the limit for $q \to \infty$.* A priori estimate in Step 3 will allow to pass to the limit in (5.2). If we choose s sufficiently large, then the strong convergence of U_ω^q in $C([0,T];H^{s-\sigma}(\mathbb{R}^2))$ gives the convergent results :

$$\begin{aligned} &M_0(U_\omega^q)\partial_t U_\omega^{q+1} \to M_0(U_\omega)\partial_t U_\omega, \quad \widetilde{M}_1(U_\omega^q)\partial_{x_1} U_\omega^{q+1} \to \widetilde{M}_1(U_\omega)\partial_{x_1} U_\omega, \\ &\widetilde{M}_2(U_\omega^q)\partial_{x_2} U_\omega^{q+1} \to \widetilde{M}_2(U_\omega)\partial_{x_2} U_\omega, \quad \widetilde{\mathcal{L}}_2(U_\omega^{q+1}) \to \widetilde{\mathcal{L}}_2(U_\omega). \end{aligned} \tag{5.13}$$

Therefore, U_ω belonging to $C([0,T];H^s(\mathbb{R}^2)) \cap C^1([0,T];H^{s-1}(\mathbb{R}^2))$ is a solution of (3.84).

Step 6. *A classical solution.* The Sobolev embedding theorem implies that

$$U_\omega \in C^1([0,T] \times \mathbb{R}^2) \tag{5.14}$$

if $s > 3$.

Step 7. *Uniqueness.* We consider the difference of two solutions of (5.1) to perform the energy estimate. Let $(U_\omega)_1$ and $(U_\omega)_2$ be two solutions satisfying $(U_\omega)_1(0,x) = U_{\omega,0}(x)$ and $(U_\omega)_2(0,x) = U_{\omega,0}(x)$. Let $H = (U_\omega)_1 - (U_\omega)_2$; we get the equation

$$M_0 \partial_t H + \widetilde{M}_1[(U_\omega)_1]\partial_{x_1} H + \widetilde{M}_2[(U_\omega)_1]\partial_{x_2} H = \widetilde{\mathcal{L}}_2(H) + F, \tag{5.15a}$$

where

$$F = \left(\widetilde{M}_1[(U_\omega)_2] - \widetilde{M}_1[(U_\omega)_1]\right)\partial_{x_1}(U_\omega)_2 + \left(\widetilde{M}_2[(U_\omega)_2] - \widetilde{M}_2[(U_\omega)_1]\right)\partial_{x_2}(U_\omega)_2. \tag{5.15b}$$

The technique of the energy estimate associated with (5.15a) is the same as that in Proposition 3.2. Thus, $(U_\omega)_1 = (U_\omega)_2$ for all $t \in [0,T]$. The classical solution is unique.

We have established the following existence and uniqueness result.

Proposition 5.1. *Let $s > 3$ and the potential $V(x)$ satisfy*

$$\frac{1}{m}\nabla V(x) - \omega^2 x \in H^s(\mathbb{R}^2).$$

Assume that the initial data $U_{\omega,0} = (a_0, b_0, u_{1,0} + \omega x_2, u_{2,0} - \omega x_1)^T \in [H^s(\mathbb{R}^2)]^4$ satisfies the uniform bound

$$\| U_{\omega,0} \|_{H^s} < C_1.$$

Then there is a time interval $[0,T]$ with $T > 0$, so that the IVP for (3.84) has a unique classical solution

$$U_\omega = (a, b, u_1 + \omega x_2, u_2 - \omega x_1)^T \in C^1([0,T] \times \mathbb{R}^2).$$

Furthermore,

$$U_\omega \in C([0,T]; H^s(\mathbb{R}^2)) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^2)),$$

and T depends on the bound C_1.

Relying on the equivalent relation between the compressible rotational Euler equation (1.7) and the quasilinear hyperbolic system (3.84), we complete the local existence and uniqueness proof of the compressible rotational Euler equation (1.7).

Theorem 5.2. *Assume that (ρ, μ_ω) is a solution of the compressible rotational Euler equation (1.7) for $0 \leq t \leq T$, $s > 3$, with initial condition*

$$\rho_0(x) = \rho(0,x) = |A_0(x)|^2\,,$$
$$\mu_{\omega,0}(x) = \mu_\omega(0,x) = |A_0(x)|^2 \left(\frac{1}{m}\nabla S_0(x) - \omega x^\perp\right).$$

Then under the hypotheses

(1) $\frac{1}{m}\nabla V(x) - \omega^2 x \in H^s(\mathbb{R}^2)$,

(2) $(\rho_0, \mu_{\omega,0}) \in L^\infty([0,T]; H^s(\mathbb{R}^2))$,

the compressible rotational Euler equation (1.7) has a unique classical solution

$$(\rho, \mu_\omega) \in C^1([0,T] \times \mathbb{R}^2)$$

on $[0,T]$. Furthermore,

$$(\rho, \mu_\omega) \in C([0,T]; H^s(\mathbb{R}^2)) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^2)).$$

CHAPTER 6

Acoustic Wave

The expansion for $\rho^\hbar$ and $u_\omega^\hbar$ takes the form

$$\rho^\hbar = 1 + \hbar\rho^{(1)} + \hbar^2\rho^{(2)} + \hbar^3\rho^{(3)} + \cdots,$$

$$u_\omega^\hbar = \hbar u^{(1)} + \hbar^2 u^{(2)} + \hbar^3 u^{(3)} + \cdots. \tag{6.1}$$

Substituting (6.1) into equations (2.12) and (2.14) and considering the $O(\hbar)$ terms, we have

$$\partial_t\rho^{(1)} + \nabla\cdot u^{(1)} = 0, \tag{6.2}$$

$$\partial_t u^{(1)} + \frac{g}{m}\nabla\rho^{(1)} + 2\omega(u^{(1)})^\perp = \omega^2 x - \frac{1}{m}\nabla V. \tag{6.3}$$

We may abbreviate this system by using the matrix

$$A = \begin{pmatrix} 0 & \nabla\cdot \\ \frac{g}{m}\nabla & 2\omega J \end{pmatrix}, \tag{6.4}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{6.5}$$

Suppose that $X = \left(\rho^{(1)}, u^{(1)}\right)^T$ and X solves the initial problem for the system . Then the system may be written as

$$\partial_t X(t,x) + AX(t,x) = G(x), \quad X(0,x) = X_0, \tag{6.6}$$

where

$$G(x) = \begin{pmatrix} 0 \\ \omega^2 x - \frac{1}{m}\nabla V \end{pmatrix}. \tag{6.7}$$

Taking the Fourier transform of both the system and the initial condition with respect to the space variables, we reduce the problem to an ordinary differential equation in the time variable

$$\partial_t\widehat{X}(t,\xi) + \widehat{A}\widehat{X}(t,\xi) = \widehat{G}(\xi), \quad \widehat{X}(0,\xi) = \widehat{X_0}, \tag{6.8}$$

where

$$\widehat{A} = \begin{pmatrix} 0 & i\xi\cdot \\ \frac{g}{m}i\xi & 4\omega\pi\delta(\xi)J \end{pmatrix}, \tag{6.9}$$

and

$$\widehat{G}(\xi) = \begin{pmatrix} 0 \\ \omega^2\widehat{x}(\xi) - \frac{1}{m}i\xi\widehat{V} \end{pmatrix}. \tag{6.10}$$

If $\xi = (\xi_1, \xi_2)$, then $\widehat{x}(\xi)$ in $\widehat{G}(\xi)$ can be expressed as

$$\widehat{x}(\xi) = \left(2\pi i\delta'(\xi_1)2\pi\delta(\xi_2), 2\pi\delta(\xi_1)2\pi i\delta'(\xi_2)\right) = -i4\pi^2\delta(\xi_1)\delta(\xi_2)\left(\frac{1}{\xi_1}, \frac{1}{\xi_2}\right). \tag{6.11}$$

The matrix $\widehat{A}$ has three distinct eigenvalues $0, \pm i\sqrt{\frac{g}{m}|\xi|^2 + 16\omega^2\pi^2\delta(\xi)}$. The space of eigenfunctions associated to 0 coincides with the null-space of $\widehat{A}$

$$\mathrm{Ker}(\widehat{A}) = \left\{(\rho^{(1)}, u^{(1)})\middle| i\xi\cdot u^{(1)} = 0, \quad \frac{g}{m}i\xi\rho^{(1)} + 4\omega\pi\delta(\xi)Ju^{(1)} = 0\right\}. \tag{6.12}$$

For each fixed $\xi \in \mathbb{R}^2$, the solution of (6.8) is given by

$$\widehat{X}(t,\xi) = e^{-\widehat{A}t}\left(\widehat{X_0} + \int_{\mathbb{R}^+} e^{\widehat{A}s}ds\widehat{G}(\xi)\right). \tag{6.13}$$

Taking the inverse Fourier transform in the ξ variables leads to

$$X(t,x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\left(e^{-\widehat{A}t}\widehat{X_0} + \int_{\mathbb{R}^+} e^{-\widehat{A}(t-s)}ds\widehat{G}(\xi)\right)e^{ix\cdot\xi}d\xi, \tag{6.14}$$

which is an explicit solution of $\rho^{(1)}$ and $u^{(1)}$.

6.1. Dispersion Limit

An alternative approach to the semiclassical limit is discussed here. Let us consider $\frac{|\psi^\hbar|^2 - 1}{\hbar}$ which is served as the density fluctuation of the sound wave. In other words, $(\rho^\hbar, \mu^\hbar)$ is near the constant state $(1,0)$. Multiplying (2.19) by $\left(\psi^\hbar\right)^*$ and (2.20) by $\psi^\hbar$ leads to the conservation law

$$\partial_t\left(\frac{|\psi^\hbar|^2 - 1}{\hbar}\right) = -\nabla\cdot\left\{\frac{1}{m}\mathrm{Im}\left[\left(\psi^\hbar\right)^*\nabla\psi^\hbar\right]\right\} + \omega\left(x^\perp\cdot\nabla\right)\left(\frac{|\psi^\hbar|^2 - 1}{\hbar}\right). \tag{6.15}$$

To know the condition of $V(x)$, equation (1.1) can be recast as

$$i\hbar\partial_t\psi^\hbar = -\frac{\hbar^2}{2m}\left(\nabla - \frac{im\omega x^\perp}{\hbar}\right)^2\psi^\hbar + \hbar g\left(\frac{|\psi^\hbar|^2-1}{\hbar}\right)\psi^\hbar$$
$$+\left\{g + m\left[\frac{1}{m}V(x) - \frac{1}{2}(\omega x^\perp)^2\right]\right\}\psi^\hbar, \tag{6.16}$$

whose Hamiltonian reads

$$E(\psi^\hbar) = \int_{\mathbb{R}^2}\frac{\hbar^2}{2m}\left|\left(\nabla - \frac{im\omega x^\perp}{\hbar}\right)\psi^\hbar\right|^2 + \frac{g}{2}\left(\frac{|\psi^\hbar|^2-1}{\hbar}\right)^2$$
$$+\left\{g + m\left[\frac{1}{m}V(x) - \frac{1}{2}(\omega x^\perp)^2\right]\right\}|\psi^\hbar|^2 dx$$
$$= \int_{\mathbb{R}^2}\frac{\hbar^2}{2m}\left|\left(\nabla - \frac{im\omega x^\perp}{\hbar}\right)\psi^\hbar\right|^2 + \frac{\hbar^2 g}{2}\left\{\left(\frac{|\psi^\hbar|^2-1}{\hbar}\right) + \frac{g + m\left[\frac{1}{m}V(x) - \frac{1}{2}\omega^2|x|^2\right]}{\hbar g}\right\}^2$$
$$+\frac{1}{2g}\left\{g^2 - m^2\left[\frac{1}{m}V(x) - \frac{1}{2}\omega^2|x|^2\right]^2\right\}dx. \tag{6.17}$$

As long as $\frac{1}{m}V(x) - \frac{1}{2}\omega^2|x|^2 > 0$, the energy $E(\psi^\hbar)$ is positive definite, and the wave-function $\psi^\hbar$ satisfies the energy inequality

$$E(t) = E(\psi^\hbar) < C. \tag{6.18}$$

We substitute ∇_ω for the operator $\nabla - \frac{im\omega x^\perp}{\hbar}$ and refer to Thierry Cazenave [**19**] for defining a new space and having its properties.

Definition 6.1. *The space H^1_ω is defined as*

$$H^1_\omega(\mathbb{R}^2) = \left\{\varphi \in L^2(\mathbb{R}^2)\middle|\quad \nabla_\omega\varphi \in L^2(\mathbb{R}^2)\right\},$$

equipped with the norm

$$\|\varphi\|^2_{H^1_\omega} = \|\nabla_\omega\varphi\|^2_{L^2} + \|\varphi\|^2_{L^2}.$$

Lemma 6.2 (Lemma 9.1.2, Thierry Cazenave [**19**])**.** *The following properties hold :*

(i) $\quad H^1_\omega \hookrightarrow L^2$.

(ii) $L^2 \hookrightarrow (H^1_\omega)^*$.

(iii) $\||\varphi|\|_{H^1} \leq \|\varphi\|_{H^1_\omega}$.

The weak formulation of (6.16) is given by

$$i\left(\psi^\hbar(t_2,\cdot) - \psi^\hbar(t_1,\cdot), \phi\right) = \frac{\hbar}{2m}\int_{t_1}^{t_2} \left(\nabla_\omega \psi^\hbar, \nabla_\omega \phi\right) dt$$

$$+ \int_{t_1}^{t_2} \left(g\left(\frac{|\psi^\hbar|^2 - 1}{\hbar}\right)\psi^\hbar, \phi\right) dt + \int_{t_1}^{t_2} \left(\frac{1}{\hbar}\left[g + m\left(\frac{1}{m}V - \frac{1}{2}\omega^2|x|^2\right)\right]\psi^\hbar, \phi\right) dt \tag{6.19}$$

for all $\phi \in C_c^\infty(\mathbb{R}^2) \cap H^1_\omega(\mathbb{R}^2)$.

Let us now search for the convergence of $\psi^\hbar$. We use Lemma 6.2 to assist us in proceeding with our work.

Lemma 6.3. *Let $T > 0$. For all $0 < \hbar \ll 1$, the sequence $\{\psi^\hbar\}_\hbar$ is a relatively compact set in $C([0,T]; L^2(\mathbb{R}^2))$; that is, there exists $\psi \in C([0,T]; L^2(\mathbb{R}^2))$ such that*

$$\psi^\hbar \to \psi \quad \textit{strongly in} \quad C([0,T]; L^2(\mathbb{R}^2)).$$

Proof. Assume that the initial data $\psi_0^\hbar$ satisfies $|\psi_0^\hbar| = 1$ almost everywhere and $\psi_0^\hbar \to \psi_0$ strongly in $H^1_\omega(\mathbb{R}^2)$ as $\hbar \to 0$; hence, $|\psi_0| = 1$ almost everywhere. We deduce from the energy inequality (6.18) that

$$\left\{\nabla_\omega \psi^\hbar\right\}_\hbar \quad \text{is bounded in} \quad L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2)), \tag{6.20}$$

$$\left\{\frac{|\psi^\hbar|^2 - 1}{\hbar}\right\}_\hbar \quad \text{is bounded in} \quad L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2)); \tag{6.21}$$

therefore,

$$\left\{\psi^\hbar\right\}_\hbar \quad \text{is bounded in} \quad L^\infty(\mathbb{R}^+; H^1_\omega(\mathbb{R}^2)). \tag{6.22}$$

It can be observed from (1.1) or (6.16) that

$$\left\{\partial_t \psi^\hbar\right\}_\hbar \quad \text{is bounded in} \quad L^\infty(\mathbb{R}^+; H^{-1}_\omega(\mathbb{R}^2)). \tag{6.23}$$

The classical compactness argument shows that there exists a function ψ satisfying

$$\psi \in L^\infty(\mathbb{R}^+; H^1_\omega(\mathbb{R}^2)), \quad \partial_t \psi \in L^\infty(\mathbb{R}^+; H^{-1}_\omega(\mathbb{R}^2)), \tag{6.24}$$

such that

$$\psi^\hbar \rightharpoonup \psi \quad \text{weakly} \quad * \quad \text{in} \quad L^\infty(\mathbb{R}^+; H^1_\omega(\mathbb{R}^2)), \tag{6.25}$$

$$\partial_t\psi^\hbar \rightharpoonup \partial_t\psi \quad \text{weakly} \quad * \quad \text{in} \quad L^\infty(\mathbb{R}^+; H^{-1}_\omega(\mathbb{R}^2)). \tag{6.26}$$

The properties (i) and (ii) in Lemma 6.2 tell that we can apply the Lions-Aubin Lemma to both (6.22) and (6.23) so that $\left\{\psi^\hbar\right\}_\hbar$ is a relatively compact set in $C([0,T]; L^2(\mathbb{R}^2))$ for $T > 0$. It is worth pointing out, in passing, that

$$|\psi^\hbar|^2 \to 1 \quad \text{a.e. and strongly in} \quad L^2(\mathbb{R}^2) \tag{6.27}$$

according to (6.21). This also implies that $|\psi|^2 = 1$. □

Next, we discuss the convergence of $\dfrac{|\psi^\hbar|^2 - 1}{\hbar}$. According to (6.21), $\dfrac{|\psi^\hbar|^2 - 1}{\hbar}$ converges weakly $*$ to some function belonging to $L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2))$.

Lemma 6.4.

$$\frac{|\psi^\hbar|^2 - 1}{\hbar} \rightharpoonup -\int \nabla \cdot \left[\frac{1}{m} Im\left(\psi^*\nabla\psi\right)\right] d\tau$$

in the sense of distributions.

Proof. Let $\partial_\tau = \partial_t - \omega\left(x^\perp \cdot \nabla\right)$; the conservation law (6.15) can be recast as

$$\partial_\tau\left(\frac{|\psi^\hbar|^2 - 1}{\hbar}\right) = -\nabla \cdot \left\{\frac{1}{m}\mathrm{Im}\left[\left(\psi^\hbar\right)^* \nabla\psi^\hbar\right]\right\}. \tag{6.28}$$

Integrating (6.28) with respect to τ and using the initial condition $|\psi^\hbar_0| = 1$, we have

$$\frac{|\psi^\hbar|^2 - 1}{\hbar} = -\int \nabla \cdot \left\{\frac{1}{m}\mathrm{Im}\left[\left(\psi^\hbar\right)^* \nabla\psi^\hbar\right]\right\} d\tau \tag{6.29}$$

along the characteristic $\dfrac{dt}{1} = \dfrac{dx}{-\omega x^\perp}$. The main step in proving Lemma 6.4 is in treating the convergence of $-\displaystyle\int \nabla\cdot\left[\left(\psi^\hbar\right)^* \nabla\psi^\hbar\right] d\tau$ in view of the weak topology. Using integration by parts and Fubini theorem yields

$$-\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int \nabla \cdot \left[\left(\psi^\hbar\right)^* \nabla\psi^\hbar\right] d\tau\phi(x)dxdt = \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int \left(\psi^\hbar\right)^* \nabla\psi^\hbar d\tau\nabla\phi(x)dxdt \tag{6.30}$$

$$= \int_{t_1}^{t_2}\iint_{\mathbb{R}^2} \left(\psi^\hbar\right)^* \nabla\psi^\hbar\nabla\phi(x)dxd\tau dt$$

for all test functions $\phi \in \mathcal{D}(\mathbb{R}^2) \cap H^1_\omega(\mathbb{R}^2)$. Provided that the phase is not singular, both the property (iii) in Lemma 6.2 and (6.20) imply that $\nabla\psi^\hbar \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2))$.

Moreover, $\nabla\psi^\hbar$ converges weakly $*$ to $\nabla\psi$ in $L^\infty(\mathbb{R}^+;L^2(\mathbb{R}^2))$. The weak convergence of $\nabla\psi^\hbar$ in $L^\infty(\mathbb{R}^+;L^2(\mathbb{R}^2))$ and the strong convergence of $\psi^\hbar$ in $C([0,T];L^2(\mathbb{R}^2))$ show that $\left(\psi^\hbar\right)^*\nabla\psi^\hbar$ converges weakly $*$ to $\psi^*\nabla\psi$ in $L^\infty([0,T];L^1(\mathbb{R}^2))$, and

$$\int -\nabla\cdot\left[\left(\psi^\hbar\right)^*\nabla\psi^\hbar\right]d\tau \to -\int \nabla\cdot\left(\psi^*\nabla\psi\right)d\tau \tag{6.31}$$

in $\mathcal{D}'([0,T]\times\mathbb{R}^2)$ for $T>0$. This completes the proof of Lemma 6.4. □

Based on the above findings, we present the results of passage to the limit.

Theorem 6.5. *Assume that $\psi_0^\hbar$ satisfies $|\psi_0^\hbar|=1$ almost everywhere and $\psi_0^\hbar\to\psi_0$ strongly in $H^1_\omega(\mathbb{R}^2)$ as $\hbar\to0$. Let $\psi^\hbar$ be the weak solution of (6.16). Then $\psi^\hbar$ converges to the weak limit ψ satisfying the wave map equation*

$$\partial_{tt}\psi-\frac{g}{m}\Delta\psi=-\psi\left(|\partial_t\psi|^2-\frac{g}{m}|\nabla\psi|^2\right),\quad |\psi|=1\quad a.e..$$

Equivalently, $\psi=e^{i\theta}$ with the phase function θ satisfies the wave equation

$$\partial_{tt}\theta-\frac{g}{m}\Delta\theta=0.$$

Proof. The strong convergence of $\psi^\hbar$ in $C([0,T];L^2(\mathbb{R}^2))$ implies that

$$\left(\psi^\hbar(t_2,\cdot),\phi\right)\to\left(\psi(t_2,\cdot),\phi\right),\quad \left(\psi^\hbar(t_1,\cdot),\phi\right)\to\left(\psi(t_1,\cdot),\phi\right). \tag{6.32}$$

The uniform boundness of $\left\{\nabla_\omega\psi^\hbar\right\}_\hbar$ in $L^\infty(\mathbb{R}^+;L^2(\mathbb{R}^2))$ implies that

$$\frac{\hbar}{2m}\int_{t_1}^{t_2}\left(\nabla_\omega\psi^\hbar,\nabla_\omega\phi\right)dt\to0. \tag{6.33}$$

We can recognize from both Lemma 6.3 and Lemma 6.4 that

$$\left(\frac{|\psi^\hbar|^2-1}{\hbar}\right)\psi^\hbar\rightharpoonup-\left\{\int\nabla\cdot\left[\frac{1}{m}\mathrm{Im}\left(\psi^*\nabla\psi\right)\right]d\tau\right\}\psi \tag{6.34}$$

in $\mathcal{D}'([0,T]\times\mathbb{R}^2)$ for $T>0$. Therefore,

$$\int_{t_1}^{t_2}\left(g\left(\frac{|\psi^\hbar|^2-1}{\hbar}\right)\psi^\hbar,\phi\right)dt\to\int_{t_1}^{t_2}\left(-g\left\{\int\nabla\cdot\left[\frac{1}{m}\mathrm{Im}\left(\psi^*\nabla\psi\right)\right]d\tau\right\}\psi,\phi\right)dt. \tag{6.35}$$

Let $V_\omega=\frac{1}{m}V-\frac{1}{2}\omega^2|x|^2$. From the energy inequality (6.18), there is further information to suggest that the quantity $\frac{1}{\hbar}\left(g+mV_\omega\right)$ is uniformly bounded. Hence,

$$\int_{t_1}^{t_2}\left(\frac{1}{\hbar}\left(g+mV_\omega\right)\psi^\hbar,\phi\right)dt\to\int_{t_1}^{t_2}\left(\frac{1}{\hbar}\left(g+mV_\omega\right)\psi,\phi\right)dt. \tag{6.36}$$

In conclusion, the wave function ψ satisfies

$$i\partial_t\psi = -g\left\{\int \nabla\cdot\left[\frac{1}{m}\mathrm{Im}\left(\psi^*\nabla\psi\right)\right]d\tau\right\}\psi + \frac{1}{\hbar}\left[g + m\left(\frac{1}{m}V - \frac{1}{2}\omega^2|x|^2\right)\right]\psi \qquad (6.37)$$

in the sense of distributions. A more clear expression of (6.37) could be showed. Differentiating (6.37) with respect to t, we have the wave map equation

$$\partial_{tt}\psi - \frac{g}{m}\Delta\psi = -\psi\left(|\partial_t\psi|^2 - \frac{g}{m}|\nabla\psi|^2\right), \quad |\psi| = 1 \quad \text{a.e..} \qquad (6.38)$$

Using the fact that $|\psi| = 1$, we write $\psi = e^{i\theta}$ and insert it into (6.37) or (6.38) to show the linear wave equation

$$\partial_{tt}\theta - \frac{g}{m}\Delta\theta = 0. \qquad (6.39)$$

□

Remark 6.6. *In* (6.38)*, the terms inside the parentheses showing up in geometrical optics is the eikonal equation.*

The dispersion limit suggests that we treat the right side of (6.2)–(6.3) as a perturbation, and we can study the linear wave equation instead of (6.2)–(6.3). Much remains to be done, but we intend to continue pursuing this interesting line of inquiry.

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