國 立 交 通 大 學
應用數學系
碩 士 論 文

多重完全圖之混合設計

Hybrid Design of the λ-fold Complete Graph

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摘要（中文）

所謂的 k 點的圖對 (G,H) 是指不同構的兩個 k 點圖，它們滿足(1) G 與 H 中都不具有孤立點，及(2) G 與 H 的圖聯集恰為 k 點的完全圖。如果我們可以把 n 點的完全圖用 G 與 H 的組合來表示，每一個至少出現一次，我們稱這樣的分割為雙重圖設計。更進一步，如果對於所有的 s 與 t，只要滿足 $\lambda\binom{n}{2} = s\cdot|E(G)|+t\cdot|E(H)|$，就可以用 s 個圖 G 與 t 個圖 H 來組合成 λK_n，則我們稱這樣的一個分割為 λ 倍的混合設計，或多重混合設計。

在這篇論文中，我們針對一個 5 點的圖對（各有 5 邊），分別建構出多重混合設計，多重混合裝填及多重混合覆蓋；後兩者是在 $\lambda\binom{n}{2}$ 不為 5 的倍數時分別討論最大裝填及最小覆蓋。

Hybrid Design of the λ-fold Complete Graph

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Abstract

By a graph-pair of order t, we mean two non-isomorphic graphs G and H on t non-isolated vertices for which $G \cup H \cong K_t$ for some integer $t \geq 4$. Given a graph-pair (G, H), if the edges of λK_n can be partitioned into s copies of G and t copies of H with $\lambda\binom{n}{2} = s \cdot |E(G)| + t \cdot |E(H)|$, $\forall s, t \in \mathbb{N} \cup \{0\}$, then we refer to this partition as a (G, H)-hybrid decomposition.

In this thesis, we consider the existence of hybrid design of λK_n for the graph-pairs of order 5 (each has 5 edges). For this graph-pair, we will also consider the maximum hybrid packings and the minimum hybrid coverings of λK_n.

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Contents

List of Figures

1 Introduction and Preliminaries

Graph decomposition is one of the most important topic in the study of graph theory and combinatorial design. There are quite a few results in the literatures which are concerned with the decomposition of a graph G into isomorphic copies of H, denoted by $H|G$. If $G = \lambda K_n$, then the decomposition is known as a λ-fold H-design of order n. Furthermore, if $G = \lambda K_n$ and $H = K_k$, then the decomposition $H|G$ is also known as a λ-fold balanced incomplete block design (BIBD) of order n with block size k, denoted by (n, k, λ)-design, see [7] for a reference.

In [2], Abueida and Daven started the research of multidesigns, i.e., decomposing λK_n into two or more different graphs, especially two non-isomorphic subgraphs H_1 and H_2 of K_5 such that $H_1 \cup H_2 = K_5$. They obtained the decomposition by requiring that each one of the two subgraphs occur at least once. In this thesis, we extend the study of multidesign by asking the number of each subgraphs being prescribed, i.e., as long as the number of edges is correct, λK_n can be decomposed into s copies of G_i and t copies of H_i where (G_i, H_i) is a graph pair. We call such a design hybrid in what follows.

Before presenting the main results, we explain the notations and ideas which we use in the proof.

1.1 Preliminaries

A path is a sequence of distinct vertices $<x_1, x_2, \ldots, x_n>$ such that consecutive vertices are adjacent, that is, there is an edge from each vertex x_i to the next vertex x_{i+1} in the path sequence. A cycle is a path that ends where it starts, that is, $x_n = x_1$. And we write $(x_1, x_2, \ldots, x_n)$ for the cycle. The path and cycle with n vertices are denoted P_n and C_n, respectively. A complete graph is a simple graph whose vertices are pairwise adjacent; the complete graph with n vertices and *n(n-1)/2* edges is denoted by K_n.

A graph G is bipartite if $V(G)$ is the union of two disjoint (possibly empty) independent sets called partite sets of G. A complete bipartite graph or bicligue is a bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the sets have size r and s, the biclique is denoted $K_{r,s}$.

An isolated vertex is a vertex of degree 0. A subgraph of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G. We then write $H \subseteq G$ and say that "G contains H". A decomposition of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list. An isomorphism from a simple graph G to a simple graph H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say "G is isomorphic *to* H", written $G \cong H$, if there is an isomorphism from G to H.

A cut-edge or cut-vertex of a graph is an edge or vertex whose deletion increases the number of components. We write $G - e$ or $G - M$ for the subgraph of G obtained by deleting an edge e or set of edges M. We write $G - v$ or $G - S$ for the subgraph obtained by deleting a vertex v or set of vertices S. An induced subgraph is a subgraph obtained by deleting a set of vertices. We write $G[T]$ for $G - \overline{T}$, where $\overline{T} = V(G) - T$; this is the subgraph of G induced by T. An add-edge of a simple graph G is an edge whose addition let G have multiple edges. We write $G+e$ or $G+M$ for the new graph obtained by adding an edge e or set of edges M.

Let S be an n-set. A *Latin square of order n based on S* is an $n \times n$ array with entries form S such that in each row and each column every element of S occurs exactly once.

The graphs we consider will be simple graphs. Let $V(K_n) = \mathbb{Z}_n$ and $V(K_{r,s}) = \mathbb{Z}_{r+s}$. If $R \subseteq \mathbb{Z}_n$, then $K_n[R]$ is the subgraph of K_n induced by the vertices in R, and if $R, S \subseteq \mathbb{Z}_n$, then $K_n[R; S]$ is the bipartite subgraph of K_n on the vertices $R \cup S$. When $r = |R|$ and $s = |S|$, it is clear that $K_n[R] \cong K_r$ and $K_n[R; S] = K_{r,s}$. Define $[a, b] = t \in \mathbb{Z}_n | a \leq t \leq b$. If $R = [a, b]$ and $S = [c, d]$, then we write $K_n[a, b]$ and $K_n[a, b; c, d]$ rather than $K_n[R]$ and $K_n[R; S]$.

The λ-fold complete graph λK_n is the graph with n vertices in which each pair of vertices is joined by exactly λ edges. A partition of the edges of λK_n into copies of G is called a *G-decomposition* or *G-design*. This situation is denoted by $\lambda K_n \rightarrow G$. A *graph-pair of order t* consists of two non-isomorphic graphs G and H on t non-isolated vertices for which $G \cup H \cong K_t$ for some integer $t \geq 4$. Given a graph-pair (G, H), we say (G, H) divides λK_n if the edges of λK_n can be partitioned into copies of G and H with at least one copy of G and at least one copy of H. We will refer to this partition as a (G, H)*-multidecomposition*. When λK_n does not admit a multidecomposition for a certain pair of subgraphs, it is natural to ask how closely we may decompose it. With a maximum multipacking of λK_n, we hope to obtain a *leave* with as few edges as possible. For a minimum multicovering, we will introduce a few extra (multiple) edges in order to cover the original edges of λK_n; necessarily, we will seek a *padding* that has as few edges as possible.

A *multidesign* is a multidecomposition, a maximum multipacking, or a minimum multicovering.

A $(G, H)-$*hybrid decomposition* of the λ-fold complete graph means that if the edges of λK_n can be partitioned into s copies of G and t copies of H, where $s \cdot e(G) + t \cdot e(H) =$

$e(\lambda K_n)$, $\forall s, t \in \mathbb{Z}^+$. When λK_n does not admit a (G, H)-hybrid decomposition, we instead find a *maximum hybrid packing* and a *minimum hybrid covering*. A *hybrid design* is a hybrid decomposition, a maximum hybrid packing, or a minimum hybrid covering.

The composition $G = G_1[G_2]$ of graphs G_1 and G_2 with disjoint point sets V_1 and V_2 and edge sets X_1 and X_2 is the graph with point vertex $V_1 \times V_2$ and $u = (u_1, u_2)$ adjacent with $v = (v_1, v_2)$ whenever u_1 is adjacent to v_1 in G_1 or $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2. It is also called the *graph lexicographic product*. For example, if $V(G_1) = \{u_1, v_1\}$, and $V(G_2) = \{u_2, v_2, w_2\}$, then $G_1[G_2]$ and $G_2[G_1]$ are listed below.

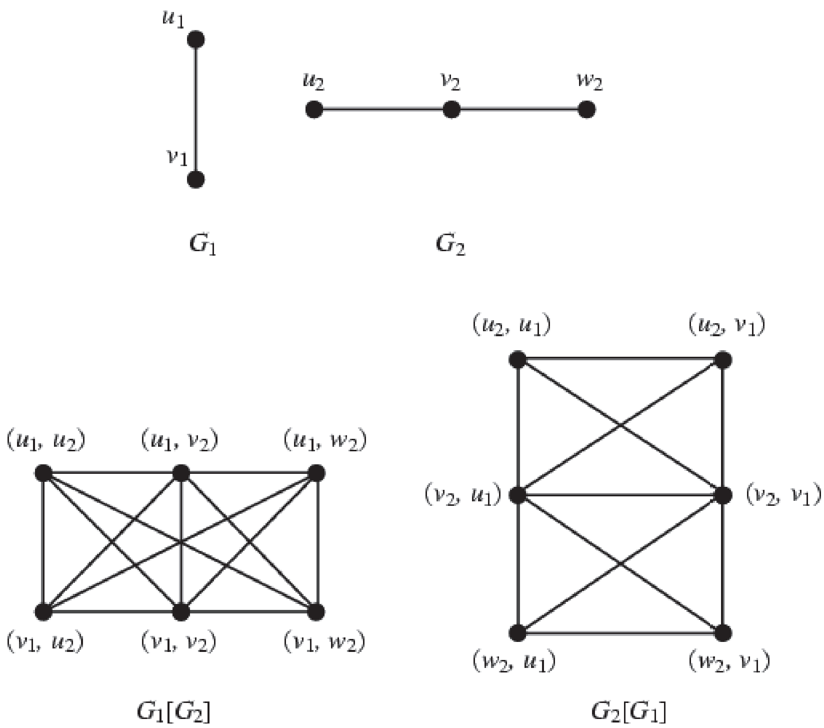

Figure 1: Graph lexicographic product.

The focus of our decomposition will be on the graph-pairs (G_i, H_i) of order 5. Figure 2 is a table of all possible graph-pairs of order 5.

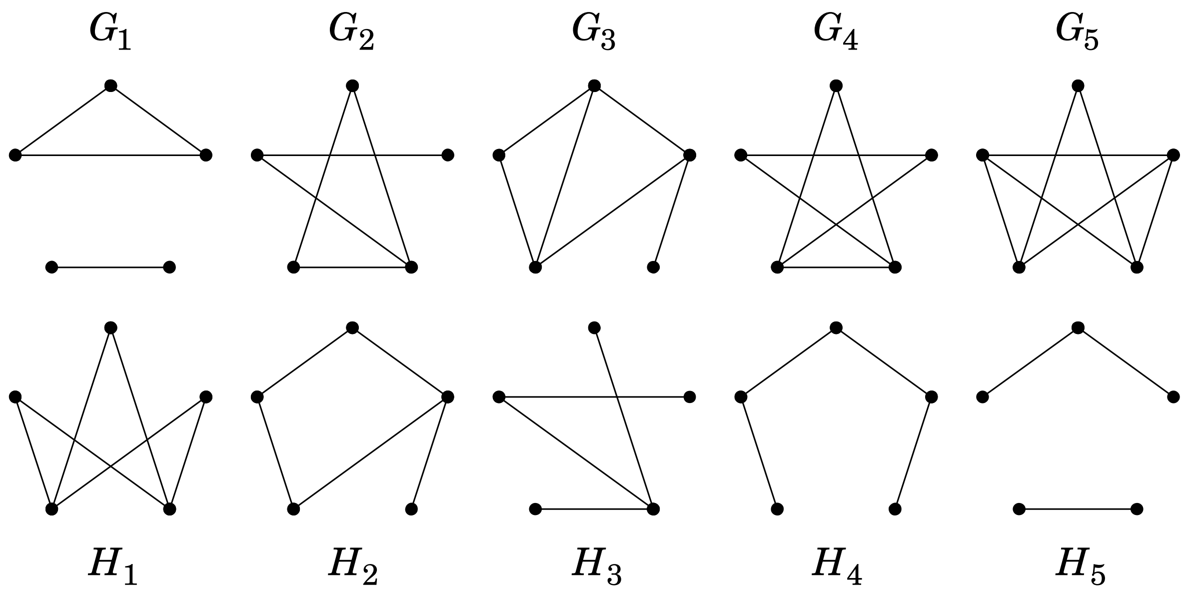

Figure 2: Graph-pairs of order 5.

In this thesis, we completely solve the Hybrid design of the λ-fold complete graph for graph-pair (G_2, H_2).

1.2 The Known Results

In [2], Abueida and Daven completely determined the values of n for which K_n admits a (G, H)-multidesign, when (G, H) is a graph-pair of order 5. The results they obtained may be summarized as follows:

Theorem 1.2.1. [2] *There is a (G_i, H_i)-multidecomposition of K_n if and only if*

(a) when $i \in \{1, 3, 4\}$, $n \equiv 0, 1 \pmod 4$, $n \geq 5$ (except for $i = 1$ and $n = 8$);

(b) when $i = 2$, $n \equiv 0, 1 \pmod 5$; and

(c) when $i = 5$, $n \neq 6, 7$.

Theorem 1.2.2. [2] *Let L be the leave from a maximum (G, H)-multipacking, and let P be the padding from a minimum (G, H)-multicovering of K_n. Then, the follows are true:*

(a) If $(G_i, H_i) \cong (G_1, H_1)$ and $n \equiv 2, 3 \pmod 4$ $(n \geq 7)$, then $L \cong P \cong K_2$;

(b) If $(G_i, H_i) \cong (G_2, H_2)$ and $n \equiv 2, 4 \pmod 5$ $(n \geq 7)$, then $L \cong K_2$ and $e(P) = 4$;

(c) If $(G_i, H_i) \cong (G_2, H_2)$ and $n \equiv 3 \pmod 5$ $(n \geq 7)$, then $L \cong K_3$ and $P \cong K_2$; and

(d) If $(G_i, H_i) \cong (G_5, H_5)$, then $e(L(K_6) = 2$, $P(K_6) \cong L(K_7) \cong P(K_7) \cong K_2$.

Theorem 1.2.3. [3] *There is a (G_2, H_2)-multidesign of λK_n. Let $1 \leq \lambda^* \leq 5$. If $n \equiv 2, 4 \bmod 5$, then $e(L(\lambda K_n)) = \lambda^*$ and $e(P(\lambda K_n)) = 5 - \lambda^*$. If $n \equiv 3 \pmod 5$, then $e(L(\lambda K_n)) = 3\lambda^* \pmod 5$ and $e(P(\lambda K_n)) = 2\lambda^* \pmod 5$.*

In order to prove the main result, we also need the followings.

Theorem 1.2.4. [6] *For each positive integer $n \neq 1$, 2 and 6, there exists a pair of orthogonal Latin squares.*

Theorem 1.2.5. [4] *$G_2|K_n$ if and only if $n \equiv 0$ or $1 \pmod 5$ and $n \geq 6$. $H_2|K_n$ if and only if $n \equiv 0$ or $1 \pmod 5$ and $n > 6$.*

Theorem 1.2.6. [4] *$K_3|K_{n,n,n}$ for each positive integer n and $K_4|K_{n,n,n,n}$ for each $n \neq 2$ or 6.*

Corollary 1.2.7. Let $1 \leq m \leq n$ and $n \neq 2$ or 6. Then, $K_{n,n,n,m}$ can be decomposed into subgraphs such that each subgraph is either a K_3 or a K_4.

Proof. First, by Theorem 1.2.6, $K_4|K_{n,n,n,n}$. Then, we delete $n - m$ vertices from one of the four partite sets. Clearly, if a K_4 contains a vertex which is deleted, then exactly one of its four vertices is deleted. This concludes the proof. ∎

Now, we are ready for the idea of our proof.

1.3 The Idea of "Proof"

Keep in mind that we plan to decompose a graph into a combination of s G_2's and t H_2's for all possible s and t. In what follows, (s,t) is called an *admissible* (ordered) pair of λK_n if $0 \le s, t \le \frac{\lambda}{5}\binom{n}{2}$ and $5(s+t) = \lambda\binom{n}{2}$. Therefore, our constructions are not just only one. So, the strategy is to decompose λK_n into a collection of t subgraphs, L_1, L_2, $\cdots$, L_t, such that each L_i can be decomposed into s_i G_2's and t_i H_2's where (s_i, t_i) is an admissible pair of L_i (hybrid (G_2, H_2)-decomposition of L_i). Now, it is not difficult to see that by combining the decompositions of the L_i's, we can construct a decomposition of λK_n into s G_2's and t H_2's for each admissible pair.

Therefore, it remains to prove that each L_i mentioned above has a hybrid (G_2, H_2)-decomposition. In case that $\lambda\binom{n}{2}$ is not a multiple of 5, then we can delete some edges from λK_n and then do the same job. So, we need the ingredients and they are constructed in Chapter 2.

The proof will be by recursive constructions, hence we need a couple of recurrence relations. (Let $\mathcal{H}$ be the collection of subgraphs which has a hybrid (G_2, H_2)-decomposition.)

Lemma 1.3.1. If K_{n_i} and $K_{n_1,n_2,\cdots,n_h}$ have an $\mathcal{H}$-decomposition for $1 \le i \le h$, then K_n has an $\mathcal{H}$-decomposition where $n = \sum_{i=1}^{h} n_i$.

Proof. The proof follows by the fact that $(\cup_{i=1}^{h} K_{n_i}) \cup K_{n_1,n_2,\cdots,n_h} = K_n$ and the combination of hybrid decompositions of K_{n_i} and $K_{n_1,n_2,\cdots,n_h}$. ∎

Lemma 1.3.2. If K_{n_i+1} and $K_{n_1,n_2,\cdots,n_h}$ have an $\mathcal{H}$-decomposition for $1 \leq i \leq h$, then K_{n+1} has an $\mathcal{H}$-decomposition where $n = (\sum_{i=1}^{h} n_i) + 1$.

Proof. Let $V(K_{n_i+1}) = X_i \cup \{\infty\}$ such that $X_i \cap X_j = \phi$ and $|X_i| = n_i$. Then $V(K_{n+1}) = (\cup_{i=1}^{h} X_i) \cup \{\infty\}$. Now, by combining the $\mathcal{H}$-decompositions of K_{n_i+1} for $1 \leq i \leq h$ and $K_{n_1,n_2,\cdots,n_h}$ we have the proof. ∎

Lemma 1.3.3. If $(K_{n_i+2} - K_2)$ has an $\mathcal{H}$-decomposition for $1 \leq i \leq h$ and $K_{n_1,n_2,\ldots,n_h}$ has an $\mathcal{H}$-decomposition, then $(K_n - K_2)$ has an $\mathcal{H}$-decomposition where $n = (\sum_{i=1}^{h} n_i) + 2$.

Proof. Let $V(K_{n_i+2}) = X_i \cup \{v_1, v_2\}$ such that $X_i \cap X_j = \phi$ and $|X_i| = n_i$. Then $V(K_{n+2}) = (\cup_{i=1}^{h} X_i) \cup \{v_1, v_2\}$. Now, by combining the $\mathcal{H}$-decompositions of $K_{n_i+2} - K_{|\{v_1,v_2\}|}$ for $1 \leq i \leq h$ and $K_{n_1,n_2,\cdots,n_h}$ we have the proof. ∎

Lemma 1.3.4. If $(K_{n_i+3} - K_3)$ has an $\mathcal{H}$-decomposition for $1 \leq i \leq h$ and $K_{n_1,n_2,\ldots,n_h}$ has an $\mathcal{H}$-decomposition, then $(K_n - K_3)$ has an $\mathcal{H}$-decomposition where $n = (\sum_{i=1}^{h} n_i) + 3$.

Proof. Let $V(K_{n_i+3}) = X_i \cup S$ where $S = \{v_1, v_2, v_3\}$ such that $X_i \cap X_j = \phi$ and $|X_i| = n_i$. Then $V(K_{n+3}) = (\cup_{i=1}^{h} X_i) \cup S$. Now, by combining the $\mathcal{H}$-decompositions of $K_{n_i+3} - K_{|S|}$ for $1 \leq i \leq h$ and $K_{n_1,n_2,\cdots,n_h}$ we have the proof. ∎

Lemma 1.3.5. If $(K_{n_i+4} - K_4)$ has an $\mathcal{H}$-decomposition for $1 \leq i \leq h$ and $K_{n_1,n_2,\ldots,n_h}$ has an $\mathcal{H}$-decomposition, then $(K_n - K_4)$ has an $\mathcal{H}$-decomposition where $n = (\sum_{i=1}^{h} n_i) + 4$.

Proof. Let $V(K_{n_i+4}) = X_i \cup S$ where $S = \{v_1, v_2, v_3, v_4\}$ such that $X_i \cap X_j = \phi$ and $|X_i| = n_i$. Then $V(K_{n+4}) = (\cup_{i=1}^{h} X_i) \cup S$. Now, by combining the $\mathcal{H}$-decompositions of $K_{n_i+4} - K_{|S|}$ for $1 \leq i \leq h$ and $K_{n_1,n_2,\cdots,n_h}$ we have the proof. ∎

Lemma 1.3.6. If $(K_{n_i+4} - K_4)$ has an $\mathcal{H}$-decomposition for $1 \leq i < h$, $(K_{n_h+4} - K_2)$ has an $\mathcal{H}$-decomposition and $K_{n_1,n_2,\ldots,n_h}$ has an $\mathcal{H}$-decomposition, then $(K_n - K_2)$ has an $\mathcal{H}$-decomposition where $n = (\sum_{i=1}^{h} n_i) + 4$.

Proof. Let $V(K_{n_i+4}) = X_i \cup S$ where $S = \{v_1, v_2, v_3, v_4\}$ such that $X_i \cap X_j = \phi$ and $|X_i| = n_i$. Then $V(K_{n+4}) = (\cup_{i=1}^{h-1} X_h) \cup S$. Now, by combining the $\mathcal{H}$-decompositions of $K_{n_i+4} - K_{|S|}$ for $1 \leq i < h$, $K_{n_h+4} - K_{|\{v_3,v_4\}|}$ and $K_{n_1,n_2,\cdots,n_h}$ we have the proof. ∎

2 The Main Results

We adopt a notation similar to that used in [4]. Given the labelling below, we denote G_2 by $[x, y, z; u, v]$ and H_2 by $[x, u, y, v; z]$ throughout of this chapter.

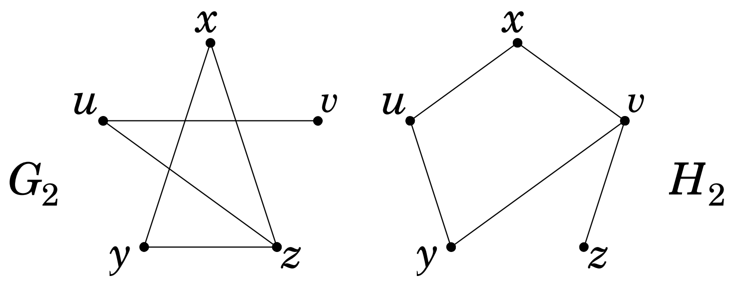

Figure 3: A graph-pair (G_2, H_2) of order 5.

2.1 (G_2, H_2)-Hybrid Design of K_n

Lemma 2.1.1. If $K_{r,r,r}$ has a hybrid (G_2, H_2)-decomposition, then $K_{ar,ar,ar}$ has a hybrid (G_2, H_2)-decomposition for $a > 0$.

Proof. Let G be a graph with vertex set $\{1, 2, \cdots, t\}$ and let $G \otimes O_n$ be the lexicographic product of G by an independent set of n elements, i.e., a graph with vertex set the disjoint union of t independent sets X_i, where $|X_i| = n$ such that two vertices are joined by an edge if and only if they belong to two different sets X_i, X_j and $\{i, j\}$ is an edge in G.

It is easy to see that $K_{ar,ar,ar} = K_{a,a,a} \otimes O_r$, which can be decomposed into subgraphs isomorphic to $K_3 \otimes O_r$ (by Theorem 1.2.6). Since $K_3 \otimes O_r = K_{r,r,r}$ which by hypothesis has a hybrid (G_2, H_2)-decomposition, the proof is concluded. ∎

Lemma 2.1.2. If $K_{r,r,r}$ has a hybrid (G_2, H_2)-decomposition and $K_{r,r,r,r}$ has a hybrid (G_2, H_2)-decomposition, then $K_{pr,pr,pr,qr}$ has a hybrid (G_2, H_2)-decomposition for integer $p \neq 2, 6$, $0 \leq q \leq p$.

Proof. Let G be a graph with vertex set $\{1, 2, \cdots, t\}$ and let $G \otimes O_n$ be the lexicographic product of G by an independent set of n elements.

It is easy to see that $K_{pr,pr,pr,qr} = K_{p,p,p,q} \otimes O_r$, which can be decomposed into subgraphs isomorphic to $K_4 \otimes O_r$ or $K_3 \otimes O_r$ for $p \neq 2, 6$, $0 \leq q \leq p$ (by Corollary 1.2.7). Since both $K_3 \otimes O_r = K_{r,r,r}$ and $K_4 \otimes O_r = K_{r,r,r,r}$ have a hybrid (G_2, H_2)-decomposition, the proof follows. ∎

Lemma 2.1.3. $K_{5,5,5}$ has a hybrid (G_2, H_2)-decomposition.

Proof. Let (s, t) be an admissible pair of $K_{5,5,5}$ and let $V(K_{5,5,5}) = X = \{x_{i,j} | i \in \mathbb{Z}_5, j \in \mathbb{Z}_3\}$. We start with $G_2 | K_{5,5,5}$ and $H_2 | K_{5,5,5}$ and then make trades from G_2's to H_2's.

(i) $K_{5,5,5} = 15G_2$. $G_{2,(i,1)} = [x_{i+1,1},\ x_{i+1,2},\ x_{i+0,0};\ x_{i+3,1},\ x_{i+1,0}]$, $G_{2,(i,2)} = [x_{i+1,2},\ x_{i+1,0},\ x_{i+0,1};\ x_{i+3,2},\ x_{i+1,1}]$, $G_{2,(i,3)} = [x_{i+1,0},\ x_{i+1,1},\ x_{i+0,2};\ x_{i+3,0},\ x_{i+1,2}]$, $\forall i \in \mathbb{Z}_5$.

(ii) $K_{5,5,5} = 15H_2$. $H_{2,(i,1)} = [x_{i+1,1},\ x_{i+0,0},\ x_{i+3,1},\ x_{i+1,0};\ x_{i+0,1}]$, $H_{2,(i,2)} = [x_{i+1,2},\ x_{i+0,1},\ x_{i+3,2},\ x_{i+1,1};\ x_{i+0,2}]$, $H_{2,(i,3)} = [x_{i+1,0},\ x_{i+0,2},\ x_{i+3,0},\ x_{i+1,2};\ x_{i+0,0}]$, $\forall i \in \mathbb{Z}_5$.

(iii) $\forall i \in \mathbb{Z}_5$. We can trade $G_{2,(i,1)} \cup G_{2,(i,2)} \cup G_{2,(i,3)}$ for $H_{2,(i,1)} \cup H_{2,(i,2)} \cup H_{2,(i,3)}$, $\forall i \in \mathbb{Z}_5$.

(iv) $\forall i \in \mathbb{Z}_5$. Let $H_{2,(i,4)} = [x_{i+1,2},\ x_{i+0,0},\ x_{i+3,1},\ x_{i+1,0};\ x_{i+0,1}]$, $H_{2,(i,5)} = [x_{i+1,2},\ x_{i+0,1},\ x_{i+3,2},\ x_{i+1,1};\ x_{i+0,0}]$. Then we can trade $G_{2,(i,1)} \cup G_{2,(i,2)} \cup G_{2,(i,3)}$ for $G_{2,(i,3)} \cup H_{2,(i,4)} \cup H_{2,(i,5)}$, $\forall i \in \mathbb{Z}_5$.

By (i), (ii), (iii), and (iv), we have $\{(s,t)|s+t=15, s,t\in\mathbb{N}\cup\{0\}, t\neq 1\}$.

(v) Let $G_{2,4}=[x_{2,1}, x_{2,2}, x_{1,0}; x_{3,1}, x_{0,0}]$, $G_{2,5}=[x_{1,2}, x_{2,1}, x_{2,0}; x_{4,1}, x_{1,0}]$, $G_{2,6}=[x_{2,2}, x_{3,0}, x_{3,1}; x_{0,2}, x_{2,1}]$, $G_{2,7}=[x_{2,1}, x_{3,0}, x_{3,2}; x_{0,0}, x_{1,1}]$, $H_{2,8}=[x_{0,0}, x_{2,2}, x_{4,0}, x_{1,2}; x_{1,1}]$. Then we can trade $G_{2,(0,1)}\cup G_{2,(1,1)}\cup G_{2,(1,3)}\cup G_{2,(2,3)}\cup G_{2,(2,2)}$ for $G_{2,4}\cup G_{2,5}\cup G_{2,6}\cup G_{2,7}\cup H_{2,8}$. Hence, $(s,t)=(14,1)$ case is done. ∎

The result for $K_{5,5,5,5}$ follows by a similar idea. But, it is more complicate.

Lemma 2.1.4. $K_{5,5,5,5}$ has a hybrid (G_2,H_2)-decomposition.

Proof. Let (s,t) be an admissible pair of $K_{5,5,5,5}$ and let $V(K_{5,5,5})=X=\{x_{i,j}|i\in\mathbb{Z}_5, j\in\mathbb{Z}_4\}$.

(i) $K_{5,5,5,5}=30G_2$. $G_{2,(i,1)}=[x_{i+0,1}, x_{i+2,2}, x_{i+2,0}; x_{i+1,3}, x_{i+3,1}]$, $G_{2,(i,2)}=[x_{i+0,1}, x_{i+1,3}, x_{i+1,0}; x_{i+3,3}, x_{i+3,0}]$, $G_{2,(i,3)}=[x_{i+0,2}, x_{i+2,1}, x_{i+2,3}; x_{i+0,1}, x_{i+1,2}]$, $G_{2,(i,4)}=[x_{i+1,3}, x_{i+2,1}, x_{i+2,2}; x_{i+3,3}, x_{i+0,2}]$, $G_{2,(i,5)}=[x_{i+1,2}, x_{i+1,3}, x_{i+0,0}; x_{i+2,1}, x_{i+1,1}]$, $G_{2,(i,6)}=[x_{i+0,2}, x_{i+1,1}, x_{i+1,0}; x_{i+3,2}, x_{i+0,0}]$, $\forall i\in\mathbb{Z}_5$

(ii) For all $i\in\mathbb{Z}_5$, let $G_{2,(i,7)}=[x_{i+0,1}, x_{i+2,2}, x_{i+2,0}; x_{i+1,3}, x_{i+3,1}]$, $G_{2,(i,8)}=[x_{i+0,2}, x_{i+2,1}, x_{i+2,3}; x_{i+0,1}, x_{i+1,2}]$, $G_{2,(i,9)}=[x_{i+1,1}, x_{i+2,3}, x_{i+2,0}; x_{i+1,2}, x_{i+2,1}]$, $G_{2,(i,10)}=[x_{i+1,3}, x_{i+2,1}, x_{i+2,2}; x_{i+3,3}, x_{i+0,2}]$, $G_{2,(i,11)}=[x_{i+2,2}, x_{i+2,3}, x_{i+1,0}; x_{i+3,1}, x_{i+2,0}]$, $H_{2,(i,12)}=[x_{i+4,2}, x_{i+1,0}, x_{i+4,3}, x_{i+2,0}; x_{i+2,1}]$. Then for all $i\in\mathbb{Z}_5$, we can trade $G_{2,(i,1)}\cup G_{2,(i,2)}\cup G_{2,(i,3)}\cup G_{2,(i,4)}\cup G_{2,(i,5)}\cup G_{2,(i,6)}$ for $G_{2,(i,7)}\cup G_{2,(i,8)}\cup G_{2,(i,9)}\cup G_{2,(i,10)}\cup G_{2,(i,11)}\cup H_{2,(i,12)}$.

(iii) For all $i\in\mathbb{Z}_5$, let $G_{2,(i,13)}=[x_{i+0,1}, x_{i+2,0}, x_{i+2,2}; x_{i+3,3}, x_{i+0,2}]$, $G_{2,(i,14)}=[x_{i+1,1}, x_{i+2,3}, x_{i+2,0}; x_{i+1,2}, x_{i+2,1}]$, $G_{2,(i,15)}=[x_{i+1,0}, x_{i+2,3}, x_{i+2,2}; x_{i+1,3}, x_{i+2,0}]$, $G_{2,(i,16)}=$

$[x_{i+0,2},\ x_{i+2,1},\ x_{i+2,3};\ x_{i+0,1},\ x_{i+1,2}]$, $H_{2,(i,17)} = [x_{i+1,3},\ x_{i+3,1},\ x_{i+2,0},\ x_{i+2,1};\ x_{i+2,2}]$, $H_{2,(i,18)} = [x_{i+4,2},\ x_{i+2,0},\ x_{i+4,3},\ x_{i+1,0};\ x_{i+3,1}]$. Then for all $i \in \mathbb{Z}_5$, we can trade $G_{2,(i,1)} \cup G_{2,(i,2)} \cup G_{2,(i,3)} \cup G_{2,(i,4)} \cup G_{2,(i,5)} \cup G_{2,(i,6)}$ for $G_{2,(i,13)} \cup G_{2,(i,14)} \cup G_{2,(i,15)} \cup G_{2,(i,16)} \cup H_{2,(i,17)} \cup H_{2,(i,18)}$.

(iv) For all $i \in \mathbb{Z}_5$, let $G_{2,(i,19)} = [x_{i+0,1},\ x_{i+2,0},\ x_{i+2,2};\ x_{i+3,3},\ x_{i+0,2}]$, $G_{2,(i,20)} = [x_{i+0,2},\ x_{i+2,1},\ x_{i+2,3};\ x_{i+1,1},\ x_{i+2,0}]$, $G_{2,(i,21)} = [x_{i+1,0},\ x_{i+2,3},\ x_{i+2,2};\ x_{i+1,3},\ x_{i+2,0}]$, $H_{2,(i,22)} = [x_{i+0,1},\ x_{i+2,3},\ x_{i+2,0},\ x_{i+1,2};\ x_{i+2,1}]$, $H_{2,(i,23)} = [x_{i+1,3},\ x_{i+3,1},\ x_{i+2,0},\ x_{i+2,1};\ x_{i+2,2}]$, $H_{2,(i,24)} = [x_{i+4,2},\ x_{i+2,0},\ x_{i+4,3},\ x_{i+1,0};\ x_{i+3,1}]$. Then for all $i \in \mathbb{Z}_5$, we can trade $G_{2,(i,1)} \cup G_{2,(i,2)} \cup G_{2,(i,3)} \cup G_{2,(i,4)} \cup G_{2,(i,5)} \cup G_{2,(i,6)}$ for $G_{2,(i,19)} \cup G_{2,(i,20)} \cup G_{2,(i,21)} \cup H_{2,(i,22)} \cup H_{2,(i,23)} \cup H_{2,(i,24)}$.

(v) For all $i \in \mathbb{Z}_5$, let $G_{2,(i,25)} = [x_{i+1,0},\ x_{i+2,3},\ x_{i+2,2};\ x_{i+3,3},\ x_{i+0,2}]$, $G_{2,(i,26)} = [x_{i+0,2},\ x_{i+2,1},\ x_{i+2,3};\ x_{i+1,1},\ x_{i+2,0}]$, $H_{2,(i,27)} = [x_{i+0,1},\ x_{i+2,0},\ x_{i+1,3},\ x_{i+2,2};\ x_{i+2,1}]$, $H_{2,(i,28)} = [x_{i+0,1},\ x_{i+2,3},\ x_{i+2,0},\ x_{i+1,2};\ x_{i+2,1}]$, $H_{2,(i,29)} = [x_{i+2,1},\ x_{i+1,3},\ x_{i+3,1},\ x_{i+2,0};\ x_{i+2,2}]$, $H_{2,(i,30)} = [x_{i+4,2},\ x_{i+2,0},\ x_{i+4,3},\ x_{i+1,0};\ x_{i+3,1}]$. Then for all $i \in \mathbb{Z}_5$, we can trade $G_{2,(i,1)} \cup G_{2,(i,2)} \cup G_{2,(i,3)} \cup G_{2,(i,4)} \cup G_{2,(i,5)} \cup G_{2,(i,6)}$ for $G_{2,(i,25)} \cup G_{2,(i,26)} \cup H_{2,(i,27)} \cup H_{2,(i,28)} \cup H_{2,(i,29)} \cup H_{2,(i,30)}$.

(vi) For all $i \in \mathbb{Z}_5$, let $G_{2,(i,31)} = [x_{i+1,0},\ x_{i+2,2},\ x_{i+2,3};\ x_{i+2,1},\ x_{i+1,2}]$, $H_{2,(i,32)} = [x_{i+0,1},\ x_{i+1,2},\ x_{i+2,0},\ x_{i+2,3};\ x_{i+1,1}]$, $H_{2,(i,33)} = [x_{i+0,2},\ x_{i+2,2},\ x_{i+1,3},\ x_{i+2,0};\ x_{i+1,1}]$, $H_{2,(i,34)} = [x_{i+2,1},\ x_{i+2,2},\ x_{i+3,3},\ x_{i+1,2};\ x_{i+2,3}]$, $H_{2,(i,35)} = [x_{i+2,1},\ x_{i+1,3},\ x_{i+3,1},\ x_{i+2,0};\ x_{i+2,2}]$, $H_{2,(i,36)} = [x_{i+4,2},\ x_{i+2,0},\ x_{i+4,3},\ x_{i+1,0};\ x_{i+3,1}]$. Then for all $i \in \mathbb{Z}_5$, we can trade $G_{2,(i,1)} \cup G_{2,(i,2)} \cup G_{2,(i,3)} \cup G_{2,(i,4)} \cup G_{2,(i,5)} \cup G_{2,(i,6)}$ for $G_{2,(i,31)} \cup H_{2,(i,32)} \cup H_{2,(i,33)} \cup H_{2,(i,34)} \cup H_{2,(i,35)} \cup H_{2,(i,36)}$.

(vii) For all $i \in \mathbb{Z}_5$, let $H_{2,(i,37)} = [x_{i+2,2},\ x_{i+1,0},\ x_{i+2,3},\ x_{i+2,1};\ x_{i+1,2}]$, $H_{2,(i,38)} = [x_{i+0,1},\ x_{i+1,2},\ x_{i+2,0},\ x_{i+2,3};\ x_{i+1,1}]$, $H_{2,(i,39)} = [x_{i+0,1},\ x_{i+2,2},\ x_{i+1,3},\ x_{i+2,0};\ x_{i+1,1}]$, $H_{2,(i,40)} = [x_{i+2,3},\ x_{i+2,2},\ x_{i+3,3},\ x_{i+0,2};\ x_{i+2,1}]$, $H_{2,(i,41)} = [x_{i+2,1},\ x_{i+1,3},\ x_{i+3,1},\ x_{i+2,0};\ x_{i+2,2}]$, $H_{2,(i,42)} = [x_{i+4,2},\ x_{i+2,0},\ x_{i+4,3},\ x_{i+1,0};\ x_{i+3,1}]$. Then for all $i \in \mathbb{Z}_5$, we can trade $G_{2,(i,1)} \cup G_{2,(i,2)} \cup G_{2,(i,3)} \cup G_{2,(i,4)} \cup G_{2,(i,5)} \cup G_{2,(i,6)}$ for $H_{2,(i,37)} \cup H_{2,(i,38)} \cup H_{2,(i,39)} \cup H_{2,(i,40)} \cup H_{2,(i,41)} \cup H_{2,(i,42)}$.

Hence, by (i), (ii), (iii), (iv), (v), (vi), and (vii), we have an admissible pair of $K_{5,5,5,5}$. ∎

We also need a couple of special decompositions.

Lemma 2.1.5. $K_{10} - K_5$ has a hybrid (G_2, H_2)-decomposition.

Proof. Let (s,t) be an admissible pair of $K_{10} - K_5$ and let $V(K_{10}) = \mathbb{Z}_{10}$.

(i) $K_{5,5}$ can be decomposed into 5 $H_2's$

$H_{2,1} = [1,9,3,8;0]$, $H_{2,2} = [2,8,4,7;1]$, $H_{2,3} = [3,7,0,6;2]$, $H_{2,4} = [1,6,4,5;3]$, $H_{2,5} = [2,5,0,9;4]$.

By assumption (observation), $K_5 = G_2 + H_2$. Hence, "$(1,6)$-case" is done.

(ii) Let $G_{2,1} = [1,2,3;4,0]$, $H_{2,6} = [4,0,8,3;1]$, $H_{2,7} = [2,3,9,1;8]$. From (i), we can trade $G_{2,1} \cup H_{2,1}$ for $H_{2,6} \cup H_{2,7}$. Hence, "$(0,7)$-case" is done.

(iii) Let $H_{2,8} = [2,4,3,1;0]$, $G_{2,2} = [0,1,8;3,4]$, $G_{2,3} = [3,9,1;2,4]$. From (i), we can trade $H_{2,1} \cup H_{2,8}$ for $G_{2,2} \cup G_{2,3}$. Hence, "$(3,4)$-case" is done.

(iv) Let $G_{2,4} = [2,4,3;1,0]$, $G_{2,5} = [1,9,3;2,4]$. From (i), we can trade $H_{2,1} \cup G_{2,4}$ for $G_{2,2} \cup G_{2,5}$. Hence, "$(2,5)$-case" is done.

(v) Let $H_{2,9} = [1,4,0,3;2]$, $G_{2,6} = [1,3,8;2,7]$, $G_{2,7} = [4,7,1;9,3]$, $G_{2,8} = [4,8,0;3,2]$. From (i), we can trade $H_{2,9}\cup H_{2,1}\cup H_{2,2}$ for $G_{2,6}\cup G_{2,7}\cup G_{2,8}$. Hence, "$(4,3)$-case" is done.

(vi) $K_{10} - K_5$ can be decomposed into $7G_2$'. $G_{2,9} = [1,0,7;3,6]$, $G_{2,10} = [2,1,6;4,5]$, $G_{2,11} = [5,2,3;0,6]$, $G_{2,12} = [4,3,9;1,8]$, $G_{2,13} = [8,4,0;5,1]$, $G_{2,14} = [7,2,4;1,3]$, $G_{2,15} = [9,0,2;8,3]$.

(vii) Let $H_{2,10} = [7,4,1,3;6]$, $G_{2,16} = [1,0,7;2,4]$. From (vi), we can trade $G_{2,9}\cup G_{2,14}$ for $H_{2,10}\cup G_{2,16}$. Hence, "$(6,1)$-case" is done.

(viii) Let $H_{2,11} = [2,1,6,4;5]$, $H_{2,12} = [1,4,7,3;6]$, $G_{2,17} = [0,1,7;2,6]$. From (vi), we can trade $G_{2,9}\cup G_{2,10}\cup G_{2,14}$ for $H_{2,11}\cup H_{2,12}\cup G_{2,17}$. Hence, "$(5,2)$-case" is done. ∎

Lemma 2.1.6. $K_{11} - K_6$ has a hybrid (G_2, H_2)-decomposition.

Proof. Let (s,t) be an admissible pair of $K_{11} - K_6$ and let $V(K_{11}) = \mathbb{Z}_{11}$.

(i) $K_{5,6}$ can be decomposed into 6 H_2's. $H_{2,1} = [6,2,7,0;10]$, $H_{2,2} = [8,0,9,3;7]$, $H_{2,3} = [6,3,10,1;9]$, $H_{2,4} = [1,8,5,7;4]$, $H_{2,5} = [8,4,10,2;9]$, $H_{2,6} = [6,4,9,5;10]$. By assumption (observation) $K_5 = G_2 + H_2$. Hence, "$(1,7)$-case" is done.

(ii) Let $G_{2,1} = [9,10,8;7,6]$, $H_{2,7} = [0,9,10,8;3]$, $H_{2,8} = [3,9,8,7;6]$. From (i), we can trade $H_{2,2}\cup$, $G_{2,1}$ for $H_{2,7}\cup H_{2,8}$. Hence, "$(0,8)$-case" is done.

(iii) Let $H_{2,9} = [6,10,9,8;7]$, $G_{2,2}$ $[7,8,3;9,0]$, $H_{2,10} = [6,10,9,8;0]$. From (i), we can trade $H_{2,2}\cup H_{2,9}$ for $G_{2,2}\cup H_{2,10}$. Hence, "$(2,6)$-case" is done.

(iv) Let $G_{2,3} = [0, 9, 8; 6, 10]$. From (i), we can trade $H_{2,2} \cup H_{2,9}$ for $G_{2,2} \cup G_{2,3}$. Hence, "(3, 5)-case" is done.

(v) Let $H_{2,11} = [7, 7, 9, 10; 8]$, $G_{2,4} = [0, 10, 6; 2, 7]$, $G_{2,5} = [7, 8, 0; 9, 10]$, $G_{2,6} = [8, 9, 3; 7, 10]$. From (i), we can trade $H_{2,2} \cup H_{2,1} \cup H_{2,11}$ for $G_{2,4} \cup G_{2,5} \cup G_{2,6}$. Hence, "(4, 4)-case" is done.

(vi) $K_{11} - K_6$ can be decomposed into 8 G_2's. $G_{2,7} = [1, 9, 8; 3, 10]$, $G_{2,8} = [0, 10, 9; 4, 8]$, $G_{2,9} = [2, 8, 10; 5, 9]$, $G_{2,10} = [6, 10, 4; 7, 0]$, $G_{2,11} = [7, 10, 1; 6, 5]$, $G_{2,12} = [0, 8, 6; 9, 7]$, $G_{2,13} = [5, 8, 7; 2, 6]$, $G_{2,14} = [6, 7, 3; 9, 2]$. Hence, "(8, 0)-case" is done.

(vii) Let $G_{2,15} = [5, 8, 7; 6, 3]$, $H_{2,12} = [7, 3, 9, 2; 6]$. From (vi), we can trade $G_{2,13} \cup G_{2,14}$ for $G_{2,15} \cup H_{2,12}$. Hence, "(7, 1)-case" is done.

(viii) Let $H_{2,13} = [5, 10, 8, 9; 1]$, $H_{2,14} = [2, 10, 3, 8; 1]$. From (vi), we can trade $G_{2,7} \cup G_{2,9}$ for $H_{2,13} \cup H_{2,14}$. Hence, "(6, 2)-case" is done.

(ix) Let $H_{2,15} = [4, 8, 10, 9, 1]$, $H_{2,16} = [0, 10, 5, 9, 8]$. From (vi), we can trade $G_{2,7} \cup G_{2,8} \cup G_{2,9}$ for $H_{2,14} \cup H_{2,15} \cup H_{2,16}$. Hence, "(5, 3)-case" is done. ∎

Theorem 2.1.7. *There exists a hybrid (G_2, H_2)-decomposition of K_n for $n \equiv 0, 1 \pmod 5$.*

Proof. By Theorem 1.2.5, we will consider only the unsolved cases for $s, t \geq 1$. Let $\mathcal{H}$ be the collection of subgraphs which has a hybrid (G_2, H_2)-decomposition. First, we show that K_{n_i} has an $\mathcal{H}$-decomposition and also $K_{n_1, n_2, \cdots, n_h}$ has an $\mathcal{H}$-decomposition for some small values of n_i.

In the Appendix, we list the hybrid (G_2, H_2)-decompositions for K_{10}, K_{11}, K_{15}, K_{16}, K_{20}, K_{21}, K_{25}, K_{26}. Therefore, for the remainder of the proof, assume $n \geq 30$.

- K_{35}

  Let (s, t) be an admissible pair of K_{35}. Let $V(K_{35}) = \{x_{i,j} | \forall i \in \mathbb{Z}_5, j \in \mathbb{Z}_7\}$, and let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$, where $j = 1, 2, \cdots, 7$.

  Since $K_{7(5)}$ can be partitioned into 7 $K_{5,5,5}$'s ($K_{|V_1|,|V_3|,|V_4|}$, $K_{|V_1|,|V_2|,|V_6|}$, $K_{|V_1|,|V_5|,|V_7|}$, $K_{|V_2|,|V_4|,|V_5|}$, $K_{|V_2|,|V_3|,|V_7|}$, $K_{|V_3|,|V_5|,|V_6|}$, $K_{|V_4|,|V_6|,|V_7|}$), K_{35} can be partitioned into 7 $(K_{15} - K_5 - K_5)$'s ($K_{|V_1|,|V_3|,|V_4|} \cup K_{|V_1|}$, $K_{|V_1|,|V_2|,|V_6|} \cup K_{|V_2|}$, $K_{|V_1|,|V_5|,|V_7|} \cup K_{|V_7|}$, $K_{|V_2|,|V_4|,|V_5|} \cup K_{|V_4|}$, $K_{|V_2|,|V_3|,|V_7|} \cup K_{|V_3|}$, $K_{|V_3|,|V_5|,|V_6|} \cup K_{|V_5|}$, $K_{|V_4|,|V_6|,|V_7|} \cup K_{|V_6|}$). From the proof of hybrid (G_2, H_2)-decomposition of K_{15}, $(K_{15} - K_5 - K_5)$ has a hybrid (G_2, H_2)-decomposition. Thus, K_{35} has a hybrid (G_2, H_2)-decomposition.

- K_{36}

  Let (s, t) be an admissible pair of K_{36}. Let $V(K_{36}) = \{x_{i,j} | \forall i \in \mathbb{Z}_5, j \in \mathbb{Z}_7\} \cup \{\infty\}$, and let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$, $j = 1, 2, \cdots, 7$.

  Since $K_{7(5)}$ can be partitioned into 7 $K_{5,5,5}$'s ($K_{|V_1|,|V_3|,|V_4|}$, $K_{|V_1|,|V_2|,|V_6|}$, $K_{|V_1|,|V_5|,|V_7|}$, $K_{|V_2|,|V_4|,|V_5|}$, $K_{|V_2|,|V_3|,|V_7|}$, $K_{|V_3|,|V_5|,|V_6|}$, $K_{|V_4|,|V_6|,|V_7|}$), and $K_{|V_1|,|V_3|,|V_4|} \cup K_{|V_1 \cup \{\infty\}|}$, $K_{|V_1|,|V_2|,|V_6|} \cup K_{|V_2 \cup \{\infty\}|}$, $K_{|V_1|,|V_5|,|V_7|} \cup K_{|V_7 \cup \{\infty\}|}$, $K_{|V_2|,|V_4|,|V_5|} \cup K_{|V_4 \cup \{\infty\}|}$, $K_{|V_2|,|V_3|,|V_7|} \cup K_{|V_3 \cup \{\infty\}|}$, $K_{|V_3|,|V_5|,|V_6|} \cup K_{|V_5 \cup \{\infty\}|}$, and $K_{|V_4|,|V_6|,|V_7|} \cup K_{|V_6 \cup \{\infty\}|}$ have a hybrid (G_2, H_2)-decomposition (from proof of K_{16}), K_{35} has a hybrid (G_2, H_2)-decomposition.

- K_{40}

  Let $V(K_{40}) = \{x_{i,j} | \forall i \in \mathbb{Z}_5, j \in \mathbb{Z}_8\}$, and let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$, $j = 1, 2, \cdots, 8$.

Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5}$ as given in Lemma 2.1.3 for each of the following subgraphs: $K_{|V_1|,|V_3|,|V_4|}$, $K_{|V_1|,|V_2|,|V_6|}$, $K_{|V_1|,|V_5|,|V_7|}$, $K_{|V_2|,|V_4|,|V_5|}$, $K_{|V_2|,|V_3|,|V_7|}$, $K_{|V_3|,|V_5|,|V_6|}$, $K_{|V_4|,|V_6|,|V_7|}$. Use a hybrid (G_2, H_2)-decomposition of K_{10} for the subgraph $K_{|V_1 \cup V_8|}$. Use a hybrid (G_2, H_2)-decomposition of $K_{10} - K_5$ as given in Lemma 2.1.5 for each of the following subgraphs: $K_{|V_2 \cup V_8|} - K_{|V_8|}$, $K_{|V_3 \cup V_8|} - K_{|V_8|}$, $K_{|V_4 \cup V_8|} - K_{|V_8|}$, $K_{|V_5 \cup V_8|} - K_{|V_8|}$, $K_{|V_6 \cup V_8|} - K_{|V_8|}$, $K_{|V_7 \cup V_8|} - K_{|V_8|}$.

- K_{41}

  Let $V(K_{41}) = \{x_{i,j} | \forall i \in \mathbb{Z}_5, j \in \mathbb{Z}_8\} \cup \{\infty\}$, and let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$, $j = 1, 2, \cdots, 8$.

  Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5}$ as given in Lemma 2.1.3 for each of the following subgraphs: $K_{|V_1|,|V_3|,|V_4|}$, $K_{|V_1|,|V_2|,|V_6|}$, $K_{|V_1|,|V_5|,|V_7|}$, $K_{|V_2|,|V_4|,|V_5|}$, $K_{|V_2|,|V_3|,|V_7|}$, $K_{|V_3|,|V_5|,|V_6|}$, $K_{|V_4|,|V_6|,|V_7|}$. Use a hybrid (G_2, H_2)-decomposition of K_{11} for the subgraph $K_{|V_1 \cup V_8 \cup \{\infty\}|}$. Use a hybrid (G_2, H_2)-decomposition of $K_{11} - K_6$ as given in Lemma 2.1.6 for each of the following subgraphs: $K_{|V_2 \cup V_8 \cup \{\infty\}|} - K_{|V_8 \cup \{\infty\}|}$, $K_{|V_3 \cup V_8 \cup \{\infty\}|} - K_{|V_8 \cup \{\infty\}|}$, $K_{|V_4 \cup V_8 \cup \{\infty\}|} - K_{|V_8 \cup \{\infty\}|}$, $K_{|V_5 \cup V_8 \cup \{\infty\}|} - K_{|V_8 \cup \{\infty\}|}$, $K_{|V_6 \cup V_8 \cup \{\infty\}|} - K_{|V_8 \cup \{\infty\}|}$, $K_{|V_7 \cup V_8 \cup \{\infty\}|} - K_{|V_8 \cup \{\infty\}|}$.

- K_{95}

  By Theorem 2.1.2 with $r = 5$, $p = 5$, $q = 4$, $K_{25,25,25,20}$ has a hybrid (G_2, H_2)-decomposition. Since K_{25}, and K_{20} have a hybrid (G_2, H_2)-decomposition, by Theorem 1.3.1, K_{95} has a hybrid (G_2, H_2)-decomposition.

- K_{96}

By Theorem 2.1.2 with $r = 5$, $p = 5$, $q = 4$, $K_{25,25,25,20}$ has a hybrid (G_2, H_2)-decomposition. Since K_{26}, and K_{21} have a hybrid (G_2, H_2)-decomposition, by Theorem 1.3.2, K_{96} has a hybrid (G_2, H_2)-decomposition.

- K_{100}

  By Theorem 2.1.2 with $r = 5$, $p = q = 5$, $K_{25,25,25,25}$ has a hybrid (G_2, H_2)-decomposition. Since K_{25} has a hybrid (G_2, H_2)-decomposition, by Theorem 1.3.1, K_{100} has a hybrid (G_2, H_2)-decomposition.

- $K_{101} = sG_2 + tH_2$, where $s \cdot e(G_2) + t \cdot e(H_2) = e(K_{101})$, $\forall s, t \in \mathbb{N} \cup \{0\}$.

  By Theorem 2.1.2 with $r = 5$, $p = q = 5$, $K_{25,25,25,25}$ has a hybrid (G_2, H_2)-decomposition. Since K_{26} has a hybrid (G_2, H_2)-decomposition, by Theorem 1.3.2, K_{101} has a hybrid (G_2, H_2)-decomposition.

Now, we are ready for the proof. As mentioned above, it suffices to decompose K_n into a collection of subgraphs which have a hybrid (G_2, H_2)-decomposition, and this can be done recursively.

Case 1. $n \equiv 0 \pmod 5$

Let $n \equiv 0, 5, 10, 15, 20, 25 \pmod{30}$. Therefore $n = 30t, 30t+5, 30t+10, 30t+15, 30t+20$, and $30t+25$ respectively. Since $30t = 3 \cdot (10t)$ and $30t + 15 = 3 \cdot (10t + 5)$, K_n can be decomposed into three copies of K_{10t} (respectively K_{10t+5}) and $K_{10t,10t,10t}$ (respectively $K_{10t+5,10t+5,10t+5}$). Hence, by the fact that both K_{10t} and $K_{5,5,5}$ have a hybrid (G_2, H_2)-decomposition, we have a hybrid (G_2, H_2)-decomposition of K_{30t} and K_{30t+15} respectively. Now, if $n = 30t+5$ (the other cases are similar), then K_{30t+5} can be decomposed into three K_{10t}'s, one K_5 and $K_{10t,10t,10t,5}$. Again, each of them has a hybrid (G_2, H_2)-decomposition, the proof of this case follows. (Note that $K_{10t,10t,10t,5}$ can be decomposed into $K_{5,5,5,5}$'s and $K_{5,5,5}$'s as long as $t \neq 1$ and 3. But, the cases K_{35} and K_{95} have been treated as special cases earlier.)

Case 2. $n \equiv 1 \pmod 5$

Let $n = 30t + 1, 30t + 6, 30t + 11, 30t + 16, 30t + 21$, and $30t + 26$. For the case $n = 30t + 1$ and $n = 30t + 16$, we can decompose K_{30t+1} (respectively K_{30t+16}) into K_{10t+1}'s (respectively K_{10t+6}'s) with one vertex in common and a $K_{10t,10t,10t}$ (respectively $K_{10t+5,10t+5,10t+5}$). Then, the proof follows by combining the hybrid (G_2, H_2)-decompositions of these subgraphs together. Now, consider $n = 30t + 6$ and the other cases are similar. Clearly, K_{30t+6} can be decomposed into three K_{10t+1}'s and one K_6 (in which they have one vertex in common), and one $K_{10t,10t,10t,5}$. Since each of them has a hybrid (G_2, H_2)-decomposition, the proof of this case follows. Again, $K_{10t,10t,10t,5}$ can be decomposed into $K_{5,5,5,5}$'s and $K_{5,5,5}$'s only if $t \neq 1$ or 3. Hence we have to deal with $n = 36$ and $n = 96$ independently, and then the "proof" follows. ∎

2.2 Maximum (G_2, H_2)-Hybrid Packing of K_n and Minimum (G_2, H_2)-Hybrid Covering of K_n

In this section, we consider the cases $n \equiv 2, 3, 4 \pmod 5$ in which there are no (G_2, H_2)-Hybrid designs of order n. Therefore, the best we can get is the maximum (G_2, H_2)-Hybrid packing, i.e., by removing the minimum leave L, $K_n - L$ has a (G_2, H_2)-Hybrid decomposition.

Lemma 2.2.1. $K_7 - K_2$ has a hybrid (G_2, H_2)-decomposition.

Proof. Let (s, t) be an admissible pair of $(K_7 - K_2)$. Let $V(K_7) = \mathbb{Z}_7$.

(i) Use the following copies of G_2:

$G_2 \cong [2,5,1;4,0],\ [2,4,6;3,1],\ [3,4,5;0,2],\ [1,6,0;3,2]$.

What remains is the edge $\{5,6\}$. Hence, "$(4,0)$-case" is done.

(ii) Use the following copies of G_2 and H_2:

$G_2 \cong [0,1,6;5,4],\ [2,4,6;3,1],\ [2,5,1;4,0];\ H_2 \cong [2,0,5,3;4]$.

What remains is the edge $\{0,3\}$. Hence, "$(3,1)$-case" is done.

(iii) Use the following copies of G_2 and H_2:

$G_2 \cong [2,5,1;4,0],\ [2,6,4;3,0];\ H_2 \cong [0,1,3,6;5],\ [0,2,3,5;4]$.

What remains is the edge $\{1,6\}$. Hence, "$(2,2)$-case" is done.

(iv) Use the following copies of G_2 and H_2:

$G_2 \cong [2,5,1;4,0];\ H_2 \cong [0,1,3,6;4],\ [2,4,5,6;1],\ [2,0,5,3;4]$.

What remains is the edge $\{0,3\}$. Hence, "$(1,3)$-case" is done.

(v) Use the following copies of H_2:

$H_2 \cong [1,5,6,0;3],\ [1,3,4,6;2],\ [1,2,5,4;0],\ [2,0,5,3;6]$.

What remains is the edge $\{0,3\}$.Hence, "$(0,4)$-case" is done. ∎

Lemma 2.2.2. $K_{12} - K_7$ has a hybrid (G_2, H_2)-decomposition.

Proof. Let (s,t) be an admissible pair of $(K_{12} - K_7)$. Let $V(K_{12}) = \mathbb{Z}_{12}$, and let $V_1 = \{0,1,2,3,4\}$, $V_2 = \{5,6,7,8,9\}$, $V_3 = \{10,11\}$.

(i) Use the following copies of H_2 on $K_{|V_1|,|V_2|}$:

$H_{2,1} = [8,0,6,1;9]$, $H_{2,2} = [7,1,5,2;8]$, $H_{2,3} = [6,2,9,3;7]$, $H_{2,4} = [5,3,8,4;6]$, $H_{2,5} = [9,4,7,0;5]$. And by Lemma 2.2.1, we have $\{(s,t)\text{-case} | s+t=9, s \le 4\}$.

(ii) Let $G_{2,1} = [0, 8, 6; 1, 9]$, $G_{2,2} = [8, 9, 11; 5, 6]$, $G_{2,3} = [7, 9, 10; 8, 1]$, $G_{2,4} = [7, 11, 6; 9, 5]$, $G_{2,5} = [6, 10, 5; 8, 7]$. From (i), we can trade $H_{2,1} \cup (K_{V_2 \cup V_3} - \{10, 11\})$ for $G_{2,1} \cup G_{2,2} \cup G_{2,3} \cup G_{2,4} \cup G_{2,5}$. Hence, "$(5, 4)$-case" is done.

(iii) Let $G_{2,6} = [8, 9, 11; 5, 2]$, $G_{2,7} = [1, 5, 7; 2, 8]$. From (i), we can trade $H_{2,1} \cup H_{2,2} \cup (K_{V_2 \cup V_3} - \{10, 11\})$ for $G_{2,1} \cup G_{2,3} \cup G_{2,4} \cup G_{2,5} \cup G_{2,6} \cup G_{2,7}$. Hence, "$(6, 3)$-case" is done.

(iv) Let $G_{2,8} = [2, 9, 6; 3, 7]$, $G_{2,9} = [6, 11, 7; 8, 5]$, $G_{2,10} = [6, 10, 5; 9, 3]$. From (i), we can trade $H_{2,1} \cup H_{2,2} \cup H_{2,3} \cup (K_{V_2 \cup V_3} - \{10, 11\})$ for $G_{2,1} \cup G_{2,3} \cup G_{2,6} \cup G_{2,7} \cup G_{2,8} \cup G_{2,9} \cup G_{2,10}$. Hence, "$(7, 2)$-case" is done.

(v) Let $G_{2,11} = [6, 11, 7; 8, 4]$, $G_{2,12} = [3, 7, 6; 2, 9]$, $G_{2,13} = [3, 8, 5; 4, 6]$. From (i), we can trade $H_{2,1} \cup H_{2,2} \cup H_{2,3} \cup H_{2,4} \cup (K_{V_2 \cup V_3} - \{10, 11\})$ for $G_{2,1} \cup G_{2,3} \cup G_{2,6} \cup G_{2,7} \cup G_{2,10} \cup G_{2,11} \cup G_{2,12} \cup G_{2,13}$. Hence, "$(8, 1)$-case" is done.

(vi) Let $G_{2,14} = [7, 9, 10; 8, 0]$, $G_{2,15} = [6, 10, 5; 4, 7]$, $G_{2,16} = [0, 5, 9; 4, 6]$, $G_{2,17} = [1, 8, 6; 0, 7]$, $G_{2,18} = [5, 8, 3; 9, 1]$. From (i)., we can trade $H_{2,1} \cup H_{2,2} \cup H_{2,3} \cup H_{2,4} \cup H_{2,5} \cup (K_{V_2 \cup V_3} - \{10, 11\})$ for $G_{2,6} \cup G_{2,7} \cup G_{2,8} \cup G_{2,11} \cup G_{2,14} \cup G_{2,15} \cup G_{2,16} \cup G_{2,17} \cup G_{2,18}$. Hence, "$(9, 0)$-case" is done. ∎

Now, we are ready to consider the case $n \equiv 2 \pmod 5$.

Theorem 2.2.3. *If $n \equiv 2 \pmod 5$, then there exists a maximum (G_2, H_2)-hybrid packing of K_n with leave $L \cong K_2$, and a minimum (G_2, H_2)-hybrid covering of K_n with padding P having 4 edges.*

Proof. By Theorem 1.2.5, we will consider only the unsolved cases for $s, t \geq 1$. Let $\mathcal{H}$ be the collection of subgraphs which has a hybrid (G_2, H_2)-decomposition. First, we show that K_{n_i} has an $\mathcal{H}$-decomposition and also $K_{n_1,n_2,\cdots,n_h}$ has an $\mathcal{H}$-decomposition for some small values of n_i.

In the Appendix, we list the maximum (G_2, H_2)-hybrid packing for K_7, K_{12}, K_{17}, K_{22}, and K_{27}. Therefore, for the remainder of the proof, assume $n \geq 30$.

- K_{37}

  Let $V(K_{37}) = \{x_{i,j} | \forall i \in \mathbb{Z}_5, j \in \mathbb{Z}_7\} \cup \{v_1, v_2\}$, and let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$ where $j = 1, 2, \cdots, 7$, and $V_8 = \{v_1, v_2\}$.

  Since $K_{7(5)}$ can be partitioned into 7 $K_{5,5,5}$'s ($K_{|V_1|,|V_3|,|V_4|}$, $K_{|V_1|,|V_2|,|V_6|}$, $K_{|V_1|,|V_5|,|V_7|}$, $K_{|V_2|,|V_4|,|V_5|}$, $K_{|V_2|,|V_3|,|V_7|}$, $K_{|V_3|,|V_5|,|V_6|}$, $K_{|V_4|,|V_6|,|V_7|}$), K_{37} can be partitioned into $K_{|V_1|,|V_3|,|V_4|} \cup (K_{|V_1 \cup V_8|} - K_{|V_8|})$, $K_{|V_1|,|V_2|,|V_6|} \cup (K_{|V_2 \cup V_8|} - K_{|V_8|})$, $K_{|V_1|,|V_5|,|V_7|} \cup (K_{|V_7 \cup V_8|} - K_{|V_8|})$, $K_{|V_2|,|V_4|,|V_5|} \cup (K_{|V_4 \cup V_8|} - K_{|V_8|})$, $K_{|V_2|,|V_3|,|V_7|} \cup (K_{|V_3 \cup V_8|} - K_{|V_8|})$, $K_{|V_3|,|V_5|,|V_6|} \cup (K_{|V_5 \cup V_8|} - K_{|V_8|})$, $K_{|V_4|,|V_6|,|V_7|} \cup (K_{|V_6 \cup V_8|} - K_{|V_8|})$, and $K_{|V_8|}$. Each of the above has a decomposition into $K_{5,5,5}$ and $K_{12} - K_7$. (except $K_{|V_8|}$) Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5}$ as given in Lemma 2.1.3 and a hybrid (G_2, H_2)-decomposition of $K_{12} - K_7$ as given in Lemma 2.2.2 for each of the above subgraphs. What remains is K_2. Cover this edge by using a copy of G_2, or H_2, then we have the proof.

- K_{42}

  Let $V(K_{42}) = \{x_{i,j} | \forall i \in \mathbb{Z}_5, j \in \mathbb{Z}_8\} \cup \{v_1, v_2\}$, and let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$ where $j = 1, 2, \cdots, 8$, and $V_9 = \{v_1, v_2\}$.

  Use a hybrid (G_2, H_2)-decomposition $K_{5,5,5}$ as given in Lemma 2.1.3 for each of the following subgraphs: $K_{|V_1|,|V_3|,|V_4|}$, $K_{|V_1|,|V_2|,|V_6|}$, $K_{|V_1|,|V_5|,|V_7|}$, $K_{|V_2|,|V_4|,|V_5|}$, $K_{|V_2|,|V_3|,|V_7|}$, $K_{|V_3|,|V_5|,|V_6|}$, $K_{|V_4|,|V_6|,|V_7|}$.

Use a hybrid (G_2, H_2)-decomposition of $K_{12} - K_7$ as given in Lemma 2.2.2 for each of the following subgraphs: $K_{|V_2 \cup V_8 \cup V_9|} - K_{|V_8 \cup V_9|}$, $K_{|V_3 \cup V_8 \cup V_9|} - K_{|V_8 \cup V_9|}$, $K_{|V_4 \cup V_8 \cup V_9|} - K_{|V_8 \cup V_9|}$, $K_{|V_5 \cup V_8 \cup V_9|} - K_{|V_8 \cup V_9|}$, $K_{|V_6 \cup V_8 \cup V_9|} - K_{|V_8 \cup V_9|}$, $K_{|V_7 \cup V_8 \cup V_9|} - K_{|V_8 \cup V_9|}$. Use a maximum (G_2, H_2)-hybrid packing of K_{12} for the subgraph $K_{|V_1 \cup V_8 \cup V_9|}$. What remains is K_2. Cover this edge by using a copy of G_2, or H_2, then we have the proof.

- K_{97}

  By Theorem 2.1.2 with $r = 5$, $p = 5$, $q = 4$, $K_{25,25,25,20}$ has a hybrid (G_2, H_2)-decomposition. Since $K_{22} - K_2$, and $K_{27} - K_2$ have a hybrid (G_2, H_2)-decomposition, by Theorem 1.3.3, $K_{97} - K_2$ has a hybrid (G_2, H_2)-decomposition. What remains is K_2. Cover this edge by using a copy of G_2, or H_2, then we have the proof.

- K_{102}

  By Theorem 2.1.2 with $r = 5$, $p = q = 5$, $K_{25,25,25,25}$ has a hybrid (G_2, H_2)-decomposition. Since $K_{27} - K_2$ has a hybrid (G_2, H_2)-decomposition, by Theorem 1.3.3, $K_{102} - K_2$ has a hybrid (G_2, H_2)-decomposition. What remains is K_2. Cover this edge by using a copy of G_2, or H_2, then we have the proof.

Now, we are ready for the proof. As mentioned above, it suffices to decompose $K_n - K_2$ into a collection of subgraphs which have a hybrid (G_2, H_2)-decomposition, and this can be done recursively.

Let $n = 30t + 2, 30t + 7, 30t + 12, 30t + 17, 30t + 22$, and $30t + 27$. For the case $n = 30t + 2$ and $n = 30t + 17$, we can decompose K_{30t+2} (respectively K_{30t+17}) into K_{10t+2}'s (respectively K_{10t+7}'s) with two vertices in common and a $K_{10t,10t,10t}$ (respectively $K_{10t+5,10t+5,10t+5}$). Since K_{10t+2} (respectively K_{10t+7}) exists a maximum (G_2, H_2)-hybrid packing with leave $L \cong K_2$ (or a minimum (G_2, H_2)-hybrid covering with padding P having 4 edges), and $K_{10t,10t,10t}$ (respectively $K_{10t+5,10t+5,10t+5}$) has a hybrid (G_2, H_2)-decomposition, by Lemma 1.3.3, we have a maximum (G_2, H_2)-hybrid packing of K_{30t+2} (respectively K_{30t+17}) with leave $L \cong K_2$, and a minimum (G_2, H_2)-hybrid covering of K_{30t+2} (respectively K_{30t+17}) with padding P having 4 edges. Now, consider $n = 30t + 7$ and the other cases are similar. Clearly, K_{30t+7} can be decomposed into three K_{10t+2}'s and one K_7 (in which they have two vertices in common), and one $K_{10t,10t,10t,5}$. Since $K_{10t+2} - K_2$ has a hybrid (G_2, H_2)-decomposition, and K_7 exists a maximum (G_2, H_2)-hybrid packing with leave $L \cong K_2$ (or a minimum (G_2, H_2)-hybrid covering with padding P having 4 edges), the proof of this case follows. Again, $K_{10t,10t,10t,5}$ can be decomposed into $K_{5,5,5,5}$'s and $K_{5,5,5}$'s only if $t \neq 1$ or 3. Hence we have to deal with $n = 37$ and $n = 97$ independently as above, and then the proof follows. ∎

Next, we consider the case $n \equiv 3 \pmod 5$.

Lemma 2.2.4. $K_8 - K_3$ has a hybrid (G_2, H_2)-decomposition.

Proof. Let (s, t) be an admissible pair of $(K_8 - K_3)$. Let $V(K_8) = \mathbb{Z}_8$.

(i) Use the following copies of G_2:

$G_2 \cong [0, 7, 5; 6, 1]$, $[3, 7, 6; 0, 4]$, $[1, 5, 4; 6, 2]$, $[2, 5, 3; 1, 7]$, $[2, 7, 4; 3, 0]$.

What remains is the complete graph $K_{|V_1|}$ where $V_1 = \{0, 1, 2\}$. Hence, "$(5, 0)$-case" is done.

(ii) Use the following copies of G_2 and H_2:

$G_2 \cong [6, 7, 3; 4, 5]$, $[1, 6, 4; 0, 5]$, $[1, 3, 5; 2, 6]$, $[4, 7, 2; 3, 0]$; $H_2 \cong [0, 6, 5, 7; 1]$.

What remains is the complete graph $K_{|V_1|}$ where $V_1 = \{0, 1, 2\}$. Hence, "$(4, 1)$-case" is done.

(iii) Use the following copies of G_2 and H_2:

$G_2 \cong [6, 7, 3; 4, 5]$, $[1, 6, 4; 7, 2]$, $[1, 3, 5; 2, 6]$; $H_2 \cong [0, 6, 5, 7; 1]$, $[3, 2, 4, 0; 5]$.

What remains is the complete graph $K_{|V_1|}$ where $V_1 = \{0, 1, 2\}$. Hence, "$(3, 2)$-case" is done.

(iv) Use the following copies of G_2 and H_2:

$G_2 \cong [6, 7, 3; 4, 5]$, $[3, 5, 1; 6, 0]$; $H_2 \cong [0, 4, 1, 7; 5]$, $[2, 7, 4, 6; 5]$, $[3, 0, 5, 2; 4]$.

What remains is the complete graph $K_{|V_1|}$ where $V_1 = \{0, 1, 2\}$. Hence, "$(2, 3)$-case" is done.

(v) Use the following copies of G_2 and H_2:

$G_2 \cong [5, 7, 3; 6, 0]$; $H_2 \cong [0, 4, 1, 7; 6]$, $[2, 7, 4, 6; 5]$, $[3, 0, 5, 2; 4]$, $[3, 4, 5, 1; 6]$.

What remains is the complete graph $K_{|V_1|}$ where $V_1 = \{0, 1, 2\}$. Hence, "$(1, 4)$-case" is done.

(vi) Use the following copies of H_2:

$H_2 \cong [0, 4, 1, 7; 6]$, $[2, 7, 5, 6; 0]$, $[3, 0, 5, 2; 4]$, $[3, 4, 5, 1; 6]$, $[6, 4, 7, 3; 5]$.

What remains is the complete graph $K_{|V_1|}$ where $V_1 = \{0, 1, 2\}$. Hence, "$(0, 5)$-case" is done. ∎

Lemma 2.2.5. $K_{13} - K_8$ has a hybrid (G_2, H_2)-decomposition.

Proof. Let (s,t) be an admissible pair of $(K_{13} - K_8)$. Let $V(K_{13}) = \mathbb{Z}_{13}$, and let $V_1 = \{0,1,2,3,4\}$, $V_2 = \{5,6,7,8,9\}$, $V_3 = \{10,11,12\}$.

(i) Use the following copies of H_2 on $K_{|V_1|,|V_2|}$:

$H_{2,1} = [6,0,8,1;5]$, $H_{2,2} = [7,1,9,2;6]$, $H_{2,3} = [8,2,5,3;7]$, $H_{2,4} = [9,3,6,4;8]$, $H_{2,5} = [5,4,7,0;9]$. And by Lemma 2.2.4, we have $\{(s,t)\text{-case } | s+t = 10, s \leq 5\}$.

(ii) Let $G_{2,1} = [1,5,8;0,6]$, $G_{2,2} = [5,6,12;9,7]$, $G_{2,3} = [5,10,9;6,1]$, $G_{2,4} = [5,7,11;9,8]$, $G_{2,5} = [8,11,6;7,10]$, $G_{2,6} = [7,12,8;10,6]$. From (i), we can trade $H_{2,1} \cup (K_{|V_2 \cup V_3|} - K_{|V_3|})$ for $G_{2,1} \cup G_{2,2} \cup G_{2,3} \cup G_{2,4} \cup G_{2,5} \cup G_{2,6}$. Hence, "$(6,4)$-case" is done.

(iii) Let $G_{2,7} = [9,10,5;6,0]$, $G_{2,8} = [5,7,1;8,10]$, $G_{2,9} = [6,9,2;7,11]$, $G_{2,10} = [6,10,7;8,0]$, $G_{2,11} = [6,11,8;9,1]$, $G_{2,12} = [8,12,5;11,9]$, $G_{2,13} = [7,9,12;6,1]$. From (i), we can trade $H_{2,1} \cup H_{2,2} \cup (K_{|V_2 \cup V_3|} - K_{|V_3|})$ for $G_{2,7} \cup G_{2,8} \cup G_{2,9} \cup G_{2,10} \cup G_{2,11} \cup G_{2,12} \cup G_{2,13}$. Hence, "$(7,3)$-case" is done.

(iv) Let $G_{2,14} = [5,10,6;7,3]$, $G_{2,15} = [6,11,8;9,10]$, $G_{2,16} = [9,11,5;2,8]$, $G_{2,17} = [5,8,12;6,0]$, $G_{2,18} = [5,7,1;8,0]$, $G_{2,19} = [7,12,9;1,6]$, $G_{2,20} = [7,10,8;3,5]$, $G_{2,21} = [6,9,2;7,11]$. From (i), we can trade $H_{2,1} \cup H_{2,2} \cup H_{2,3} \cup (K_{|V_2 \cup V_3|} - K_{|V_3|})$ for $G_{2,14} \cup G_{2,15} \cup G_{2,16} \cup G_{2,17} \cup G_{2,18} \cup G_{2,19} \cup G_{2,20} \cup G_{2,21}$. Hence, "$(8,2)$-case" is done.

(v) Let $G_{2,22} = [2,8,5;3,7]$, $G_{2,23} = [8,11,6;1,9]$, $G_{2,24} = [5,11,9;3,6]$, $G_{2,25} = [6,7,10;8,3]$, $G_{2,26} = [8,9,4;6,0]$, $G_{2,27} = [5,6,12;8,7]$, $G_{2,28} = [7,12,9;10,5]$. From (i), we can trade $H_{2,1}\cup$ $H_{2,2}\cup$ $H_{2,3}\cup$ $H_{2,4}\cup$ $(K_{|V_2\cup V_3|} - K_{|V_3|})$ for $G_{2,18}\cup$ $G_{2,21}\cup$ $G_{2,22}\cup$ $G_{2,23}\cup$ $G_{2,24}\cup$ $G_{2,25}\cup$ $G_{2,26}\cup$ $G_{2,27}\cup$ $G_{2,28}$. Hence, "$(9,1)$-case" is done.

(vi) Let $G_{2,29} = [7,8,0;9,3]$, $G_{2,30} = [5,7,1;8,2]$, $G_{2,31} = [5,9,11;7,2]$, $G_{2,32} = [3,6,5;2,7]$, $G_{2,33} = [8,9,4;5,0]$, $G_{2,34} = [2,9,6;4,7]$. From (i), we can trade $H_{2,1}\cup$ $H_{2,2}\cup$ $H_{2,3}\cup$ $H_{2,4}\cup$ $H_{2,5}\cup$ $(K_{|V_2\cup V_3|} - K_{|V_3|})$ for $G_{2,17}\cup$ $G_{2,23}\cup$ $G_{2,25}\cup$ $G_{2,28}\cup$ $G_{2,29}$ $G_{2,30}\cup$ $G_{2,31}\cup$ $G_{2,32}\cup$ $G_{2,33}\cup$ $G_{2,34}$. Hence, "$(10,0)$-case" is done. ∎

Theorem 2.2.6. *If* $n \equiv 3 \pmod 5$*, then there exists a maximum* (G_2, H_2)*-hybrid packing of* K_n *with leave* $L \cong K_3$*, and a minimum* (G_2, H_2)*-hybrid covering of* K_n *with padding* P *having* 2 *edges.*

Proof. By Theorem 1.2.5, we will consider only the unsolved cases for $s, t \geq 1$. Let $\mathcal{H}$ be the collection of subgraphs which has a hybrid (G_2, H_2)-decomposition. First, we show that K_{n_i} has an $\mathcal{H}$-decomposition and also $K_{n_1,n_2,\cdots,n_h}$ has an $\mathcal{H}$-decomposition for some small values of n_i.

In the Appendix, we list the maximum (G_2, H_2)-hybrid packing for K_8, K_{13}, K_{18}, K_{23}, and K_{28}. Therefore, for the remainder of the proof, assume $n \geq 30$.

- K_{38}

  Let $V(K_{38}) = \{x_{i,j} | \forall i \in \mathbb{Z}_5, j \in \mathbb{Z}_7\} \cup \{v_1, v_2, v_3\}$, and let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$ where $j = 1, 2, \cdots, 7$, $V_8 = \{v_1, v_2, v_3\}$.

  Since $K_{7(5)}$ can be partitioned into 7 $K_{5,5,5}$'s ($K_{|V_1|,|V_3|,|V_4|}$, $K_{|V_1|,|V_2|,|V_6|}$, $K_{|V_1|,|V_5|,|V_7|}$, $K_{|V_2|,|V_4|,|V_5|}$, $K_{|V_2|,|V_3|,|V_7|}$, $K_{|V_3|,|V_5|,|V_6|}$, $K_{|V_4|,|V_6|,|V_7|}$), K_{38} can be partitioned into $K_{|V_1|,|V_3|,|V_4|}\cup$ $(K_{|V_1\cup V_8|} - K_{|V_8|})$, $K_{|V_1|,|V_2|,|V_6|} \cup (K_{|V_2\cup V_8|} - K_{|V_8|})$, $K_{|V_1|,|V_5|,|V_7|} \cup (K_{|V_7\cup V_8|} - K_{|V_8|})$,

$K_{|V_2|,|V_4|,|V_5|} \cup (K_{|V_4 \cup V_8|} - K_{|V_8|})$, $K_{|V_2|,|V_3|,|V_7|} \cup (K_{|V_3 \cup V_8|} - K_{|V_8|})$, $K_{|V_3|,|V_5|,|V_6|} \cup (K_{|V_5 \cup V_8|} - K_{|V_8|})$, $K_{|V_4|,|V_6|,|V_7|} \cup (K_{|V_6 \cup V_8|} - K_{|V_8|})$, and $K_{|V_8|}$. Each of the above has a decomposition into $K_{5,5,5}$ and $K_{13} - K_8$. (except $K_{|V_8|}$) Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5}$ as given in Lemma 2.1.3 and a hybrid (G_2, H_2)-decomposition of $K_{13} - K_8$ as given in Lemma 2.2.5 for each of the above subgraphs. What remains is K_3. Cover this edge by using a copy of G_2, then we have the proof.

- K_{43}

Let $V(K_{43}) = \{x_{i,j} | \forall i \in \mathbb{Z}_5, j \in \mathbb{Z}_8\} \cup \{v_1, v_2, v_3\}$, and let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$ where $j = 1, 2, \cdots, 8$, $V_9 = \{v_1, v_2, v_3\}$.

Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5}$ as given in Lemma 2.1.3 for each of the following subgraphs: $K_{|V_1|,|V_3|,|V_4|}$, $K_{|V_1|,|V_2|,|V_6|}$, $K_{|V_1|,|V_5|,|V_7|}$, $K_{|V_2|,|V_4|,|V_5|}$, $K_{|V_2|,|V_3|,|V_7|}$, $K_{|V_3|,|V_5|,|V_6|}$, $K_{|V_4|,|V_6|,|V_7|}$. Use a hybrid (G_2, H_2)-decomposition of $K_{13} - K_8$ as given in Lemma 2.2.5 for each of the following subgraphs: $K_{|V_1 \cup V_8 \cup V_9|} - K_{|V_8 \cup V_9|}$, $K_{|V_2 \cup V_8 \cup V_9|} - K_{|V_8 \cup V_9|}$, $K_{|V_3 \cup V_8 \cup V_9|} - K_{|V_8 \cup V_9|}$, $K_{|V_4 \cup V_8 \cup V_9|} - K_{|V_8 \cup V_9|}$, $K_{|V_5 \cup V_8 \cup V_9|} - K_{|V_8 \cup V_9|}$, $K_{|V_6 \cup V_8 \cup V_9|} - K_{|V_8 \cup V_9|}$, $K_{|V_7 \cup V_8 \cup V_9|} - K_{|V_8 \cup V_9|}$. What remains is K_3. Cover this edge by using a copy of G_2, then we have the proof.

- K_{98}

By Theorem 2.1.2 with $r = 5$, $p = 5$, $q = 4$, $K_{25,25,25,20}$ has a hybrid (G_2, H_2)-decomposition. Since $K_{23} - K_3$, and $K_{28} - K_3$ have a hybrid (G_2, H_2)-decomposition, by Theorem 1.3.4, $K_{98} - K_3$ has a hybrid (G_2, H_2)-decomposition. What remains is K_3. Cover this edge by using a copy of G_2, then we have the proof.

- K_{103}

  By Theorem 2.1.2 with $r = 5$, $p = q = 5$, $K_{25,25,25,25}$ has a hybrid (G_2, H_2)-decomposition. Since $K_{28} - K_3$ has a hybrid (G_2, H_2)-decomposition, by Theorem 1.3.4, K_{103} has hybrid (G_2, H_2)-decomposition. What remains is K_3. Cover this edge by using a copy of G_2, then we have the proof.

Now, we are ready for the proof. As mentioned above, it suffices to decompose $K_n - K_3$ into a collection of subgraphs which have a hybrid (G_2, H_2)-decomposition, and this can be done recursively.

Let $n = 30t + 3, 30t + 8, 30t + 13, 30t + 18, 30t + 23$, and $30t + 28$. For the case $n = 30t + 3$ and $n = 30t + 18$, we can decompose K_{30t+3} (respectively K_{30t+18}) into K_{10t+3}'s (respectively K_{10t+8}'s) with three vertices in common and a $K_{10t,10t,10t}$ (respectively $K_{10t+5,10t+5,10t+5}$). Since K_{10t+3} (respectively K_{10t+8}) exists a maximum (G_2, H_2)-hybrid packing with leave $L \cong K_3$ (or a minimum (G_2, H_2)-hybrid covering with padding P having 2 edges), and $K_{10t,10t,10t}$ (respectively $K_{10t+5,10t+5,10t+5}$) has a hybrid (G_2, H_2)-decomposition, by Lemma 1.3.3, we have a maximum (G_2, H_2)-hybrid packing of K_{30t+3} (respectively K_{30t+18}) with leave $L \cong K_3$, and a minimum (G_2, H_2)-hybrid covering of K_{30t+3} (respectively K_{30t+18}) with padding P having 2 edges. Now, consider $n = 30t + 8$ and the other cases are similar. Clearly, K_{30t+8} can be decomposed into three K_{10t+3}'s and one K_8 (in which they have three vertices in common), and one $K_{10t,10t,10t,5}$. Since $K_{10t+3} - K_3$ has a hybrid (G_2, H_2)-decomposition, and K_8 exists a maximum (G_2, H_2)-hybrid packing with leave $L \cong K_3$ (or a minimum (G_2, H_2)-hybrid covering with padding P having 2 edges), the proof of this case follows. Again, $K_{10t,10t,10t,5}$ can be decomposed into $K_{5,5,5,5}$'s and $K_{5,5,5}$'s only if $t \neq 1$ or 3. Hence we have to deal with $n = 38$ and $n = 98$ independently as above, and then the proof follows. ∎

Finally, we consider the case $n \equiv 4 \pmod 5$.

Lemma 2.2.7. There exists a hybrid (G_2, H_2)-decomposition of $K_9 - K_4$.

Proof. Let (s,t) be an admissible pair of $(K_9 - K_4)$ and let $V(K_9) = \mathbb{Z}_9$.

(i) Use the following copies of G_2:

$G_2 \cong [0,3,6;2,1]$, $[0,2,5;1,6]$, $[1,3,8;4,6]$, $[3,5,4;2,8]$, $[4,7,1;0,8]$, $[2,3,7;0,4]$.

Hence, "$(6,0)$-case" is done.

(ii) Use the following copies of G_2 and H_2:

$G_2 \cong [3,6,0;2,1]$, $[1,3,8;4,6]$, $[3,5,4;2,8]$, $[4,7,1;0,8]$, $[2,3,7;0,4]$; $H_2 \cong [1,6,2,5;0]$.

Hence, "$(5,1)$-case" is done.

(iii) Use the following copies of G_2 and H_2:

$G_2 \cong [1,5,4;2,3]$, $[6,7,1;3,5]$, $[5,8,7;3,4]$, $[3,6,8;4,7]$; $H_2 \cong [6,0,7,2;5]$, $[4,6,5,0;3]$.

Hence, "$(4,2)$-case" is done.

(iv) Use the following copies of G_2 and H_2:

$G_2 \cong [4,5,1;3,0]$, $[7,8,3;2,4]$, $[1,7,6;3,5]$; $H_2 \cong [0,7,2,6;8]$, $[0,5,6,4;3]$, $[7,4,8,5;2]$.

Hence, "$(3,3)$-case" is done.

(v) Use the following copies of G_2 and H_2:

$G_2 \cong [7,8,3;1,5]$, $[1,7,6;3,4]$; $H_2 \cong [6,2,7,0;3]$, $[2,3,5,4;1]$, $[7,4,8,5;2]$, $[4,0,5,6;8]$.

Hence, "$(2,4)$-case" is done.

(vi) Use the following copies of G_2 and H_2:

$G_2 \cong [6,7,1;5,2]$; $H_2 \cong [6,2,7,0;3]$, $[0,5,6,4;3]$, $[2,3,5,4;1]$, $[4,7,5,8;3]$, $[6,8,7,3;1]$.

Hence, "$(1,5)$-case" is done.

(vii) Use the following copies of H_2:

$H_2 \cong [0,6,2,7;8]$, $[4,8,5,7;1]$, $[1,6,8,3;4]$, $[4,1,5,2;3]$, $[3,6,4,0;5]$, $[3,7,6,5;4]$.

Hence, "$(0,6)$-case" is done. ∎

Lemma 2.2.8. There exists a hybrid (G_2, H_2)-decomposition of $K_n - K_4$ for $n \equiv 4 \pmod 5$.

Proof. By Theorem 1.2.5, we will consider only the unsolved cases for $s, t \geq 1$. Let $\mathcal{H}$ be the collection of subgraphs which has a hybrid (G_2, H_2)-decomposition. First, we show that K_{n_i} has an $\mathcal{H}$-decomposition and also $K_{n_1,n_2,\cdots,n_h}$ has an $\mathcal{H}$-decomposition for some small values of n_i.

In the Appendix, we list the hybrid (G_2, H_2)-decompositions for $K_9 - K_4$, $K_{14} - K_4$, $K_{19} - K_4$, $K_{24} - K_4$, and $K_{29} - K_4$. Therefore, for the remainder of the proof, assume $n \geq 30$.

- $K_{39} - K_4$

  Let $V(K_{39}) = \{x_{i,j} | \forall i \in \mathbb{Z}_5, j \in \mathbb{Z}_7\} \cup \{x_1, x_2, x_3, x_4\}$, and let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$ where $j = 1, 2, \cdots, 7$, and $V_8 = \{x_1, x_2, x_3, x_4\}$.

  Since $K_{7(5)}$ can be partitioned into 7 $K_{5,5,5}$'s ($K_{|V_1|,|V_3|,|V_4|}$, $K_{|V_1|,|V_2|,|V_6|}$, $K_{|V_1|,|V_5|,|V_7|}$, $K_{|V_2|,|V_4|,|V_5|}$, $K_{|V_2|,|V_3|,|V_7|}$, $K_{|V_3|,|V_5|,|V_6|}$, $K_{|V_4|,|V_6|,|V_7|}$), $K_{39} - K_4$ can be partitioned into $K_{|V_1|,|V_3|,|V_4|} \cup (K_{|V_1 \cup V_8|} - K_{|V_8|})$, $K_{|V_1|,|V_2|,|V_6|} \cup (K_{|V_2 \cup V_8|} - K_{|V_8|})$, $K_{|V_1|,|V_5|,|V_7|} \cup$

$(K_{|V_7 \cup V_8|} - K_{|V_8|})$, $K_{|V_2|,|V_4|,|V_5|} \cup (K_{|V_4 \cup V_8|} - K_{|V_8|})$, $K_{|V_2|,|V_3|,|V_7|} \cup (K_{|V_3 \cup V_8|} - K_{|V_8|})$, $K_{|V_3|,|V_5|,|V_6|} \cup (K_{|V_5 \cup V_8|} - K_{|V_8|})$, $K_{|V_4|,|V_6|,|V_7|} \cup (K_{|V_6 \cup V_8|} - K_{|V_8|})$. Each of the above has a decomposition into $K_{5,5,5}$ and $K_9 - K_4$. Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5}$ as given in Lemma 2.1.3 and a hybrid (G_2, H_2)-decomposition of $K_9 - K_4$ as given in Lemma 2.2.7 for each of the above subgraphs.

- $K_{44} - K_4$

  Let $V(K_{44}) = \{x_{i,j} | \forall i \in \mathbb{Z}_5, j \in \mathbb{Z}_8\} \cup \{x_1, x_2, x_3, x_4\}$, and let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$ where $j = 1, 2, \cdots, 8$, and $V_9 = \{x_1, x_2, x_3, x_4\}$.

  Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5}$ as given in Lemma 2.1.3 for each of the following subgraphs: $K_{|V_1|,|V_2|,|V_4|}$, $K_{|V_2|,|V_3|,|V_5|}$, $K_{|V_3|,|V_4|,|V_6|}$, $K_{|V_4|,|V_5|,|V_7|}$, $K_{|V_5|,|V_6|,|V_8|}$, $K_{|V_6|,|V_7|,|V_1|}$, $K_{|V_7|,|V_8|,|V_2|}$, $K_{|V_8|,|V_1|,|V_3|}$. Use a hybrid (G_2, H_2)-decomposition of $K_{14} - K_4$ for each of the following subgraphs: $K_{|V_1 \cup V_5 \cup V_9|} - K_{|V_9|}$, $K_{|V_2 \cup V_6 \cup V_9|} - K_{|V_9|}$, $K_{|V_3 \cup V_7 \cup V_9|} - K_{|V_9|}$, $K_{|V_4 \cup V_8 \cup V_9|} - K_{|V_9|}$.

- $K_{99} - K_4$

  By Theorem 2.1.2 with $r = 5$, $p = 5$, $q = 4$, $K_{25,25,25,20}$ has a hybrid (G_2, H_2)-decomposition. Since $K_{24} - K_4$, and $K_{29} - K_4$ have a hybrid (G_2, H_2)-decomposition, by Theorem 1.3.5, K_{99} is a hybrid (G_2, H_2)-decomposition.

- $K_{104} - K_4$

  By Theorem 2.1.2 with $r = 5$, $p = q = 5$, $K_{25,25,25,25}$ has a hybrid (G_2, H_2)-decomposition. Since $K_{29} - K_4$ has a hybrid (G_2, H_2)-decomposition, by Theorem 1.3.5, K_{104} is a hybrid (G_2, H_2)-decomposition.

Now, we are ready for the proof. As mentioned above, it suffices to decompose $K_n - K_4$ into a collection of subgraphs which have a hybrid (G_2, H_2)-decomposition, and this can be done recursively.

Let $n = 30t + 4, 30t + 9, 30t + 14, 30t + 19, 30t + 24,$ and $30t + 29$. For the case $n = 30t+4$ and $n = 30t+19$, we can decompose $K_{30t+4} - K_4$ (respectively $K_{30t+19} - K_4$) into $K_{10t+4} - K_4$'s (respectively $K_{10t+9} - K_4$'s) with four vertices in common and a $K_{10t,10t,10t}$ (respectively $K_{10t+5,10t+5,10t+5}$). Then, the proof follows by combining the hybrid (G_2, H_2)-decompositions of these subgraphs together. Now, consider $n = 30t+9$ and the other cases are similar. Clearly, K_{30t+9} can be decomposed into three $K_{10t+4} - K_4$'s and one $K_9 - K_4$ (in which they have four vertices in common), and one $K_{10t,10t,10t,5}$. Since each of them has a hybrid (G_2, H_2)-decomposition, the proof of this case follows. Again, $K_{10t,10t,10t,5}$ can be decomposed into $K_{5,5,5,5}$'s and $K_{5,5,5}$'s only if $t \neq 1$ or 3. Hence we have to deal with $n = 39$ and $n = 99$ independently as above, and then we have proof. ∎

Theorem 2.2.9. *If* $n \equiv 4 \pmod 5$*, then there exists a maximum* (G_2, H_2)*-hybrid packing of* K_n *with leave* $L \cong K_2$*, and a minimum* (G_2, H_2)*-hybrid covering of* K_n *with padding* P *having* 4 *edges.*

Proof. By Theorem 1.2.5, we will consider only the unsolved cases for $s, t \geq 1$. Let $\mathcal{H}$ be the collection of subgraphs which has a hybrid (G_2, H_2)-decomposition. First, we show that K_{n_i} has an $\mathcal{H}$-decomposition and also $K_{n_1,n_2,\cdots,n_h}$ has an $\mathcal{H}$-decomposition for some small values of n_i.

In the Appendix, we list the maximum (G_2, H_2)-hybrid packing for K_9, K_{14}, K_{19}, K_{24}, and K_{29}. Therefore for the remainder of the proof, assume $n \geq 30$.

- K_{39}

  Let $V(K_{39}) = \{x_{i,j} | \forall i \in \mathbb{Z}_5, j \in \mathbb{Z}_7\} \cup \{x_1, x_2, x_3, x_4\}$, and let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$ where $j = 1, 2, \cdots, 7$, and $V_8 = \{x_1, x_2, x_3, x_4\}$.

  Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5}$ as given in Lemma 2.1.3 for each of the following subgraphs:

  $K_{|V_1|,|V_3|,|V_4|}$, $K_{|V_1|,|V_2|,|V_6|}$, $K_{|V_1|,|V_5|,|V_7|}$, $K_{|V_2|,|V_4|,|V_5|}$, $K_{|V_2|,|V_3|,|V_7|}$, $K_{|V_3|,|V_5|,|V_6|}$, $K_{|V_4|,|V_6|,|V_7|}$.

  Use a hybrid (G_2, H_2)-decomposition of $K_9 - K_4$ as given in Lemma 2.2.7 for each of the following subgraphs: $K_{|V_2 \cup V_8|} - K_{|V_8|}$, $K_{|V_3 \cup V_8|} - K_{|V_8|}$, $K_{|V_4 \cup V_8|} - K_{|V_8|}$, $K_{|V_5 \cup V_8|} - K_{|V_8|}$, $K_{|V_6 \cup V_8|} - K_{|V_8|}$, $K_{|V_7 \cup V_8|} - K_{|V_8|}$.

  Use a maximum (G_2, H_2)-hybrid packing of K_9 for $K_{|V_1 \cup V_8|}$. What remains is a single edge which we may cover it by using a copy of G_2 or H_2, then we have the proof.

- K_{44}

  Let $V(K_{44}) = \{x_{i,j} | \forall i \in \mathbb{Z}_5, j \in \mathbb{Z}_8\} \cup \{x_1, x_2, x_3, x_4\}$, and let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$ where $j = 1, 2, \cdots, 8$, and $V_9 = \{x_1, x_2, x_3, x_4\}$.

  Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5}$ as given in Lemma 2.1.3 for each of the following subgraphs: $K_{|V_1|,|V_2|,|V_7|}$, $K_{|V_2|,|V_3|,|V_8|}$, $K_{|V_3|,|V_4|,|V_1|}$, $K_{|V_4|,|V_5|,|V_2|}$, $K_{|V_5|,|V_6|,|V_3|}$, $K_{|V_6|,|V_7|,|V_4|}$, $K_{|V_7|,|V_8|,|V_5|}$, $K_{|V_8|,|V_1|,|V_6|}$. Use a hybrid (G_2, H_2)-decomposition of $K_{14} - K_4$ for each of the following subgraphs: $K_{|V_2 \cup V_6 \cup V_9|} - K_{|V_9|}$, $K_{|V_3 \cup V_7 \cup V_9|} - K_{|V_9|}$, $K_{|V_4 \cup V_8 \cup V_9|} - K_{|V_9|}$. Use a maximum (G_2, H_2)-hybrid packing of K_{14} for $K_{|V_1 \cup V_5 \cup V_9|}$. What remains is a single edge which we may cover it by using a copy of G_2 or H_2, then we have the proof.

- K_{99}

  By Theorem 2.1.2 with $r = 5$, $p = 5$, $q = 4$, $K_{25,25,25,20}$ has a hybrid (G_2, H_2)-decomposition. Since K_{24} has a maximum (G_2, H_2)-hybrid packing with leave K_2, and $K_{29} - K_4$ has a hybrid (G_2, H_2)-decomposition, by Theorem 1.3.6, K_{99} has a maximum (G_2, H_2)-hybrid packing with leave $L \cong K_2$, then we have the proof.

- K_{104}

  By Theorem 2.1.2 with $r = 5$, $p = q = 5$, $K_{25,25,25,25}$ has a hybrid (G_2, H_2)-decomposition. Since K_{29} has a maximum (G_2, H_2)-hybrid packing with leave K_2, and $K_{29} - K_4$ has a hybrid (G_2, H_2)-decomposition, by Theorem 1.3.6, K_{104} has a maximum (G_2, H_2)-hybrid packing with leave $L \cong K_2$, then we have the proof.

Now, we are ready for the proof. As mentioned above, it suffices to decompose $K_n - K_2$ into a collection of subgraphs which have a hybrid (G_2, H_2)-decomposition, and this can be done recursively.

Let $n = 30t + 4, 30t + 9, 30t + 14, 30t + 19, 30t + 24$, and $30t + 29$. For the case $n = 30t + 4$ and $n = 30t + 19$, we can decompose K_{30t+4} (respectively K_{30t+19}) into three copies of K_{10t+4} (respectively K_{10t+9}) with four vertices in common and a $K_{10t,10t,10t}$ (respectively $K_{10t+5,10t+5,10t+5}$). Since $K_{10t+4} - K_4$ (respectively $K_{10t+9} - K_4$) has a hybrid (G_2, H_2)-decomposition, and K_{10t+4} (respectively K_{10t+9}) exists a maximum (G_2, H_2)-hybrid packing with leave $L \cong K_2$ (or a minimum (G_2, H_2)-hybrid covering with padding P having 4 edges), and $K_{10t,10t,10t}$ (respectively $K_{10t+5,10t+5,10t+5}$) has a hybrid (G_2, H_2)-decomposition, by Lemma 1.3.6, we have a maximum (G_2, H_2)-hybrid packing of K_{30t+4} (respectively K_{30t+19}) with leave $L \cong K_2$, and a minimum (G_2, H_2)-hybrid covering of

K_{30t+4} (respectively K_{30t+19}) with padding P having 4 edges. Now, consider $n = 30t + 9$ and the other cases are similar. Clearly, K_{30t+9} can be decomposed into three K_{10t+4}'s and one K_9 (in which they have four vertices in common), and one $K_{10t,10t,10t,5}$. Since $K_{10t+4} - K_4$ has a hybrid (G_2, H_2)-decomposition, and K_9 exists a maximum (G_2, H_2)-hybrid packing with leave $L \cong K_2$ (or a minimum (G_2, H_2)-hybrid covering with padding P having 4 edges), the proof of this case follows. Again, $K_{10t,10t,10t,5}$ can be decomposed into $K_{5,5,5,5}$'s and $K_{5,5,5}$'s only if $t \neq 1$ or 3. Hence we have to deal with $n = 39$ and $n = 99$ independently as above, and then we have the proof. ∎

2.3 (G_2, H_2)-Hybrid Decomposition of λK_n

We start with several lemmas.

Theorem 2.3.1. *There is a (G_2, H_2)-hybrid decomposition of λK_n for all $\lambda \geq 1$ and $n \equiv 0, 1 \pmod 5$.*

Proof. By Theorem 2.1.7, there is an exact (G_2, H_2)-hybrid decomposition for $n \equiv 0, 1 \pmod 5$ on K_n, $n \geq 5$. So there is a hybrid decomposition for any λK_n for those same values of n. ∎

For $n \equiv 2, 4 \pmod 5$, we have the following:

Theorem 2.3.2. *If $n \equiv 2, 4 \pmod 5$, then $\forall \lambda \geq 1$, there is a maximum (G_2, H_2)-hybrid packing of λK_n with $e(L(\lambda K_n)) = \lambda'$, and a minimum (G_2, H_2)-hybrid covering of λK_n with $e(P(\lambda K_n)) = 5 - \lambda'$ where $1 \leq \lambda' \leq 5$ and $\lambda' \equiv \lambda \pmod 5$.*

Proof. Since $e(G_2) = 5$ and $e((H_2)) = 5$, it suffices to consider $1 \leq \lambda' \leq 5$ where $\lambda' \equiv \lambda \pmod 5$.

By Theorems 2.2.3 and 2.2.9, it was shown that a (G_2, H_2)-hybrid packing of K_n with leave $L \cong K_2$. Since each K_n can be packed independently, we may assume the leave be $\{1,2\}$, $\{2,0\}$, $\{1,0\}$, $\{0,3\}$ and $\{3,4\}$ respectively for different K_n, or $\{1,2\}$, $\{2,3\}$, $\{3,0\}$, $\{0,1\}$ and $\{0,4\}$ as the case may be. Then, as $\lambda' = 5$, we have an extra G_2 or H_2 and λK_n can be decomposed completely with G_2 and H_2. Thus, the proof follows. ∎

For $n \equiv 3$ mod 5, we have the following:

Theorem 2.3.3. *Let $1 \leq \lambda^* \leq 5$. If $n \equiv 3 \pmod 5$, then $\forall \lambda \geq 1$ and $\lambda \equiv \lambda^* \pmod 5$, there is a maximum (G_2, H_2)-hybrid packing of λK_n with $e(L(\lambda K_n)) = 3\lambda^*$, and a minimum (G_2, H_2)-hybrid covering of λK_n with $e(P(\lambda K_n)) = 2\lambda^*$.*

Proof. Since $e(G_2) = 5$ and $e((H_2)) = 5$, it suffices to consider $1 \leq \lambda^* \leq 5$ where $\lambda^* \equiv \lambda \pmod 5$.

By Theorems 2.2.6, it was shown that a (G_2, H_2)-hybrid packing of K_n with leave $L \cong K_3$. Since each K_n can be packed independently, we may assume the leave be $K_{|\{0,1,2\}|}$, $K_{|\{1,3,4\}|}$, $K_{|\{0,1,4\}|}$, $K_{|\{2,3,4\}|}$ and $K_{|\{1,2,3\}|}$ respectively for different K_n. Then, as $\lambda' = 5$, we have three extra G_2 and λK_n can be decomposed completely with G_2 and H_2. Thus, the proof follows. ∎

Conclusion

As can be seen from Chapter 2, it takes a lot of effort to obtain a hybrid decomposition, since there are too many cases to consider. But, indeed, we believe that for each graph pair of order 5, similar results can be obtained as we have done in this thesis. It is more complicate due to the reason that we have to trade $3G_1$ for $2H_1$ if it is possible, so are the other graph pairs not considered here. We do expect that the other hybrid decompositions can be obtained in the near future.

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Appendix

1. Hybrid (G_2, H_2)-decomposition of K_{10}

Let (s,t) be an admissible pair of K_{10}.

(i) By Theorem 1.2.5, "$(9,0)$-case", and "$(0,9)$-case" hold.

(ii) By Lemma 2.1.5, $K_{10} - K_5$ has a hybrid (G_2, H_2)-decomposition. And $K_5 = 1G_2 + 1H_2$ (observation), we have "$\{(s,t)$-case $|s,t \geq 1\}$".

2. Hybrid (G_2, H_2)-decomposition of K_{11}

Let (s,t) be an admissible pair of K_{11}.

(i) By Theorem 1.2.5, "$(11,0)$-case", and "$(0,11)$-case" hold.

(ii) By Lemma 2.1.6, $K_{11} - K_6$ has a hybrid (G_2, H_2)-decomposition. And $K_6 = 2G_2 + 1H_2 = 1G_2 + 2H_2 = 3G_2$:

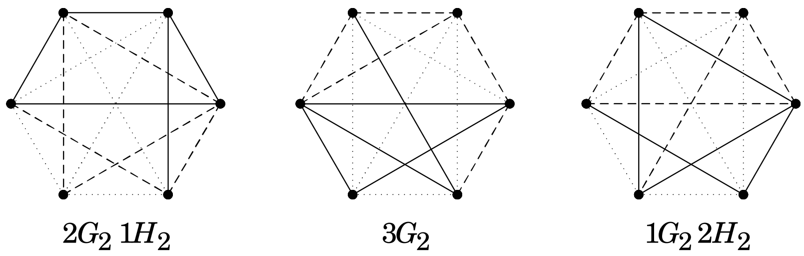

Figure 4: Hybrid (G_2, H_2)-decomposition of K_6

Hence, "$\{(s,t)$-case $|s+t = 11, s,t \in \mathbb{N}\}$" hold.

3. Hybrid (G_2, H_2)-decomposition of K_{15}

Let (s,t) be an admissible pair of K_{15}. Let $V(K_{15}) = \{x_{i,j} | \forall i \in \mathbb{Z}_5, j \in \mathbb{Z}_3\}$, and let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$, $j = 1,2,3$.

(i) By Lemma 2.1.3, and $K_5 = 1G_2 + 1H_2$ (observation), "$\{(s,t)|s+t=21, s,t \geq 3\}$" hold.

(ii) Let $K_{5,5,5} = 15G_2$, $G_{2,(i,1)} = [x_{i+1,1}, x_{i+1,2}, x_{i+0,0}; x_{i+3,1}, x_{i+1,0}]$, $G_{2,(i,2)} = [x_{i+1,2}, x_{i+1,0}, x_{i+0,1}; x_{i+3,2}, x_{i+1,1}]$, $G_{2,(i,3)} = [x_{i+1,0}, x_{i+1,1}, x_{i+0,2}; x_{i+3,0}, x_{i+1,2}]$, $\forall i \in \mathbb{Z}_5$. Let $G_{2,4} = [x_{1,1}, x_{1,2}, x_{0,0}; x_{2,0}, x_{4,0}]$, $G_{2,5} = [x_{0,0}, x_{3,1}, x_{1,0}; x_{4,0}, x_{3,0}]$, $G_{2,6} = [x_{1,0}, x_{2,0}, x_{3,0}; x_{0,0}, x_{4,0}]$, $G_{2,7} = [x_{1,2}, x_{1,0}, x_{0,1}; x_{2,1}, x_{4,1}]$, $G_{2,8} = [x_{0,1}, x_{3,2}, x_{1,1}; x_{4,1}, x_{3,1}]$, $G_{2,9} = [x_{1,1}, x_{2,1}, x_{3,1}; x_{0,1}, x_{4,1}]$, $G_{2,10} = [x_{1,0}, x_{1,1}, x_{0,2}; x_{2,2}, x_{4,2}]$, $G_{2,11} = [x_{0,2}, x_{3,0}, x_{1,2}; x_{4,2}, x_{3,2}]$, $G_{2,12} = [x_{1,2}, x_{2,2}, x_{3,2}; x_{0,2}, x_{4,2}]$. We can trade $G_{2,(0,1)} \cup K_{V_1}$ for $G_{2,4} \cup G_{2,5} \cup G_{2,6}$. Hence, "$(19,2)$-case" is done.

(iii) From (ii), we can trade $G_{2,(0,2)} \cup K_{|V_2|}$ for $G_{2,7} \cup G_{2,8} \cup G_{2,9}$. Hence, "$(20,1)$-case" is done.

(iv) From (iii), we can trade $G_{2,(0,3)} \cup K_{|V_3|}$ for $G_{2,10} \cup G_{2,11} \cup G_{2,12}$. Hence, "$(21,0)$-case" is done.

(v) Let $K_{5,5,5} = 15H_2$. $H_{2,(i,1)} = [x_{i+1,1}, x_{i+0,0}, x_{i+3,1}, x_{i+1,0}; x_{i+0,1}]$, $H_{2,(i,2)} = [x_{i+1,2}, x_{i+0,1}, x_{i+3,2}, x_{i+1,1}; x_{i+0,2}]$, $H_{2,(i,3)} = [x_{i+1,0}, x_{i+0,2}, x_{i+3,0}, x_{i+1,2}; x_{i+0,0}]$, $\forall i \in \mathbb{Z}_5$. Let $H_{2,4} = [x_{0,0}, x_{3,1}, x_{2,1}, x_{1,1}; x_{1,0}]$, $H_{2,5} = [x_{1,0}, x_{0,1}, x_{4,1}, x_{3,1}; x_{1,1}]$, $H_{2,6} = [x_{1,1}, x_{4,1}, x_{2,1}, x_{0,1}; x_{3,1}]$, $H_{2,7} = [x_{0,1}, x_{3,2}, x_{2,2}, x_{1,2}; x_{1,1}]$, $H_{2,8} = [x_{1,1}, x_{0,2}, x_{4,2}, x_{3,2}; x_{1,2}]$, $H_{2,9} = [x_{1,2}, x_{4,2}, x_{2,2}, x_{0,2}; x_{3,2}]$, $H_{2,10} = [x_{0,2}, x_{3,0}, x_{2,0}, x_{1,0}; x_{1,2}]$, $H_{2,11} = [x_{1,2}, x_{0,0}, x_{4,0}, x_{3,0}; x_{1,0}]$, $H_{2,12} = [x_{1,0}, x_{4,0}, x_{2,0}, x_{0,0}; x_{3,0}]$. We can trade $H_{2,(0,1)} \cup K_{V_2}$ for $H_{2,4} \cup H_{2,5} \cup H_{2,6}$. Hence, "$(2,19)$-case" is done.

(vi) From (v), we can trade $H_{2,(0,2)} \cup K_{|V_3|}$ for $H_{2,7} \cup H_{2,8} \cup H_{2,9}$. Hence, "$(1,20)$-case" is done.

(vii) From (vi)., we can trade $H_{2,(0,3)} \cup K_{|V_1|}$ for $H_{2,10} \cup H_{2,11} \cup H_{2,12}$. Hence, "$(0, 21)$-case" is done.

4. Hybrid (G_2, H_2)-decomposition of K_{16}

Let (s, t) be an admissible pair of K_{16}. Let $V(K_{16}) = \{x_{i,j} | \forall i \in \mathbb{Z}_5, j \in \mathbb{Z}_3\} \cup \{\infty\}$, and let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$, $j = 1, 2, 3$.

(i) Since $K_{5,5,5}$ has a hybrid (G_2, H_2)-decomposition (by Lemma 2.1.3) and $K_6 = 2G_2 + 1H_2 = 1G_2 + 2H_2 = 3G_2$. By Theorem 1.3.2, "$\{(s, t) | s + t = 21, s \geq 3\}$" are done.

(ii) Let $K_{5,5,5} = 15H_2$. $H_{2,(i,1)} = [x_{i+1,1},\ x_{i+0,0},\ x_{i+3,1},\ x_{i+1,0};\ x_{i+0,1}]$, $H_{2,(i,2)} = [x_{i+1,2},\ x_{i+0,1},\ x_{i+3,2},\ x_{i+1,1};\ x_{i+0,2}]$, $H_{2,(i,3)} = [x_{i+1,0},\ x_{i+0,2},\ x_{i+3,0},\ x_{i+1,2};\ x_{i+0,0}]$, $\forall i \in \mathbb{Z}_5$. Let $G_{2,4} = [x_{1,1},\ x_{2,1},\ x_{3,1};\ x_{4,1},\ x_{0,1}]$ $(\leq K_{V_2})$, $H_{2,5} = [x_{0,0},\ x_{3,1},\ x_{2,1},\ x_{1,1};\ x_{1,0}]$, $H_{2,6} = [x_{1,0},\ x_{0,1},\ x_{4,1},\ x_{3,1};\ x_{1,1}]$, $G_{2,7} = [x_{1,2},\ x_{2,2},\ x_{3,2};\ x_{4,2},\ x_{0,2}]$ $(\leq K_{V_3})$, $H_{2,8} = [x_{0,1},\ x_{3,2},\ x_{2,2},\ x_{1,2};\ x_{1,1}]$, $H_{2,9} = [x_{1,1},\ x_{0,2},\ x_{4,2},\ x_{3,2};\ x_{1,2}]$, $G_{2,10} = [x_{1,0},\ x_{2,0},\ x_{3,0};\ x_{4,0},\ x_{0,0}]$ $(\leq K_{V_1})$, $H_{2,11} = [x_{0,2},\ x_{3,0},\ x_{2,0},\ x_{1,0};\ x_{1,2}]$, $H_{2,12} = [x_{1,2},\ x_{0,0},\ x_{4,0},\ x_{3,0};\ x_{1,0}]$. We can trade $H_{2,(0,1)} \cup G_{2,4}$ for $H_{2,5} \cup H_{2,6}$. Hence, "$(2, 22)$-case" is done.

(iii) From (ii), We can trade $H_{2,(0,2)} \cup G_{2,7}$ for $H_{2,8} \cup H_{2,9}$. Hence, "$(1, 23)$-case" is done.

(iv) From (iii), We can trade $H_{2,(0,3)} \cup G_{2,10}$ for $H_{2,11} \cup H_{2,12}$. Hence, "$(0, 24)$-case" is done.

5. Hybrid (G_2, H_2)-decomposition of K_{20}

Let (s,t) be an admissible pair of K_{20}. Let $V(K_{20}) = \{x_{i,j} | \forall i \in \mathbb{Z}_5, j \in \mathbb{Z}_4\}$, and let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$, $j = 1,2,3,4$. Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5}$ as given in Lemma 2.1.3 for the subgraph: $K_{|V_1|,|V_2|,|V_3|}$. Use a hybrid (G_2, H_2)-decomposition of K_{10} for the subgraph: $K_{|V_3 \cup V_4|}$. Use a hybrid (G_2, H_2)-decomposition of $K_{10} - K_5$ as given in Lemma 2.1.5 for the subgraph: $K_{|V_1 \cup V_4|} - K_{|V_4|}$, and $K_{|V_2 \cup V_4|} - K_{|V_4|}$. Hence, "$\{(s,t)$-case $|s+t=38\}$" are done.

6. Hybrid (G_2, H_2)-decomposition of K_{21}

Let (s,t) be an admissible pair of K_{21}. Let $V(K_{21}) = \{x_{i,j} | \forall i \in \mathbb{Z}_5, j \in \mathbb{Z}_4\} \cup \{\infty\}$, and let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$, $j = 1,2,3,4$.

(i) Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5,5}$ as given in Lemma 2.1.4 for the subgraph: $K_{|V_1|,|V_2|,|V_3|,|V_4|}$.

And $K_{|V_1 \cup \{\infty\}|}$ $K_{|V_2 \cup \{\infty\}|}$, $K_{|V_3 \cup \{\infty\}|}$, and $K_{|V_4 \cup \{\infty\}|}$ be a subgraph isomorphic to K_6. Then by Lemma 1.3.2, we have $\{(s,t) | s+t = 42, s \geq 3\}$.

(ii) Let $K_{|V_1|,|V_2|,|V_3|,|V_4|} = 30H_2$. $H_{2,(i,1)} = [x_{i+1,1},\ x_{i+0,0},\ x_{i+3,1},\ x_{i+1,0};\ x_{i+0,1}]$, $H_{2,(i,2)} = [x_{i+1,2},\ x_{i+0,1},\ x_{i+3,2},\ x_{i+1,1};\ x_{i+0,2}]$, $H_{2,(i,3)} = [x_{i+1,3},\ x_{i+0,2},\ x_{i+3,3},\ x_{i+1,2};\ x_{i+0,3}]$, $H_{2,(i,4)} = [x_{i+1,0},\ x_{i+0,3},\ x_{i+3,0},\ x_{i+1,3};\ x_{i+0,0}]$, $H_{2,(i,5)} = [x_{i+1,0},\ x_{i+0,2},\ x_{i+3,0},\ x_{i+1,2};\ x_{i+0,0}]$, $H_{2,(i,6)} = [x_{i+1,1}, x_{i+0,3}, x_{i+3,1}, x_{i+1,3}; x_{i+0,1}]$ $\forall i \in \mathbb{Z}_5$. Then from the proof of K_{16}, we can trade $H_{2,(0,1)} \cup K_{|V_2|} = 1G_2 + 2H_2$ for $4H_2$. Hence, "$(2,40)$-case" is done.

(iii) From (ii), we can trade $H_{2,(0,2)} \cup K_{|V_3|} = 1G_2 + 2H_2$ for $4H_2$. Hence, "$(1,41)$-case" is done.

(iv) From (ii), we can trade $H_{2,(0,3)} \cup K_{|V_4|} = 1G_2 + 2H_2$ for $4H_2$. Hence, "$(0, 42)$-case" is done.

7. Hybrid (G_2, H_2)-decomposition of K_{25}

Let (s, t) be an admissible pair of K_{25}. Let $V(K_{25}) = \{x_{i,j} | \forall i \in \mathbb{Z}_5, j \in \mathbb{Z}_5\}$, and let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$, $j = 1, 2, \cdots, 5$. Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5,5}$ as given in Lemma 2.1.4 for the subgraph: $K_{|V_1|,|V_2|,|V_3|,|V_4|}$. Use a hybrid (G_2, H_2)-decomposition of K_{10} for the subgraph: $K_{|V_1 \cup V_5|}$. Use a hybrid (G_2, H_2)-decomposition of $K_{10} - K_5$ as given in Lemma 2.1.5 for each of the following subgraphs: $K_{|V_2 \cup V_5|} - K_{|V_5|}$, $K_{|V_3 \cup V_5|} - K_{|V_5|}$, and $K_{|V_4 \cup V_5|} - K_{|V_5|}$. Hence, "$\{(s, t)$-case $| s + t = 60\}$" are done.

8. Hybrid (G_2, H_2)-decomposition of K_{26}

Let (s, t) be an admissible pair of K_{26} Let $V(K_{26}) = \{x_{i,j} | \forall i \in \mathbb{Z}_5, j \in \mathbb{Z}_5\} \cup \{\infty\}$, and let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$, $j = 1, 2, \cdots, 5$. Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5,5}$ as given in Lemma 2.1.4 for the subgraph: $K_{|V_1|,|V_2|,|V_3|,|V_4|}$. Use a hybrid (G_2, H_2)-decomposition of K_{11} for the subgraph: $K_{|V_1 \cup V_5 \cup \{\infty\}|}$. Use a hybrid (G_2, H_2)-decomposition of $K_{11} - K_6$ as given in Lemma 2.1.6 for each of the following subgraphs: $K_{|V_2 \cup V_5 \cup \{\infty\}|} - K_{|V_5 \cup \{\infty\}|}$, $K_{|V_3 \cup V_5 \cup \{\infty\}|} - K_{|V_5 \cup \{\infty\}|}$, and $K_{|V_4 \cup V_5 \cup \{\infty\}|} - K_{|V_5 \cup \{\infty\}|}$. Hence, "$\{(s, t)$-case $| s + t = 65\}$" are done.

9. Maximum (G_2, H_2)-hybrid packing of K_7

By Lemma 2.2.1, $K_7 - K_2$ has a hybrid (G_2, H_2)-decomposition. What remains is K_2. Cover this edge by using a copy of G_2 or H_2.

10. Maximum (G_2, H_2)-hybrid packing of K_{12}

Let $V(K_{12}) = \mathbb{Z}_{12}$, and let $V_1 = \{0, 1, 2, 3, 4\}$, $V_2 = \{5, 6, 7, 8, 9\}$, $V_3 = \{10, 11\}$. Since $K_{12} = K_{V_1 \cup V_3} + (K_{V_1 \cup V_2 \cup V_3} - K_{V_1 \cup V_3})$, by Lemmas 2.2.1 and 2.2.2, $K_{12} = sG_2 + tH_2$ where $s + t = 13$, $\forall s, t \in \mathbb{N} \cup \{0\}$. What remains is K_2. Cover this edge by using a copy of G_2 or H_2.

11. Maximum (G_2, H_2)-hybrid packing of K_{17}

Let $V(K_{17}) = \mathbb{Z}_{17}$. Let $V_1 = \{0, 1, 2, 3, 4\}$, $V_2 = \{5, 6, 7, 8, 9\}$, $V_3 = \{10, 11, 12, 13, 14, \}$, $V_4 = \{15, 16\}$.

Then $K_{17} = K_{|V_1|,|V_2|,|V_3|} + K_{|V_1 \cup V_4|} + (K_{|V_2 \cup V_4|} - K_{|V_4|}) + (K_{|V_3 \cup V_4|} - K_{|V_4|})$. Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5}$ on $K_{|V_1|,|V_2|,|V_3|}$ and a hybrid (G_2, H_2)-decomposition of $(K_7 - K_2)$ on $(K_{|V_2 \cup V_4|} - K_{|V_4|})$, $(K_{|V_3 \cup V_4} - K_{V_4|})$. Use a maximum (G_2, H_2)-hybrid packing of K_7 on $K_{|V_1 \cup V_4|}$. What remains is K_2. Cover this edge by using a copy of G_2, or H_2.

12. Maximum (G_2, H_2)-hybrid packing of K_{22}

Let $V(K_{22}) = \{x_{i,j} | \forall i \in \mathbb{Z}_5, j \in \mathbb{Z}_4\} \cup \{v_1, v_2\}$, and let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$ where $j = 1, 2, 3, 4$, and $V_5 = \{v_1, v_2\}$.

Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5,5}$ as given in Lemma 2.1.4 for $K_{|V_1|,|V_2|,|V_3|,|V_4|}$.

Use a hybrid (G_2, H_2)-decomposition of $(K_7 - K_2)$ as given in Lemma 2.2.1 for each of the following subgraphs:

$(K_{|V_2 \cup V_5|} - K_{|V_5|})$, $(K_{|V_3 \cup V_5|} - K_{|V_5|})$, $(K_{|V_4 \cup V_5|} - K_{|V_5|})$. Use a maximum (G_2, H_2)-hybrid packing of K_7 for $K_{|V_1 \cup V_5|}$. What remains is K_2. Cover this edge by using a copy of G_2, or H_2.

13. Maximum (G_2, H_2)-hybrid packing of K_{27}

Let $V(K_{27}) = \{x_{i,j} | \forall i \in \mathbb{Z}_5, j \in \mathbb{Z}_5\} \cup \{v_1, v_2\}$, and let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$ where $j = 1, 2, \cdots, 5$, and $V_6 = \{v_1, v_2\}$.

Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5,5}$ as given in Lemma 2.1.4 for $K_{|V_1|,|V_2|,|V_3|,|V_4|}$. Use a hybrid (G_2, H_2)-decomposition of $(K_7 - K_2)$ as given in Lemma 2.2.1 for each of the following subgraphs:

$(K_{|V_2 \cup V_6|} - K_{|V_6|})$, $(K_{|V_3 \cup V_6|} - K_{|V_6|})$, $(K_{|V_4 \cup V_6|} - K_{|V_6|})$. Use a maximum (G_2, H_2)-hybrid packing of K_{12} for $K_{|V_1 \cup V_5 \cup V_6|}$. What remains is K_2. Cover this edge by using a copy of G_2, or H_2.

14. Maximum (G_2, H_2)-hybrid packing of K_8

By Lemma 2.2.4, $K_8 - K_3$ has a hybrid (G_2, H_2)-decomposition. What remains is K_3. Cover this edge by using a copy of G_2.

15. Maximum (G_2, H_2)-hybrid packing of K_{13}

Let $V(K_{13}) = \mathbb{Z}_{13}$, and let $V_1 = \{0, 1, 2, 3, 4\}$, $V_2 = \{5, 6, 7, 8, 9\}$, $V_3 = \{10, 11, 12\}$. Use a hybrid (G_2, H_2)-decomposition of $(K_8 - K_3)$ as given in Lemma 2.2.4 for $(K_{|V_1 \cup V_3|} - K_{|V_3|})$. Use a hybrid (G_2, H_2)-decomposition of $(K_{13} - K_8)$ as given in Lemma 2.2.5 for $(K_{|V_1 \cup V_2 \cup V_3|} - K_{|V_1 \cup V_3|})$. What remains is K_3. Cover this edge by using a copy of G_2.

16. Maximum (G_2, H_2)-hybrid packing of K_{18}

Let $V(K_{18}) = \mathbb{Z}_{18}$. Let $V_1 = \{0, 1, 2, 3, 4\}$, $V_2 = \{5, 6, 7, 8, 9\}$, $V_3 = \{10, 11, 12, 13, 14, \}$, $V_4 = \{15, 16, 17\}$.

Then $K_{18} = K_{|V_1|,|V_2|,|V_3|} + K_{|V_1 \cup V_4|} + (K_{|V_2 \cup V_4|} - K_{|V_4|}) + (K_{|V_3 \cup V_4|} - K_{|V_4|})$. Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5}$ on $K_{|V_1|,|V_2|,|V_3|}$ and a hybrid (G_2, H_2)-decomposition of $(K_8 - K_3)$ on $(K_{|V_1 \cup V_4|} - K_{|V_4|})$, $(K_{|V_2 \cup V_4|} - K_{|V_4|})$, $(K_{|V_3 \cup V_4|} - K_{|V_4|})$. What remains is K_3. Cover this edge by using a copy of G_2.

17. Maximum (G_2, H_2)-hybrid packing of K_{23}

Let $V(K_{23}) = \{x_{i,j} | \forall i \in \mathbb{Z}_5, j \in \mathbb{Z}_4\} \cup \{v_1, v_2, v_3\}$, and let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$ where $j = 1, 2, 3, 4$, and $V_5 = \{v_1, v_2, v_3\}$.

Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5,5}$ as given in Lemma 2.1.4 for $K_{|V_1|,|V_2|,|V_3|,|V_4|}$.

Use a hybrid (G_2, H_2)-decomposition of $(K_8 - K_3)$ as given in Lemma 2.2.4 for each of the following subgraphs:

$(K_{|V_1 \cup V_5|} - K_{|V_5|})$, $(K_{|V_2 \cup V_5|} - K_{|V_5|})$, $(K_{|V_3 \cup V_5|} - K_{|V_5|})$, $(K_{|V_4 \cup V_5|} - K_{|V_5|})$. What remains is K_3. Cover this edge by using a copy of G_2.

18. Maximum (G_2, H_2)-hybrid packing of K_{28}

Let $V(K_{28}) = \{x_{i,j} | \forall i \in \mathbb{Z}_5, j \in \mathbb{Z}_5\} \cup \{v_1, v_2, v_3\}$, and let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$ where $j = 1, 2, \cdots, 5$, and $V_6 = \{v_1, v_2, v_3\}$.

Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5,5}$ as given in Lemma 2.1.4 for $K_{|V_1|,|V_2|,|V_3|,|V_4|}$.

Use a hybrid (G_2, H_2)-decomposition of $(K_8 - K_3)$ as given in Lemma 2.2.4 for each of the following subgraphs:

$(K_{|V_1 \cup V_6|} - K_{|V_6|})$, $(K_{|V_2 \cup V_6|} - K_{|V_6|})$, $(K_{|V_3 \cup V_6|} - K_{|V_6|})$, $(K_{|V_4 \cup V_6|} - K_{|V_6|})$. What remains is K_3. Cover this edge by using a copy of G_2.

19. Hybrid (G_2, H_2)-decomposition of $K_{14} - K_4$

Let (s,t) be an admissible pair of $K_{14}-K_4$. Let $V(K_{14})=\mathbb{Z}_{14}$, and let $V_1=\{0,1,2,3,4\}$, $V_2=\{5,6,7,8,9\}$, $V_3=\{10,11,12\}$.

(i) Use a hybrid (G_2,H_2)-decomposition of K_9-K_4 for each of the following subgraphs: $K_{|V_1\cup V_3|}-K_{|V_3|}$, $K_{|V_2\cup V_3|}-K_{|V_3|}$. For $K_{|V_1|,|V_2|}$, use the following copies of H_2: $H_{2,1}=[6,0,8,1;5]$, $H_{2,2}=[7,1,9,2;6]$, $H_{2,3}=[8,2,5,3;7]$, $H_{2,4}=[9,3,6,4;8]$, $H_{2,5}=[5,4,7,0;9]$. Hence, $\{(s,t)$-case $|s+t=17, s\leq 12\}$ are done.

(ii) Let $G_{2,1}=[5,6,1;8,0]$, $G_{2,2}=[9,10,5;11,7]$, $G_{2,3}=[8,11,6;7,10]$, $G_{2,4}=[5,12,8;9,11]$, $G_{2,5}=[7,9,12;6,0]$, $G_{2,6}=[6,9,13;5,7]$, $G_{2,7}=[7,13,8;10,6]$. Then we trade $H_{2,1}\cup$ $(K_{V_2\cup V_3}-K_{V_3})$ for $G_{2,1}\cup G_{2,2}\cup G_{2,3}\cup G_{2,4}\cup G_{2,5}\cup G_{2,6}\cup G_{2,7}$. Hence, "$(13,4)$-case" is done.

(iii) Let $G_{2,8}=[7,13,8;10,5]$, $G_{2,9}=[6,11,8;9,5]$, $G_{2,10}=[7,10,6;1,9]$, $G_{2,11}=[5,8,12;6,0]$, $G_{2,12}=[5,6,13;9,10]$, $G_{2,13}=[5,7,1;8,0]$, $G_{2,14}=[6,9,2;7,11]$, $G_{2,15}=[7,12,9;11,5]$. Then we trade $H_{2,1}\cup H_{2,2}\cup$ $(K_{V_2\cup V_3}-K_{V_3})$ for $G_{2,8}\cup G_{2,9}\cup G_{2,10}\cup G_{2,11}\cup G_{2,12}\cup G_{2,13}\cup G_{2,14}\cup G_{2,15}$. Hence, "$(14,3)$-case" is done.

(iv) Let $G_{2,16}=[9,11,5;3,7]$, $G_{2,17}=[6,7,10;8,3]$, $G_{2,18}=[7,12,9;10,5]$, $G_{2,19}=[5,6,13;9,8]$, $G_{2,20}=[8,11,6;1,9]$, $G_{2,21}=[7,13,8;2,5]$. Then we trade $H_{2,1}\cup H_{2,2}\cup H_{2,3}\cup$ $(K_{V_2\cup V_3}-K_{V_3})$ for $G_{2,11}\cup G_{2,13}\cup G_{2,14}\cup G_{2,16}\cup G_{2,17}\cup G_{2,18}\cup G_{2,19}\cup G_{2,20}\cup G_{2,21}$. Hence, "$(15,2)$-case" is done.

(v) Let $G_{2,22}=[8,9,4;6,3]$, $G_{2,23}=[5,6,13;9,3]$. Then we trade $H_{2,1}\cup H_{2,2}\cup H_{2,3}\cup H_{2,4}\cup$ $(K_{V_2\cup V_3}-K_{V_3})$ for $G_{2,11}\cup G_{2,13}\cup G_{2,14}\cup G_{2,16}\cup G_{2,17}\cup G_{2,18}\cup G_{2,20}\cup G_{2,21}\cup G_{2,22}\cup G_{2,23}$. Hence, "$(16,1)$-case" is done.

(vi) Let $G_{2,24} = [3, 6, 4; 7, 0]$, $G_{2,25} = [8, 9, 4; 5, 0]$, $G_{2,26} = [1, 4, 13; 0, 9]$, $G_{2,27} = [3, 12, 0; 2, 11]$, $G_{2,28} = [4, 10, 0; 1, 12]$, $G_{2,29} = [3, 11, 1; 2, 10]$, $G_{2,30} = [2, 12, 4; 11, 0]$, $G_{2,31} = [2, 13, 3; 10, 1]$. Then we trade $H_{2,1}\cup H_{2,2}\cup H_{2,3}\cup H_{2,4}\cup H_{2,5}\cup (K_{V_1\cup V_3}- K_{V_3})\cup (K_{V_2\cup V_3}- K_{V_3})$ for $G_{2,11}\cup G_{2,13}\cup G_{2,14}\cup G_{2,16}\cup G_{2,17}\cup G_{2,18}\cup G_{2,20}\cup G_{2,21}\cup G_{2,23}\cup G_{2,24}\cup G_{2,25}\cup G_{2,26}\cup G_{2,27}\cup G_{2,28}\cup G_{2,29}\cup G_{2,30}\cup G_{2,31}$. Hence, "$(17, 0)$-case" is done.

20. Hybrid (G_2, H_2)-decomposition of $K_{19} - K_4$

Let $V(K_{19}) = \{x_{i,j}|i \in \mathbb{Z}_5, j \in \mathbb{Z}_3\}\cup \{x_1, x_2, x_3, x_4\}$. Let $V_j = \{x_{i,j-1}|i \in \mathbb{Z}_5\}$ where $j = 1, 2, 3$, and $V_4 = \{x_1, x_2, x_3, x_4\}$.

Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5}$ as given in Lemma 2.1.3 for the subgraph: $K_{|V_1|,|V_2|,|V_3|}$. Use a hybrid (G_2, H_2)-decomposition of $K_9 - K_4$ as given in Lemma 2.2.7 for each of the following subgraphs: $K_{|V_1\cup V_4|} - K_{|V_4|}$, $K_{|V_2\cup V_4|} - K_{|V_4|}$, $K_{|V_3\cup V_4|} - K_{|V_4|}$.

21. Hybrid (G_2, H_2)-decomposition of $K_{24} - K_4$

Let $V(K_{24}) = \{x_{i,j}|i \in \mathbb{Z}_5, j \in \mathbb{Z}_4\}\cup \{x_1, x_2, x_3, x_4\}$. Let $V_j = \{x_{i,j-1}|i \in \mathbb{Z}_5\}$ where $j = 1, 2, 3, 4$, and $V_5 = \{x_1, x_2, x_3, x_4\}$.

Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5,5}$ as given in Lemma 2.1.4 for the subgraph: $K_{|V_1|,|V_2|,|V_3|,|V_4|}$. Use a hybrid (G_2, H_2)-decomposition of $K_9 - K_4$ as given in Lemma 2.2.7 for each of the following subgraphs: $K_{|V_1\cup V_5|} - K_{|V_5|}$, $K_{|V_2\cup V_5|} - K_{|V_5|}$, $K_{|V_3\cup V_5|} - K_{|V_5|}$, $K_{|V_4\cup V_5|} - K_{|V_5|}$.

22. Hybrid (G_2, H_2)-decomposition of $K_{29} - K_4$

Let $V(K_{29}) = \{x_{i,j} | i \in \mathbb{Z}_5, j \in \mathbb{Z}_5\} \cup \{x_1, x_2, x_3, x_4\}$. Let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$ where $j = 1, 2, \cdots, 5$, and $V_6 = \{x_1, x_2, x_3, x_4\}$.

Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5}$ as given in Lemma 2.1.3 for each of the following subgraphs: $K_{|V_1|,|V_2|,|V_3|}$, $K_{|V_1|,|V_4|,|V_5|}$. Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,4}$ (by Appendix 29.) for each of the following subgraphs: $K_{|V_2|,|V_4|,|V_6|}$, $K_{|V_3|,|V_5|,|V_6|}$. Use a hybrid (G_2, H_2)-decomposition of K_{10} for each of the following subgraphs: $K_{|V_2 \cup V_5|}$, $K_{|V_3 \cup V_4|}$. Use a hybrid (G_2, H_2)-decomposition of $K_9 - K_4$ as given in Lemma 2.2.7 for the subgraph: $K_{|V_1 \cup V_6|} - K_{|V_6|}$.

23. Maximum (G_2, H_2)-hybrid packing of K_9

Let (s, t) be an admissible pair of K_9. Let $V(K_9) = \mathbb{Z}_9$.

(i) Use the following copies of G_2:

$G_2 \cong [1, 2, 0; 8, 6]$, $[2, 8, 7; 0, 6]$, $[1, 5, 4; 2, 3]$, $[0, 3, 4; 8, 5]$, $[1, 7, 6; 3, 5]$, $[2, 5, 6; 4, 7]$, $[1, 8, 3; 7, 5]$.

What remains is the edge $\{0, 5\}$. Hence, "$(7, 0)$-case" is done.

(ii) Use the following copies of G_2 and H_2:

$G_2 \cong [1, 2, 0; 8, 6]$, $[0, 3, 4; 6, 5]$, $[1, 7, 6; 3, 5]$, $[1, 5, 4; 2, 3]$, $[2, 5, 8; 4, 7]$, $[1, 8, 3; 7, 5]$; $H_2 \cong [0, 6, 2, 7; 8]$.

What remains is the edge $\{0, 5\}$. Hence, "$(6, 1)$-case" is done.

(iii) Use the following copies of G_2 and H_2:

$G_2 \cong [1, 2, 0; 8, 6]$, $[1, 5, 4; 2, 3]$, $[2, 5, 8; 4, 7]$, $[1, 7, 6; 3, 5]$, $[1, 8, 3; 7, 5]$; $H_2 \cong [0, 6, 2, 7; 8]$, $[0, 5, 6, 4; 3]$.

What remains is the edge $\{0, 3\}$. Hence, "$(5, 2)$-case" is done.

(iv) Use the following copies of G_2 and H_2:

$G_2 \cong [6,7,1;2,0]$, $[2,5,8;4,7]$, $[1,5,4;2,3]$, $[1,8,3;7,5]$; $H_2 \cong [0,6,2,7;8]$, $[0,5,6,4;3]$, $[3,6,8,0;1]$.

What remains is the edge $\{3,5\}$. Hence, "$(4,3)$-case" is done.

(v) Use the following copies of G_2 and H_2:

$G_2 \cong [6,7,1;2,0]$, $[2,5,8;4,7]$, $[1,8,3;7,5]$; $H_2 \cong [0,6,2,7;8]$, $[0,5,6,4;3]$, $[3,6,8,0;1]$, $[2,3,5,4;1]$.

What remains is the edge $\{1,5\}$. Hence, "$(3,4)$-case" is done.

(vi) Use the following copies of G_2 and H_2:

$G_2 \cong [6,7,1;2,0]$, $[1,3,8;2,5]$; $H_2 \cong [0,6,2,7;8]$, $[0,5,6,4;3]$, $[3,6,8,0;1]$, $[2,3,5,4;1]$, $[4,8,5,7;3]$.

What remains is the edge $\{1,5\}$. Hence, "$(2,5)$-case" is done.

(vii) Use the following copies of G_2 and H_2:

$G_2 \cong [6,7,1;2,0]$; $H_2 \cong [0,6,2,7;8]$, $[0,5,6,4;3]$, $[3,6,8,0;1]$, $[2,3,5,4;1]$, $[4,8,5,7;3]$, $[5,2,8,1;3]$.

What remains is the edge $\{3,8\}$. Hence, "$(1,6)$-case" is done.

(viii) Use the following copies of H_2:

$H_2 \cong [6,0,7,2;1]$, $[4,0,5,6;3]$, $[3,0,8,7;6]$, $[1,6,8,3;4]$, $[1,0,2,4;5]$, $[5,1,8,2;3]$, $[7,4,8,5;3]$.

What remains is the edge $\{1,7\}$. Hence, "$(0,7)$-case" is done.

24. Maximum (G_2, H_2)-hybrid packing of K_{14}

Let (s,t) be an admissible pair of K_{14}. Let $V(K_{14}) = \mathbb{Z}_{14}$, and let $V_1 = \{0,1,2,3,4\}$,

$V_2 = \{5, 6, 7, 8, 9\}$, $V_3 = \{10, 11, 12\}$.

(i) Use a maximum (G_2, H_2)-hybrid packing of K_9 for the subgraph: $K_{|V_1 \cup V_3|}$. What remains is a single edge. We may assume L is the edge $\{3, 4\}$.

Use a hybrid (G_2, H_2)-decomposition of $K_9 - K_4$ for the subgraph: $K_{|V_2 \cup V_3|} - K_{|V_3|}$. For $K_{|V_1|,|V_2|}$, use the following copies of H_2: $H_{2,1} = [6, 0, 8, 1; 5]$, $H_{2,2} = [7, 1, 9, 2; 6]$, $H_{2,3} = [8, 2, 5, 3; 7]$, $H_{2,4} = [9, 3, 6, 4; 8]$, $H_{2,5} = [5, 4, 7, 0; 9]$. Hence, $\{(s,t)\text{-case } |s + t = 18, s \leq 13\}$ are done.

(ii) Let $G_{2,1} = [5, 6, 1; 8, 0]$, $G_{2,2} = [9, 10, 5; 11, 7]$, $G_{2,3} = [8, 11, 6; 7, 10]$, $G_{2,4} = [5, 12, 8; 9, 11]$, $G_{2,5} = [7, 9, 12; 6, 0]$, $G_{2,6} = [6, 9, 13; 5, 7]$, $G_{2,7} = [7, 13, 8; 10, 6]$. Then we trade $H_{2,1} \cup (K_{|V_2 \cup V_3|} - K_{|V_3|})$ for $G_{2,1} \cup G_{2,2} \cup G_{2,3} \cup G_{2,4} \cup G_{2,5} \cup G_{2,6} \cup G_{2,7}$. Hence, "$(14, 4)$-case" is done.

(iii) Let $G_{2,8} = [7, 13, 8; 10, 5]$, $G_{2,9} = [6, 11, 8; 9, 5]$, $G_{2,10} = [7, 10, 6; 1, 9]$, $G_{2,11} = [5, 8, 12; 6, 0]$, $G_{2,12} = [5, 6, 13; 9, 10]$, $G_{2,13} = [5, 7, 1; 8, 0]$, $G_{2,14} = [6, 9, 2; 7, 11]$, $G_{2,15} = [7, 12, 9; 11, 5]$. Then we trade $H_{2,1} \cup H_{2,2} \cup (K_{|V_2 \cup V_3|} - K_{|V_3|})$ for $G_{2,8} \cup G_{2,9} \cup G_{2,10} \cup G_{2,11} \cup G_{2,12} \cup G_{2,13} \cup G_{2,14} \cup G_{2,15}$. Hence, "$(15, 3)$-case" is done.

(iv) Let $G_{2,16} = [9, 11, 5; 3, 7]$, $G_{2,17} = [6, 7, 10; 8, 3]$, $G_{2,18} = [7, 12, 9; 10, 5]$, $G_{2,19} = [5, 6, 13; 9, 8]$, $G_{2,20} = [8, 11, 6; 1, 9]$, $G_{2,21} = [7, 13, 8; 2, 5]$. Then we trade $H_{2,1} \cup H_{2,2} \cup H_{2,3} \cup (K_{|V_2 \cup V_3|} - K_{|V_3|})$ for $G_{2,11} \cup G_{2,13} \cup G_{2,14} \cup G_{2,16} \cup G_{2,17} \cup G_{2,18} \cup G_{2,19} \cup G_{2,20} \cup G_{2,21}$. Hence, "$(16, 2)$-case" is done.

(v) Let $G_{2,22} = [8, 9, 4; 6, 3]$, $G_{2,23} = [5, 6, 13; 9, 3]$. Then we trade $H_{2,1} \cup H_{2,2} \cup H_{2,3} \cup H_{2,4} \cup (K_{|V_2 \cup V_3|} - K_{|V_3|})$ for $G_{2,11} \cup G_{2,13} \cup G_{2,14} \cup G_{2,16} \cup G_{2,17} \cup G_{2,18} \cup G_{2,20} \cup G_{2,21} \cup G_{2,22} \cup G_{2,23}$. Hence, "$(17, 1)$-case" is done.

(vi) Let $G_{2,24} = [3, 6, 4; 7, 0]$, $G_{2,25} = [8, 9, 4; 5, 0]$. Then we trade $H_{2,1} \cup H_{2,2} \cup H_{2,3} \cup H_{2,4} \cup H_{2,5} \cup (K_{|V_2 \cup V_3|} - K_{|V_3|}) \cup \{3, 4\}$ for $G_{2,11} \cup G_{2,13} \cup G_{2,14} \cup G_{2,16} \cup G_{2,17} \cup G_{2,18} \cup G_{2,20} \cup G_{2,21} \cup G_{2,23} \cup G_{2,24} \cup G_{2,25} \cup \{0, 9\}$. Hence, "$(18, 0)$-case" is done, and $L = \{0, 9\}$. Cover this edge using a copy of G_2 or H_2.

25. Maximum (G_2, H_2)-hybrid packing of K_{19}

Let $V(K_{19}) = \{x_{i,j} | i \in \mathbb{Z}_5, j \in \mathbb{Z}_3\} \cup \{x_1, x_2, x_3, x_4\}$. Let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$, $j = 1, 2, 3$, and $V_4 = \{x_1, x_2, x_3, x_4\}$.

Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5}$ as given in Lemma 2.1.3 for the subgraph: $K_{|V_1|,|V_2|,|V_3|}$. Use a hybrid (G_2, H_2)-decomposition of $K_9 - K_4$ as given in Lemma 2.2.7 for each of the following subgraphs: $K_{|V_2 \cup V_4|} - K_{|V_4|}$, $K_{|V_3 \cup V_4|} - K_{|V_4|}$. Use a maximum (G_2, H_2)-hybrid packing of K_9 for the subgraph $K_{|V_1 \cup V_4|}$. What remains is a single edge, which we may cover using a copy of G_2 or H_2.

26. Maximum (G_2, H_2)-hybrid packing K_{24}

Let $V(K_{24}) = \{x_{i,j} | i \in \mathbb{Z}_5, j \in \mathbb{Z}_4\} \cup \{x_1, x_2, x_3, x_4\}$. Let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$, $j = 1, 2, 3, 4$, and $V_5 = \{x_1, x_2, x_3, x_4\}$.

Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5,5}$ as given in Lemma 2.1.4 for the subgraph: $K_{|V_1|,|V_2|,|V_3|,|V_4|}$. Use a hybrid (G_2, H_2)-decomposition of $K_9 - K_4$ as given in Lemma 2.2.7 for each of the following subgraphs: $K_{|V_2 \cup V_5|} - K_{|V_5|}$, $K_{|V_3 \cup V_5|} - K_{|V_5|}$, $K_{|V_4 \cup V_5|} - K_{|V_5|}$. Use a maximum (G_2, H_2)-hybrid packing of K_9 for the subgraph $K_{|V_1 \cup V_5|}$. What remains is a single edge, which we may cover using a copy of G_2 or H_2.

27. Maximum (G_2, H_2)-hybrid packing of K_{29}

Let $V(K_{29}) = \{x_{i,j} | i \in \mathbb{Z}_5, j \in \mathbb{Z}_5\} \cup \{x_1, x_2, x_3, x_4\}$. Let $V_j = \{x_{i,j-1} | i \in \mathbb{Z}_5\}$, $j = 1, 2, \cdots, 5$, and $V_6 = \{x_1, x_2, x_3, x_4\}$.

Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,5}$ as given in Lemma 2.1.3 for each of the following subgraphs: $K_{|V_1|,|V_2|,|V_3|}$, $K_{|V_1|,|V_4|,|V_5|}$. Use a hybrid (G_2, H_2)-decomposition of $K_{5,5,4}$ for each of the following subgraphs: $K_{|V_2|,|V_4|,|V_6|}$, $K_{|V_3|,|V_5|,|V_6|}$. Use a hybrid (G_2, H_2)-decomposition of K_{10} for each of the following subgraphs: $K_{|V_2 \cup V_5|}$, $K_{|V_3 \cup V_4|}$. Use a maximum (G_2, H_2)-hybrid packing of K_9 for the subgraph $K_{V_1 \cup V_6}$. What remains is a single edge, which we may cover using a copy of G_2 or H_2.

28. Hybrid (G_2, H_2)-decomposition of $K_{5,5,4}$.

Let (s, t) be an admissible pair of $K_{5,5,4}$. Let $V_1 = \{0, 1, 2, 3, 4\}$, $V_2 = \{5, 6, 7, 8, 9\}$, $V_3 = \{10, 11, 12, 13\}$.

(i) Let $H_1 = [6, 0, 8, 1; 5]$, $H_2 = [7, 1, 9, 2; 6]$, $H_3 = [8, 2, 5, 3; 7]$, $H_4 = [9, 3, 6, 4; 8]$, $H_5 = [5, 4, 7, 0; 9]$, $H_6 = [0, 11, 5, 10; 7]$, $H_7 = [1, 11, 6, 10; 2]$, $H_8 = [3, 10, 8, 11; 2]$, $H_9 = [4, 10, 9, 11; 7]$, $H_{10} = [0, 13, 5, 12; 7]$, $H_{11} = [1, 13, 6, 12; 2]$, $H_{12} = [3, 12, 8, 13; 7]$, $H_{13} = [4, 12, 9, 12; 2]$. Hence, "$(0, 13)$-case" is done.

(ii) Let $G_1 = [3, 7, 13; 8, 12]$, $H_{14} = [5, 2, 8, 3; 12]$. Then we can trade $H_3 \cup H_{12}$ for $G_1 \cup H_{14}$. Hence, "$(1, 12)$-case" is done.

(iii) Let $G_2 = [3, 7, 13; 8, 2]$, $G_3 = [8, 12, 3; 5, 2]$. Then we can trade $H_3 \cup H_{12}$ for $G_2 \cup G_3$. Hence, "$(2, 11)$-case" is done.

(iv) Let $G_4 = [2, 8, 13; 4, 12]$, $G_5 = [3, 7, 13; 9, 12]$. Then we can trade $H_3 \cup H_{12} \cup H_{13}$ for $G_3 \cup G_4 \cup G_5$. Hence, "$(3, 10)$-case" is done.

(v) Let $G_6 = [9, 13, 4; 6, 3]$, $G_7 = [2, 13, 8; 4, 12]$, $G_8 = [7, 13, 3; 9, 12]$. Then we can trade $H_3 \cup H_{12} \cup H_{13} \cup H_4$ for $G_3 \cup G_6 \cup G_7 \cup G_8$. Hence, "$(4, 9)$-case" is done.

(vi) Let $G_9 = [9, 12, 3; 8, 11]$, $G_{10} = [4, 12, 8; 10, 3]$, $G_{11} = [8, 13, 2; 5, 3]$, $G_{12} = [7, 13, 3; 11, 2]$. Then we can trade $H_3 \cup H_{12} \cup H_{13} \cup H_4 \cup H_8$ for $G_6 \cup G_9 \cup G_{10} \cup G_{11} \cup G_{12}$. Hence, "$(5, 8)$-case" is done.

(vii) Let $G_{13} = [3, 12, 9; 1, 7]$, $G_{14} = [3, 13, 7; 2, 6]$, $G_{15} = [3, 8, 11; 2, 9]$. Then we can trade $H_3 \cup H_{12} \cup H_{13} \cup H_4 \cup H_8 \cup H_2$ for $G_6 \cup G_{10} \cup G_{11} \cup G_{13} \cup G_{14} \cup G_{15}$. Hence, "$(6, 7)$-case" is done.

(viii) Let $G_{16} = [3, 12, 9; 2, 6]$, $G_{17} = [3, 13, 7; 1, 9]$, $G_{18} = [3, 8, 11; 4, 10]$, $G_{19} = [2, 7, 11; 9, 10]$. Then we can trade $H_3 \cup H_{12} \cup H_{13} \cup H_4 \cup H_8 \cup H_2 \cup H_9$ for $G_6 \cup G_{10} \cup G_{11} \cup G_{16} \cup G_{17} \cup G_{18} \cup G_{19}$. Hence, "$(7, 6)$-case" is done.

(ix) Let $G_{20} = [4, 9, 13; 1, 12]$, $G_{21} = [3, 12, 9; 2, 5]$, $G_{22} = [2, 8, 13; 6, 4]$, $G_{23} = [2, 12, 6; 3, 5]$. Then we can trade $H_3 \cup H_{12} \cup H_{13} \cup H_4 \cup H_8 \cup H_2 \cup H_9 \cup H_{11}$ for $G_{10} \cup G_{17} \cup G_{18} \cup G_{19} \cup G_{20} \cup G_{21} \cup G_{22} \cup G_{23}$. Hence, "$(8, 5)$-case" is done.

(x) Let $G_{24} = [4, 9, 13; 0, 12]$, $G_{25} = [3, 7, 13; 1, 9]$, $G_{26} = [1, 7, 12; 5, 13]$. Then we can trade $H_3 \cup H_{12} \cup H_{13} \cup H_4 \cup H_8 \cup H_2 \cup H_9 \cup H_{11} \cup H_{10}$ for $G_{10} \cup G_{18} \cup G_{19} \cup G_{21} \cup G_{22} \cup G_{23} \cup G_{24} \cup G_{25} \cup G_{26}$. Hence, "$(9, 4)$-case" is done.

(xi) Let $G_{27} = [4, 13, 9; 1, 5]$, $G_{28} = [8, 12, 4; 6, 0]$, $G_{29} = [3, 7, 13; 0, 12]$, $G_{30} = [2, 13, 8; 10, 3]$, $G_{31} = [6, 13, 1; 8, 0]$. Then we can trade $H_3 \cup H_{12} \cup H_{13} \cup H_4 \cup H_8 \cup H_2 \cup H_9 \cup H_{11} \cup H_{10} \cup H_1$ for $G_{18} \cup G_{19} \cup G_{21} \cup G_{23} \cup G_{26} \cup G_{27} \cup G_{28} \cup G_{29} \cup G_{30} \cup G_{31}$. Hence, "$(10, 3)$-case" is done.

(xii) Let $G_{32} = [9, 12, 3; 5, 2]$, $G_{33} = [2, 13, 8; 10, 1]$, $G_{34} = [3, 6, 10; 2, 9]$, $G_{35} =$

$[2, 12, 6; 11, 1]$. Then we can trade $H_3 \cup H_{12} \cup H_{13} \cup H_4 \cup H_8 \cup H_2 \cup H_9 \cup H_{11} \cup H_{10} \cup H_1 \cup H_7$ for $G_{18} \cup G_{19} \cup G_{26} \cup G_{27} \cup G_{28} \cup G_{29} \cup G_{31} \cup G_{32} \cup G_{33} \cup G_{34} \cup G_{35}$. Hence, "$(11, 2)$-case" is done.

(xiii) Let $G_{36} = [3, 13, 7; 10, 0]$, $G_{37} = [2, 13, 8; 0, 11]$, $G_{38} = [1, 6, 13; 0, 12]$, $G_{39} = [1, 8, 10; 5, 11]$. Then we can trade $H_3 \cup H_{12} \cup H_{13} \cup H_4 \cup H_8 \cup H_2 \cup H_9 \cup H_{11} \cup H_{10} \cup H_1 \cup H_7 \cup H_6$ for $G_{18} \cup G_{19} \cup G_{26} \cup G_{27} \cup G_{28} \cup G_{32} \cup G_{34} \cup G_{35} \cup G_{36} \cup G_{37} \cup G_{38} \cup G_{39}$. Hence, "$(12, 1)$-case" is done.

(xiv) Let $G_{40} = [3, 13, 7; 4, 5]$, $G_{41} = [2, 13, 8; 0, 9]$, $G_{42} = [2, 12, 6; 11, 0]$, $G_{43} = [1, 8, 10; 7, 0]$, $G_{44} = [0, 10, 5; 11, 1]$. Then we can trade $H_3 \cup H_{12} \cup H_{13} \cup H_4 \cup H_8 \cup H_2 \cup H_9 \cup H_{11} \cup H_{10} \cup H_1 \cup H_7 \cup H_6 \cup H_5$ for $G_{18} \cup G_{19} \cup G_{26} \cup G_{27} \cup G_{28} \cup G_{32} \cup G_{34} \cup G_{38} \cup G_{40} \cup G_{41} \cup G_{42} \cup G_{43} \cup G_{44}$. Hence, "$(13, 0)$-case" is done.