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Scalar-tensor theory and the anisotropic perturbations of the inflationary universe

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Abstract Inflationary higher derivative scalar-tensor theory is analyzed in this paper in a de Sitter background space. A useful model-independent formula of the Friedmann equation is derived and used to study the stability problem associated with the anisotropic perturbations of the inflationary solution. The stability conditions of the de Sitter solution are derived for a general class of models. For a simple demonstration, an induced gravity model is considered in this paper for the effects of the higher derivative interactions including a cubic term.

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1 Introduction

The physical universe is a highly homogeneous and isotropic [1, 2] space known as the Friedmann–Robertson–Walker (FRW) space [3–6]. The cosmological problems, such as the flatness, the monopole, and the horizon problem, associated with the standard big bang model can be resolved by a successful inflationary mechanism [7–10].

Moreover, the Einstein–Hilbert models are expected to acquire higher derivative modifications near the Planck scale [11, 12]. For example, the quantum gravity and the string theories both show that the higher derivative terms could have interesting cosmological implications [11, 12] in the high energy domain. On the other hand, the higher derivative terms can also be interpreted as the quantum corrections of the matter fields [13–15]. Therefore, the possibility of deriving inflation from the higher derivative corrections has been a focus of research interest for a long time [14–18]. In addition, a general analysis on the stability conditions of the gravity theories is also useful in the search of the compatible physical models with our physical universe. For example, the stability conditions for a variety of pure gravity theories as a potential candidate of inflationary universe in the flat Friedmann–Robertson–Walker (FRW) space has been discussed in detail in Refs. [16, 17, 19–24].

It is known that any stable isotropically expanding solution should also be stable against any anisotropic perturbation. In fact, our physical universe could be anisotropic in the early stage of the evolution. It is therefore interesting to study the stability conditions derived from the anisotropic perturbations against a de Sitter expanding space during the early epoch. For instance, it has been shown that an inflationary solution does exist for an NS-NS model with a metric, a dilaton, and an axion field [25] in a Bianchi space. This inflationary solution can be shown to be stable against small anisotropic perturbations [26]. Similar analysis has also been studied for a variety of models [27].

Recently, there are growing interests in the study of the Kantowski–Sachs (KS) type spaces [28–30]. We will hence propose to study the existence and stability problem of the inflationary solution in a KS space. In particular, we will focus on the effects of the higher derivative terms in the KS space. Note that the stability analysis for a large class of pure gravity models admitting an inflationary KS/FRW solution was presented in Refs. [31, 32]. It is shown that the stability conditions of the de Sitter background space are closely related to the choice of the coupling constants in these models. For later convenience, any KS type solution that approaches asymptotically to a FRW final de Sitter state will be referred to as the KS/FRW solution in this paper.

The perturbation equations for any small anisotropic perturbations in a KS type space are identical to the perturbation equations for any small isotropic perturbations against the isotropic de Sitter space in the inflationary phase [16, 17, 19, 20]. Therefore, the existence of an unstable mode of the perturbation equations will ensure a proper resolution for the graceful exit problem both for the anisotropic perturbations and the isotropic perturbations in the inflationary phase as long as they both approach the same final de Sitter

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state. In certain models, unstable modes may not exist for the pure gravity models. A slow roll-over scalar field could hopefully provide a possible alternative to this problem. The slow roll-over scalar field will hold the de Sitter phase stable for a brief moment before the inflationary phase comes to an end. The anisotropic perturbations of these models may, however, not be compatible with the slow changing scalar field. Therefore, we need to solve the perturbation equations carefully in order to find out possible constraints on the coupling constants in these models. Once the inflation is over, the stable modes will ensure that the de Sitter space can remain stable and anisotropy will not grow out of control. Therefore, the absence of an unstable mode in the post inflationary era is also critical to the stability of the de Sitter background.

To be more specifically, an inflationary de Sitter solution in a scalar-tensor theory must have at least an unstable mode for the perturbation in $\delta\phi$ or some linear combinations of the δH and $\delta\phi$. Accordingly, the inflationary era will come to an end once the unstable mode takes over after a brief period Δt of inflationary expansion. If this period Δt is not long enough to derive 60-e fold inflationary expansion, the inflationary phase will be ended and wipe out the effect of the slow-rolling scalar field. Therefore, a unstable mode with a reasonable large unstable period Δt is needed for the graceful exit problem. We will show that the scalar field does provide a proper resolution to the graceful exit problem in this paper for in the higher derivative induced gravity models.

This paper will be organized as follows: (1) the derivation of a simple and model-independent formula of the Friedmann equation for a pure gravity theory will be reviewed briefly in Sect. 2; (2) a more general and model-independent stability analysis of the higher derivative scalar tensor models will be presented in Sect. 3; (3) in Sect. 4, we will focus on the higher derivative induced model with a cubic Lagrangian as an explicit example; (4) the conclusions will be presented in Sect. 5.

2 The Friedmann equation and the Bianchi identity in a KS space

The metric of the Kantowski–Sachs type space can be written as

$$ds^2 = -dt^2 + c^2(t)\,dr^2 + a^2(t)\big(d^2\theta + f^2(\theta)\,d\varphi^2\big) \tag{1}$$

with $f(\theta) = (\theta, \sinh\theta, \sin\theta)$ denoting the flat, open and close anisotropic space. More specifically, the Bianchi I (BI), III (BIII), and Kantowski–Sachs (KS) space corresponds to the flat, open and closed model respectively. This metric can also be written as

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\,d\theta^2\right) + a_z^2(t)\,dz^2 \tag{2}$$

with r and θ the polar coordinates, and z as the z-coordinate. Note that $k = 0, 1, -1$ stands for the flat, open and closed universes similar to the FRW space when $a = a_z$.

Writing $H_{\mu\nu} \equiv G_{\mu\nu} - T_{\mu\nu}$, the Einstein equation can be written as $D_\mu H^{\mu\nu} = 0$ by incorporating the Bianchi identity, $D_\mu G^{\mu\nu} = 0$, and the energy momentum conservation, $D_\mu T^{\mu\nu} = 0$. Here $G^{\mu\nu}$ and $T^{\mu\nu}$ represent the Einstein tensor and the energy momentum tensor coupled to the system respectively. With the metric (2), it can be shown that the r component of the equation $D_\mu H^{\mu\nu} = 0$ implies that $H^r_r = H^\theta_\theta$. This result also says that any matter coupled to the system must have the symmetric property $T^r_r = T^\theta_\theta$. In addition, the equations $D_\mu H^{\mu\theta} = 0$ and $D_\mu H^{\mu z} = 0$ both vanish identically for all kinds of energy momentum tensors. The most interesting information comes from the t component of this equation. It says that $(\partial_t + 3H)H^t_t = 2H_1 H^r_r + H_z H^z_z$. This equation asserts that (i) $H^t_t = 0$ implies that $H^r_r = H^z_z = 0$ and (ii) $H^r_r = H^z_z = 0$ only implies $(\partial_t + 3H)H^t_t = 0$ instead of $H^t_t = 0$. The case (ii) can be solved to give $H^t_t = \text{constant} \times \exp[-a^2 a_z]$ that approaches zero when $a^2 a_z \to \infty$.

For an anisotropic KS space, the metric contains two independent variables a and a_z. The Einstein field equations have, however, three non-vanishing components, i.e., $H^t_t = 0$, $H^r_r = H^\theta_\theta = 0$ and $H^z_z = 0$. The Bianchi identity implies that the tt component is not redundant and hence must be retained for a complete analysis. Ignoring either one of the rr or zz components will not, however, affect the final result of the system. In short, the $H^t_t = 0$ equation, known as the generalized Friedmann equation, is a non-redundant field equation as compared to the $H^r_r = 0$ and $H^z_z = 0$ equations.

By restoring the g_{tt} component $b^2(t) = 1/B_1$ will be helpful in deriving the non-redundant field equation associated with G_{tt} that will be shown shortly. More specifically, we will introduce the generalized KS metric:

$$ds^2 = -b^2(t)\,dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\,d\theta^2\right) + a_z^2(t)\,dz^2 \tag{3}$$

for the following reasons. In principle, the Lagrangian of the system should reduce from a functional of the metric $g_{\mu\nu}$, or equivalently $\mathcal{L}(g_{\mu\nu})$, to a simpler function of $a(t)$ and $a_z(t)$, namely $L(a(t), a_z(t)) \equiv a^2 a_z \mathcal{L}(g_{\mu\nu}(a(t), a_z(t)))$. The equation of motion as a function of $a(t)$ and $a_z(t)$ should be derivable from the variation of the effective Lagrangian $L(t)$ with respect to the variable a and a_z. The result is, however, incomplete because the variation of a and a_z are related to the variation of g_{rr} and g_{zz} respectively. The field equation from the variation of g_{tt} cannot be derived from the effective Lagrangian without restoring the variable $b(t)$ in advance. This is the motivation to introduce the metric (3) such that the effective Lagrangian $L(t) \equiv ba^2 a_z \mathcal{L}(g_{\mu\nu}(b(t), a(t), a_z(t)))$ restores the non-redundant information hidden in the $H^t_t = 0$ equation associated with the variation of the g_{tt} equation. The non-redundant Friedmann equation can hence be reproduced by setting $b = 1$ after the variation of $b(t)$ has been done.

Note that all non-vanishing components of the curvature tensor can be computed as [31, 32]

$$R^{ti}_{tj} = \left[\frac{1}{2}\dot{B}_1 H_i + B_1\left(\dot{H}_i + H_i^2\right)\right]\delta^i_j \tag{4}$$

$$R^{ij}_{kl} = B_1 H_i H_j \epsilon^{ijm}\epsilon_{klm} + \frac{k}{a^2}\epsilon^{ijz}\epsilon_{klz} \tag{5}$$

with $H_i \equiv (\dot{a}/a, \dot{a}/a, \dot{a}_z/a_z) \equiv (H_1, H_2 = H_1, H_z)$ for r, θ, and z component respectively.

Given a pure gravity model with a reduced Lagrangian $L = \sqrt{g}\mathcal{L} = L(b(t), a((t), a_z(t))$, it can be shown that

$$L = \frac{a^2 a_z}{\sqrt{B_1}}\mathcal{L}\left(R^{ti}_{tj}, R^{ij}_{kl}\right) = \frac{a^2 a_z}{\sqrt{B_1}}\mathcal{L}\left(H_i, \dot{H}_i, a^2\right) \tag{6}$$

with $B_1 = b^{-2}$ for convenience. As mentioned earlier the Friedmann equation can be derived from the variational equation with respect to the $\delta B_1 (= \delta b^{-2} = -2\delta b/b^3)$-equation of the reduced Lagrangian L. Our task here is to replace all δB_1 and $\delta\dot{B}_1$ effectively with δH_i and $\delta\dot{H}_i$ such that before we can set $B_1 = 1$ freely without any trouble and write the Friedmann equation free of the function $b(t)$. As a result, we can derive the field equations directly from the g_{ij} components more easily without bothering the restoration of the g_{tt} information any more. As a result, the Friedmann equation can be obtained from the above method by replacing $\delta L/\delta B_1$ and $\delta L/\delta\dot{B}_1$ with some proper combinations of $\delta L/\delta H_i$ and $\delta L/\delta\dot{H}_i$.

As a result, the Friedmann equation for the pure gravity model L can be shown to be [31, 32]

$$DL \equiv L + H_i\left(\frac{d}{dt} + 3H\right)L^i - H_i L_i - \dot{H}_i L^i = 0 \tag{7}$$

$$D_z L \equiv L + \left(\frac{d}{dt} + 3H\right)^2 L^z - \left(\frac{d}{dt} + 3H\right)L_z = 0 \tag{8}$$

Here $L_i \equiv \delta\mathcal{L}/\delta H_i$, $L^i \equiv \delta\mathcal{L}/\delta\dot{H}_i$, and $3H \equiv \sum_i H_i$. The second equation is derived from the variation equation δa_z. For simplicity, we have written $\mathcal{L}$ as L in the above equations. Note again that the δa_1 equation is redundant following the Bianchi identity shown above.

The proof follows from an observation that $\dot{B}_1$ always shows up as a combination of $\dot{B}_1 H_i + 2B_1(\dot{H}_i + H_i^2)$. Therefore $\delta L/\delta\dot{B}_1 = H_i\delta L/[2\delta\dot{H}_i]$. Here we have set $B_1 = 1$ whenever it will not affect the final result. Moreover, the summation over repeated indices is not written explicitly. In addition, $\delta L/\delta B_1 = H_i\delta L/[2\delta H_i] + \dot{H}_i\delta L/\delta\dot{H}_i$ if $L = L(B_1(a^i\dot{H}_i + a^{ij}H_i H_j))$ for any arbitrary "constant" coefficients a^i and a^{ij}. In fact, it can be shown that this result holds for all anisotropic Bianchi type spaces including the KS type spaces shown in (4)–(5). Indeed, the term $B_1\dot{H}_i$ will always show up together with $B_1 H_i H_j$ from the dimension analysis. Therefore the Friedmann equation derived above is a universal formula holds for all homogeneous Bianchi type spaces.

3 Higher derivative scalar tensor model

With an additional scalar field Lagrangian L_ϕ coupled to the scalar tensor Lagrangian L_g, we will have

$$\begin{aligned} L &= \sum_a f_a(\phi)L_{(a)} + L_\phi \\ &\equiv \sum_a f_a(\phi)L_{(a)}(H_i, \dot{H}_i) - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \end{aligned} \tag{9}$$

with $f_a(\phi)$ some polynomial functions of ϕ and $L_{(a)}$ some ath order pure gravity Lagrangian. These models are also known as modified gravity theories. For example, $f_1(\phi) = \epsilon\phi^2/2$, $L_{(1)} = -R$, $f_2 = -\alpha$, $L_{(2)} = R^2$, $f_3(\phi) = \gamma\phi^{-2}$ and $L_{(3)} = R^3$ stand for the induced gravity model of the Einstein–Hilbert action, the quadratic term and the cubic Lagrangian of the system. Here $V(\phi)$ denotes the scalar field potential coupled to the gravitational system.

The Friedmann equation becomes

$$\begin{aligned} &D\left[\sum_i f_a(\phi)L_{(a)}\right] \\ &\quad = \sum_a f_a(\phi)DL_{(a)} + \left[\sum_a H_i f'_a(\phi)\dot{\phi}\right]L^i_{(a)} \\ &\quad = \frac{1}{2}\dot{\phi}^2 + V(\phi) \end{aligned} \tag{10}$$

for this model. In addition, the scalar field equation can be shown to be

$$\ddot{\phi} + 3H_0\dot{\phi} + V' = \sum_a f'_a(\phi)L_{(a)} \tag{11}$$

We will focus on the stability problem of an inflationary de Sitter background solution $H_i = H_0$ and $\phi = \phi_0$ with a constant Hubble parameter H_0 and a constant initial scalar field ϕ_0. Let $H_i = H_0 + \delta H_i$ and $\phi = \phi_0 + \delta\phi$ be the anisotropic perturbations against the constant de Sitter background space. As a result, we have

$$\sum_a f_a(\phi_0)DL_{(a)}(H_i = H_0) = V(\phi_0) \tag{12}$$

$$V'(\phi_0) = \sum_a f'_a(\phi_0)L_{(a)} \tag{13}$$

as the leading zeroth-order equations of the perturbation equations.

The first-order perturbation equations of the pure gravity part of DL can be shown to be

$$\delta(DL) = \langle H_i L^{ij}\delta\ddot{H}_j\rangle + 3H\langle H_i L^{ij}\delta\dot{H}_j\rangle + 3H\langle (H_i L^i_j + L^j)\delta H_j\rangle + \langle H_i L^i\rangle\delta(3H) - \langle H_i L_{ij}\delta H_j\rangle \quad (14)$$

for any $DL_{(a)}$ defined by (7) with all functions of H_i evaluated in the de Sitter background $H_i = H_0$. From now on, the notation $\langle A_i B_i\rangle \equiv \sum_{i=1,z} A_i B_i$ denotes the summation over $i = 1$ and z for repeated indices. Note that we have absorbed the information of $i = 2$ into $i = 1$. They contributes equally to the field equations in the KS type spaces. In addition, $L^i_j \equiv \delta^2 L_{(a)}/\delta\dot{H}_i\delta H_j$ and similarly for L_{ij} and L^{ij}. Here the upper index ^{i} and the lower index $_j$ denote the variation with respect to $\dot{H}_i$ and H_j respectively for convenience. Note that the perturbation equation associated with (8) can also be shown to be the same as (14) in the de Sitter space due to the symmetry of the de Sitter background space [7–10].

In addition, it can be shown that $\langle H_i L^{i1}\rangle = 2\langle H_i L^{iz}\rangle$, $\langle H_i L^i_1\rangle = 2\langle H_i L^i_z\rangle$, $L^1 = 2L^z$, $\langle H_i L_{i1}\rangle = 2\langle H_i L_{iz}\rangle$, and $L_1 = 2L_z$ for a KS type space approaching the inflationary de Sitter background metric with $H_i = H_0$. As a result, the stability equations (14) can be greatly simplified. For convenience, we will also define the operator $\mathcal{D}_L$ as

$$\mathcal{D}_L\delta H \equiv \langle H_i L^{i1}\rangle\delta\ddot{H} + 3H\langle H_i L^{i1}\rangle\delta\dot{H} + 3H\langle H_i L^i_1 + L^1\rangle\delta H + 2\langle H_i L^i\rangle\delta H - \langle H_i L_{i1}\rangle\delta H \quad (15)$$

This equation hence becomes

$$\mathcal{D}_L\delta H = H_0\big[\langle L^{i1}\rangle\delta\ddot{H} + 3H_0\langle L^{i1}\rangle\delta\dot{H} + \big(3\langle H_0 L^i_1 + L^1\rangle + 2\langle L^i\rangle - \langle L_{i1}\rangle\big)\delta H\big] \quad (16)$$

$$= H_0\langle L^{i1}\rangle\left[\delta\ddot{H} + 3H_0\delta\dot{H} + \left(\frac{3\langle H_0 L^i_1 + L^1\rangle + 2\langle L^i\rangle - \langle L_{i1}\rangle}{\langle L^{i1}\rangle}\right)\delta H\right] \quad (17)$$

when the constant Hubble parameter is written explicitly. For convenience, we will also write $\mathcal{D}_{L_{(a)}}\delta H = \mathcal{D}_a\delta H$. As a result, the stability equation can be written as

$$\delta(DL_{(a)}) = \mathcal{D}_a\left(\delta H_1 + \frac{\delta H_z}{2}\right) = \frac{3}{2}\mathcal{D}_a(\delta H) \quad (18)$$

with $H = (2H_1 + H_z)/3$ as the mean value of all H_i.

Hence the first-order perturbation equation in δH and $\delta\phi$ of the Friedmann equation can be shown to be

$$\frac{3}{2}\sum_a f_a(\phi_0)\mathcal{D}_a\delta H = \left[V'(\phi_0) - \sum_a f'_a(\phi_0)DL_{(a)}\right]\delta\phi - \frac{3}{2}\left[\sum_a H_0 f'_a(\phi_0)\right]L^1_{(a)}\delta\dot{\phi} \quad (19)$$

Therefore, we will be solving the following equation:

$$\frac{3}{2}H_0\langle L^{i1}\rangle\big[\delta\ddot{H} + 3H_0\delta\dot{H} + KH_0^2\delta H\big] = \left[V'(\phi_0) - \sum_a f'_a(\phi_0)DL_{(a)}\right]\delta\phi - \frac{3}{2}\left[\sum_a H_0 f'_a(\phi_0)\right]L^1_{(a)}\delta\dot{\phi} \quad (20)$$

with

$$K \equiv \frac{3\langle H_0 L^i_1 + L^1\rangle + 2\langle L^i\rangle - \langle L_{i1}\rangle}{\langle L^{i1}\rangle H_0^2},$$

and $L = \sum_a f_a(\phi_0)L_{(a)}$ the total coefficient K and the total gravitational Lagrangian respectively in the constant ϕ_0 and H_0 background space. The explicit expression of K depends on the models being considered. The values of K plays, however, some crucial rolls in the stability problem of the corresponding de Sitter universe. Some general selection rules can be obtained in a straightforward way. Similarly, the first order perturbation equation of the scalar field can be shown to be

$$\delta\ddot{\phi} + 3H_0\delta\dot{\phi} + JH_0^2\delta\phi = \frac{3}{2}\sum_a f'_a(\phi_0)\big[L^1_{(a)}\delta\dot{H} + L_{(a)1}\delta H\big] \quad (21)$$

with

$$J = \left[V''_0 - \sum_a f''_a(\phi_0)L_{(a)}\right]H_0^{-2}$$

In addition, the variational equation of δa_z can be shown explicitly to be redundant in the limit $H_i = H_0 + \delta H_i$ and $\phi = \phi_0 + \delta\phi$ following the Bianchi identity. In summary, the values of J and K will affect the stability of the de Sitter solution. The above equations hence provide a model-independent method in determining whether a model is compatible with the inflationary de Sitter universe.

Indeed, by assuming that $\delta H = \exp[hH_0t]\delta H_0$ and $\delta\phi = \exp[pH_0t]\delta\phi_0$ for some constants h and p, we can write the above equations as

$$\frac{3}{2}H_0^3\langle L^{i1}\rangle\big[h^2 + 3h + K\big]\delta H$$

$$= -\frac{3}{2}\sum_a H_0^2 f_a'(\phi_0)L^1_{(a)}[p - J_1]\delta\phi \tag{22}$$

$$H_0^2\big(p^2 + 3p + J\big)\delta\phi = \frac{3}{2}\sum_a f_a'(\phi_0)L^1_{(a)}H_0[h + K_1]\delta H \tag{23}$$

with $K_1 = [\sum_a f_a'(\phi_0)L_{(a)1}]/[\sum_a f_a'(\phi_0)L^1_{(a)}H_0]$ and

$$J_1 = 2\frac{V'(\phi_0) - \sum_a f_a'(\phi_0)DL_{(a)}}{3\sum_a H_0^2 f_a'(\phi_0)L^1_{(a)}}$$

Note that the perturbation equation of δH shown on the left-hand side of (22) is the same as the pure gravity model with similar coupling constants. This equation is also the same as the perturbation equation in their isotropic limit.

We will show that the stability of the anisotropic space depends on the coefficient K. Indeed, (22) and (23) indicate that there are two decaying modes for δH and $\delta\phi$ with

$$2h = -3 \pm \sqrt{9 - 4K} = -2K_1 \tag{24}$$

$$2p = -3 \pm \sqrt{9 - 4J} = 2J_1 \tag{25}$$

As a result the modified gravity models are subjected to strong constraints in order to accommodate a consistent perturbative de Sitter inflationary solution:

$$H_i = H_0 + A_i \exp[-K_1 H_0 t] \tag{26}$$

$$\phi = \phi_0 + \delta\phi_0 \exp[J_1 H_0 t] \tag{27}$$

Here A_i and $\delta\phi_0$ are small initial perturbations at $t = 0$. Otherwise, the only consistent perturbative solution would be the trivial solution with $\delta\phi = 0$ and/or $\delta H_i = 0$.

Note that an unstable mode with $p = J_1 H_0$ indicates that the perturbative solution will remain stable for a brief period of time of the order of $1/(J_1 H_0)$. This means that the de Sitter solution will not be stable once $\Delta t > 1/(J_1 H_0)$. The exact decaying process will, however, also depend on the dynamics of the scalar field. For a slow roll-over scalar field, the system may remain close to the de Sitter phase for a brief period of time in competition with the instability period $\Delta t \sim 1/(J_1 H_0)$ derived from the unstable mode $p = J_1 H_0$.

4 Higher derivative induced gravity model

For a simple demonstration in this section, we will focus on the higher derivative induced gravity model given by

$$L = -\frac{\epsilon}{2}\phi^2 R - \alpha R^2 - \beta R^\mu_\nu R^\nu_\mu + \frac{\gamma}{\phi^2}R^{\mu\nu}_{\beta\gamma}R^{\beta\gamma}_{\sigma\rho}R^{\sigma\rho}_{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$$
$$\equiv \frac{\epsilon}{2}\phi^2 L_1 + L_2 + \frac{\gamma}{\phi^2}L_3 + L_\phi \tag{28}$$

with $L_1 = -R$, $L_2 = -\alpha R^2 - \beta R^\mu_\nu R^\nu_\mu$, $L_3 = R^{\mu\nu}_{\beta\gamma}R^{\beta\gamma}_{\sigma\rho}R^{\sigma\rho}_{\mu\nu}$ and $L_\phi = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$ denoting the lowest order curvature coupling, the higher order terms, and the scalar field Lagrangian, respectively. By definition the induced gravity models assume that all dimensionful parameters and all coupling constants, except the symmetry breaking scale parameter ϕ_0, are induced by some proper choices of the dynamical fields. For example, the gravitational constant is replaced by $8\pi G = 2/(\epsilon\phi^2)$ as a dynamical field. In addition, the cosmological constant becomes $V(\phi)$ in this model. There is no need for any induced parameters for the quadratic terms R^2 and $R^2_{\mu\nu}$ because the coupling constants α and β are both dimensionless by themselves. The action of this system is also invariant under the global scale transformation $g_{\mu\nu} \to \Lambda^{-2}g_{\mu\nu}$ and $\phi \to \Lambda\phi$ with some arbitrary constant parameter Λ.

The corresponding Lagrangian can be shown to be

$$\begin{aligned} L = {} & \epsilon\phi^2(2A + B + 2C + D) \\ & - 4\alpha\big[4A^2 + B^2 + 4C^2 + D^2 + 4AB + 8AC + 4AD \\ & + 4BC + 2BD + 4CD\big] - 2\beta\big[3A^2 + B^2 + 3C^2 + D^2 \\ & + 2AB + 2AC + 2AD + 2BC + 2CD\big] \\ & + 8\frac{\gamma}{\phi^2}\big[2A^3 + B^3 + 2C^3 + D^3\big] \\ & + \frac{1}{2}\dot\phi^2 - V(\phi) \end{aligned} \tag{29}$$

in the Kantowski–Sachs type spaces. Here $A = \dot H_1 + H_1^2$, $B = H_1^2 + k/a^2$, $C = H_1 H_z$, $D = \dot H_z + H_z^2$. This Lagrangian can be shown to reproduce the de Sitter models when we set $H_i \to H_0$ in the isotropic limit.

The Friedmann equation (10) reads

$$\frac{1}{2}\epsilon\phi^2 DL_1 + DL_2 + \frac{\gamma}{\phi^2}DL_3 + \epsilon\phi\dot\phi H_i L^i_1 - 2\frac{\gamma}{\phi^3}\dot\phi H_i L^i_3 = \frac{1}{2}\dot\phi^2 + V(\phi) \tag{30}$$

for the induced gravity model. In addition, the scalar field equation (11) can be shown to be:

$$\ddot\phi + 3H_0\dot\phi + V' = \epsilon\phi L_1 - 2\frac{\gamma}{\phi^3}L_3 \tag{31}$$

As a result, the leading order Friedmann equation and the scalar field equation can be shown to be

$$V_0 \equiv V(\phi_0) = 3\epsilon_0\phi_0^2 H_0^2 \tag{32}$$

$$V'(\phi_0) = 12\epsilon_0\phi_0 H_0^2 \tag{33}$$

in the presence of the de Sitter solution with $\phi = \phi_0$ and $H_i = H_0$ for all directions. Here $\epsilon_0 \equiv \epsilon[1 - 8\gamma H_0^4/(\epsilon\phi_0^4)]$.

The conventional approach assumes that the scalar field is a slow roll-over field obeying $\ddot{\phi} \ll V'$ and $H_0\dot{\phi} \ll V'$ near the inflationary phase. It can be shown that in the de Sitter inflationary phase, the dynamical part of the scalar field equation evolves as $\ddot{\phi} + 3H_0\dot{\phi} \sim 0$. This equation leads to the approximate solution

$$\phi \sim \phi_0 + \frac{\dot{\phi}_0}{3H_0}\left[1 - \exp(-3H_0 t)\right] \tag{34}$$

This result is clearly consistent with the slow roll-over assumption we just made. In summary, the zeroth order equations lead to a few constraints on the field parameters:

$$4V_0 = \phi_0 \frac{\partial V}{\partial \phi}(\phi = \phi_0) = 12\epsilon_0 \phi_0^2 H_0^2 \tag{35}$$

An appropriate effective spontaneously symmetry breaking potential V of the form

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \phi_0^2\right)^2 + 6\epsilon_0 H_0^2\left(\phi^2 - \phi_0^2\right) + 3\epsilon_0 H_0^2 \phi_0^2 \tag{36}$$

with arbitrary coupling constant λ, can be shown to be a good candidate satisfying all the scaling conditions (35).

The value of H_0 can be chosen to induce enough inflation for a brief moment as long as the slow roll-over scalar field remains close to the initial state $\phi = \phi_0$. The de Sitter phase will hence remain valid and drive the inflationary process for a brief moment governed by the decaying speed of the scalar field.

The inflationary Hubble parameter H_0 is related to γ, ϵ_0, and V_0 by the following equation:

$$H_0^6 - \frac{\epsilon \phi_0^2}{8\gamma} H_0^2 + \frac{V_0}{3\epsilon \phi_0^2} = 0 \tag{37}$$

This equation can be solved to give

$$H_0^2 = \sqrt{\frac{\epsilon \phi_0^2}{6\gamma}} \cos\left[\frac{\theta_0 \mp \pi}{3}\right] \tag{38}$$

with $\cos\theta_0 \equiv \sqrt{6\gamma/\epsilon}\, V_0/[\epsilon \phi_0^3]$. As a result,

$$\epsilon_0 = \epsilon\left[1 - \frac{4}{3\phi_0^2}\cos^2\left(\frac{\theta_0 \mp \pi}{3}\right)\right] \tag{39}$$

A different choice of ϵ_0 is therefore equivalent to a different choice of initial state V_0 and vice versa. For the practical reasons, we can take either ϵ_0 or V_0 related by above equation as a free parameter.

Note that the local extremum of this effective potential can be shown to be $\phi = 0$ (local maximum) and $\phi^2 = \phi_m^2 = \phi_0^2 - 12\epsilon_0 H_0^2/\lambda < \phi_0^2$ (local minimum). In addition the minimum value of the effective potential can be shown to be

$$V_m = V_0 - 36\epsilon_0^2 H_0^4/\lambda < V_0 \tag{40}$$

The constraint $V_m > 0$ implies that $\lambda \phi_0^2 > 12\epsilon_0 H_0^2$. Or equivalently, it implies that $\phi_m^2 > 0$. In addition, we will set $\epsilon \phi_m^2/2 = 1/(8\pi G) = 1$ in Planck units for convenience in this paper.

When the scalar field settles down to the local minimum ϕ_m of the effective potential at large time in the post inflationary era, it will oscillate around the local minimum and kick off the reheating process. The scalar field will eventually become a constant background field and induces a small cosmological constant $V_m = V(\phi_m)$.

The final state $\phi = \phi_m$ requires the identity

$$\epsilon_0 H_0^2 = \epsilon_m H_m^2 \tag{41}$$

for the consistency of a stable final state. Here $\epsilon_m \equiv \epsilon[1 - 8\gamma H_m^4/(\epsilon \phi_m^4)]$. By solving H_m^2 as a function of H_0^2, we can obtain

$$H_m^2 = \sqrt{\frac{\epsilon \phi_m^4}{6\gamma}} \cos\left[\frac{\theta_m \mp \pi}{3}\right] \tag{42}$$

with the following constraint:

$$\cos\theta_m = \left[\frac{54\gamma}{\epsilon \phi_m^4}\right]^{1/2}\left[1 - \frac{8\gamma H_0^4}{\epsilon \phi_0^4}\right] \le 1 \tag{43}$$

Hence we have

$$\frac{\epsilon_0}{\epsilon} \le \left[\frac{\epsilon \phi_m^4}{54\gamma}\right]^{1/2} = \left[\frac{2}{27\epsilon\gamma}\right]^{1/2} \tag{44}$$

This implies the inequality $(27\epsilon_0^2 - 16H_0^4/\phi_0^4)\gamma \le 2\epsilon_0$. Therefore, we have

$$\gamma \le \frac{2\epsilon_0}{27\epsilon_0^2 - 16H_0^4/\phi_0^4} \tag{45}$$

if the denominator $27\epsilon_0^2 - 16H_0^4/\phi_0^4$ is positive. Otherwise, the inequality $(27\epsilon_0^2 - 16H_0^4/\phi_0^4)\gamma \le 2\epsilon_0$ is automatically satisfied. In addition, the leading order perturbation equation in δH and $\delta\phi$ of the Friedmann equation in this model can be shown to be

$$4\left(3\alpha + \beta - 6\gamma\frac{H_0^2}{\phi_0^2}\right)\left(\delta\ddot{H} + 3H_0\delta\dot{H} + KH_0^2\delta H\right) = \epsilon\left(1 - 24\frac{\gamma H_0^4}{\epsilon\phi_0^4}\right)\phi_0[\delta\dot{\phi} - H_0\delta\phi] \tag{46}$$

with

$$K = \frac{24\gamma H_0^4/\phi_0^2 - \epsilon\phi_0^2}{4(3\alpha + \beta - 6\gamma H_0^2/\phi_0^2)H_0^2}$$

Similarly, the leading perturbation equation of the scalar field can be shown to be

$$\delta\ddot{\phi} + 3H_0\delta\dot{\phi} + JH_0^2\delta\phi = 6\epsilon_0\phi_0(\delta\dot{H} + 4H_0\delta H) \tag{47}$$

with

$$J=\left(V''-12\epsilon_0 H_0^2-384\gamma\frac{H_0^6}{\phi_0^4}\right)H_0^{-2}$$

The variational equation of a_z can be shown explicitly to be redundant in the limit $H_i=H_0+\delta H_i$ and $\phi=\phi_0+\delta\phi$ following the Bianchi identity.

Assuming that $\delta H=\exp[hH_0t]\delta H_0$ and $\delta\phi=\exp[pH_0t]\delta\phi_0$ for some constants h and p, we can write the above equations as

$$\begin{aligned}&\epsilon\left(1-24\frac{\gamma H_0^4}{\epsilon\phi_0^4}\right)\phi_0[p-1]\delta\phi\\&\quad=4\left(3\alpha+\beta-6\gamma\frac{H_0^2}{\phi_0^2}\right)H_0\left[h^2+3h+K\right]\delta H\end{aligned}\tag{48}$$

$$H_0\left[p^2+3p+J\right]\delta\phi=6\epsilon_0\phi_0[h+4]\delta H\tag{49}$$

These equations are consistent when all coefficients vanish simultaneously. This implies that $h=-4$ and $p=1$. This set of solution $(h,p)=(-4,1)$ hence imposes two additional constraints:

$$\begin{aligned}&\epsilon-16(3\alpha+\beta)\frac{H_0^2}{\phi_0^2}+72\gamma\frac{H_0^4}{\phi_0^4}\\&\quad=\epsilon_0-16(3\alpha+\beta)\frac{H_0^2}{\phi_0^2}+80\gamma\frac{H_0^4}{\phi_0^4}=0\end{aligned}\tag{50}$$

$$\lambda=192\gamma\frac{H_0^6}{\phi_0^6}-2\frac{H_0^2}{\phi_0^2}\tag{51}$$

with $2\lambda\phi_0^2=V_0''-12\epsilon_0 H_0^2$.

The coupling constant λ has to positive in order for the effective potential $V(\phi)$ to be free from run-away negative global minimum at $\phi\to\infty$. As a result, the constraints $\epsilon_0>0$ and $\lambda>0$ imply that

$$\frac{\phi_0^4}{96H_0^4}<\gamma<\frac{(3\alpha+\beta)\phi_0^2}{5H_0^2}\tag{52}$$

Together with the constraint (45),

$$\gamma\le\frac{2\epsilon_0}{27\epsilon_0^2-16H_0^4/\phi_0^4}$$

the physical parameters such as γ can be chosen properly to accommodate a large class of solutions to the evolution of our physical universe. As a result, the inflationary phase will remain stable against small perturbation along the $\delta H(=\exp[-4H_0t]\delta H_0)$ direction. On the other hand, the inflationary phase also has an unstable mode when we perturb the system along the $\delta\phi(=\exp[H_0t]\delta\phi_0)$ direction that will hold the de Sitter phase stable only for a brief moment $\Delta t\sim 1/(pH_0)=1/H_0$. This brief period is apparently not enough for a complete inflationary phase. As indicated from (34), ϕ does not evolve appreciably during the inflationary phase if the unstable mode will not break the stability of the system. In short, enough inflation will require an unstable mode with a long enough Δt before the exit of the inflationary phase.

In addition to the above trivial solution, there are some other perturbation solutions. Note that the perturbation equations can also be cast in the form

$$\begin{aligned}D\delta\Psi&=D\begin{pmatrix}\delta H\\ \delta\phi\end{pmatrix}\\&=\begin{pmatrix}A_1(h^2+3h+K)&-C_1(h-1)\\B_1(h+4)&-(h^2+3h+J)\end{pmatrix}\begin{pmatrix}\delta H\\ \delta\phi\end{pmatrix}\\&=0\end{aligned}\tag{53}$$

with $\delta H\equiv k_H\exp[hH_0t]$, $\delta\phi\equiv k_\phi\exp[hH_0t]$. Here we have assumed that $\delta H=\sum_i k_i\exp[h_iH_0t]$ and $\delta H=\sum_i j_i\exp[h_iH_0t]$ such that

$$D\delta\Psi=\sum_i D\begin{pmatrix}k_i\\ j_i\end{pmatrix}\exp[h_iH_0t]=0\tag{54}$$

Hence solving the perturbation equations amounts to solving the eigenvalue problem given by (53). It is also understood that h written in (53) represents the eigenvalue h of the operator ∂_t operating on its eigenstate, namely, $\partial_t\exp[hH_0t]=hH_0\exp[hH_0t]$. In order to simplify the derivation, we will extract all dimensionful parameters by defining $\varphi_0=\phi_0^2/H_0^2$, $\beta_1=(3\alpha+\beta)/\varphi_0$, $\gamma_1=\gamma/\varphi_0^{-2}$ and $\lambda_1=\lambda\varphi_0$. As a result, the coefficients A_1, B_1, C_1, J and K can be written as

$$A_1=4(\beta_1-6\gamma_1)\varphi_0\tag{55}$$

$$B_1=6(\epsilon-8\gamma_1)\sqrt{\varphi_0}\tag{56}$$

$$C_1=(\epsilon-24\gamma_1)\sqrt{\varphi_0}\tag{57}$$

$$J=2\lambda_1-384\gamma_1\tag{58}$$

$$K=\frac{24\gamma_1-\epsilon}{4(\beta_1-6\gamma_1)}\tag{59}$$

The perturbation equations have a non-trivial solution only when $\det D=0$, which can be written as

$$\Delta^2+F\Delta+G=0\tag{60}$$

with $\Delta\equiv h^2+3h$ and

$$F=\frac{(24\gamma_1-\epsilon)(1+6\epsilon-48\gamma_1)}{4(\beta_1-6\gamma_1)}+2\lambda_1-384\gamma_1\tag{61}$$

$$G=\frac{(24\gamma_1-\epsilon)(\lambda_1-12\epsilon-96\gamma_1)}{2(\beta_1-6\gamma_1)}\tag{62}$$

Therefore, we can solve the perturbation equations and obtain the eigenvalue h as

$$h = \frac{-3 \pm \sqrt{9+4\Delta}}{2} \tag{63}$$

with

$$\Delta = \frac{-F \pm \sqrt{F^2-4G}}{2} \tag{64}$$

Hence we find four independent solutions to the perturbation equations (53). The graceful exit requires the exist of at least an unstable mode with $h > 0$. This will be the case if $\Delta > 0$. In fact, an inflationary phase for a period $\Delta t \sim 60/H_0$ is required for the universe to undergo enough expansion of roughly exp[60] times before the end of the inflationary phase. This in turns requires that $h \sim 1/60$. It is easy to show that this condition is equivalent to the constraint $\Delta \sim 1/5$. This condition can be shown to be $F + 5G \sim -1/5$. Hence it can also be written explicitly as

$$\lambda_1 \sim \big(4(\beta_1 - 6\gamma_1)(1920\gamma_1 - 1) + 5(24\gamma_1 - \epsilon) \times (1080\gamma_1 + 114\epsilon - 1)\big)/\big(10(96\gamma_1 - 5\epsilon + 4\beta_1)\big) \tag{65}$$

as a constraint on λ_1. Note that in the limit $\gamma_1 = 0$, we have

$$\lambda_1 \sim \frac{5\epsilon(1-114\epsilon) - 4\beta_1}{10(4\beta_1 - 5\epsilon)} \tag{66}$$

Together with the constraint (45), which is equivalent to

$$2\varphi_0^2 \geq 27\gamma_1 \left(\epsilon - 16\gamma_1 + 64\frac{\gamma_1^2}{\epsilon}\right) \tag{67}$$

the unstable mode can be managed to provide reasonable resolution to the graceful exit problem for the inflationary models. For example, we can choose

$$\gamma_1 > \frac{1-114\epsilon}{1080} \tag{68}$$

and either

$$\frac{\beta_1}{6} > \gamma_1 > \frac{\epsilon}{24} > \frac{1}{1920} \tag{69}$$

or

$$\frac{\beta_1}{6} > \gamma_1 > \frac{1}{1920} > \frac{\epsilon}{24} \tag{70}$$

as the constraint on the coupling constants ϵ, β_1, and γ_1 to ensure that $\lambda_1 > 0$. Note that the inequality $\beta_1/6 > \gamma_1 > \epsilon/24$ can be shown to imply that $96\gamma_1 - 5\epsilon + 4\beta_1 > 0$. We can also show that (69) implies that $\epsilon > 1/80$. As a result, the inequality (68) implies that

$$\gamma_1 > -\frac{17}{43200} > \frac{1-114\epsilon}{1080}$$

On the other hand, the inequality (70) implies that $\epsilon < 1/80$. As a result, the inequality (68) implies that

$$\gamma_1 > \frac{1-114\epsilon}{1080} > -\frac{17}{43200}$$

Therefore, we can indeed choose proper constraints on the coupling constants to ensure the resolution of graceful exit problem in the inflationary universe. Both constraints shown above can be realized with reasonably chosen coupling constants. Therefore, the scalar field does provide a useful tool both in inducing proper inflation and providing a natural mechanism for the graceful exit problem.

Note again that when the scalar field settles down to the local minimum ϕ_m of the effective potential at large time in the post-inflationary era, it will oscillate around the local minimum and kick off the reheating process. The scalar field will eventually become a constant background field with a small cosmological constant $V_m = V(\phi_m)$. The stability of the system will then be dominated by the evolution of the scale factor a. Therefore, we end up with a stable de Sitter background space in the large time region.

The reason that only the special combination $3\alpha + \beta$ shows up in the stability equation is that two identities connect the quadratic curvature terms in the 4-dimensional space time. Indeed, there are a Gauss–Bonnet invariant E and an additional conformal Weyl invariant at our disposal:

$$E = R^{ab}_{cd} R^{cd}_{ab} - 4R^a_b R^b_a + R^2 \tag{71}$$

$$C^2 \equiv C^{ab}_{cd} C^{cd}_{ab} = R^{ab}_{cd} R^{cd}_{ab} - 2R^a_b R^b_a + \frac{1}{3}R^2 \tag{72}$$

As a result, we can write

$$3\alpha R^2 + 3\beta R^a_b R^b_a = (3\alpha + \beta)R^2 + \frac{3\beta}{2}\big(C^2 - E\big) \tag{73}$$

The Gauss–Bonnet term $\sqrt{g}E$ is an Euler invariant known to be a total derivative. Therefore, it will not contribute to the field equation. In addition, the FRW space is known to be conformally flat. Hence the conformal Weyl invariant C^2 will not contribute to the field equations either. Therefore, the stability equation will depend only on the combination $3\alpha + \beta$.

The reason that the quadratic terms do not affect the scale of inflation H_0 can be checked readily by showing that any quadratic Lagrangian of the combinations $l_1 \dot{H}^2 + l_2(\dot{H}H^2 + H^4)$ will not contribute to the Friedmann equation. Here l_i are constants. Both R^2 and $R^a_b R^a_b$ are of this form, hence they will not contribute to the background Friedmann equation. Note that the curvature term is assumed to be negligible in this phase. Alternatively, we can focus on the flat homogeneous space for simplicity.

Indeed, the quadratic terms will contribute to the Friedmann equation as a combination of

$$E_2 = L + H_i\left(\frac{d}{dt} + 3H\right)L^i - H_i L_i - \dot{H}_i L^i$$
$$\to L + 3H^2 L_{\dot{H}} - H L_H \tag{74}$$

in the de Sitter background with $L_H \equiv \delta L/\delta H$ and $L_{\dot{H}} \equiv \delta L/\delta \dot{H}$. It is clear that the l_1 term vanishes in the de Sitter space. Furthermore, $\dot{H}H^2$ terms will not contribute to the E_2 except through the effect of $L_{\dot{H}}$. Hence $HL_H \to 4L$ in the de Sitter space for the quadratic Lagrangian. As a result $E_2 \to 3(H^2 L_{\dot{H}} - L)$. Therefore, $E_2 = 0$ if and only if the contributions of $\dot{H}H^2$ and H^4 in the quadratic Lagrangian are equal as stated in the form $l_2(\dot{H}H^2 + H^4)$ shown above.

In the de Sitter background space, the Riemannian curvature component functions A, B, C, D for the KS type space defined earlier are related to each other by $A = D$ and $B = C$ when the curvature term is negligible in the inflationary phase. Therefore any combinations of the forms $A^2 + B^2$ and AB all fall into the class of $l_2(\dot{H}H^2 + H^4)$. Therefore it is straightforward to verify that the quadratic Lagrangian does not contribute to the Friedmann equation in the de Sitter background.

5 Einstein gravity and induced gravity

In order to compare and clarify the differences of the stability conditions contributed from the higher derivative terms and the induced gravity models with respect to the Einstein gravity, we will also study the stability conditions of the Einstein theory and induced gravity model without higher derivative terms in this section.

5.1 Leading order induced gravity model

For the induced gravity model, the Lagrangian of the system is

$$L = -\frac{\epsilon}{2}\phi^2 R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{75}$$

The Friedmann equation (10) reads

$$\frac{1}{2}\epsilon\phi^2 D L_1 + \epsilon\phi\dot{\phi}H_i L_1^i = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{76}$$

for the induced gravity model (75) with $L_1 = -R$. In addition, the scalar field equation (11) can be shown to be:

$$\ddot{\phi} + 3H\dot{\phi} + V' = -\epsilon\phi R \tag{77}$$

As a result, we also end up with the constraint (35),

$$4V_0 = \phi_0 \frac{\partial V}{\partial \phi}(\phi = \phi_0) = 12\epsilon_0\phi_0^2 H_0^2$$

in the presence of the de Sitter background solution with $\phi = \phi_0$ and $H_i = H_0$ for all directions.

In addition, the slow roll-over field obeys $\ddot{\phi} \ll V'$ and $H_0\dot{\phi} \ll V'$ near the inflationary phase. It can be shown that in the de Sitter inflationary phase, the dynamical part of the scalar field equation also evolves as $\ddot{\phi} + 3H_0\dot{\phi} \sim 0$. This equation leads to the approximate solution (34)

$$\phi \sim \phi_0 + \frac{\dot{\phi}_0}{3H_0}\left[1 - \exp(-3H_0 t)\right]$$

This result is clearly consistent with the slow roll-over assumption we just made. An appropriate effective spontaneously symmetry breaking potential V is therefore the same as the one given in (36):

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \phi_0^2\right)^2 + 6\epsilon_0 H_0^2\left(\phi^2 - \phi_0^2\right) + 3\epsilon_0 H_0^2\phi_0^2$$

with arbitrary coupling constant λ. As a result, the stability conditions for the models without higher derivative terms will therefore be the $\alpha = \beta = \gamma = 0$ limit of the higher derivative models discussed in Sect. 4. For example, the stability equations become

$$-\epsilon\phi_0^2\delta H = \epsilon\phi_0[\delta\dot{\phi} - H_0\delta\phi] \tag{78}$$

$$\delta\ddot{\phi} + 3H_0\delta\dot{\phi} + J_0 H_0^2\delta\phi = 6\epsilon_0\phi_0(\delta\dot{H} + 4H_0\delta H) \tag{79}$$

with $J_0 = (V'' - 12\epsilon H_0^2)H_0^{-2}$. Similarly, the variational equation of a_z can be shown explicitly to be redundant in the limit $H_i = H_0 + \delta H_i$ and $\phi = \phi_0 + \delta\phi$ following the Bianchi identity. In this case, the trivial solution $(h, p) = (-4, 1)$ will not survive for the independent perturbations $\delta H = k_H \exp[hH_0 t]$ and $\delta\phi = k_\phi \exp[pH_0 t]$. Instead, there is a consistent solution of the form with $\delta H \equiv k_H \exp[hH_0 t]$, $\delta\phi \equiv k_\phi \exp[hH_0 t]$. Note that the perturbation equations can also be cast in the form

$$D\delta\Psi = D\begin{pmatrix}\delta H \\ \delta\phi\end{pmatrix} = \begin{pmatrix}\phi_0/H_0 & (h-1) \\ 6\epsilon\phi_0(h+4) & -H_0(h^2 + 3h + J_0)\end{pmatrix}\begin{pmatrix}\delta H \\ \delta\phi\end{pmatrix} = 0 \tag{80}$$

Note that non-trivial solutions to the above equation exist only when $\det D = 0$. Therefore, we can derive the following stability equation for h:

$$h^2 + 3h + K_0 = 0 \tag{81}$$

with $K_0 = [2\lambda\phi_0^2/H_0^2 - 24\epsilon]/[1 + 6\epsilon]$. As a result, the solution h to above stability equation can be shown to be

$$h = h_\pm = \frac{1}{2}\left\{-3 \pm \left[9 - 8\frac{\lambda\phi_0^2 - 24\epsilon H_0^2}{(1 + 6\epsilon)H_0^2}\right]^{1/2}\right\} \tag{82}$$

Consequently, an unstable mode, $h_+ > 0$, exists when $\lambda\phi_0^2 < 24\epsilon H_0^2$. Therefore, the unstable mode can provide a natural way to end the inflationary phase. Note that the stability equation (81) is a polynomial equations of degree 2, a simplified version of the degree 4 polynomial equation (60) for the higher derivative models. It is therefore straightforward to observe the critical role of the higher derivative terms by comparing these two equations.

5.2 Einstein theory with a scalar field

Let us consider further the Einstein theory with a coupled scalar field in the absence of the higher derivative terms:

$$L = -\frac{1}{2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{83}$$

The Friedmann equation reads

$$H_1^2 + 2H_1 H_z = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{84}$$

for the induced gravity model (83). Similarly, we have ignored the curvature term during the inflationary era. In addition, the scalar field equation can be shown to be

$$\ddot{\phi} + 3H_0\dot{\phi} + V' = 0 \tag{85}$$

As a result, we also end up with the leading order Friedmann equation and the scalar field equation as

$$V_0 = 3H_0^2 \tag{86}$$

$$V'(\phi_0) = 0 \tag{87}$$

in the presence of the de Sitter solution with $\phi = \phi_0$ and $H_i = H_0$ for all directions. The perturbation equations are $\delta H = 0$ and

$$\left(p^2 + 3p + V_0''/H_0^2\right)\delta\phi = 0$$

for $\delta\phi \sim \exp[pH_0 t]$. Note that h-mode and p-mode decouple in this set of equations. Therefore, there are only trivial solutions for these models. Indeed, the h-mode equation implies that the h-mode is a stable mode. Therefore, the scalar p-mode will have to take care of the graceful exit mechanism in these models. For example, the model with a symmetry breaking scalar potential

$$V = \lambda\left(\phi^2 - v^2\right)^2/4 \tag{88}$$

will ensure the constraints (86) and (87) remain consistent if the evolution starts with $\phi_0 = 0$. Here v is a constant denoting the symmetry breaking scale given by the relation $\lambda v^4/4 = 3H_0^2$. Indeed, the constrain $V'(\phi_0) = 0$ indicates that the scalar field in the inflationary era has to start off from the local maximum, $\phi = 0$, of the scalar potential and rolls slowly down toward the local minimum, $\phi = v$, of the scalar potential. Hence the solution of the $\delta\phi$ equation is

$$p = p_\pm = \frac{1}{2}\left\{-3 \pm \left[9 + 4\lambda v^2\right]^{1/2}\right\} \tag{89}$$

As a result, $p_+ > 0$ is an unstable mode. Hence the graceful exit can occur if the evolution starts from a field configuration H_0 and $\phi_0 = 0$ under the effect of the symmetry breaking potential (88).

Einstein–Hilbert model

If we turn off the scalar field in (83), the system will become the Einstein–Hilbert model with a cosmological constant Λ:

$$L = -\frac{1}{2}R - \Lambda \tag{90}$$

The situation is similar to the model with scalar field. The Friedmann equation reads

$$H_1^2 + 2H_1 H_z = \Lambda \tag{91}$$

for the pure gravity model (90). Therefore, the system will remain stable for a long time by itself. This is also the reason why a scalar field or the higher derivative term is needed for the graceful exit.

6 Conclusion

The existence of a stable de Sitter background is closely related to the choices of the coupling constants in the system. The pure higher derivative gravity model with the quadratic terms and a cubic interaction is known to admit a stable inflationary solution with a proper choice of the field parameters [31, 32]. Indeed, proper choice of the coupling constants enables the existence of a de Sitter phase that is stable against any small isotropic and anisotropic perturbations. In many cases, there also exists another unstable mode that will be acting in favor of the graceful exit of the inflationary models.

It is shown that any small perturbation (against the isotropic FRW background space) and any small perturbation (against the anisotropic KS type background space) obey the same perturbation equations. This is also true for modified gravity models. Therefore, the stable modes will act in favor of the stability of the background de Sitter space. These stable modes will also ensure that the anisotropy of the de Sitter space will not grow out of control. On the other hand, the unstable mode indicates that the isotropic background is unstable against any small isotropic or anisotropic perturbations. Therefore, only a small anisotropy in the early universe could be generated by arbitrary small anisotropic perturbations in both of these models. Hence we are looking for constraints on the field parameters that will ensure that the system admits at least an unstable mode for the resolution of the graceful exit problem for the inflationary solution.

Indeed, we have shown that various constraints must be observed for the existence of an unstable mode in the modified gravity models. In particular, we show that, for induced gravity models, an unstable mode does exist with properly chosen constraints. As a result, only a small anisotropy against the de Sitter background can grow during the inflationary phase for this induced model. Indeed, an explicit model with a spontaneously symmetry breaking ϕ^4 potential is presented as an example for a simple demonstration. Accordingly, various constraints are also derived for this model. In addition, we also compare the higher derivative models with the models without higher derivative terms. The differences with respect to the Einstein gravity are also clarified in previous section. As a result, the effect of the higher derivative terms become more transparent by these comparisons.

In summary, we have shown that an unstable mode for a small (an)isotropic perturbation against the de Sitter background does exist for the induced gravity model The problem of a graceful exit can be achieved counting on the unstable mode of the scalar field perturbation. In addition, we also explain explicitly the reason that the quadratic terms will not affect the inflationary solution characterized by the Hubble parameter H_0. These quadratic terms play, however, a critical role in the stability problem of the de Sitter background in the modified gravity models.

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References

1. S. Gulkis, P.M. Lubin, S.S. Meyer, R.F. Silverberg, Sci. Am. **262**(1), 122–129 (1990)
2. D.J. Fixsen et al., Astrophys. J. **473**, 576–587 (1996)
3. S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972)
4. C.W. Misner, K. Thorne, T.A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973)
5. R.M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984)
6. E.W. Kolb, M.S. Turner, *The Early Universe* (Addison-Wesley, Reading, 1990)
7. A.H. Guth, Phys. Rev. D **23**, 347 (1981)
8. E.W. Kolb, M.S. Turner, Ann. Rev. Nucl. Part. Sci. **33**, 645 (1983)
9. F.S. Accetta, D.J. Zoller, M.S. Turner, Phys Rev. D **31**, 3046 (1985)
10. W.F. Kao, Class. Quantum Gravity **24**, 4295 (2007). hep-th/0605078
11. M.B. Green, J.H. Schwartz, E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, 1986)
12. A.S. Goncharov, A.D. Linde, V.F. Mukhanov, Int. J. Mod. Phys. A **2**, 561 (1987)
13. N.D. Birrell, P.C.W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982)
14. G.F. Chapline, N.S. Manton, Phys. Lett. B **120**, 105 (1983)
15. C.G. Callan, D. Friedan, E.J. Martinec, M.J. Perry, Nucl. Phys. B **262**, 593 (1985). gr-qc/0102077
16. A. Dobado, A. Lopez, Phys. Lett. B **316**, 250 (1993)
17. A. Dobado, A.L. Maroto, Phys. Rev. D **52**, 1895–1901 (1995). Erratum: *ibid.* D **53** (1996) 2262. hep-ph/9406409
18. P. Kanti, J. Rizos, K. Tamvakis, Phys. Rev. D **59**, 083512 (1999)
19. W.F. Kao, U.-L. Pen, Phys. Rev. D **44**, 3974–3977 (1991)
20. W.F. Kao, U.-L. Pen, P. Zhang, Phys. Rev. D **63**, 127301 (2001). gr-qc/9911116
21. K.I. Maeda, Phys. Rev. D **39**, 3159 (1989)
22. R. Wald, Phys. Rev. D **28**, 2118 (1983)
23. J.D. Barrow, S. Hervik, Phys. Rev. D **73**, 023007 (2006)
24. J.D. Barrow, S. Hervik, Phys. Rev. D **74**, 124017 (2006)
25. C.-M. Chen, T. Harko, M.K. Mak, Phys. Rev. D **63**, 064002 (2001). hep-th/0005236
26. C.-M. Chen, W.F. Kao, Phys. Rev. D **64**, 124019 (2001). hep-th/0104101
27. S.A. Abel, G. Servant, Nucl. Phys. B **597**, 3–22 (2001). hep-th/0009089
28. X.-z. Li, J.-g. Hao, Phys. Rev. D **68**, 083512 (2003). astro-ph/0308288
29. S. Shu, NCTU, Master thesis
30. S. Nojiri, O. Obregon, S.D. Odintsov, K.E. Osetrin, Phys. Rev. D **60**, 00 (1999). hep-th/9902035
31. W.F. Kao, Phys. Rev. D **74**, 043522 (2006)
32. W.F. Kao, Eur. Phys. J. C **53**, 87 (2008)