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Chaotic synchronization in lattices of two-variable maps coupled with one variable

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In this paper, we study chaotic synchronization in 1D lattices of two-variable maps coupled with one variable. We give a rigourous proof for the occurrence of chaotic synchronization of spatially homogeneous solutions in such coupled map lattices (CMLs) of lattice size $n = 4$ with suitable coupling coefficients. For the case of lattice size $n > 4$, we demonstrate numerical results of synchronized chaotic behaviour of the CMLs. Moreover, we show numerically that the difference between two variables manifests chaotic behaviour. This behaviour combined with the special coupling method in the CMLs guarantees high security in applications using our new model.

Keywords: chaotic synchronization; coupled map lattices; Lyapunov method; hyperchaos.

1. Introduction

Secure communication faces more and more serious challenges. In recent years, decryption techniques have been developed very rapidly. For example, as an Internet standard, MD5 (message-digest algorithm 5) has been employed in a wide variety of security applications and is also commonly used to check the integrity of files. Wang & Yu (2005) demonstrated collision attacks against MD5, SHA-0 (SHA stands for secure hash algorithm) and other related hash functions. Later, Wang *et al.* (2005) found a method to find collisions in the SHA-1 hash function, which is used in many of today's mainstream security products. Their attack is estimated to require far fewer operations than previously thought needed to find a collision in SHA-1. Although no attacks have yet been reported on the SHA-2 variants, which are algorithmically similar to SHA-1, a new hash function, to be known as SHA-3, is currently under development. It shows the necessity of developing alternative methods in secure communication.

With the combination of synchronization and unpredictability, chaotic synchronization has attracted a lot of attention since 1990 for its promising potential in secure communication. A secret message can

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be modulated on the chaotic signal of a sender, and a receiver with an identical system which is driven by the modulated signal can decrypt this message. Many encryption models based on chaotic synchronization have been proposed (see Pecora & Carroll, 1990; Vohra *et al.*, 1992; Cuomo & Oppenheim, 1992, 1993; Wu & Chua, 1994; Heagy *et al.*, 1995; Pecora *et al.*, 1997).

On the other hand, it has been pointed out that the proposed chaos-based communication systems have many flaws and need to be improved (see Pérez & Cerdeira, 1995; Yang *et al.*, 1998; Short & Parker, 1998; Zhou & Lai, 1999; Li *et al.*, 2005; Hu & Guo, 2008). Prompted by these decryption methods, many countermeasures have been developed to improve the security of communication systems based on chaotic synchronization. Although some of them have been shown to be insecure still, more and more complicated and effective countermeasures have been proposed. For example, Kanter *et al.* (2008a) showed that for non-identical partners which use private commutative filters and can synchronize, it may be difficult for the attacker to synchronize and to reveal the time-dependent output signal of the parties. Another work of Kanter *et al.* (2008b) even maps the task of the attacker onto the nondeterministic polynomial time-complete problems, for which all known deterministic algorithms require running time that is exponential with some tunable parameters of the problem. Thus, it is computationally infeasible for an attacker to extract the message from the transmitted signal.

These works stimulated intensive research on communication with synchronized chaos which is still ongoing. For example, communication with chaos synchronization has recently been demonstrated with semiconductor lasers which were synchronized over a distance of 120 km in a public fiber network in Greece (see Argyris *et al.*, 2005).

In this paper, we consider chaotic synchronization in coupled map lattices (CMLs) which can be considered as systems of interacting maps, where the individual map is characterized not only by its internal state but also by the position in the physical space. CMLs are, in general, the intermediate between partial differential equations (PDEs) and cellular automata which form a wide class of extended dynamical systems. PDEs are usually used to describe the physical phenomenon of spatial-temporal dynamical systems. However, the analytic study of solutions of PDEs suffers from extreme difficulty with complex behaviour. On the other hand, the computer simulation is utilized as an effective and powerful tool to study dynamical systems with complex behaviour. In such a study, the dynamical system shall be discretized in space as well as time. This is one of the motivations to introduce new models of CMLs (see Afraimovich & Bunimovich, 1993; Bunimovich, 1997; Bunimovich & Carlen, 1995; Giberti & Vernia, 1994; Kaneko, 1993).

The simplest type of chaotic synchronization of CMLs occurs in stable spatially homogeneous regimes corresponding to the existence of attractive spatially homogeneous solutions. In other words, in such cases, there is a large (open) set of initial conditions such that a solution starting from an initial condition in the set becomes spatially homogeneous as the discrete time k becomes very large, i.e. the coordinates of the individual maps become almost equal to each other (and the differences approach to 0 as $k \to \infty$). In established regimes, individual maps become indistinguishable and we observe exact perfect synchronization. Recently, synchronization in a lattice of one-variable maps has been studied in Lin *et al.* (1999), Lin & Wang (2002) and Jost & Joy (2002). The model in these 1D CMLs is given by

$$x_i(k+1) = f(x_i(k)) + c(f(x_{i-1}(k)) + f(x_{i+1}(k)) - 2f(x_i(k))) \tag{1.1}$$

for $1 \leqslant i \leqslant n$, with periodic boundary conditions $f(x_0(k)) = f(x_n(k))$ and $f(x_{n+1}(k)) = f(x_1(k))$. Here, $f\colon [0,1] \to [0,1]$ is a 1D map. For instance, f is usually chosen to be the well-known logistic map:

$$x(k+1) = f(x(k)) = \gamma x(k)(1 - x(k)), \quad 0 < \gamma \leqslant 4. \tag{1.2}$$

It is known that (Gleick, 1987) the logistic map (1.2) has a chaotic attractor for $\gamma \in (\gamma_\infty \approx 3.57, 4]$. Lin *et al.* (1999) gave a rigourous proof for chaotic synchronization of (1.1) and (1.2) with $n = 2, 3, 4$ and $\gamma \in (\gamma_\infty, 3.82] \in (\gamma_\infty, 4]$, provided the coupling coefficient c is sufficiently close to $1/3$. The result is generalized by Lin & Wang (2002) for $\gamma \in (\gamma_\infty, 4]$ by the Lyapunov function method. Lin *et al.* (1999) also provided a complete numerical experiment for chaotic synchronization of 1D and 2D CMLs of (1.1) and (1.2) with various lattice sizes. Jost & Joy (2002) gave a necessary and sufficient condition for the occurrence of local synchronization as well as a sufficient condition for the occurrence of global synchronization of (1.1) with more general one-variable maps.

In the following, we propose a model on synchronization of discrete hyperchaotic systems:

$$\begin{cases} x_i(k+1) = g(x_i(k), y_i(k)) + c(g(x_{i-1}(k), y_{i-1}(k)) \\ \qquad\qquad + g(x_{i+1}(k), y_{i+1}(k)) - 2g(x_i(k), y_i(k))), \\ y_i(k+1) = h(x_i(k), y_i(k)), \quad \text{for } 1 \leqslant i \leqslant n, \end{cases} \tag{1.3}$$

with periodic boundary conditions $(x_0(k), y_0(k)) = (x_n(k), y_n(k))$ and $(x_1(k), y_1(k)) = (x_{n+1}(k), y_{n+1}(k))$, where

$$\begin{cases} g(x, y) = f_\gamma(x) + \theta(f_\delta(y) - f_\gamma(x)), \\ h(x, y) = f_\delta(y) + \theta(f_\gamma(x) - f_\delta(y)) \end{cases} \tag{1.4}$$

defined on $[0, 1]^2$ with $0 < \theta < 1$, $1 < \delta, \gamma < 4$ and $\delta \neq \gamma$, in which $f_\gamma(x) = \gamma x(1 - x)$ and $f_\delta(y) = \delta y(1 - y)$ are the logistic maps.

In the CMLs of (1.3), we put two-variable maps of (1.4) on the ith node of a circle lattice, $i = 1, \ldots, n$, and only couple the x_i-variable with x_{i-1}- and x_{i+1}-variables of its two neighbours. In other words, in the CMLs of (1.3), the y_i-variable connects only with the x_i-variable in the ith node, and the coupling occurs only through the x_i-variable with the nearest nodes. The topological structure of the CMLs of (1.3) with lattice size $n = 4$ is shown in Fig. 1.

In (1.4), we construct a two-variable map by connecting two logistic maps with the parameter $\theta \in (0, 1)$. We shall prove that the two-variable system (1.4) is chaotic in the type of snap-back repeller (Marotto, 1978) for some suitable θ and show the fast fourier transformation (FFT) values of the difference of $x(k)$ and $y(k)$ as $k \to \infty$ which forms a chaotic behaviour. We shall also prove the occurrence of chaotic synchronization of (1.3), i.e.

$$\lim_{k\to\infty} (|x_i(k) - x_j(k)| + |y_i(k) - y_j(k)|) = 0, \tag{1.5}$$

for $i, j = 1, 2, \ldots, n$, with some suitable coupling strengths c and the lattice size $n = 4$.

It is worth pointing out that usually it is a difficult task to find an analytic proof for globally chaotic synchronization in CMLs. In fact, the study of an uncoupled discrete chaotic dynamical system itself is still a challenge to mathematicians. For example, one of the most important works of the Wolf prize winner Carleson is Benedicks & Carleson (1985), a partial result on the logistic map. Moreover, in CMLs, one cannot obtain synchronization by increasing the coupling strength, which is often the reason for the occurrence of synchronization in coupled continuous systems. Thus, the proof for the occurrence of global synchronization in CMLs seems more difficult. We note that by now most of the mathematical results in this area focus on the local stability of the synchronous manifold. Thus, from the point of view of mathematics, these results cannot predict whether or when synchronization will occur.

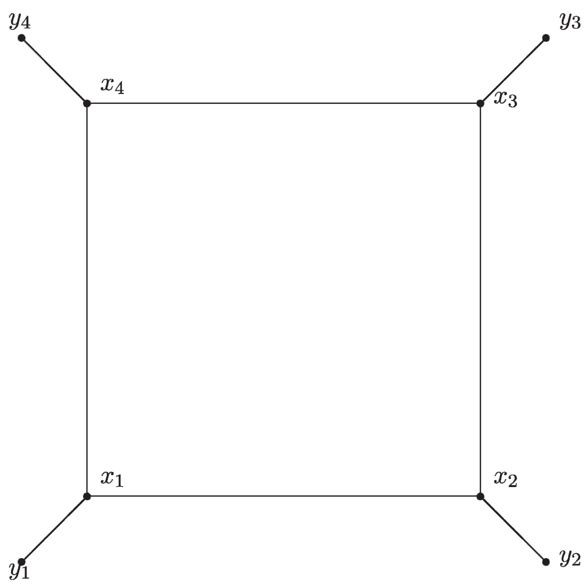

FIG. 1. Topological structure of CMLs with the lattice size $n = 4$.

Here are some motivations for the study of chaotic synchronization of the CMLs (1.3) and (1.4).

(a) In many applications, such as in secure communication, in contrast to the CMLs of (1.1) and (1.2), the duplexing coupling of x_i-variables in (1.3) induces the chaotic synchronization of y_i-variables which can be used to make a chaotic mask of message and send it out to the neighbours via y_i-variables. Then, the secret message can be decoded by the synchronization of y_i-variables. For example, when synchronization is obtained, Node 1 encodes the secret message m_k by y-variable to obtain the signal $\tilde{m}_k = m_k + y_1(k)$ and sends it to Node 2. Then, Node 2 can recover the message easily by $\tilde{m}_k - y_2(k)$ since $y_1(k)$ and $y_2(k)$ are synchronized with each other. On the other hand, since only x-variables of all nodes are transmitted to induce synchronization, an eavesdropper knows nothing about y-variables. Thus, he cannot recover the message.

(b) The logistic map used in (1.4) is a well-studied simple model which has chaotic behaviour over a wide range of parameters in $(\gamma_\infty, 4]$.

(c) In contrast to the other two-variable maps, such as the Hénon (1976) map, the differences of $x_i(k)$ and $y_i(k)$ in (1.3) form a chaotic behaviour. Thus, one channel (duplexingly coupled with x_i-variables) makes the CMLs of (1.3) synchronized and the other channel (simplexingly connected with y_i-variables) is used to realize secure communication. On the contrary, the Hénon map has the relation $x_i(k+1) = y_i(k)$ which cannot be used in secure communication because the values of $y_i(k)$ can be encoded by the duplexing coupling of x_i-variables.

In practice, we have further measures against general attacks. For example, a variational logistic map (VLM) has been proposed (Chen *et al.*, 2008) with a large parameter space without windows. The VLM with a disturbing method can pass the most stringent statistical testing suite in TestU01. With up to 3200 Mbps throughput and complex output properties, VLM is suitable for security applications. A chaotic cryptographical scheme (Schneier, 1996), constructed by coupling four VLMs, generates the output sequence with a minimal length equal to 2128 by a 128-bit external key.

Besides secure communication, chaotic synchronization has the same importance in biology and life science, which is another reason why we focus on this area. For example, people found that fireflies are able to synchronize the timing of their light emission within a flashing population by adjusting the frequency and phase of their own flashing (see Mirollo & Strogatz, 1990). For fireflies, this kind of capability plays a critical role in the processing of mating. People believe that modelling networks after such biological systems may potentially be more efficient than current networking schemes allow. In the last decade, many people have pointed out that synchronization among large groups of neurons is a fundamental mechanism that allows us to understand how the brain solves the binding problem. For instance, Parkinsonian tremor and epileptic seizures are believed to be caused by such a mechanism (see Gray, 1999; Haken, 2002; Singer, 1999a,b; Tass *et al.*, 1998). Recently, Kaneko and his coauthors obtained a lot of results in studying a series of biology-related problems with chaotic synchronization theory, such as the origin of heredity, cell differentiation, universal features of a cell with recursive growth, stability and irreversibility in the development of cell societies, pattern formation and the origins of positional information and multicellular organisms, etc. (see Furusawa & Kaneko, 2000, 2001, 2003; Kaneko & Yomo, 1997, 2002).

This paper is organized as follows. Marotto (1978) introduced the 'snap-back repeller' of a differentiable map and proved that the existence of a snap-back repeller is sufficient to imply chaotic behaviour of the map. In Section 2, based on the theorem of Marotto (1978) and a generalized version (Shiraiwa & Kurata, 1979), we give a rigourous proof for the chaotic behaviour of the CMLs (1.3) and (1.4) for $3.678 < \gamma \approx \delta < 4$. In Section 3, we prove that the system (1.3) and (1.4) is synchronized, i.e. the conditions in (1.5) hold or a spatially homogenous solution of (1.3) exists for $n = 4$, $c \in (0.41, 0.43)$, $\theta \in (0.62, 0.64)$ and $\gamma \approx \delta \in (3.7 - \epsilon, 3.7 + \epsilon)$ with $0 < \epsilon << 1$. In Section 4, we show numerical results for the chaotic synchronized behaviour of (1.3) and (1.4).

2. Chaos for the two-variable map

In this section, we shall prove the chaotic behaviour for a two-variable map of (1.4). The proof is based on a theorem of Marotto (1978) and a generalized version in Shiraiwa & Kurata (1979).

DEFINITION 2.1 (Marotto) Let $F: \mathbb{R}^N \to \mathbb{R}^N$ be a $\mathbb{C}^1$-map. Let z^* be a fixed point of F such that all the eigenvalues of $\mathcal{D}F(z^*)$ have absolute values larger than 1. Then, z^* is called a snap-back repeller if there exists a point z_0 in $W^u_{\text{loc}}(z^*)$, the local unstable set of z^*, and some integer m such that $F^m(z_0) = z^*$ and $\det \mathcal{D}F^m(z_0) \neq 0$.

THEOREM 2.1 (Marotto) Let $F: \mathbb{R}^N \to \mathbb{R}^N$ be a $\mathbb{C}^1$-map. Let z^* be a snap-back repeller of F. Then, the following holds:

(i) There is a positive integer p_0 such that for each $p > p_0$, F has a point of period p.

(ii) There is an uncountable set $S \subset \mathbb{R}^N$ containing no periodic points of F such that

(iia) $F(S) \subset S$;

(iib) for every $\xi, \eta \in S$ with $\xi \neq \eta$,

$$\limsup_{k\to\infty} |F^k(\xi) - F^k(\eta)| > 0;$$

(iic) for every $\xi \in S$ and any periodic point η of F,

$$\limsup_{k\to\infty} |F^k(\xi) - F^k(\eta)| > 0;$$

(iii) There is an uncountable subset $S_0 \subset S$ such that for every $\xi, \eta \in S_0$,

$$\liminf_{k\to\infty} |F^k(\xi) - F^k(\eta)| = 0.$$

Conditions (i)–(iii) were first defined as 'chaos' of a one-variable map and proved as necessary conditions of a 'period-3' map by Li & Yorke (1975).

Note that the original proof of Marotto (1978) has some logical error which has been corrected recently by Chen *et al.* (1998).

REMARK Shiraiwa & Kurata (1979) proved that conditions (i)–(iii) in Theorem 2.1 hold by modifying the assumption as follows:
'Let $z^* \in \mathbb{R}^N$ be a hyperbolic fixed point of F such that

(1) there exists a point $z_1 \in W^u_{\text{loc}}(z^*)$ $(z_1 \neq z^*)$ and a positive integer m such that $F^m(z_1) \in W^s_{\text{loc}}(z^*)$;

(2) there exists a u-dimensional disk B^u embedded in $W^u_{\text{loc}}(z^*)$ such that B^u is a neighbourhood of z_1 in $W^u_{\text{loc}}(z^*)$, $F^m|B^u\colon B^u \to \mathbb{R}^N$ is an embedding and $F^m(B^u)$ intersects $W^s_{\text{loc}}(z^*)$ transversely at $F^m(z_1)$, where $u = \dim W^u_{\text{loc}}(z^*) > 0$'.

In case $u = \dim \mathbb{R}^N$ and $f^m(z_1) = z^*$, the above assumptions reduce to the snap-back repeller of the original Marotto's theorem.

In the following, we use the generalized version of Shiraiwa & Kurata (1979) to prove the existence of chaotic behaviour of (1.4).

2.1 *Two-variable map connected with logistic maps*

Consider a special case of a two-variable map connected with logistic maps as in (1.4):

$$F(x, y) = \begin{pmatrix} (1-\theta) f_\gamma(x) + \theta f_\gamma(y) \\ (1-\theta) f_\gamma(y) + \theta f_\gamma(x) \end{pmatrix} \tag{2.1}$$

with $\gamma = \delta$.

We give an elementary stability analysis of fixed points of (2.1), which is useful in Section 2.2 to determine if a snap-back repeller exists for a two-variable map (1.4).

LEMMA 2.1 In the invariant region $[0, 1] \times [0, 1]$, the fixed point $\left(\frac{\gamma-1}{\gamma}, \frac{\gamma-1}{\gamma}\right)$ of (2.1) exists for all $1 \leqslant \gamma \leqslant 4$, $0 \leqslant \theta \leqslant 1$.

Proof. Obvious. □

LEMMA 2.2

(i) If $\gamma < 3$, then the fixed point $\left(\frac{\gamma-1}{\gamma}, \frac{\gamma-1}{\gamma}\right)$ is a stable point.

(ii) If $\gamma > 3$ and $0 < \theta < \frac{\gamma-3}{2(\gamma-2)}$ or $1 > \theta > \frac{\gamma-1}{2(\gamma-2)}$, then the fixed point $\left(\frac{\gamma-1}{\gamma}, \frac{\gamma-1}{\gamma}\right)$ is a repelling fixed point.

(iii) If $\gamma > 3$ and $\frac{\gamma-1}{2(\gamma-2)} > \theta > \frac{\gamma-3}{2(\gamma-2)}$, then the fixed point $\left(\frac{\gamma-1}{\gamma}, \frac{\gamma-1}{\gamma}\right)$ is a saddle fixed point of the map F as in (2.1).

Proof. The Jacobian matrix at $\left(\frac{\gamma-1}{\gamma}, \frac{\gamma-1}{\gamma}\right)$ of (2.1) is

$$J = DF = \begin{bmatrix} (\theta-1)(-2+\gamma) & -\theta(-2+\gamma) \\ -\theta(-2+\gamma) & (\theta-1)(-2+\gamma) \end{bmatrix}.$$

The eigenvalues λ_1 and λ_2 of J can be computed by

$$\lambda_1 = 2-\gamma, \quad \lambda_2 = (-1+2\theta)(-2+\gamma).$$

Therefore, we have the following:

(i) $|\lambda_1| < 1, |\lambda_2| < 1$ for $\gamma < 3$, i.e. the fixed point $\left(\frac{\gamma-1}{\gamma}, \frac{\gamma-1}{\gamma}\right)$ is a stable point.

(ii) $|\lambda_1| > 1, |\lambda_2| > 1$ for $\gamma > 3$ and $\theta > \frac{\gamma-1}{2(\gamma-2)}$ or $\theta < \frac{\gamma-3}{2(\gamma-2)}$, i.e. the fixed point $\left(\frac{\gamma-1}{\gamma}, \frac{\gamma-1}{\gamma}\right)$ is a repelling fixed point.

(iii) $|\lambda_1| > 1, |\lambda_2| < 1$ for $\gamma > 3$ and $\frac{\gamma-1}{2(\gamma-2)} > \theta > \frac{\gamma-3}{2(\gamma-2)}$, i.e. the fixed point $\left(\frac{\gamma-1}{\gamma}, \frac{\gamma-1}{\gamma}\right)$ is a saddle fixed point.

□

2.2 *Snap-back repeller of two-variable maps*

In this section, we shall prove the existence of a snap-back repeller of (1.4).
We first prove that the fixed point $x^* = \frac{\gamma-1}{\gamma}$ of the logistic map is a snap-back repeller for $\gamma > \gamma^* \approx 3.678$.

Let $\xi = f_\gamma(x) = \gamma x(1-x)$. Then,

$$x = \frac{\gamma \pm \sqrt{\gamma^2 - 4\gamma\xi}}{2\gamma}.$$

We choose pre-images of the fixed point x^* from backward orbits (if they exist) by the following 'best' way:

$$x_{-1} = \frac{\gamma - \sqrt{\gamma^2 - 4\gamma x^*}}{2\gamma} \in f^{-1}(x^*),$$
$$x_{-(j+1)} = \frac{\gamma + \sqrt{\gamma^2 - 4\gamma x_{-j}}}{2\gamma} \in f^{-1}(x_{-j}), \quad \text{for } j = 1, 2, \ldots. \tag{2.2}$$

REMARK The above way for choosing pre-images of x^* is the best in the sense that if we choose $\tilde{x}_{-1} = \frac{\gamma-\sqrt{\gamma^2-4\gamma x^*}}{2\gamma}$, $\tilde{x}_{-2} = \frac{\gamma-\sqrt{\gamma^2-4\gamma\tilde{x}_{-1}}}{2\gamma}$, $\tilde{x}_{-(j+1)} = \frac{\gamma+\sqrt{\gamma^2-4\gamma\tilde{x}_{-j}}}{2\gamma}$, for $j = 2, 3, \ldots$, then it is easy to show that $\tilde{x}_{-3} > x_{-2}$. Since a point $x \in (0, 1)$ has no pre-image if and only if $x \in (\gamma/4, 1)$, it is easily seen that if x_{-k}, chosen by the best way, does not exist for some k, then $\tilde{x}_{-(k+1)}$ does not exist either.

From the best way for the choice of $\{x_{-j}\}$, we have $x_{-j} \geqslant \frac{1}{2}$ as $j \geqslant 2$ (if they exist). Moreover, $x_{-1} = \frac{1}{\gamma}$ and $x_{-2} = \frac{\gamma+\sqrt{\gamma^2-4}}{2\gamma}$ always exist. However, x_{-j} may not exist as $j \geqslant 3$. The following lemma is a criteria for the existence of x_{-3}.

LEMMA 2.3 If $\gamma > 3.678$, then $x_{-2} = \frac{\gamma+\sqrt{\gamma^2-4\gamma x_{-1}}}{2\gamma} < \frac{\gamma}{4}$, where $x_{-1} = \frac{1}{\gamma}$.

Proof. It is easily seen that $x_{-2} < \frac{\gamma}{4}$ if and only if $f(x_{-2}) > f\left(\frac{\gamma}{4}\right)$, i.e. $x_{-1} = f(x_{-2}) > f\left(\frac{\gamma}{4}\right)$, which is equivalent to $\frac{\gamma-1}{\gamma} > f^2\left(\frac{\gamma}{4}\right) = f^3\left(\frac{1}{2}\right)$. By direct computation, $\frac{\gamma-1}{\gamma} > f^3\left(\frac{1}{2}\right)$ is equivalent to $\gamma - 1 - \gamma f^3\left(\frac{1}{2}\right) > 0$. Denote $\Gamma(\gamma) = \gamma - 1 - \gamma^3 f\left(\frac{1}{2}\right)$. Then,

$$\begin{aligned}\Gamma(\gamma) = \gamma - 1 - \gamma^3 f\left(\frac{1}{2}\right) &= \gamma - 1 - \frac{\gamma^4}{4}\left(1-\frac{\gamma}{4}\right)\left[1 - \frac{\gamma^2}{4}\left(1-\frac{\gamma}{4}\right)\right] \\ &= \gamma - 1 - \frac{1}{4}\gamma^4 + \frac{1}{16}\gamma^6 - \frac{1}{32}\gamma^5 + \frac{1}{16}\gamma^5 + \frac{1}{256}\gamma^8 \\ &= \frac{1}{256}(\gamma^8 - 8\gamma^7 + 16\gamma^6 + 16\gamma^5 - 64\gamma^4 + 256\gamma - 256) \\ &= \frac{1}{256}(\gamma+2)(\gamma^3 - 2\gamma^2 - 4\gamma - 8)(\gamma-2)^4.\end{aligned}$$

Since $\gamma \in [0, 4]$, we have $\Gamma(\gamma) > 0$ if and only if $\gamma^3 - 2\gamma^2 - 4\gamma - 8 > 0$ and $\gamma \neq 2$. Denote $\Gamma_1(\gamma) = \gamma^3 - 2\gamma^2 - 4\gamma - 8$. Then, $\Gamma_1'(\gamma) = 3\gamma^2 - 4\gamma - 4 = 0$ implies that $\gamma = 2$ or $\gamma = -\frac{2}{3}$. Obviously, $\Gamma_1''(2) > 0$ and $\Gamma_1''\left(-\frac{2}{3}\right) < 0$. Since $\Gamma_1(0) = -8$, $\Gamma_1(2) = -16$ and $\Gamma_1(4) > 0$, by the intermediate value theorem, there exists a $\gamma^* \in (2, 4)$ such that $\Gamma_1(\gamma^*) = 0$. By numerical computation, we have $\gamma^* \approx 3.678$. So $\Gamma_1(\gamma) > 0$, for $\gamma \in (\gamma^*, 4]$ and $\Gamma_1(\gamma) \leqslant 0$, for $\gamma \in [0, \gamma^*]$. □

THEOREM 2.2 If $\gamma > 3.678$, then $x^* = \frac{\gamma-1}{\gamma}$ is a snap-back repeller of the logistic map $f_\gamma(x)$.

Proof. We prove that x^* satisfies the conditions as in Definition 2.1.

(i) x^* is a fixed point of f_γ, i.e. $|f_\gamma'(x^*)| > 1$.
(ii) For all $\epsilon > 0$, there exists a $\xi \in B(x^*, \varepsilon)$ such that $f_\gamma^m(\xi) = x^*$ for some m.
(iii) $|(f_\gamma^m(x^*))'| \neq 0$.

Condition (i) is easy to check. To prove (ii) and (iii), we perform the following six steps.

Step 1: Since $f_\gamma^{-1}(x^*) = \left\{\frac{1}{\gamma}, x^*\right\}$, from the best way we choose, we choose $x_{-1} = \frac{1}{\gamma} < \frac{\gamma-1}{\gamma} = x^*$.

Step 2: Since $f_\gamma^{-1}(x_{-1}) = \{x_{-2}, 1 - x_{-2}\}$, where $x_{-2} > \frac{1}{2}$ and f_γ is strictly increasing on $[0, 1/2]$ with $f_\gamma\left(\left[0, \frac{1}{2}\right]\right) = \left[0, \frac{\gamma}{4}\right]$, there exists a $\xi^* \in \left[0, \frac{1}{\gamma}\right]$ such that $f_\gamma(\xi^*) = \frac{1}{\gamma} = x_{-1}$. It is easily seen that $\xi^* = 1 - x_{-2}$ and $0 < 1 - x_{-2} < \frac{1}{\gamma}$. This implies that $x^* < x_{-2} < 1$.

Step 3: Since $x_{-3} = \frac{\gamma+\sqrt{\gamma^2-4\gamma x_{-2}}}{2\gamma}$ and $f_\gamma'(x) < 0$ for $x \in [x^*, 1]$, by Lemma 2.2, we have $\frac{1}{2} < x_{-3} < x^*$.

Step 4: Since $f_\gamma^{-1}(x_{-3}) = \{x_{-4}, 1 - x_{-4}\}$, $f_\gamma^{-1}\left(\frac{1}{2}\right) \cap \left[0, \frac{1}{2}\right] < 1 - x_{-4} < \frac{1}{r}$ and $f_\gamma^{-1}\left(\frac{1}{\gamma}\right) \cap \left[0, \frac{1}{2}\right] < f_\gamma^{-1}\left(\frac{1}{2}\right) \cap \left[0, \frac{1}{2}\right]$, we have $x_{-2} > x_{-4} > x^*$.

Step 5: Suppose $x_{-3} \geqslant x_{-5} \geqslant x^*$. Then, $x_{-2} = f_\gamma(x_{-3}) \leqslant f_\gamma(x_{-5}) \leqslant f_\gamma(x^*) = x^*$, which contradicts that $x_{-2} > x^*$. So $\frac{1}{2} < x_{-3} < x_{-5}$.

Step 6: Since $f_\gamma^{-1}(x_{-5}) = \{x_{-6}, 1 - x_{-6}\}$ and $f_\gamma^{-1}(x_{-5}) \cap \left[0, \frac{1}{2}\right] < 1 - x_{-6} < \frac{1}{r}$, we have $x_{-4} > x_{-6} > x^*$.

According to the above steps, we have

$$x_{-2} > x_{-4} > x_{-6} > \cdots > x^*,$$

$$x_{-1} > x_{-3} < x_{-5} < \cdots < x^*.$$

It is easily shown that $\lim_{n\to\infty} x_{-2n} = x^*$ and $\lim_{n\to\infty} x_{-(2n-1)} = x^*$. For any $\epsilon > 0$, there exists a $\xi \in B(x^*, \varepsilon)$ such that $f_\gamma^m(\xi) = x^*$ for some m, thus (ii) holds. Since $f_\gamma'(x) = 0$ if and only if $x = \frac{1}{2}$, condition (iii) is satisfied. □

In the following, we shall prove that the two-variable map (2.1) has a snap-back repeller.

THEOREM 2.3 If $\gamma > 3.678$, then $\left(\frac{\gamma-1}{\gamma}, \frac{\gamma-1}{\gamma}\right)$ is a snap-back repeller of the two-variable map (2.1).

Proof. We prove that $\left(\frac{\gamma-1}{\gamma}, \frac{\gamma-1}{\gamma}\right)$ satisfies (i)–(iii) as in Theorem 1. From Lemma 2.2, we know that for $\gamma > 3.678$, $u = \dim W_{\text{loc}}^u\left(\frac{\gamma-1}{\gamma}, \frac{\gamma-1}{\gamma}\right) \geqslant 1$, so condition (i) is satisfied. Obviously, $F\left(\frac{\gamma-1}{\gamma}, \frac{\gamma-1}{\gamma}\right) = \begin{pmatrix} f_\gamma\left(\frac{\gamma-1}{\gamma}\right) \\ f_\gamma\left(\frac{\gamma-1}{\gamma}\right) \end{pmatrix}$. In Theorem 2.5, we proved that the fixed point $x^* = \frac{\gamma-1}{\gamma}$ of $f_\gamma(x)$ is a snap-back repeller. So for any $\epsilon > 0$, there exists a $\xi \in B(x^*, \epsilon/2)$ such that $f_\gamma^m(\xi) = \frac{\gamma-1}{\gamma} = x^*$ for some m. Therefore, $(\xi, \xi) \in B\left(\left(\frac{\gamma-1}{\gamma}, \frac{\gamma-1}{\gamma}\right), \epsilon\right)$ such that $\mathcal{F}^m(\xi, \xi) = \begin{pmatrix} \frac{\gamma-1}{\gamma} \\ \frac{\gamma-1}{\gamma} \end{pmatrix}$. Hence, condition (ii) is satisfied. From Lemma 2.3, it follows that $W_{\text{loc}}^u(x^*, x^*)$ and $W_{\text{loc}}^s(x^*, x^*)$ are small deformations of the manifold of $\{(x, y) | \, x = y\}$ and the manifold of $\{(x, y) | x + y = 1\}$, respectively, on a neighbourhood of (x^*, x^*). It is easily seen that condition (iii) is satisfied. Hence, we complete the proof. □

Now, we consider (1.4) with $\gamma \neq \delta$. Rewrite (1.4) as

$$\tilde{F}((x, y), \gamma, \delta) = \begin{pmatrix} (1-\theta) f_\gamma(x) + \theta f_\delta(y) \\ (1-\theta) f_\delta(y) + \theta f_\gamma(x) \end{pmatrix}, \tag{2.3}$$

where $f_\gamma(x) = \gamma x(1 - x)$ and $f_\delta(y) = \delta y(1 - y)$.

THEOREM 2.4 If $\gamma \approx \delta > 3.678$, then the fixed point 'near' $\left(\frac{\gamma-1}{\gamma}, \frac{\gamma-1}{\gamma}\right)$ is a snap-back repeller of (2.3).

Proof. We shall check that the three conditions (i)–(iii) as in Theorem 2.2 hold. If $\gamma = \delta > 3.678$, then from Theorem 2.6, we proved that the fixed point $\left(\frac{\gamma-1}{\gamma}, \frac{\gamma-1}{\gamma}\right) = (x^*, x^*)$ is a snap-back repeller of (2.3). Therefore, when $\gamma = \delta$, we have the following:

(a) $\mathcal{D}\tilde{F}(x^*, x^*, \gamma, \gamma)$ has an eigenvalue with absolute value larger than 1, i.e. $\dim W_{\text{loc}}^u(x^*, x^*) \geqslant 1$.

(b) For any $\epsilon > 0$, there exists a point $(\xi_1^*, \xi_2^*) \in B_{\text{loc}}^u(x^*, x^*, \epsilon)$ such that $\tilde{F}^m(\xi_1^*, \xi_2^*) = (x^*, x^*)$ for some m.

(c) $\det \mathcal{D}\tilde{F}^m((\xi_1^*, \xi_2^*), \gamma, \gamma) \neq 0$.

For a fixed γ, let $\delta = \gamma + \eta$ and we define $Z((x, y), \eta)$ by

$$Z((x, y), \eta) = \tilde{F}((x, y), \eta) - (x, y) = \tilde{F}((x, y), \gamma, \gamma + \eta) - (x, y). \tag{2.4}$$

It is easy to check that

$$Z \in \mathcal{C}^1, \quad Z((x^*, x^*), 0) = \tilde{F}((x^*, x^*), 0) - (x^*, x^*) = (0, 0) \tag{2.5}$$

and the matrix

$$\mathcal{D}_{(x,y)} Z((x, y), \eta) = \mathcal{D}_{(x,y)} \tilde{F}((x, y), \eta) - I_2$$

is invertible at $(x, y) = (x^*, x^*)$ and $\eta = 0$.

By the implicit function theorem and (a), for sufficiently small η, there exists a $q_1 > 0$ and a function ζ^* on $(-q_1, q_1)$ such that $Z(\zeta^*(\eta), \eta) = 0$ for $\eta \in (-q_1, q_1)$, i.e. $\tilde{F}(\zeta^*(\eta), \eta) = \zeta^*(\eta)$ with $\zeta^*(0) = (x^*, x^*)$ and $\mathcal{D}_{(x,y)} Z(\zeta^*(\eta), \eta)$ has a positive eigenvalue, i.e. $\mathcal{D}_{(x,y)} \tilde{F}(\zeta^*(\eta), \eta)$ has an eigenvalue with absolute value larger than 1 for $\eta \in (-q_1, q_1)$. Thus, condition (i) holds. Next, we define a function $W((x, y), \eta)$ by

$$W((x, y), \eta) = \tilde{F}^m((x, y), \eta) - \zeta^*(\eta). \tag{2.6}$$

From (b), we have

$$W \in \mathcal{C}^1, \quad W((x^*, x^*), 0) = \tilde{F}^m((x^*, x^*), 0) - \xi^*(0) = (0, 0)$$

and

$$\mathcal{D}_{(x,y)} W((x^*, x^*), 0) = D_{(x,y)} \tilde{F}^m((x^*, x^*), 0) \text{ invertible.}$$

By the implicit function theorem, there exists a q_2 with $0 < q_2 < q_1$ and a function ω defined on $(-q_2, q_2)$ such that $\omega \in \mathcal{C}^1$ with $\omega(0) = (x^*, x^*)$ and $W(\omega(\eta), \eta) = 0$ for $\eta \in (-q_2, q_2)$, i.e. $\tilde{F}^m(\omega(\eta), \eta) = \zeta^*(\eta)$. Thus, condition (ii) is satisfied. Since $\tilde{F}^m$and $\omega \in \mathcal{C}^1$, from (c), there is a q_3 with $0 < q_3 < q_2$ such that $\det \mathcal{D}_{(x,y)} \tilde{F}^m(\zeta^*(\eta)) \neq 0$ and $\omega(\eta) \in (-q_1, q_1)$, for all $\eta \in (-q_3, q_3)$. If we choose $\eta < q_3$, then condition (iii) is satisfied. We complete the proof. □

3. Synchronization for 1D CMLs of two-variable maps coupled with one variable

In this section, we shall prove that the chaotic synchronization occurs for 1D CMLs of two-variable maps coupled with one variable as in (1.3) and (1.4) with $n = 4$.
First, we state a proposition and a lemma.

PROPOSITION 3.1 For $c \in [0, 1]$, $d \in [0, 1/2]$ and every $(x_1(0), y_1(0), x_2(0), y_2(0), x_3(0), y_3(0), x_4(0), y_4(0)) \in (0, 1)^8$, there exists a $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$,

$$(x_1(k), y_1(k), x_2(k), y_2(k), x_3(k), y_3(k), x_4(k), y_4(k))$$

generated by (1.3) and (1.4) lie in

$$D_0 = [(4 - \max(\gamma, \delta))/4, \max(\gamma, \delta)/4]^8.$$

Proof. The proof is similar to Theorem 2.3 in Lin *et al.* (1999). □

From this proposition, we w.l.o.g. assume that (1.3) and (1.4) are defined on D_0.

LEMMA 3.1 Consider the map

$$X_i(k+1) = f_i(X(k), C_i) \tag{3.1}$$

with $i = 1, 2, \dots, N$, where C_i are parameters, $X(k) = (X_1(k), \dots, X_N(k))$ with $X_i(k) \in \mathbb{R}^M$ and the differential function vector $f = (f_1, \dots, f_N)$ possess the proposition: if $X_1 = \dots = X_N$ and $C_1 = \dots = C_N$, then $f_i = f_j$, $i, j = 1, 2, \dots, N$. Then, for any $l \in \mathbb{N}$, $X(k) = (X_1(k), \dots, X_N(k))$ and $C = (C_1, \dots, C_N)$, there exists a $c_l' > 0$ such that it holds that

$$|X_i(k+l) - X_j(k+l)| \leqslant c_l'(|X_i(k) - X_j(k)| + |C_i - C_j|).$$

Proof. Since $f_i(X_1, \dots, X_1, C_i) = f_{i+1}(X_1, \dots, X_1, C_i)$, we have

$$\begin{aligned} &|X_i(k+1) - X_j(k+1)| \\ &\quad = |f_i(X(k), C_i) - f_i(X_1, \dots, X_1, C_i) + f_j(X_1, \dots, X_1, C_i) - f_j(X(k), C_j)| \\ &\quad \leqslant |f_i(X(k), C_i) - f_i(X_1, \dots, X_1, C_i)| + |f_j(X_1, \dots, X_1, C_i) - f_j(X(k), C_j)|, \end{aligned}$$

which implies this lemma for $l = 1$. The case for $l \geqslant 2$ is similar. □

Define the set $N_{\eta,\eta'}$ to be the subset of D_0 which satisfies the following:

(i) $|x_i - x_j| + |y_i - y_j| < \eta^2$ for $|i - j|$ is even,
(ii) $|x_i - x_j| + |y_i - y_j| < \eta$ for $|i - j|$ is odd and
(iii) $|x_i - y_j| < \eta'$ for $1 \leqslant i, j \leqslant 4$,

where $i, j = 1, 2, 3, 4$.

THEOREM 3.1 Assume $\gamma, \delta \in (3.7 - \epsilon, 3.7 + \epsilon)$ with $\epsilon > 0$, the connected parameter $\theta \in (0.62, 0.64)$ and the coupling coefficient $c \in (0.41, 0.43)$, respectively, where $\epsilon > 0$. Then, there exist an $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, the spatially homogeneous chaotic solutions for (1.3) and (1.4) with $n = 4$ are stable, i.e. there exist an $\eta_0 > 0$ and an $\eta_0' > 0$ such that for any initial points in $N_{\eta_0,\eta_0'}$, it holds that

$$\lim_{k\to\infty} |x_i(k) - x_j(k)| = 0, \quad \lim_{k\to\infty} |y_i(k) - y_j(k)| = 0, \quad i, j = 1, 2, 3, 4. \tag{3.2}$$

REMARK

(i) Here, the spatially homogeneous solution is of the form

$$\{(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4) \in D_0 |\, x_i = x_j, y_i = y_j, i, j = 1, 2, 3, 4\}.$$

(ii) From $\epsilon_0 \ll 1$, we have $\delta \approx \gamma$. For this case, almost synchronization can occur between x-variables and y-variables, i.e. $|x_i(k) - y_j(k)| \ll 1$ for k large enough. However, perfect synchronization never occurs between them if $\delta \neq \gamma$. In fact, numerical results in Section 4 show that $|x_i(k) - y_j(k)|$ is chaotic, which shows that almost synchronization between x-variables and y-variables does not influence the security of the CMLs in (1.3) and (1.4).

The proof of Theorem 3.1 can be reduced to the following theorem.

THEOREM 3.2 There exist $K_0 \in \mathbb{N}$, $\epsilon_1 > 0$, $\eta_1 > 0$ and $\eta_1' > 0$ such that for every $0 \leqslant \epsilon < \epsilon_1$, $0 \leqslant \eta < \eta_1$ and $0 \leqslant \eta' < \eta_1'$, we have $\Phi^{K_0}(N_{\eta,\eta'}) \in N_{\eta/2,\eta'}$, where Φ is the map defined by (1.3) and (1.4).

Proof of Theorem 3.1. Define K_0, η_1 and η_1' as in Theorem 3.2. From (1.3) and (1.4), we have

$$|x_i(k+1) - x_j(k+1)| + |y_i(k+1) - y_j(k+1)| \leqslant c_{i,j}(|x_i(k) - x_j(k)| + |y_i(k) - y_j(k)|)$$

for $|i - j|$ is even and

$$|x_i(k+1) - x_j(k+1)| + |y_i(k+1) - y_j(k+1)| \leqslant c_{i,j} \sum_{i,j=1}^{4} (|x_i(k) - x_j(k)| + |y_i(k) - y_j(k)|),$$

where $c_{i,j}$ are constants independent of k.

From Lemma 3.1, for the map Φ, we obtain that there exist an $\eta_2 > 0$ and an $\eta_2' > 0$ such that for any $0 < \eta < \eta_2$, $0 < \eta' < \eta_2'$ and $k \leqslant K_0$, there exists a $c_8 > 0$ independent of η and η' such that for any initial point $(x_1(0), y_1(0), \ldots, x_4(0), y_4(0))$ in $N_{\eta,\eta'}$, it holds that

$$\begin{aligned} |x_i(k) - x_j(k)| &\leqslant c_{11}^2 \eta^2, && \text{for } |i-j| \text{ even},\\ |x_i(k) - x_j(k)| &\leqslant c_8 \eta, && \text{for } |i-j| \text{ odd}. \end{aligned} \tag{3.3}$$

Set $\eta_0 = \max(\eta_1, \eta_2)$ and $\eta_0' = \max(\eta_1', \eta_2')$. From Theorem 3.2, we have $\Phi^{(K_0)}(N_{\eta,\eta'}) \subset N_{\eta/2,\eta'}$ for $0 \leqslant \eta < \eta_0$ and $0 \leqslant \eta' < \eta_0'$. By iteration, we have $\Phi^{(lK_0)}(N_{\eta,\eta'}) \subset N_{\frac{\eta}{2^l},\eta'}$. Hence, for any given $\epsilon > 0$, there exists an $l_0 \in \mathbb{N}$ such that if $l > l_0$, then $\Phi^{(lK_0)}(N_{\eta,\eta'}) \subset N_{\epsilon/c_8,\eta'}$. Combining this with (3.3), we have

$$\begin{aligned} |x_i(k) - x_j(k)| &\leqslant \epsilon^2, && \text{for } |i-j| \text{ even},\\ |x_i(k) - x_j(k)| &\leqslant \epsilon, && \text{for } |i-j| \text{ odd}, \end{aligned} \tag{3.4}$$

for every $i \geqslant l_0 K_0$. This completes the proof of Theorem 3.1. □

The remaining part of this section is devoted to the proof of Theorem 3.2. Following the idea in Lin & Wang (2002), we shall use the Lyapunov method to show the synchronization for 1D CMLs. Due to the complicated topological structure of (1.3) and (1.4), the construction of the appropriate Lyapunov function is much more complicated than that of Lin & Wang (2002), and thus, we can only obtain the local synchronization of (1.3) and (1.4).

By direct computation, we have

$$\begin{aligned} x_1(k+1) - x_3(k+1) &= (1-2c)(g(x_1(k), y_1(k)) - g(x_3(k), y_3(k)))\\ &= (1-2c)(1-\theta)\gamma(1 - (x_1(k) + x_3(k)))(x_1(k) - x_3(k))\\ &\quad + (1-2c)\theta\delta(1 - (y_1(k) + y_3(k)))(y_1(k) - y_3(k)) \end{aligned}$$

and

$$\begin{aligned} y_1(k+1) - y_3(k+1) &= h(x_1(k), y_1(k)) - h(x_3(k), y_3(k))\\ &= (1-\theta)\delta(1 - (y_1(k) + y_3(k)))(y_1(k) - y_3(k))\\ &\quad + \theta\gamma(1 - (x_1(k) + x_3(k)))(x_1(k) - x_3(k)). \end{aligned}$$

From $|\gamma - \delta| < 2\epsilon$, we obtain that

$$
\begin{aligned}
&x_1(k+2) - x_3(k+2) \\
&\quad = (1-2c)^2(1-\theta)^2\gamma^2(1-(x_1(k+1)+x_3(k+1)))(1-(x_1(k)+x_3(k)))(x_1(k)-x_3(k)) \\
&\qquad + (1-2c)^2\theta(1-\theta)\gamma^2(1-(x_1(k+1)+x_3(k+1)))(1-(y_1(k+1)-y_3(k+1)))(y_1(k)-y_3(k)) \\
&\qquad + (1-2c)\theta(1-\theta)\gamma^2(1-(y_1(k+1)-y_3(k+1)))(1-(y_1(k)+y_3(k)))(y_1(k)-y_3(k)) \\
&\qquad + (1-2c)\theta^2\gamma^2(1-(y_1(k+1)-y_3(k+1)))(1-(x_1(k)+x_3(k)))(x_1(k)-x_3(k)) + \epsilon c_1 \mathrm{dist}_{13}(k),
\end{aligned}
$$

where $\mathrm{dist}_{13}(k) = |(x_1(k) - x_3(k))| + |(y_1(k) - y_3(k))|$ and c_1 is a constant independent of ϵ. In the last inequality, we use the fact that $|\gamma - \delta| < 2\epsilon$. Similarly, we have

$$
\begin{aligned}
&y_1(k+2) - y_3(k+2) \\
&\quad = (1-\theta)^2\delta^2(1-(y_1(k+1)+y_3(k+1)))(1-(y_1(k)+y_1(k)))(y_1(k)-y_3(k)) \\
&\qquad + (1-\theta)\theta\gamma^2(1-(y_1(k+1)+y_3(k+1)))(1-(x_1(k)+x_3(k)))(x_1(k)-x_3(k)) \\
&\qquad + (1-2c)(1-\theta)\theta\gamma^2(1-(x_1(k+1)+x_3(k+1)))(1-(x_1(k)+x_3(k)))(x_1(k)-x_3(k)) \\
&\qquad + (1-2c)\theta^2\delta^2(1-(x_1(k+1)+x_3(k+1)))(1-(y_1(k)+y_3(k)))(y_1-y_3) + \epsilon c_2 \mathrm{dist}_{13}(k),
\end{aligned}
$$

where c_2 is a constant independent of ϵ.

After direct computation by using the definition of $N_{\eta,\eta'}$, we have

$$
\begin{aligned}
x_1(k+3) - x_3(k+3) &= A(x_1(k)-x_3(k)) + B(y_1(k)-y_3(k)) + c_3(\epsilon+\eta+\eta')\mathrm{dist}_{13}(k), \\
y_1(k+3) - y_3(k+3) &= C(x_1(k)-x_3(k)) + D(y_1(k)-y_3(k)) + c_4(\epsilon+\eta+\eta')\mathrm{dist}_{13}(k)
\end{aligned} \tag{3.5}
$$

with $\mathrm{dist}_{13}(k) = |x_1(k) - x_3(k)| + |y_1(k) - y_3(k)|$, where c_3 and c_4 are constants independent of ϵ, η and η' and A, B, C and D are of the form:

$$
\begin{aligned}
A &= m_A\gamma^3(1-(x_1(k+2)+x_3(k+2)))(1-(x_1(k+1)+x_3(k+1)))(1-(x_1(k)+x_3(k))), \\
B &= m_B\gamma^3(1-(y_1(k+2)+y_3(k+2)))(1-(y_1(k+1)+y_3(k+1)))(1-(y_1(k)+y_3(k))), \\
C &= m_C\gamma^3(1-(x_1(k+2)+x_3(k+2)))(1-(x_1(k+1)+x_3(k+1)))(1-(x_1(k)+x_3(k))), \\
D &= m_D\gamma^3(1-(y_1(k+2)+y_3(k+2)))(1-(y_1(k+1)+y_3(k+1)))(1-(y_1(k)+y_3(k))),
\end{aligned} \tag{3.6}
$$

in which

$$
\begin{aligned}
m_A &= (1-2c)(1-\theta)[(1-2c)^2(1-\theta)^2 + 2(1-2c)\theta^2 + \theta^2], \\
m_B &= (1-2c)^3(1-\theta)^2\theta + (1-2c)^2(1-\theta)^2\theta + (1-2c)\theta(1-\theta)^2 + (1-2c)^2\theta^3, \\
m_C &= \theta(1-\theta)^2 + \theta(1-\theta)^2(1-2c) + \theta(1-\theta)^2(1-2c)^2 + \theta^3(1-2c), \\
m_D &= (1-\theta)^3 + 2(1-\theta)\theta^2(1-2c) + (1-\theta)\theta^2(1-2c)^2.
\end{aligned}
$$

For any point $\zeta(k) = (x_1(k), y_1(k), x_2(k), y_2(k), x_3(k), y_3(k), x_4(k), y_4(k)) \in N_{\eta,\eta'}$, we define a Lyapunov function L_{13} as follows:

$$L_{13}(\zeta(k)) = \frac{|x_1(k) - x_3(k)|}{\sqrt{(x_1(k) + x_3(k))(2 - (x_1(k) + x_3(k)))}} + \frac{|y_1(k) - y_3(k)|}{\sqrt{(y_1(k) + y_3(k))(2 - (y_1(k) + y_3(k)))}}. \tag{3.7}$$

Substituting (3.5) into (3.7), we obtain that

$$\begin{aligned}
L_{13}(\zeta(k+3)) &= \frac{|x_1(k+3) - x_3(k+3)|}{\sqrt{(x_1(k+3) + x_3(k+3))(2 - (x_1(k+3) + x_3(k+3)))}} \\
&\quad + \frac{|y_1(k+3) - y_3(k+3)|}{\sqrt{(y_1(k+3) + y_3(k+3))(2 - (y_1(k+3) + y_3(k+3)))}} \\
&\leqslant \frac{|(m_A + m_C)\gamma^3(1 - (x_1(k+2) + x_3(k+2)))(1 - (x_1(k+1) + x_3(k+1)))(1 - (x_1(k) + x_3(k)))|}{\sqrt{(x_1(k+3) + x_3(k+3))(2 - (x_1(k+3) + x_3(k+3)))}} \\
&\quad \times |x_1(k) - x_3(k)| \\
&\quad + \frac{|(m_B + m_D)\gamma^3(1-(y_1(k+2) + y_3(k+2)))(1 - (y_1(k+1) + y_3(k+1)))(1-(y_1(k) + y_3(k)))|}{\sqrt{(y_1(k+3) + y_3(k+3))(2 - (y_1(k+3) + y_3(k+3)))}} \\
&\quad \times |y_1(k) - y_3(k)| \\
&\leqslant \frac{(m_A + m_C)\gamma^3(1 - 2x_1(k+2))(1 - 2x_1(k+1))(1 - 2x_1(k))}{\sqrt{2x_1(k+3)(2 - 2x_1(k+3))}} |x_1(k) - x_3(k)| \\
&\quad + \frac{(m_B + m_D)\gamma^3(1 - 2y_1(k+2))(1 - 2y_1(k+1))(1 - 2y_1(k))}{\sqrt{2y_1(k+3)(2 - 2y_1(k+3))}} |y_1(k) - y_3(k)| \\
&\quad + c_5(\epsilon + \eta + \eta')\mathrm{dist}_{13}(k) \\
&\leqslant \frac{(m_A + m_C)\gamma^{3/2}(1 - 2x_1(k))(1 - 2f_\gamma(x_1(k)))(1 - 2f_\gamma(f_\gamma(x_1(k))))}{\sqrt{(1 - f_\gamma(x_1(k)))(1 - f_\gamma(f_\gamma(x_1(k))))(1 - f_\gamma(f_\gamma(f_\gamma(x_1(k)))))}} \\
&\quad \times \frac{|x_1(k) - x_3(k)|}{\sqrt{(x_1(k) + x_3(k))(2 - x_1(k) - x_3(k))}} \\
&\quad + \frac{(m_B + m_D)\gamma^{3/2}(1 - 2y_1(k))(1 - 2f_\gamma(y_1(k)))(1 - 2f_\gamma(f_\gamma(y_1(k))))}{\sqrt{(1 - f_\gamma(y_1(k)))(1 - f_\gamma(f_\gamma(y_1(k))))(1 - f_\gamma(f_\gamma(f_\gamma(y_1(k)))))}} \\
&\quad \times \frac{|y_1(k) - y_3(k)|}{\sqrt{(y_1(k) + y_3(k))(2 - y_1(k) - y_3(k))}} + c_5(\epsilon + \eta + \eta')\mathrm{dist}_{13}(k),
\end{aligned}$$

where c_5 is a constant independent of ϵ, η and η'. In the last inequality, we use the definition of $N_{\eta,\eta'}$.

LEMMA 3.2 There exist an $\epsilon_2 > 0$, an $\eta_2 > 0$ and an $\eta_2' > 0$ such that for any $0 < \epsilon < \epsilon_2$, $0 \leqslant \eta < \eta_2$, $0 \leqslant \eta' < \eta_2'$ and any $\zeta(k) \in N_{\eta,\eta'}$, there exists a $\lambda \in (0, 1)$ such that

$$L_{13}(\zeta(k+3)) \leqslant \lambda \cdot L_{13}(\zeta(k)).$$

Proof. We first estimate the following two numbers:

$$\lambda_1 = (m_A + m_C)\gamma^{3/2} \max_{1-\gamma/4 \leqslant x \leqslant \gamma/4} \frac{(1-2x)(1-2f_\gamma(x))(1-2f_\gamma(f_\gamma(x)))}{\sqrt{(1-f_\gamma(x))(1-f_\gamma(f_\gamma(x)))(1-f_\gamma(f_\gamma(f_\gamma(x))))}},$$

$$\lambda_2 = (m_B + m_D)\gamma^{3/2} \max_{1-\gamma/4 \leqslant y \leqslant \gamma/4} \frac{(1-2y)(1-2f_\gamma(y))(1-2f_\gamma(f_\gamma(y)))}{\sqrt{(1-f_\gamma(y))(1-f_\gamma(f_\gamma(y)))(1-f_\gamma(f_\gamma(f_\gamma(y))))}}.$$

Since

$$\max_{1-\gamma/4 \leqslant x \leqslant \gamma/4} \frac{(1-2x)(1-2f_\gamma(x))(1-2f_\gamma(f_\gamma(x)))}{\sqrt{(1-f_\gamma(x))(1-f_\gamma(f_\gamma(x)))(1-f_\gamma(f_\gamma(f_\gamma(x))))}} \leqslant 0.6$$

and

$$m_A + m_C \approx 0.169,$$

we have $\lambda_1 \approx 0.9 < 1$. Similarly, we can prove that $\lambda_2 \approx 0.9 < 1$. Let

$$\lambda = \max(\lambda_1, \lambda_2) + 2c_5(\epsilon + \eta + \eta')/\mu,$$

where

$$\mu = \min_{1-\gamma/4 \leqslant x \leqslant \gamma/4} \sqrt{(1-f_\gamma(x_1))(1-f_\gamma(f_\gamma(x_1)))(1-f_\gamma(f_\gamma(f_\gamma(x_1))))} > 0. \tag{3.8}$$

It is easily seen that $L_{13}(\zeta(k+3)) \leqslant \lambda L_{13}(\zeta(k))$.
Obviously, if

$$\epsilon_2 + \eta_2 + \eta_2' < (1 - \max(\lambda_1, \lambda_2))\mu/(2c_5), \tag{3.9}$$

then $0 < \lambda < 1$. This completes the proof of the lemma. □

By direct computation, we have the following equalities:

$$x_1(k+2) - x_2(k+2) = E(x_1(k) - x_2(k)) + F(y_1(k) - y_2(k)) + c_6(\epsilon + \eta + \eta^2)\mathrm{dist}_{12}(k),$$
$$y_1(k+2) - y_2(k+2) = G(x_1(k) - x_2(k)) + H(y_1(k) - y_2(k)) + c_7(\epsilon + \eta + \eta^2)\mathrm{dist}_{12}(k). \tag{3.10}$$

Here, we use the fact that $|x_i(k) - x_j(k)| + |y_i(k) - y_j(k)| < \eta^2$ if $|i - j|$ is even and c_6 and c_7 are constants independent of ϵ and η and

$$E = m_E\gamma^2(1 - (x_1(k+1) + x_2(k+1)))(1 - (x_1(k) + x_2(k))),$$
$$F = m_F\gamma^2(1 - (y_1(k+1) + y_2(k+1)))(1 - (y_1(k) + y_2(k))),$$
$$G = m_G\gamma^2(1 - (x_1(k+1) + x_2(k+1)))(1 - (x_1(k) + x_2(k))),$$
$$H = m_H\gamma^2(1 - (y_1(k+1) + y_2(k+1)))(1 - (y_1(k) + y_2(k))),$$

in which

$$m_E = (1-4c)^2(1-\theta)^2 + (1-4c)\theta^2,$$
$$m_F = 2(1-4c)(1-\theta)\theta(1-2c),$$
$$m_G = 2(1-\theta)\theta(1-2c),$$
$$m_H = (1-\theta)^2 + 2c\theta(1-4c).$$

We define a Lyapunov function L_{12} in N_η for the first and the second nodes:

$$L_{12}(\zeta(k)) = \frac{|x_1(k)-x_2(k)|}{\sqrt{(x_1(k)+x_2(k))(2-(x_1(k)+x_2(k)))}} + \frac{|y_1(k)-y_2(k)|}{\sqrt{(y_1(k)+y_2(k))(2-(y_1(k)+y_2(k)))}}. \tag{3.11}$$

In a similar way to that of the above discussion, we can also prove the following lemma.

LEMMA 3.3 There exist $0 < \epsilon_3 < \epsilon_2$, $0 < \eta_3 < \eta_2$, $0 < \eta_3' < \eta_2'$ and $0 < \tilde{\lambda} < 1$ such that for any $\zeta(k) \in N_{\eta_3,\eta_3'}$, it holds that

$$L_{12}(\zeta(k+2)) \leqslant \tilde{\lambda} L_{12}(\zeta(k)).$$

LEMMA 3.4 Consider the map

$$x(k+1) = (1-\theta) f_\gamma(x(k)) + \theta f_\delta(y(k)), \quad y(k+1) = (1-\theta) f_\delta(y(k)) + \theta f_\gamma(x(k)), \tag{3.12}$$

where $\theta \in (0.62, 0.64)$, $\gamma, \delta \in (3.7-\epsilon', 3.7+\epsilon')$ and f_γ and f_δ are logistic maps. Then, for every $\epsilon > 0$, there exists an $\epsilon_0' > 0$ such that for every $0 < \epsilon' < \epsilon_0'$ it holds that if $|x(0)-y(0)| > \epsilon$, then $|x(k)-y(k)|$ decreases exponentially as k increases until it becomes less than ϵ.

Proof. Define the Lyapunov function for (3.12):

$$L(x,y) = \frac{(x-y)^2}{(x+y)(2-x-y)}.$$

Then, there exists a k_0 such that $L(x(k_0), y(k_0)) < \lambda' L(x(0), y(0))$ with $0 < \lambda' < 1$ and $(x(0), y(0)) \in (1-\gamma/4, \gamma/4)^2$. Hence, if $\epsilon' = 0$, then $L(x(k), y(k))$ decreases exponentially to zero, which implies the exponential decrease of $|x(k)-y(k)|$ to zero. For $\epsilon' > 0$, we have

$$L(x(k_0), y(k_0)) < \lambda' L(x(0), y(0)) + c_{13}\epsilon'.$$

For any $\epsilon > 0$, let $\epsilon_0' < \frac{(1-\lambda')\nu\epsilon}{c_{13}}$. Then, $L(x(k_0), y(k_0))$ will decrease exponentially until it becomes less than $\nu\epsilon$, which implies the exponential decrease of $|x(k)-y(k)|$ until it becomes less than ϵ. □

The following lemma is useful later.

LEMMA 3.5 For any $4 \leqslant k \in \mathbb{N}$, there exist an η_k and an η_k' such that $\Phi^i\left(N_{\eta_k,\eta_k'}\right) \subset N_{\eta_3,\eta_3'}$ for $1 \leqslant i \leqslant k$.

Proof. It is easy to see that for any initial point in $N_{\eta_4,\eta_4'}$, where $\eta_4 < 1$ and $\eta_4' < 1$ are small positive numbers determined later, we have

$$\begin{aligned}|x_1(k+1) - x_3(k+1)| + |y_1(k+1) - y_3(k+1)| &\leqslant c_8(|x_1(k) - x_3(k)| + |y_1(k) - y_3(k)|)\\ &\leqslant c_8\eta_4^2,\end{aligned}$$

and c_9 is independent of k and

$$\begin{aligned}&|x_1(k+1) - x_2(k+1)| + |y_1(k+1) - y_2(k+1)|\\ &\quad\leqslant c_{10}(|x_1(k) - x_2(k)| + |y_1(k) - y_2(k)|) + c_{11}(|x_1(k) - x_3(k)| + |y_1(k) - y_3(k)| + |x_2(k) - x_4(k)|\\ &\qquad + |y_2(k) - y_4(k)|)\\ &\quad\leqslant c_{10}\eta_4 + 3c_{11}\eta_4^2,\end{aligned}$$

where we use the definition of $N_{\eta,\eta'}$ and c_9, c_{10} and c_{11} are independent of k. Hence, if $\eta_4 < \frac{\eta_0}{1+c_9+c_{10}+3c_{11}} = \frac{\eta_0}{C_4}$, then

$$|x_1(k+1) - x_3(k+1)| + |y_1(k+1) - y_3(k+1)| \leqslant \eta_3^2$$

and

$$|x_1(k+1) - x_2(k+1)| + |y_1(k+1) - y_2(k+1)| \leqslant \eta_3.$$

Similarly, we have

$$\begin{aligned}|x_i(k+1) - y_i(k+1)| &\leqslant |g(x_i(k), y_i(k)) - h(x_i(k), y_i(k))| + c|g(x_i(k), y_i(k)) - g(x_{i-1}(k), y_{i-1}(k))|\\ &\qquad + c|g(x_i(k), y_i(k)) - g(x_{i+1}(k), y_{i+1}(k))|\\ &\leqslant c_{12}\eta_4 + c_{13}\eta_4'.\end{aligned}$$

If $\eta_4 < \frac{\eta_3'}{2c_{12}}$ and $\eta_4' < \frac{\eta_3'}{C_4} = \frac{\eta_3'}{2c_{13}}$, then we have

$$|x_i(k+1) - y_i(k+1)| \leqslant \eta_3'.$$

It is easy to obtain the similar estimates for

$$|x_i(k+1) - x_j(k+1)| + |y_i(k+1) - y_j(k+1)|$$

and

$$|x_i(k+1) - y_j(k+1)|$$

for other $i, j = 1, 2, 3, 4$.

Let $\eta_4 = \frac{\eta_3}{C_4}$ and $\eta_4' = \frac{\eta_3'}{C_4'}$. The above inequalities imply that $\Phi\left(N_{\eta_1,\eta_1'}\right) \in N_{\eta_0,\eta_0'}$. By induction, assume that for $k \in \mathbb{N}$, there exist a C_k and a C_k' such that $\Phi^i\left(N_{\eta_k,\eta_k'}\right) \in N_{\eta_3,\eta_3'}$ with $\eta_k = \frac{\eta_0}{C_k}$ and $\eta_k' = \frac{\eta_0'}{C_k'}$, for $1 \leqslant i \leqslant k$. Then, similarly we can find C_{k+1} and C_{k+1}' such that $\Phi\left(N_{\eta_{k+1},\eta_{k+1}'}\right) \in N_{\eta_k,\eta_k'}$ with $\eta_{k+1} = \frac{\eta_0}{C_{k+1}}$ and $\eta_{k+1}' = \frac{\eta_0'}{C_{k+1}'}$. Thus, we have $\Phi^{k+1}\left(N_{\eta_{k+1},\eta_{k+1}'}\right) \in N_{\eta_3,\eta_3'}$. This completes the proof. □

From Lemmas 3.2–3.5, we have the following lemma.

LEMMA 3.6 There exist $c_{13} > 0$, $K_1 \in \mathbb{N}$, $0 < \bar{\eta} < \eta_3$ and $0 < \bar{\eta}' < \eta_3'$ such that for any $0 \leqslant \eta < \bar{\eta}$, $0 \leqslant \eta' < \bar{\eta}'$ and initial points in $N_{\eta,\eta'}$, it holds that

(i) $|x_i(K_1) - x_j(K_1)| + |y_i(K_1) - y_j(K_1)| < \frac{\nu^4\eta^2}{4}$ for $|i-j|$ even,

(ii) $|x_i(K_1) - x_j(K_1)| + |y_i(K_1) - y_j(K_1)| < \frac{\nu^2\eta}{2}$ for $|i-j|$ odd and

(iii) $|x_i(K_1) - y_j(K_1)| < c_{13}^{K_1}\eta'$.

Proof of Theorem 3.2. Define K_1, $\bar{\eta}$ and $\bar{\eta}'$ as in Lemma 3.5. Obviously, for $k \geqslant K_1$, if only $|x_i(k) - y_j(k)| < \eta_3'$, $i, j = 1, 2, 3, 4$, then

$$|x_i(k) - x_j(k)| + |y_i(k) - y_j(k)| \leqslant \eta_3^2/4 \text{ for } |i-j| \text{ even}$$

and

$$|x_i(k) - x_j(k)| + |y_i(k) - y_j(k)| \leqslant \eta_3/2 \text{ for } |i-j| \text{ odd.}$$

Let K_2 be a small positive integer such that $\lambda'^{K_2} c_{13}^{K_1} < 1/2$. Define $\eta_0' = \frac{1}{2} c_{13}^{-(K_1+k_0)} \eta_3'$. Then, we have

$$|x_i(k) - y_j(k)| < \eta_3',$$

where $k = K_1 + 1, \ldots, K_1 + k_0$ and

$$|x_i(K_1 + k_0) - y_j(K_1 + k_0)| < \lambda'|x_i(K_1) - y_j(K_1)|.$$

Hence, we have

$$|x_i(K_1 + k_0K_2) - y_j(K_1 + k_0K_2)| < \nu'^{K_2}|x_i(K_1) - y_j(K_1)| < \frac{1}{2}|x_i(0) - y_j(0)|.$$

Thus, defining $K_0 = K_1 + k_0K_2$, $\eta_0 = \bar{\eta}$ and η_0' as above, we finish the proof of Theorem 3.2. □

4. Numerical results

In this section, we present some numerical results to illustrate the chaotic synchronization behaviour of (1.3) and (1.4) in established regimes (see the last pages). In Figs 2–4, we show the regions of parameters θ and the coupling coefficients c for which synchronization occurs in 1D CMLs (1.3) and (1.4) with fixed γ, δ and $n = 4$. Figure 5 shows the region of θ and c for which synchronization of (1.3) and (1.4) occurs is very small with the lattice $n = 8$. In Figs 6–7, we show the regions of parameters γ and δ for some fixed θ and c. In Fig. 8, we present the difference of $x(k) - y(k)$ for CMLs of (1.3) and (1.4) with $n = 4$ and plot the FFT of $|x(k) - y(k)|$. The numerical behaviour shows that the difference of $x(k)$ and $y(k)$ forms a chaotic behaviour.

5. Conclusion

In this paper, we have designed CMLs of two-variable maps (connected with two logistic maps) coupled with one variable. We have proved that our 1D CMLs with the lattice size $n = 4$ have chaotic synchronized behaviour for some suitable coupling coefficients. We also present the numerical results

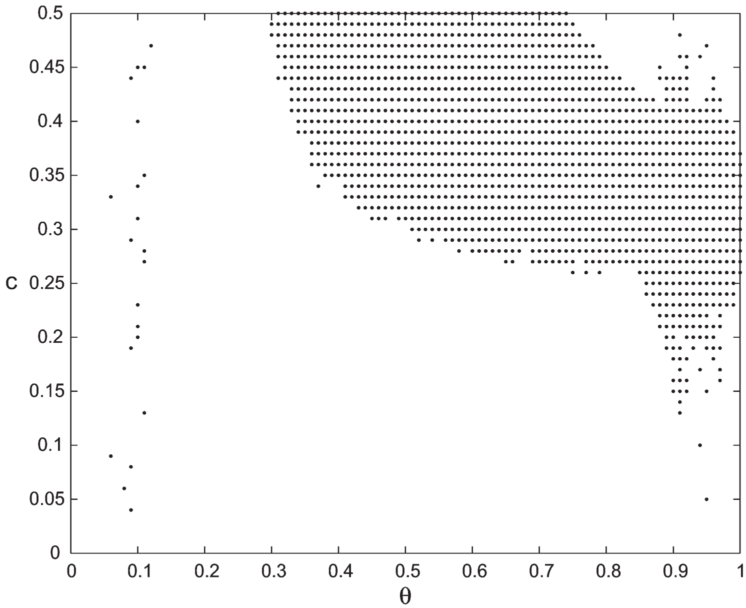

FIG. 2. Range of c and θ for 1D CMLs with $\gamma = 3.685$, $\delta = 3.68$ and $n = 4$.

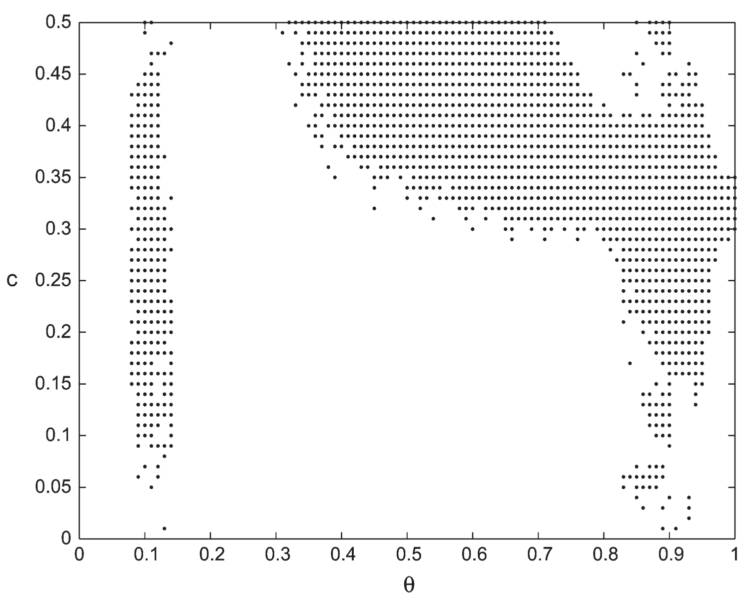

FIG. 3. Range of c and θ for 1D CMLs with $\gamma = 3.9$, $\upsilon\delta = 3.75$ and $n = 4$.

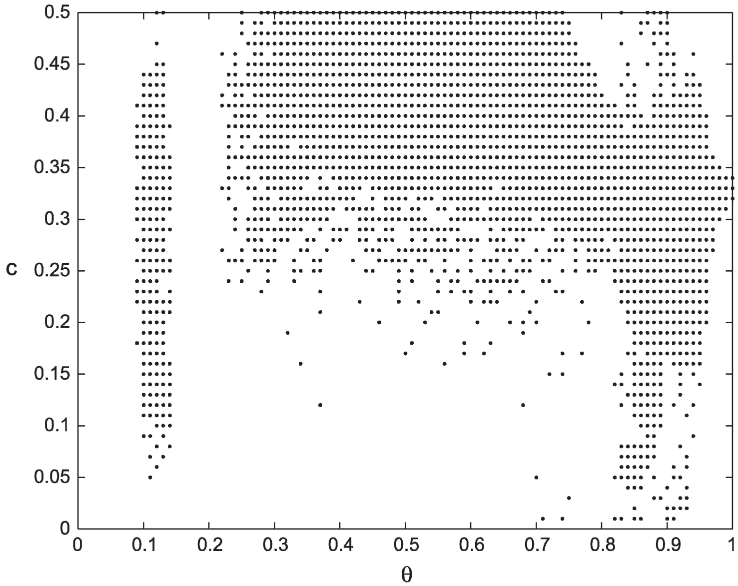

FIG. 4. Range of c and θ for 1D CMLs with $\gamma = 3.95$, $\delta = 3.75$ and $n = 4$.

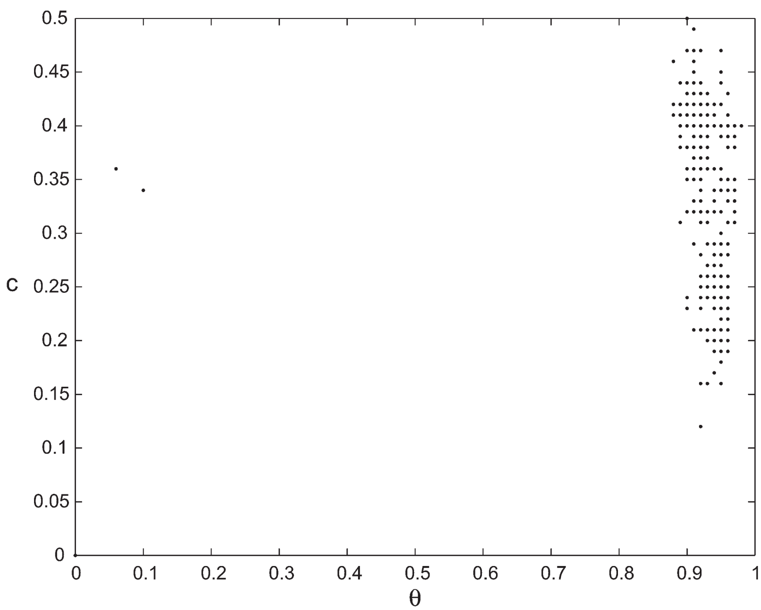

FIG. 5. Range of c and θ for 1D CMLs with $\gamma = 3.685$, $\delta = 3.68$ and $n = 8$.

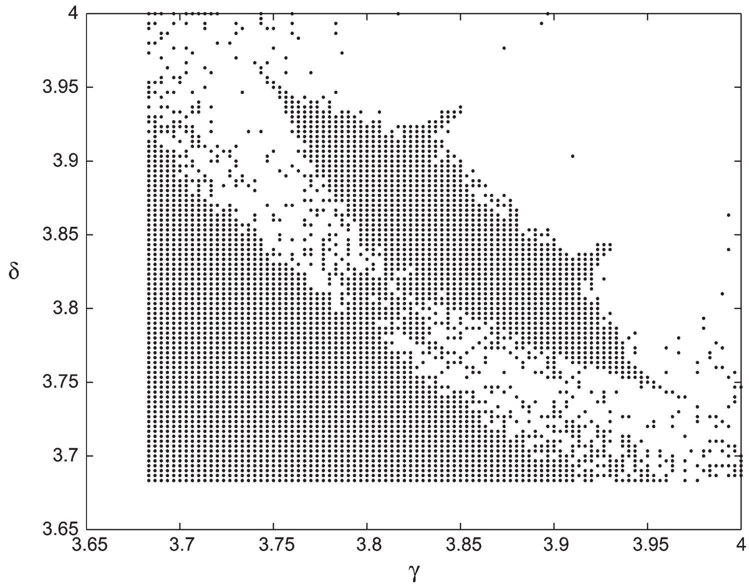

FIG. 6. Range of γ and δ for 1D CMLs with $c = 0.8$, $\theta = 0.42$ and $n = 4$.

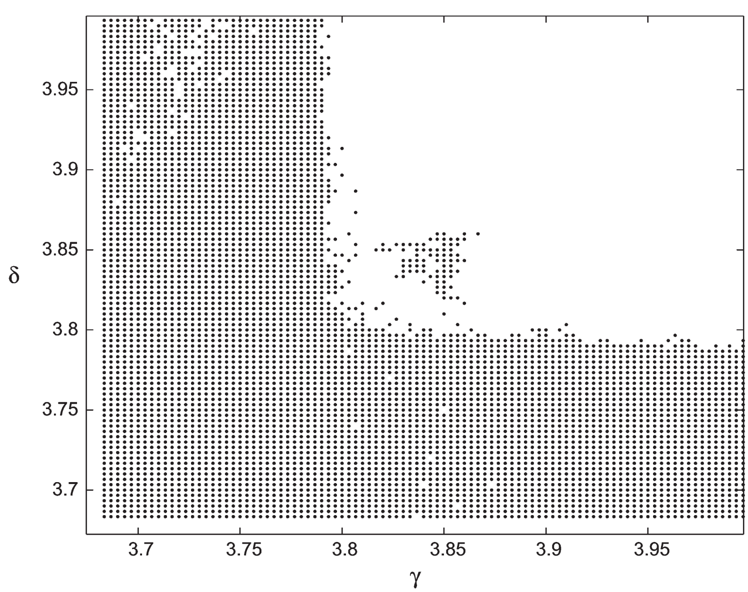

FIG. 7. Range of γ and δ for 1D CMLs with $c = 0.95$, $\theta = 0.25$ and $n = 4$.

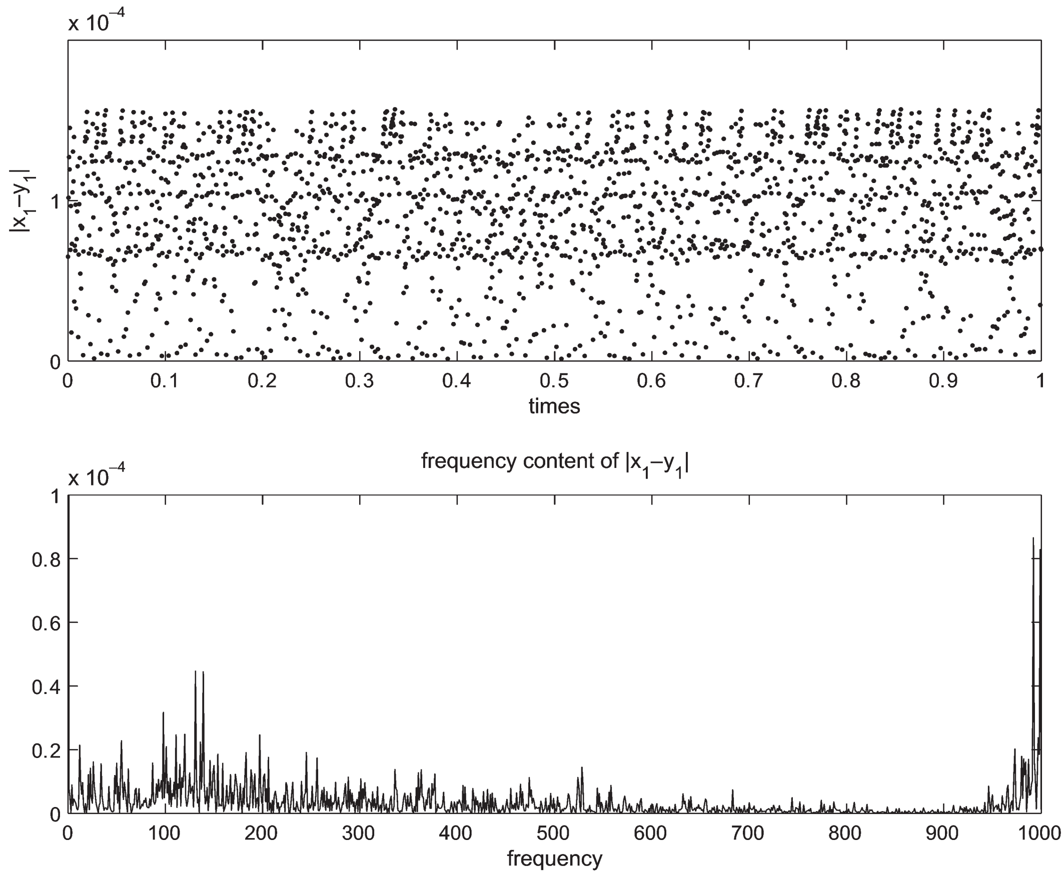

FIG. 8. The difference of $x(k)$ and $y(k)$ in CMLs (1.4) with $\gamma = 3.68$, $\delta = 3.681$, $c = 1/3$ and $\theta = 2/3$. The below picture is the FFT of $|x(k) - y(k)|$.

of synchronization of 1D cases with various coupling coefficients, connected parameters and lattice sizes. The two-variable map as in (1.4) connected with logistic maps produces chaotic behaviour over a certain wide connecting range. Due to the special topological structure of security in private communication, the new designed topological structure of (1.3) and (1.4) appears to be attractive from both theoretical and practical points of view.

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