Nonlinear Analysis 71 (2009) 4047–4059

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Chaos control of new Mathieu–Van der Pol systems with new Mathieu–Duffing systems as functional system by GYC partial region stability theory

Zheng-Ming Ge *, Shih-Yu Li [1]

Department of Mechanical Engineering, National Chiao Tung University, Hsinchu, Taiwan, ROC

ARTICLE INFO

Article history:
Received 21 March 2008
Accepted 18 February 2009

Keywords:
Chaos control
Partial region stability theory
New Mathieu–Van der Pol system
New Mathieu–Duffing system

ABSTRACT

In this paper, a new strategy by using GYC partial region stability theory is proposed to achieve chaos control. Using the GYC partial region stability theory, the new Lyapunov function used is a simple linear homogeneous function of error states and the lower order controllers are much more simple and introduce less simulation error. Numerical simulations are given for new Mathieu–Van der Pol system and new Mathieu–Duffing system to show the effectiveness of this strategy.

1. Introduction

Since Ott et al. [1] gave the famous OGY control method in 1990, the applications of the various methods to control a chaotic behavior in natural sciences and engineering are well known. For example, the adaptive control [2–5], the method of chaos control based on sampled data [6], the method of pulse feedback of systematic variable [7], the active control [8,9] and linear error feedback control [10,11]. However, when Lyapunov stability of zero solution of states is studied, the stability of solutions on the whole neighborhood region of the origin is demanded.

In this paper, a new strategy to achieve chaos control by GYC partial region stability theory is proposed [12,13]. Using the GYC partial region stability theory, the new Lyapunov function is a simple linear homogeneous function of error states and the lower order controllers are much more simple and introduce less simulation error.

The layout of the rest of the paper is as follows. In Section 2, chaos control scheme by GYC partial region stability theory is proposed. In Section 3, new Mathieu–Van der pol system and new Mathieu–Duffing system are presented. In Section 4, three simulation examples are given. In Section 5, conclusions are drawn. The partial region stability theory is enclosed in Appendix.

2. Chaos control scheme

Consider the following chaotic system

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}) \tag{2.1}$$

where $\mathbf{x} = [x_1, x_2, \dots, x_n]^T \in R^n$ is a state vector, $\mathbf{f} : R_+ \times R^n \to R^n$ is a vector function.

* Corresponding address: Department of Mechanical Engineering, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu 300, Taiwan, ROC. Tel.: +886 3 5712874; fax: +886 3 5720634.
E-mail address: zmg@cc.nctu.edu.tw (Z.-M. Ge).

[1] Tel.: +886 3 5712121x55179.

0362-546X/$ – see front matter
doi:10.1016/j.na.2009.02.095

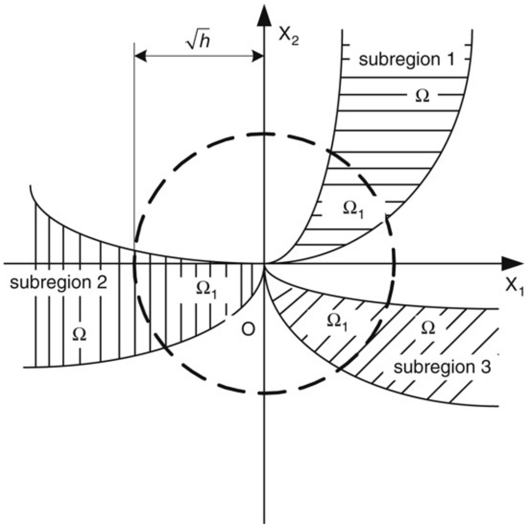

Fig. 1. Partial regions Ω and Ω_1.

The goal system which can be either chaotic or regular, is

$$\dot{\mathbf{y}} = \mathbf{g}(t, \mathbf{y}) \tag{2.2}$$

where $\mathbf{y} = [y_1, y_2, \ldots, y_n]^T \in R^n$ is a state vector, $\mathbf{g} : R_+ \times R^n \to R^n$ is a vector function.

In order to make the chaos state $\mathbf{x}$ approaching the goal state $\mathbf{y}$, define $\mathbf{e} = \mathbf{x} - \mathbf{y}$ as the state error. The chaos control is accomplished in the sense that [13–22]:

$$\lim_{t\to\infty} \mathbf{e} = \lim_{t\to\infty} (\mathbf{x} - \mathbf{y}) = 0. \tag{2.3}$$

In this paper, we will use examples in which the error dynamics always happens in the first quadrant of coordinate system and use GYC partial region stability theory which is enclosed in the Appendix. The Lyapunov function is a simple linear homogeneous function of error states and the controllers are simpler because they are in lower order than that of traditional controllers.

3. New Chaotic Mathieu–Van der pol system and new chaotic Mathieu–Duffing system

This section introduces new Mathieu–van der Pol system and new Mathieu–Duffing system, respectively.

3.1. New Mathieu–Van der Pol system

Mathieu equation and van der Pol equation are two typical nonlinear nonautonomous systems:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -(a + b\sin\omega t)x_1 - (a + b\sin\omega t)x_1^3 - cx_2 + d\sin\omega t \end{cases} \tag{3.1}$$

$$\begin{cases} \dot{x}_3 = x_4 \\ \dot{x}_4 = -ex_3 + f(1 - x_3^2)x_4 + g\sin\omega t. \end{cases} \tag{3.2}$$

Exchanging $\sin\omega t$ in Eq. (3.1) with x_3 and $\sin\omega t$ in Eq. (3.2) with x_1, we obtain the autonomous new Mathieu–Van der Pol system:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -(a + bx_3)x_1 - (a + bx_3)x_1^3 - cx_2 + dx_3 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = -ex_3 + f(1 - x_3^2)x_4 + gx_1 \end{cases} \tag{3.3}$$

where a, b, c, d, e, f, g are uncertain parameters. This system exhibits chaos when the parameters of system are $a = 10$, $b = 3, c = 0.4, d = 70, e = 1, f = 5, g = 0.1$ and the initial states of system are $(x_{10}, x_{20}, x_{30}, x_{40}) = (0.1, -0.5, 0.1, -0.5)$. Its phase portraits are shown in Fig. 2.

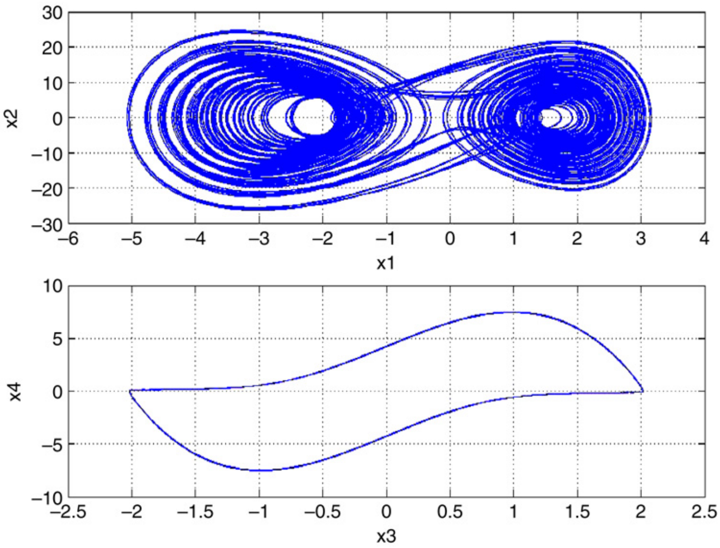

Fig. 2. Chaotic phase portraits for new Mathieu–Van der Pol system.

3.2. New Mathieu–Duffing system

Mathieu equation and Duffing equation are two typical nonlinear nonautonomous systems:

$$\begin{cases}\dot{z}_1 = z_2 \\ \dot{z}_2 = -(a_1 + b_1 \sin\omega t)z_1 - (a_1 + b_1 \sin\omega t)z_1^3 - c_1 z_2 + d_1 \sin\omega t\end{cases} \tag{3.4}$$

$$\begin{cases}\dot{z}_3 = z_4 \\ \dot{z}_4 = -z_3 - z_3^3 - e_1 z_4 + f_1 \sin\omega t.\end{cases} \tag{3.5}$$

Exchanging $\sin\omega t$ in Eq. (3.4) with z_3 and $\sin\omega t$ in Eq. (3.5) with z_1, we obtain the autonomous master new Mathieu–Duffing system:

$$\begin{cases}\dot{z}_1 = z_2 \\ \dot{z}_2 = -(a_1 + b_1 z_3)z_1 - (a_1 + b_1 z_3)z_1^3 - c_1 z_2 + d_1 z_3 \\ \dot{z}_3 = z_4 \\ \dot{z}_4 = -z_3 - z_3^3 - e_1 z_4 + f_1 z_1\end{cases} \tag{3.6}$$

where a_1, b_1, c_1, d_1, e_1 and f_1 are uncertain parameters. This system exhibits chaos when the parameters of system are $a_1 = 20.30, b_1 = 0.5970, c_1 = 0.005, d_1 = -24.441, e_1 = 0.002, f_1 = 14.63$ and initial states is $(-2, 10, -2, 10)$. Its phase portraits are shown in Fig. 3.

4. Numerical simulations

The following chaotic system

$$\begin{cases}\dot{x}_1 = x_2 - 200 \\ \dot{x}_2 = -(a + b(x_3 - 200))(x_1 - 200) - (a + b(x_3 - 200))(x_1 - 200)^3 - c(x_2 - 200) + d(x_3 - 200) \\ \dot{x}_3 = (x_4 - 200) \\ \dot{x}_4 = -e(x_3 - 200) + f(1 - (x_3 - 200)^2)(x_4 - 200) + g(x_1 - 200)\end{cases} \tag{4.1}$$

is the new Mathieu–Van der pol system of which the old origin is translated to $(x_1, x_2, x_3, x_4) = (200, 200, 200, 200)$ in order that the error dynamics happens always in the first quadrant of error state coordinate system. This translated new Mathieu–Van der pol system presents chaotic motion when initial conditions is $(x_{10}, x_{20}, x_{30}, x_{40}) = (210.1, 209.5, 210.1, 209.5)$ and the parameters are $a = 10, b = 3, c = 0.4, d = 70, e = 1, f = 5, g = 0.1$.

In order to lead (x_1, x_2, x_3, x_4) to the goal, we add control terms u_1, u_2, u_3 and u_4 to each equation of Eq. (4.1), respectively.

$$\begin{cases}\dot{x}_1 = x_2 - 200 + u_1 \\ \dot{x}_2 = -(a + b(x_3 - 200))(x_1 - 200) - (a + b(x_3 - 200))(x_1 - 200)^3 - c(x_2 - 200) + d(x_3 - 200) + u_2 \\ \dot{x}_3 = (x_4 - 200) + u_3 \\ \dot{x}_4 = -e(x_3 - 200) + f(1 - (x_3 - 200)^2)(x_4 - 200) + g(x_1 - 200) + u_4.\end{cases} \tag{4.2}$$

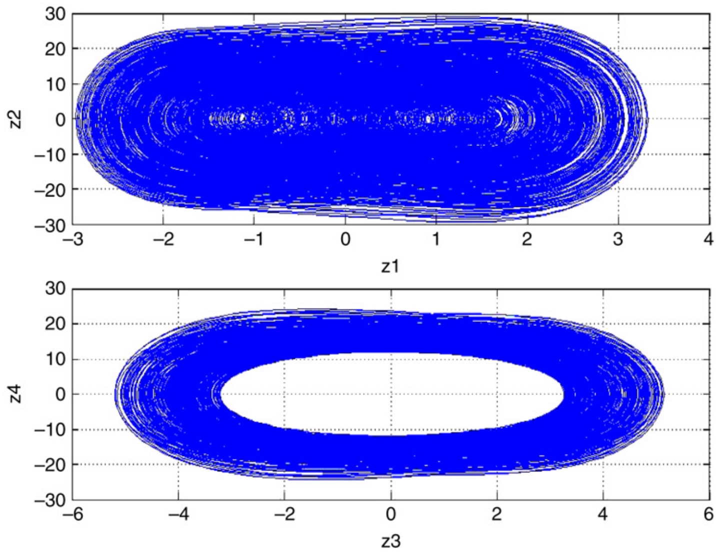

Fig. 3. Chaotic phase portraits for new Mathieu–Duffing system in the first quadrant.

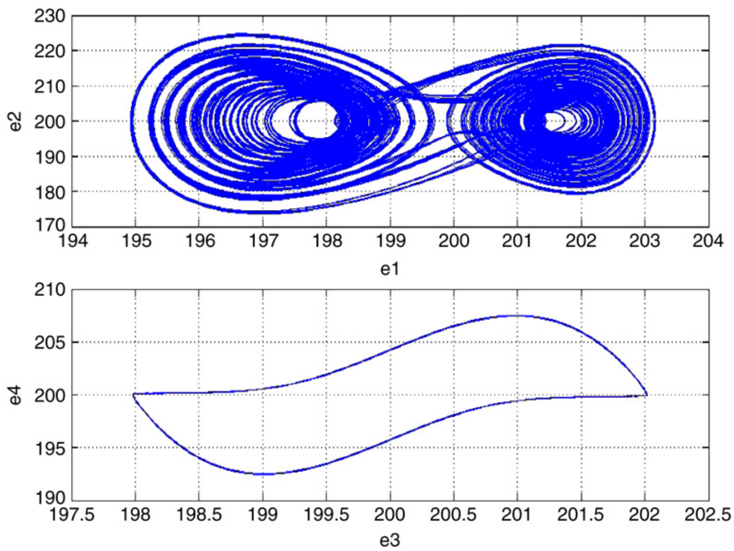

Fig. 4. Phase portrait of error dynamics for Case I.

CASE I. Control the chaotic motion to zero.

In this case we will control the chaotic motion of the new Mathieu–Van der pol system (4.1) to zero. The goal is $y = 0$. The state error is $e_i = x_i - y_i = x_i$, $(i = 1, 2, 3, 4)$ and error dynamics becomes

$$\begin{cases} \dot{e}_1 = \dot{x}_1 = x_2 - 200 + u_1 \\ \dot{e}_2 = \dot{x}_2 = -(a + b(x_3 - 200))(x_1 - 200) - (a + b(x_3 - 200))(x_1 - 200)^3 \\ \qquad - c(x_2 - 200) + d(x_3 - 200) + u_2 \\ \dot{e}_3 = \dot{x}_3 = (x_4 - 200) + u_3 \\ \dot{e}_4 = \dot{x}_4 = -e(x_3 - 200) + f(1 - (x_3 - 200)^2)(x_4 - 200) + g(x_1 - 200) + u_4. \end{cases} \tag{4.3}$$

In Fig. 4, we can see that the error dynamics always exists in first quadrant.

By GYC partial region asymptotical stability theorem, one can easily choose a Lyapunov function in the form of a positive definite function in first quadrant as:

$$V = e_1 + e_2 + e_3 + e_4. \tag{4.4}$$

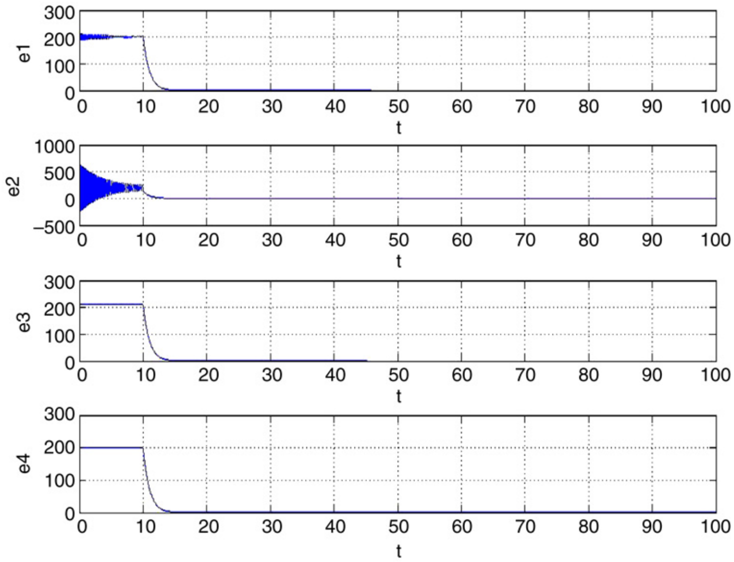

Fig. 5. Time histories of errors for Case I.

Its time derivative through error dynamics (4.3) is

$$\begin{aligned}\dot{V} &= \dot{e}_1 + \dot{e}_2 + \dot{e}_3 + \dot{e}_4\\ &= (x_2 - 200 + u_1) + (-(a + b(x_3 - 200))(x_1 - 200)\\ &\quad - (a + b(x_3 - 200))(x_1 - 200)^3 - c(x_2 - 200) + d(x_3 - 200) + u_2) + (x_4 - 200 + u_3)\\ &\quad + (-e(x_3 - 200) + f(1 - (x_3 - 200)^2)(x_4 - 200) + g(x_1 - 200) + u_4).\end{aligned} \tag{4.5}$$

Choose

$$\begin{aligned}u_1 &= -(x_2 - 200) - e_1\\ u_2 &= (-(a + b(x_3 - 200))(x_1 - 200) - (a + b(x_3 - 200))(x_1 - 200)^3 - c(x_2 - 200) + d(x_3 - 200)) - e_2\\ u_3 &= -(x_4 - 200) - e_3\\ u_4 &= (-e(x_3 - 200) + f(1 - (x_3 - 200)^2)(x_4 - 200) + g(x_1 - 200)) - e_4.\end{aligned} \tag{4.6}$$

We obtain

$$\dot{V} = \dot{e}_1 + \dot{e}_2 + \dot{e}_3 + \dot{e}_4 < 0$$

which is negative definite function in first quadrant. The numerical results are shown in Fig. 5. After 10 s, the error trajectories approach the origin.

CASE II. Control the chaotic motion to a regular function.

In this case we will control the chaotic motion of the new Mathieu–Van der pol system (4.1) to regular function of time. The goal is $y_i = F_i e^{\sin \omega t}$, $(i = 1, 2, 3, 4)$. The error equation

$$e_i = x_i - y_i = x_i - F_i e^{\sin \omega t}, \quad (i = 1, 2, 3, 4) \tag{4.7}$$

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty} (x_i - F_i e^{\sin \omega t}) = 0, \quad (i = 1, 2, 3, 4)$$

where $F_1 = F_2 = F_3 = F_4 = F = 10$ and $\omega = 0.5$.

The error dynamics is

$$\begin{cases}\dot{e}_1 = x_2 - 200 + u_1 - F_1 \omega e^{\sin \omega t}(\cos \omega t)\\ \dot{e}_2 = -(a + b(x_3 - 200))(x_1 - 200) - (a + b(x_3 - 200))(x_1 - 200)^3\\ \qquad - c(x_2 - 200) + d(x_3 - 200) + u_2 - F_2 \omega e^{\sin \omega t}(\cos \omega t)\\ \dot{e}_3 = (x_4 - 200) + u_3 - F_3 \omega e^{\sin \omega t}(\cos \omega t)\\ \dot{e}_4 = -e(x_3 - 200) + f(1 - (x_3 - 200)^2)(x_4 - 200) + g(x_1 - 200) + u_4 - F_4 \omega e^{\sin \omega t}(\cos \omega t).\end{cases} \tag{4.8}$$

In Fig. 6, the error dynamics always exists in first quadrant.

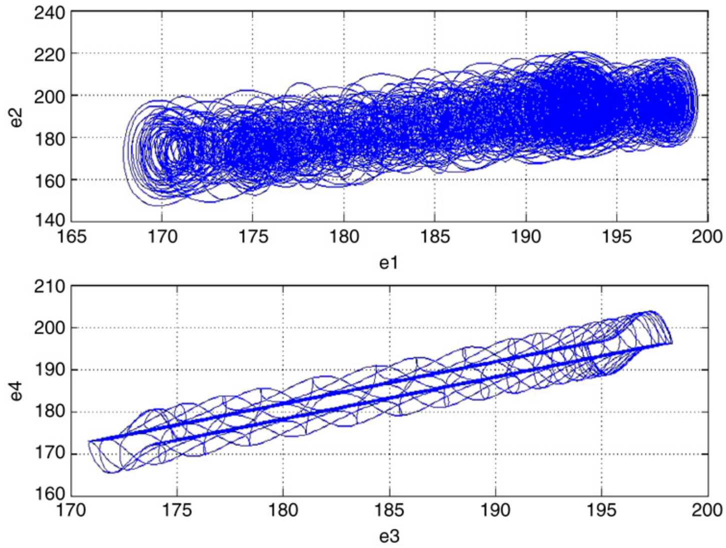

Fig. 6. Phase portraits of error dynamics for Case II.

By GYC partial region asymptotical stability theorem, one can easily choose a Lyapunov function in the form of a positive definite function in first quadrant as:

$$V = e_1 + e_2 + e_3 + e_4.$$

Its time derivative is

$$\begin{aligned} V &= \dot{e}_1 + \dot{e}_2 + \dot{e}_3 + \dot{e}_4 = (x_2 - 200 + u_1 - F_1\omega e^{\sin\omega t}(\cos\omega t)) + (-(a + b(x_3 - 200))(x_1 - 200) \\ &\quad - (a + b(x_3 - 200))(x_1 - 200)^3 - c(x_2 - 200) + d(x_3 - 200) + u_2 - F_2\omega e^{\sin\omega t}(\cos\omega t)) \\ &\quad + ((x_4 - 200) + u_3 - F_3\omega e^{\sin\omega t}(\cos\omega t)) + (-e(x_3 - 200) + f(1 - (x_3 - 200)^2)(x_4 - 200) \\ &\quad + g(x_1 - 200) + u_4 - F_4\omega e^{\sin\omega t}(\cos\omega t)). \end{aligned} \tag{4.9}$$

Choose

$$\begin{aligned} u_1 &= -(x_2 - 200 - F_1\omega e^{\sin\omega t}(\cos\omega t)) - e_1 \\ u_2 &= -(-(a + b(x_3 - 200))(x_1 - 200) - (a + b(x_3 - 200))(x_1 - 200)^3 \\ &\quad - c(x_2 - 200) + d(x_3 - 200) - F_2\omega e^{\sin\omega t}(\cos\omega t)) - e_2 \\ u_3 &= -((x_4 - 200) - F_3\omega e^{\sin\omega t}(\cos\omega t)) - e_3 \\ u_4 &= -(-e(x_3 - 200) + f(1 - (x_3 - 200)^2)(x_4 - 200) + g(x_1 - 200) - F_4\omega e^{\sin\omega t}(\cos\omega t)) - e_4. \end{aligned} \tag{4.10}$$

We obtain

$$\dot{V} = -e_1 - e_2 - e_3 - e_4 < 0$$

which is a negative definite function in first quadrant. The numerical results are shown in Figs. 7 and 8. After 10 s, the errors approach zero and the chaotic trajectories approach to regular motion.

CASE III. Control the chaotic motion of the new Mathieu–Van der pol system to chaotic motion of the new Mathieu–Duffing system.

In this case we will control chaotic motion of the new Mathieu–Van der pol system (4.1) to that of the new chaotic Mathieu–Duffing system. The goal system for control is new Mathieu–Duffing system with initial states (−2, 10, −2, 10), system parameters $a_1 = 20.30$, $b_1 = 0.5970$, $c_1 = 0.005$, $d_1 = -24.441$, $e_1 = 0.002$ and $f_1 = 14.63$.

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = -(a_1 + b_1 z_3)z_1 - (a_1 + b_1 z_3)z_1^3 - c_1 z_2 + d_1 z_3 \\ \dot{z}_3 = z_4 \\ \dot{z}_4 = -z_3 - z_3^3 - e_1 z_4 + f_1 z_1. \end{cases} \tag{4.11}$$

The error equation is $e_i = x_i - z_i$, $(i = 1, 2, 3, 4)$. Our aim is $\lim_{t\to\infty} e_i = 0$, $(i = 1, 2, 3, 4)$.

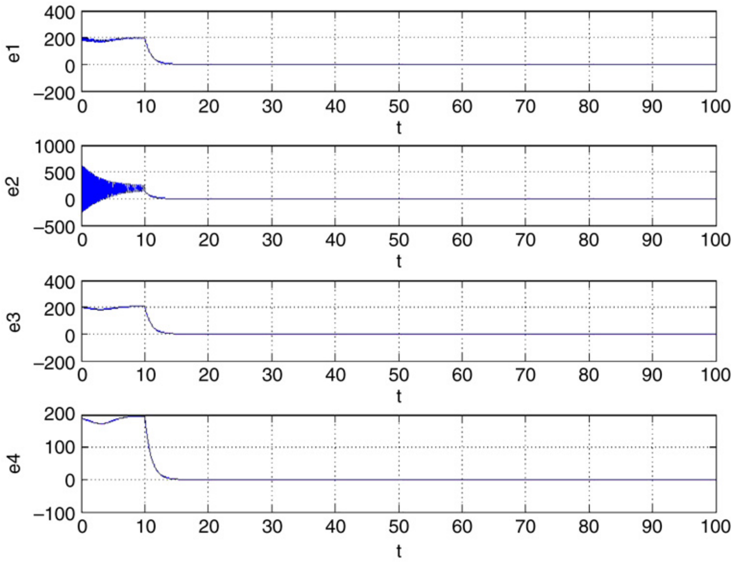

Fig. 7. Time histories of errors for Case II.

The error dynamics becomes

$$\begin{cases}\dot{e}_1 = \dot{x}_1 - \dot{z}_1 = (x_2 - 200 - z_2) + u_1 \\ \dot{e}_2 = \dot{x}_2 - \dot{z}_2 = (-(a + b(x_3 - 200))(x_1 - 200) - (a + b(x_3 - 200))(x_1 - 200)^3 \\ \qquad - c(x_2 - 200) + d(x_3 - 200) - (-(a_1 + b_1 z_3) z_1 - (a_1 + b_1 z_3) z_1^3 - c_1 z_2 + d_1 z_3)) + u_2 \\ \dot{e}_3 = \dot{x}_3 - \dot{z}_3 = (x_4 - 200 - z_4) + u_3 \\ \dot{e}_4 = \dot{x}_4 - \dot{z}_4 = (-e(x_3 - 200) + f(1 - (x_3 - 200)^2)(x_4 - 200) \\ \qquad + g(x_1 - 200) - (-z_3 - z_3^3 - e_1 z_4 + f_1 z_1)) + u_4. \end{cases} \quad (4.12)$$

In Fig. 9, the error dynamics always exists in first quadrant.

By GYC partial region asymptotical stability theorem, one can easily choose a Lyapunov function in the form of a positive definite function in first quadrant as:

$$V = e_1 + e_2 + e_3 + e_4.$$

Its time derivative is

$$\begin{aligned}\dot{V} &= \dot{e}_1 + \dot{e}_2 + \dot{e}_3 + \dot{e}_4 = ((x_2 - 200 - z_2) + u_1) + ((-(a + b(x_3 - 200))(x_1 - 200) \\ &\quad - (a + b(x_3 - 200))(x_1 - 200)^3 - c(x_2 - 200) + d(x_3 - 200) - (-(a_1 + b_1 z_3) z_1 - (a_1 + b_1 z_3) z_1^3 \\ &\quad - c_1 z_2 + d_1 z_3)) + u_2) + ((x_4 - 200 - z_4) + u_3) + ((-e(x_3 - 200) + f(1 - (x_3 - 200)^2)(x_4 - 200) \\ &\quad + g(x_1 - 200) - (-z_3 - z_3^3 - e_1 z_4 + f_1 z_1)) + u_4).\end{aligned} \quad (4.13)$$

Choose

$$\begin{aligned} u_1 &= -(x_2 - 200 - z_2) - e_1 \\ u_2 &= -(-(a + b(x_3 - 200))(x_1 - 200) - (a + b(x_3 - 200))(x_1 - 200)^3 - c(x_2 - 200) \\ &\quad + d(x_3 - 200) - (-(a_1 + b_1 z_3) z_1 - (a_1 + b_1 z_3) z_1^3 - c_1 z_2 + d_1 z_3)) - e_2 \\ u_3 &= -(x_4 - 200 - z_4) - e_3 \\ u_3 &= -(-e(x_3 - 200) + f(1 - (x_3 - 200)^2)(x_4 - 200) + g(x_1 - 200) - (-z_3 - z_3^3 - e_1 z_4 + f_1 z_1)) - e_4. \end{aligned} \quad (4.14)$$

We obtain

$$\dot{V} = -e_1 - e_2 - e_3 - e_4 < 0$$

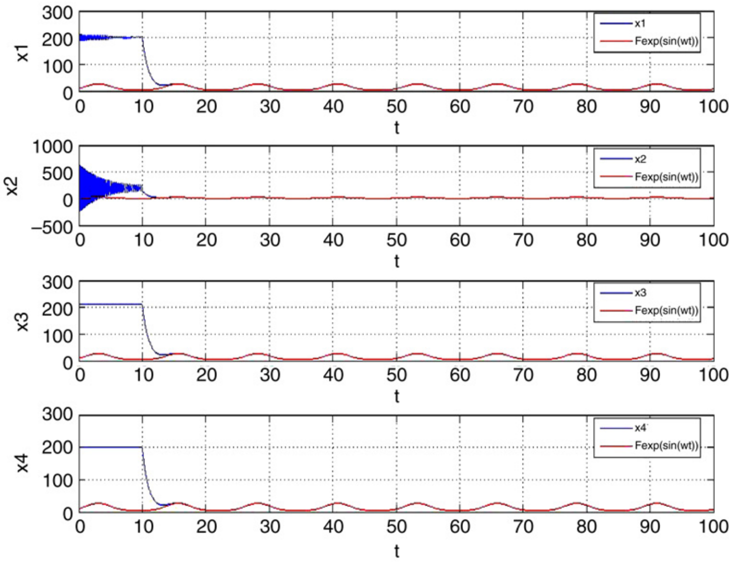

Fig. 8. Time histories of x_1, x_2, x_3, x_4 for Case II.

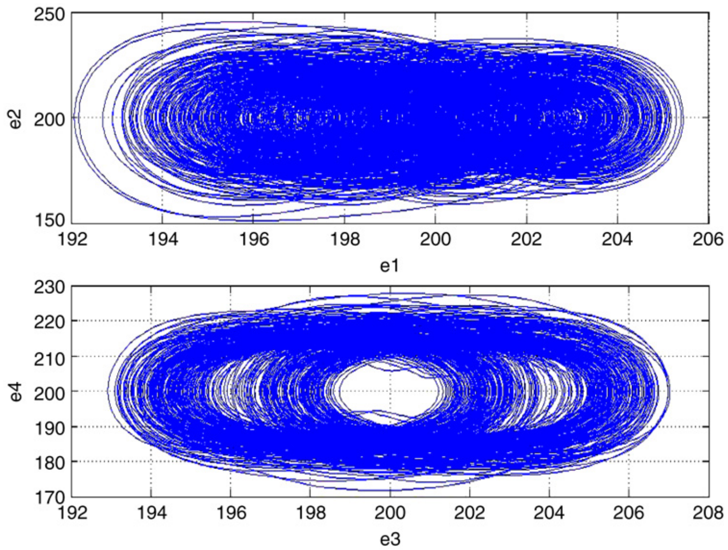

Fig. 9. Phase portraits of error dynamics for Case III.

which is negative definite function in first quadrant. The numerical results are shown in Figs. 10 and 11. After 10 s, the errors approach zero and the chaotic trajectories of the new Mathieu–Van der pol system approach to that of the new Mathieu–Duffing system.

5. Conclusions

In this paper, a new strategy by using GYC partial region stability theory is proposed to achieve chaos control. Using the GYC partial region stability theory, the new Lyapunov function used is a simple linear homogeneous function of states and the lower order controllers are much more simple and introduce less simulation error. The new chaotic Mathieu–Van der pol system and new chaotic Mathieu–Duffing system system are used as simulation examples which confirm the scheme effectively.

Acknowledgment

This research was supported by the National Science Council, Republic of China, under Grant Number NSC 96-2221-E-009-145-MY3.

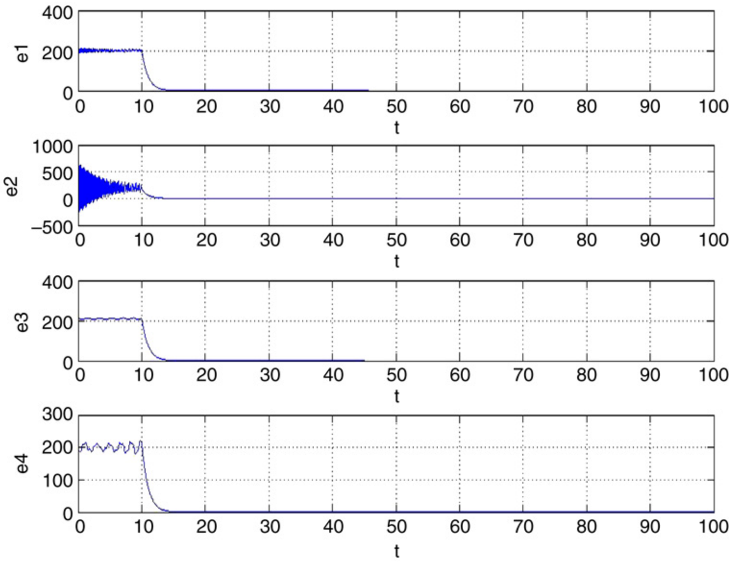

Fig. 10. Time histories of errors for Case III.

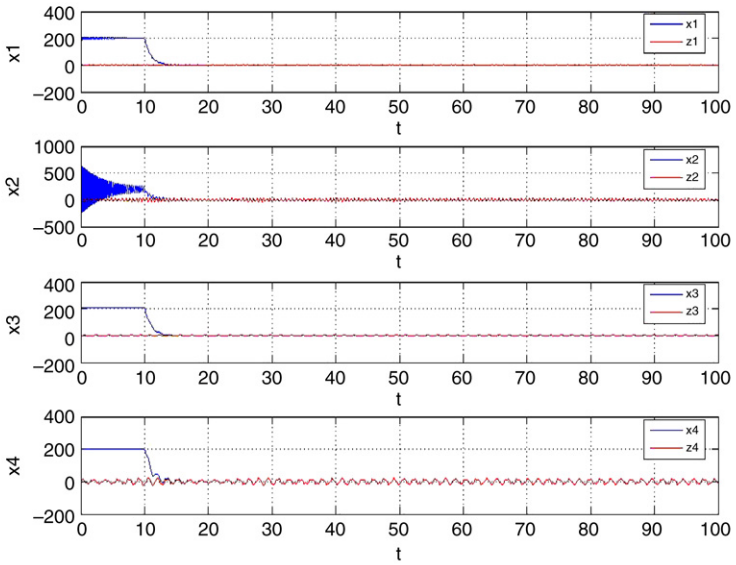

Fig. 11. Time histories of x_1, x_2, x_3, x_4 and z_1, z_2, z_3, z_4 for Case III.

Appendix. GYC partial region stability theory

A.1. Definition of the stability on partial region

Consider the differential equations of disturbed motion of a nonautonomous system in the normal form

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = X_s(t, x_1, \ldots, x_n), \quad (s = 1, \ldots, n) \tag{A.1}$$

where the function X_s is defined on the intersection of the partial region Ω (shown in Fig. 1) and

$$\sum_s x_s^2 \leq H \tag{A.2}$$

and $t > t_0$, where t_0 and H are certain positive constants. X_s which vanishes when the variables x_s are all zero, is a real-valued function of $t, x_1, \ldots, x_n$. It is assumed that X_s is smooth enough to ensure the existence, uniqueness of the solution of the initial value problem. When X_s does not contain t explicitly, the system is autonomous.

Obviously, $x_s = 0$ $(s = 1, \ldots, n)$ is a solution of Eq. (A.1). We are interested to the asymptotical stability of this zero solution on partial region Ω (including the boundary) of the neighborhood of the origin which in general may consist of several subregions (Fig. 1).

Definition 1. For any given number $\varepsilon > 0$, if there exists a $\delta > 0$, such that on the closed given partial region Ω when

$$\sum_s x_{s0}^2 \leq \delta, \quad (s = 1, \ldots, n) \tag{A.3}$$

for all $t \geq t_0$, the inequality

$$\sum_s x_s^2 < \varepsilon, \quad (s = 1, \ldots, n) \tag{A.4}$$

is satisfied for the solutions of Eq. (A.1) on Ω, then the zero solution $x_s = 0$ $(s = 1, \ldots, n)$ is stable on the partial region Ω.

Definition 2. If the undisturbed motion is stable on the partial region Ω, and there exists a $\delta' > 0$, so that on the given partial region Ω when

$$\sum_s x_{s0}^2 \leq \delta', \quad (s = 1, \ldots, n). \tag{A.5}$$

The equality

$$\lim_{t\to\infty} \left(\sum_s x_s^2 \right) = 0 \tag{A.6}$$

is satisfied for the solutions of Eq. (A.1) on Ω, then the zero solution $x_s = 0$ $(s = 1, \ldots, n)$ is asymptotically stable on the partial region Ω.

The intersection of Ω and region defined by Eq. (A.5) is called the region of attraction.

Definition of functions $V(t, x_1, \ldots, x_n)$: Let us consider the functions $V(t, x_1, \ldots, x_n)$ given on the intersection Ω_1 of the partial region Ω and the region

$$\sum_s x_s^2 \leq h, \quad (s = 1, \ldots, n) \tag{A.7}$$

for $t \geq t_0 > 0$, where t_0 and h are positive constants. We suppose that the functions are single-valued and have continuous partial derivatives and become zero when $x_1 = \cdots = x_n = 0$.

Definition 3. If there exist $t_0 > 0$ and a sufficiently small $h > 0$, so that on partial region Ω_1 and $t \geq t_0$, $V \geq 0$ (or ≤ 0), then V is a positive (or negative) semidefinite, in general semidefinite, function on the Ω_1 and $t \geq t_0$.

Definition 4. If there exists a positive (negative) definite function $W(x_1 \ldots x_n)$ on Ω_1, so that on the partial region Ω_1 and $t \geq t_0$

$$V - W \geq 0 \text{ (or} - V - W \geq 0), \tag{A.8}$$

then $V(t, x_1, \ldots, x_n)$ is a positive definite function on the partial region Ω_1 and $t \geq t_0$.

Definition 5. If $V(t, x_1, \ldots, x_n)$ is neither definite nor semidefinite on Ω_1 and $t \geq t_0$, then $V(t, x_1, \ldots, x_n)$ is an indefinite function on partial region Ω_1 and $t \geq t_0$. That is, for any small $h > 0$ and any large $t_0 > 0$, $V(t, x_1, \ldots, x_n)$ can take either positive or negative value on the partial region Ω_1 and $t \geq t_0$.

Definition 6. Bounded function V.

If there exist $t_0 > 0$, $h > 0$, so that on the partial region Ω_1, we have

$$|V(t, x_1, \ldots, x_n)| < L$$

where L is a positive constant, then V is said to be bounded on Ω_1.

Definition 7. Function with infinitesimal upper bound.

If V is bounded, and for any $\lambda > 0$, there exists $\mu > 0$, so that on Ω_1 when $\sum_s x_s^2 \leq \mu$, and $t \geq t_0$, we have

$$|V(t, x_1, \ldots, x_n)| \leq \lambda$$

then V admits an infinitesimal upper bound on Ω_1.

A.2. GYC theorem of stability and asymptotical stability on partial region

Theorem 1. *If there can be found a definite function $V(t, x_1, \ldots, x_n)$ on the partial region for Eq. (A.1), and the derivative with respect to time based on these equations are:*

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial t} + \sum_{s=1}^{n} \frac{\partial V}{\partial x_s} X_s. \tag{A.9}$$

Then, it is a semidefinite function on the partial region whose sense is opposite to that of V, or if it becomes zero identically, then the undisturbed motion is stable on the partial region.

Proof. Let us assume for the sake of definiteness that V is a positive definite function. Consequently, there exists a sufficiently large number t_0 and a sufficiently small number $h < H$, such that on the intersection Ω_1 of partial region Ω and

$$\sum_s x_s^2 \le h, \quad (s = 1, \ldots, n)$$

and $t \ge t_0$, the following inequality is satisfied

$$V(t, x_1, \ldots, x_n) \ge W(x_1, \ldots, x_n),$$

where W is a certain positive definite function which does not depend on t. Besides that, Eq. (A.9) may assume only negative or zero value in this region. □

Let ε be an arbitrarily small positive number. We shall suppose that in any case $\varepsilon < h$. Let us consider the aggregation of all possible values of the quantities $x_1, \ldots, x_n$, which are on the intersection ω_2 of Ω_1 and

$$\sum_s x_s^2 = \varepsilon, \tag{A.10}$$

and let us designate by $l > 0$ the precise lower limit of the function W under this condition. By virtue of Eq. (A.8), we shall have

$$V(t, x_1, \ldots, x_n) \ge l \quad \text{for } (x_1, \ldots, x_n) \text{ on } \omega_2. \tag{A.11}$$

We shall now consider the quantities x_s as functions of time which satisfy the differential equations of disturbed motion. We shall assume that the initial values x_{s0} of these functions for $t = t_0$ lie on the intersection Ω_2 of Ω_1 and the region

$$\sum_s x_s^2 \le \delta, \tag{A.12}$$

where δ is so small that

$$V(t_0, x_{10}, \ldots, x_{n0}) < l. \tag{A.13}$$

By virtue of the fact that $V(t_0, 0, \ldots, 0) = 0$, such a selection of the number δ is obviously possible. We shall suppose that in any case the number δ is smaller than ε. Then the inequality

$$\sum_s x_s^2 < \varepsilon, \tag{A.14}$$

being satisfied at the initial instant will be satisfied, in the very least, for a sufficiently small $t - t_0$, since the functions $x_s(t)$ very continuously with time. We shall show that these inequalities will be satisfied for all values $t > t_0$. Indeed, if these inequalities were not satisfied at some time, there would have to exist such an instant $t = T$ for which this inequality would become an equality. In other words, we would have

$$\sum_s x_s^2(T) = \varepsilon,$$

and consequently, on the basis of Eq. (A.11)

$$V(T, x_1(T), \ldots, x_n(T)) \ge l. \tag{A.15}$$

On the other hand, since $\varepsilon < h$, the inequality (Eq. (A.7)) is satisfied in the entire interval of time $[t_0, T]$, and consequently, in this entire time interval $\frac{\mathrm{d}V}{\mathrm{d}t} \le 0$. This yields

$$V(T, x_1(T), \ldots, x_n(T)) \le V(t_0, x_{10}, \ldots, x_{n0}),$$

which contradicts Eq. (A.14) on the basis of Eq. (A.13). Thus, the inequality (Eq. (A.4)) must be satisfied for all values of $t > t_0$, hence follows that the motion is stable.

Finally, we must point out that from the view-point of mathematics, the stability on partial region in general does not relate logically to the stability on the whole region. If an undisturbed solution is stable on a partial region, it may be either stable or unstable on the whole region and vice versa. In specific practical problems, we do not study the solution starting within Ω_2 and running out of Ω.

Theorem 2. *If in satisfying the conditions of Theorem 1, the derivative $\frac{dV}{dt}$ is a definite function on the partial region with opposite sign to that of V and the function V itself permits an infinitesimal upper limit, then the undisturbed motion is asymptotically stable on the partial region.*

Proof. Let us suppose that V is a positive definite function on the partial region and that consequently, $\frac{dV}{dt}$ is negative definite. Thus on the intersection Ω_1 of Ω and the region defined by Eq. (A.7) and $t \geq t_0$ there will be satisfied not only the inequality (Eq. (A.8)), but the following inequality as well:

$$\frac{dV}{dt} \leq -W_1(x_1, \ldots, x_n), \tag{A.16}$$

where W_1 is a positive definite function on the partial region independent of t.

Let us consider the quantities x_s as functions of time which satisfy the differential equations of disturbed motion assuming that the initial values $x_{s0} = x_s(t_0)$ of these quantities satisfy the inequalities (Eq. (A.12)). Since the undisturbed motion is stable in any case, the magnitude δ may be selected so small that for all values of $t \geq t_0$ the quantities x_s remain within Ω_1. Then, on the basis of Eq. (A.16) the derivative of function $V(t, x_1(t), \ldots, x_n(t))$ will be negative at all times and, consequently, this function will approach a certain limit, as t increases without limit, remaining larger than this limit at all times. We shall show that this limit is equal to some positive quantities different from zero. Then for all values of $t \geq t_0$ the following inequality will be satisfied:

$$V(t, x_1(t), \ldots, x_n(t)) > \alpha \tag{A.17}$$

where $\alpha > 0$.

Since V permits an infinitesimal upper limit, it follows from this inequality that

$$\sum_s x_s^2(t) \geq \lambda, \quad (s = 1, \ldots, n), \tag{A.18}$$

where λ is a certain sufficiently small positive number. Indeed, if such a number λ did not exist, that is, if the quantity $\sum_s x_s(t)$ is smaller than any preassigned number no matter how small, then the magnitude $V(t, x_1(t), \ldots, x_n(t))$, as follows from the definition of an infinitesimal upper limit, would also be arbitrarily small, which contradicts Eq. (A.17).

If for all values of $t \geq t_0$ the inequality (Eq. (A.18)) is satisfied, then Eq. (A.16) shows that the following inequality will be satisfied at all times:

$$\frac{dV}{dt} \leq -l_1,$$

where l_1 is a positive number different from zero which constitutes the precise lower limit of the function $W_1(t, x_1(t), \ldots, x_n(t))$ under condition (Eq. (A.18)). Consequently, for all values of $t \geq t_0$ we shall have:

$$V(t, x_1(t), \ldots, x_n(t)) = V(t_0, x_{10}, \ldots, x_{n0}) + \int_{t_0}^{t} \frac{dV}{dt} dt \leq V(t_0, x_{10}, \ldots, x_{n0}) - l_1(t - t_0),$$

which is, obviously, in contradiction with Eq. (A.17). The contradiction thus obtained shows that the function $V(t, x_1(t), \ldots, x_n(t))$ approaches zero as t increases without limit. Consequently, the same will be true for the function $W(x_1(t), \ldots, x_n(t))$ as well, from which it follows directly that

$$\lim_{t\to\infty} x_s(t) = 0, \quad (s = 1, \ldots, n),$$

which proves the theorem. □

References

[1] E. Ott, C. Grebogi, J.A. Yorke, Controlling chaos, Physical Review Letters 64 (1990) 1196–1199.
[2] H.Y. Hu, An adaptive control scheme for recovering periodic motion of chaotic systems, Journal of Sound and Vibration 199 (1997) 269–274.
[3] Jun-Juh Yan, Meei-Ling Hung, Teh-Lu Liao, Adaptive sliding mode control for synchronization of chaotic gyros with fully unknown parameters, Journal of Sound and Vibration 298 (2006) 298–306.
[4] Heng-Hui Chen, Adaptive synchronization of chaotic systems via linear balanced feedback control, Journal of Sound and Vibration 306 (2007) 865–876.
[5] Mei Sun, Lixin Tian, Shumin Jiang, Jun Xu, Feedback control and adaptive control of the energy resource chaotic system, Chaos, Solitons and Fractals 32 (2007) 1725–1734.
[6] T. Yang, L.B. Yang, C.M. Yang, Theory of control of chaos using sample data, Physics Letters A 246 (1998) 284–288.
[7] T. Yang, C.M. Yang, L.B. Yang, Control of Rossler system to periodic motions using impulsive control method, Physics Letters A 232 (1997) 356–361.
[8] M.T. Yassen, Chaos synchronization between two different chaotic system using active control, Chaos, Solitons and Fractals 23 (2005) 131–140.
[9] Tang Fang, Ling Wang, An adaptive active control for the modified Chua's circuit, Physics Letters A 346 (2005) 342–346.
[10] Marat Rafikov, José Manoel, Balthazar, On control and synchronization in chaotic and hyperchaotic systems via linear feedback control, Communications in Nonlinear Science and Numerical Simulation 13 (2008) 1246–1255.
[11] Z.-M. Ge, H.-H. Chen, Double degeneracy and chaos in a rate gyro with feedback control, Journal of Sound and Vibration 209 (1998) 753–769.
[12] Z.-M. Ge, C.-W. Yao, H.-K. Chen, Stability on partial region in dynamics, Journal of Chinese Society of Mechanical Engineer 15 (1994) 140–151.
[13] Z.-M. Ge, H.-K. Chen, Three asymptotical stability theorems on partial region with applications, Japanese Journal of Applied Physics 37 (1998) 2762–2773.

[14] Z.-M. Ge, C.-H. Yang, H.-H. Chen, S.-C. Lee, Non-linear dynamics and chaos control of a physical pendulum with vibrating and rotation support, Journal of Sound and Vibration 242 (2001) 247–264.
[15] Z.-M. Ge, J.-K. Yu, Pragmatical asymptotical stability theorem on partial region and for partial variable with applications to gyroscopic systems, The Chinese Journal of Mechanics 16 (2000) 179–187.
[16] Z.-M. Ge, C.-M. Chang, Chaos synchronization and parameters identification of single time scale brushless DC motors, Chaos, Solitons and Fractals 20 (2004) 883–903.
[17] F. Liu, Y. Ren, X. Shan, Z. Qiu, A linear feedback synchronization theorem for a class of chaotic systems, Chaos, Solitons and Fractals 13 (2002) 723–730.
[18] Z.-M. Ge, C.-H. Yang, Generalized synchronization of quantum-CNN chaotic oscillator with different order systems, Chaos, Solitons and Fractals 35 (2008) 980–990.
[19] A. Krawiecki, A. Sukiennicki, Generalizations of the concept of marginal synchronization of chaos, Chaos, Solitons and Fractals 11 (2000) 1445–1458.
[20] Z.-M. Ge, C.-H. Yang, Synchronization of complex chaotic systems in series expansion form, Chaos, Solitons, and Fractals 34 (2007) 1649–1658.
[21] Z.-M. Ge, Y.-S. Chen, Synchronization of unidirectional coupled chaotic systems via partial stability, Chaos, Solitons and Fractals 21 (2004) 101.
[22] Samuel Bowong, F.M. Moukam Kakmeni, Dimi Jean Luc, Chaos control in the uncertain Duffing oscillator, Journal of Sound and Vibration 292 (2006) 869–880.