國立交通大學

應用數學系

碩　士　論　文

2 維有限型的子移位之混合性質

The mixing property of

2-dimensional subshift of finite type

研 究 生：蔡佩君

指導老師：林松山　教授

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國 立 交 通 大 學
應 用 數 學 系
碩 士 論 文

A Thesis

Submitted to Department of Applied Mathematics

College of Science

National Chiao Tung University

in Partial Fulfillment of the Requirements

for the Degree of

Master

in

Applied Mathematics

June 2005

Hsinchu, Taiwan, Republic of China

中 華 民 國 九 十 四 年 六 月

2維有限型的子移位之混合性質

學生：蔡佩君　　　　　　　　　　　　指導老師：林松山 教授

國立交通大學應用數學系(研究所)碩士班

摘　　　要

在這篇論文中，討論n階置換矩陣A_n的原始性質。而這些主題與2維有限型的移位之混合性質有關。

我們的目的是給定2階置換矩陣A_2的某些必備條件，進而証得矩陣A_n的原始性質。這些結果可以被提供去研究有安全符號的矩陣A_n之原始性質。在這篇論文中，我們也檢查一些與有限型的矩陣子移位$\Lambda = \Omega(A_2)$是弱擴張混合相關的例子，對於這些例子，我們都可証得所有$n \geq 2$的n階矩陣A_n是原始的。最後，我們描述一些與A_n的原始性質有關的結果。

The mixing property of
2-dimensional subshift of finite type

Student：Pei-Jiun Tsai　　　　　　　　　　Advisors：Song-Sun Lin

Department（Institute）of Applied Mathematics
National Chiao Tung University

ABSTRACT

In this paper, the primitivity of n-th order transition matrices A_n defined on $Z_{2\times n}$ are studied, this topics related to the mixing property of 2-dimensional shift of finite type.

Our propose is to give some necessary conditions for A_2 to guarantee the p-rimitivity of A_n. The results can be applied to study the primitivity of A_n which has safe symbol. In the paper, we also check some examples related to the matr-ix subshift of finite type $\Lambda = \Omega(A_2)$ be extensively weak mixing, and for these examples, we all show that A_n is primitive for all $n \geq 2$. Finally, we describe some results related to the primitivity of A_n.

誌　　　　謝

這篇論文的完成必須感謝許多協助與支持我的人。首先，我想要感謝的人就是林松山教授，您兩年來耐心的指導與勉勵，讓我得以順利完成此篇論文，此外在學問與待人處世方面，對我也有很大的啟發。謹此，致上我最誠摯的敬意與謝意。還有要感謝班榮超學長和林吟衡學姊，常常給予適時的幫助，告訴我該如何著手，也會提供一些想法來解決問題，尤其是班榮超學長，從他身上看到了對數學的熱情，豐富的知識，在在令人佩服。

接著，我要感謝我最親愛的父母親 蔡進先生和陳素霞女士，因為他們偉大無私的父母愛，讓我得以成長茁壯至今。在求學的過程中，他們總是在背後默默的支持著我，給我最大的空間，讓我做任何我想做的事，讓我可以無後顧之憂的學習。另外，也謝謝哥哥、姊姊和弟弟，在我需要幫助時，永遠是第一個伸出雙手，深濃的兄弟姐妹之情，是支持我向前的原動力。謹此，獻上我最真誠感恩的心，因為沒有他們，就沒有現在的我。

最後，感謝我在梧棲國小的同事們，香吟、耀仁、婉娥、亞民、靜宜、雪甄、至芸…，在兩年的碩士班生涯中，給我幫助與鼓勵，讓我能兼顧工作與學業。在我心情低落時，大家陪我一起喝茶、聊天；在我茫然不知所措時，大家給我開導與建議，因為有你們，豐富了我的學習路程，謝謝你們，你們永遠是我最要好的支持者。

目　錄

1 Introduction

Many systems have been studied as models for spatial pattern formation in biology, chemistry, engineering and physics. Lattices play important roles in modeling underlying spatial structures. We mention some works arising in biology([1], [3], [22], [24], [25], [28], [29], [30]), chemical reaction and phase transitions([4], [10], [15], [16], [17], [19], [23], [32], [36]), image processing and pattern recognition([11], [13], [15], [16], [18], [20], [21], [26], [35]), as well as materials science([12], [14], [27]). In Lattice Dynamical Systems(LDS), especially Cellular Neural Networks(CNN), the complexity of the set of all global patterns has received considerable attention in recent years([2], [5], [6], [9]). One of the interesting problem comes from the statistic mechanism, is d-demensional shift of finite type, state as follows, given a list of patterns with shape $F \in \mathbb{Z}^d$, consider the set

$$X = X_{\mathcal{L}} = \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} \;\middle|\; \text{for all } n \in \mathbb{Z}^d, \text{ and } \sigma^n(x)\middle| F \in \mathcal{L} \right\} \tag{1.1}$$

where $\mathcal{A}$ is a finite set, we call it symbol, and without loss of generality, F is d-demensional cube, i.e., $F = \{(n_1, ..., n_d) \mid 1 \leq n_k \leq k,\ \forall k = 1, ..., d\}$, many invariants related to the shift of finite will discussed likewise in [31], e.g., the topological entropy, measure-theoretical entropy, variational principle, mixing property, and extension problem. Unfortunately, unlike the one dimensional case, it's extremely difficulty to compute and check those invariants, for example, only a very few example of entropy of 2-dimensional shift of finite type can be computed explicitly, also for mixing property. In this Paper we start to study the mixing property of d-dimension shift of finite type, and we focus on d=2. In [7], the authors construct a finite approximation scheme of higher dimensional shift of finite type, and call it the series of transition matrices in multi-dimensional lattice model in $\mathbb{Z}^2$, we are going to use the structure of such transition matrices to study the mixing property of higher dimensional shift of finite type.

We first recall some results in [7], which are crucial in this study. For simplicity, we only consider two symbols which are given on 2×2 lattice $Z_{2\times 2}$. We begin with a consideration of given horizontal transition matrix

$$H_2 = \begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{pmatrix} \tag{1.2}$$

which is related to a set of admissible local patterns on $Z_{2\times 2}$, and

$$h_{ij} \in \{0,1\} \text{ for } 1 \leq i,j \leq 4 \tag{1.3}$$

The associated vertical transition matrix V_2 is defined by

$$V_2 = \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{pmatrix} \tag{1.4}$$

In 2-dimensional shift of finite type, one can immediate construct the H_2 according to the list of pattern with shape $F = \{(n_1, n_2) \ \mid 1 \leq n_i \leq 2, \ \forall i = 1,2\}$. In [7], H_2 and V_2 possess the following property to each other

$$\mathbb{H}_2 = \begin{pmatrix} v_{11} & v_{12} & v_{21} & v_{22} \\ v_{13} & v_{14} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{41} & v_{42} \\ v_{33} & v_{34} & v_{43} & v_{44} \end{pmatrix} = \begin{pmatrix} H_{2;1} & H_{2;2} \\ H_{2;3} & H_{2;4} \end{pmatrix}, \tag{1.5}$$

and

$$\mathbb{V}_2 = \begin{pmatrix} h_{11} & h_{12} & h_{21} & h_{22} \\ h_{13} & h_{14} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{41} & h_{42} \\ h_{33} & h_{34} & h_{43} & h_{44} \end{pmatrix} = \begin{pmatrix} V_{2;1} & V_{2;2} \\ V_{2;3} & V_{2;4} \end{pmatrix}. \tag{1.6}$$

The recursive formula for n-th order horizontal transition matrices H_n defined on $Z_{2\times n}$ has been obtained [7] by the following procedure:

$$\mathbb{H}_{k+1} = \begin{pmatrix} v_{11}H_{k;1} & v_{12}H_{k;2} & v_{21}H_{k;1} & v_{22}H_{k;2} \\ v_{13}H_{k;3} & v_{14}H_{k;4} & v_{23}H_{k;3} & v_{24}H_{k;4} \\ v_{31}H_{k;1} & v_{32}H_{k;2} & v_{41}H_{k;1} & v_{42}H_{k;2} \\ v_{33}H_{k;3} & v_{34}H_{k;4} & v_{43}H_{k;3} & v_{44}H_{k;4} \end{pmatrix} \tag{1.7}$$

whenever

$$\mathbb{H}_k = \begin{pmatrix} H_{k;1} & H_{k;2} \\ H_{k;3} & H_{k;4} \end{pmatrix} \tag{1.8}$$

for $k \geq 2$. The number of all admissible patterns defined on $Z_{m\times n}$ which can be generated from $\mathbb{H}_2$ is now defined by

$$\begin{aligned} \Gamma_{m\times n}(H_2) &= \left|H_n^{m-1}\right| \\ &= \text{the summation of all entries in } H_n^{m-1} \end{aligned} \tag{1.9}$$

The quantitative properties of $\mathbb{H}_n$ for $n \geq 2$ are interesting problem in matrix theory and combinatorial dynamics, the most important one is the primitive property, in matrix analysis, the primitivity of a nonnegative matrix will guarantee the positivity of the maximal eigenvalue of a given matrix, and according to the discussion above, if some $\mathbb{H}_2$ is induced from some 2-dimensional shift of finite type, then primitivity of $\mathbb{H}_2$ demonstrate the shift is mixing. And some interesting dynamics will appear therein, for example, the periodic orbits is dense, and there exists a unique measure of maximal entropy. Thus, it give rise to the study the primitivity of $\mathbb{H}_n, \forall n \geq 2$.

The difficulties of this study is that the size of $\mathbb{H}_n$ grows exponentially, i.e., $\mathbb{H}_n \in \mathbf{M}_{2^n \times 2^n}$, then it's of nature and interesting to ask that which kind of sufficient conditions will guarantee the primitivity for $\mathbb{H}_n$. To overcome this problem, the powerful tool s_n, S_n, R_n and C_n will be introduced, thus we obtain some checkable conditions of $\mathbb{H}_2$ to guarantee the primitivity for $\mathbb{H}_n, \forall n \geq 2$.

The paper is organized as follows. Section 2 introduce some definitions, s_n, S_n, R_n and C_n, the main result and proof will presented in section 3, section 4 included some examples of 2 symbols in $\mathbb{Z}^2$ and some results related to the primitivity of $\mathbb{H}_n, \forall n \geq 2$.

2 Preliminaries

2.1 Definitions

In this section, we give some standard definitions in matrix analysis related to our study. As mentioned in the introduction, horizontal transition matrix $\mathbb{H}_2$ and vertical transition matrix $\mathbb{V}_2$ are related to each other. However, in application, usually it is better working on one matrix then the other. Therefore, we use $\mathbb{A}_2$ and $\mathbb{B}_2$ to replace $\mathbb{H}_2$ and $\mathbb{V}_2$ throughout this paper, i.e., if $\mathbb{A}_2 = \mathbb{H}_2$ then $\mathbb{B}_2 = \mathbb{V}_2$ and if $\mathbb{A}_2 = \mathbb{V}_2$ then $\mathbb{B}_2 = \mathbb{H}_2$. Therefore, for simplicity, only $\mathbb{A}_2$ is stated herein.

First, we define the non-compressible property for a matrix.

Definition 1 *We say a matrix $A \in \mathbf{M}_{n\times n}(\mathbb{Z})$ is non-compressible if no column and row of A are all zero.*

In the other word ,if A is compressible, i.e., at least one column or row of A is all zero.

Next, we follow the notation from [7] to denote the recursive formulae for n-th order transition matrices $\mathbb{A}_n$ defined on $\mathbb{Z}_{2\times n}$ (or $\mathbb{Z}_{n\times 2}$) , by

$$\mathbb{A}_n = (\mathbb{A}_{n-1})_{2^{n-1}\times 2^{n-1}} \circ \left(E_{2^{n-2}\times 2^{n-2}} \otimes \begin{pmatrix} A_{2;1} & A_{2;2} \\ A_{2;3} & A_{2;4} \end{pmatrix} \right)_{2^{n-1}\times 2^{n-1}}, \tag{2.1}$$

for $n > 2$, where

$$\begin{pmatrix} A_{2;1} & A_{2;2} \\ A_{2;3} & A_{2;4} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{pmatrix} = \mathbb{A}_2$$

and $A_{2;\alpha} \in \mathbf{M}_{2\times 2}(\mathbb{Z})$, $\forall \alpha \in \{1,2,3,4\}$.

Then, we define the r_n and c_n below.

Definition 2 *If $A \in \mathbf{M}_{n\times n}(\mathbb{Z})$, we define*

$$r(A) = \{ i \mid A(i,j) = 0,\ \forall j \}$$

and

$$c(A) = \{ i \mid A(j,i) = 0,\ \forall j \},$$

and from (2.1), we denote r_n and c_n for given $\mathbb{A}_2$ as $r_n(\mathbb{A}_2) = r(\mathbb{A}_n)$ and $c_n(\mathbb{A}_2) = c(\mathbb{A}_n)$.

Therefore, we define the safe symbol of a matrix, and the conception of safe symbol comes from [34].

Definition 3 *If $A \in \mathbf{M}_{n\times n}(\mathbb{Z})$, we say index i is a safe symbol if*

$$A(j,i) = 1,\ \forall j \in \{1,...,n\} \backslash r(A),$$

and

$$A(i,j) = 1,\ \forall j \in \{1,...,n\} \backslash c(A).$$

Example 4 *If* $\mathbb{A}_2 = \begin{pmatrix} \boxed{1} & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, *then from Definition2 and 3, for $n = 2$, we have*

$$\begin{aligned} r_2 &= \{4\} \\ c_2 &= \{4\} \end{aligned}$$

and

$$i = 1 \text{ is a safe symbol.}$$

2.2 s_n, S_n, R_n and C_n

In this section, s_n, S_n, R_n and C_n are introduced, these four concepts are defined in, and is crucial for our study. First, we define the s_n.

Definition 5 *If $A = (a_{ij})_{i,j=1}^{n} \in \mathbf{M}_{n\times n}(\mathbb{Z})$ has a safe symbol i, we denote $s(A) = a_{ii}$, and from (2.1), we denote s_n for given $\mathbb{A}_2$ as $s_n(\mathbb{A}_2) = s(\mathbb{A}_n)$.*

Next, for (2.1), we define the S_n, R_n and C_n below.

Definition 6 *From (2.1), if $\mathbb{A}_n \in \mathbf{M}_{2^n\times 2^n}(\mathbb{Z})$ has a safe symbol, we denote $S(\mathbb{A}_n) = A_{2;\alpha_S}$, for $\alpha_S \in \{1,2,3,4\}$, be the 2×2 block that $s(\mathbb{A}_n)$ is inside and we denote S_n for given $\mathbb{A}_2$ as $S_n(\mathbb{A}_2) = S(\mathbb{A}_n)$.*

Definition 7 *From (2.1), if $\mathbb{A}_n \in \mathbf{M}_{2^n\times 2^n}(\mathbb{Z})$ has a safe symbol, we define*

$$R(\mathbb{A}_n) = \{A_{2;\alpha} \mid A_{2;\alpha} \text{ and } S(\mathbb{A}_n) \text{ are on the same row}\},$$

$$C(\mathbb{A}_n) = \{A_{2;\alpha} \mid A_{2;\alpha} \text{ and } S(\mathbb{A}_n) \text{ are on the same column}\}$$

and we denote R_n and C_n for given $\mathbb{A}_2$ as $R_n(\mathbb{A}_2) = R(\mathbb{A}_n)$ and $C_n(\mathbb{A}_2) = C(\mathbb{A}_n)$.

Next, from (2.1) and Definition7, if $\mathbb{A}_2$ has a safe symbol, i.e., s_2 exists, we define the proposition below.

Proposition 8 *For* $A_{2;\alpha} = \begin{pmatrix} b_{\alpha 1} & b_{\alpha 2} \\ b_{\alpha 3} & b_{\alpha 4} \end{pmatrix}, \forall \alpha \in \{1,2,3,4\}$, *we say* $A_{2;\alpha}$ *has property R if it satisfied one of the follow situations:* $\forall i \in \{1,2\} \backslash r(A_{2;\alpha})$,

$$\begin{aligned}
(1)\ \textit{if } s_2 &= b_{11},\ \textit{then } A_{2;\alpha}(i,1) = 1, \forall \alpha \in \{1,3\};\\
(2)\ \textit{if } s_2 &= b_{14},\ \textit{then } A_{2;\alpha}(i,2) = 1, \forall \alpha \in \{1,3\};\\
(3)\ \textit{if } s_2 &= b_{41},\ \textit{then } A_{2;\alpha}(i,1) = 1, \forall \alpha \in \{2,4\};\\
(4)\ \textit{if } s_2 &= b_{44},\ \textit{then } A_{2;\alpha}(i,2) = 1, \forall \alpha \in \{2,4\},
\end{aligned}$$

and we say $A_{2;\alpha}$ *has property C if it satisfied one of the follow situations:* $\forall j \in \{1,2\} \backslash c(A_{2;\alpha})$,

$$\begin{aligned}
(1)\ \textit{if } s_2 &= b_{11},\ \textit{then } A_{2;\alpha}(1,j) = 1, \forall \alpha \in \{1,2\};\\
(2)\ \textit{if } s_2 &= b_{14},\ \textit{then } A_{2;\alpha}(2,j) = 1, \forall \alpha \in \{1,2\};\\
(3)\ \textit{if } s_2 &= b_{41},\ \textit{then } A_{2;\alpha}(1,j) = 1, \forall \alpha \in \{3,4\};\\
(4)\ \textit{if } s_2 &= b_{44},\ \textit{then } A_{2;\alpha}(2,j) = 1, \forall \alpha \in \{3,4\}.
\end{aligned}$$

Example 9 *If* $\mathbb{A}_2 = \begin{pmatrix} b_{11} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{pmatrix} = \begin{pmatrix} A_{2;1} & A_{2;2} \\ A_{2;3} & A_{2;4} \end{pmatrix} = \begin{pmatrix} \boxed{1} & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,

then $\mathbb{A}_2$ *has a safe symbol. Therefore, from Definition5, 6, 7 and Proposition9, for* $n = 2$, *we have*

$$s_2 = b_{11},$$

$$S_2 = A_{2;\alpha_s} = A_{2;1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{aligned}
R_2 &= \{A_{2;1}, A_{2;2}\}\\
C_2 &= \{A_{2;1}, A_{2;3}\}.
\end{aligned}$$

and

$$\begin{aligned}
A_{2;\alpha} \textit{ has property } R,\ \forall \alpha &\in \{1,3\}\\
A_{2;\alpha} \textit{ has property } C,\ \forall \alpha &\in \{1,2\}
\end{aligned}$$

3 Main Theorem

In this section, we will formulate the main theorems of our study and proof. First, we define the primitive property for a matrix.

Definition 10 *Let $A \in \mathbf{M}_{n\times n}(\mathbb{Z})$ is called primitive if there exists an integer $k \geq 1$ such that $A^k \geq E_{n\times n}$ (full matrix), and let $\tau(A)$ be the minimum number of such k, i.e.,*

$$\tau(A) \equiv min\left\{k : A^k \geq E_{n\times n}\right\}.$$

In this paper, we use the "generalized" primitive property for a matrix, i.e., let $A \in \mathbf{M}_{n\times n}(\mathbb{Z})$ is called "generalized" primitive if there exists an integer $k \geq 1$ such that

$$A^k(i,j) \geq 1, \forall i \in \{1,...,n\} \backslash r(A), \ \forall j \in \{1,...,n\} \backslash c(A)$$

Before proving the main Theorem, we show lemma first.

Lemma 11 *If $A \in \mathbf{M}_{n\times n}(\mathbb{Z})$ has at least one safe symbol then A is primitive.*
Proof. *By the definition of primitive, it suffices to show that there exists an integer $k \geq 1$ such that*

$$A^k(i,j) \geq 1, \forall i \in \{1,...,n\} \backslash r(A), \ \forall j \in \{1,...,n\} \backslash c(A). \tag{3.1}$$

Since $A \in \mathbf{M}_{n\times n}(\mathbb{Z})$ has at least one safe symbol, we let $A = (a_{ij})_{i,j=1}^{n}$ and index $i = m$ be a safe symbol. Then

$$a_{mj} = 1, \ \forall j \in \{1,...,n\} \backslash c(A), \tag{3.2}$$

and

$$a_{jm} = 1, \ \forall j \in \{1,...,n\} \backslash r(A). \tag{3.3}$$

Indeed, let $A^2 = (\delta_{ij})_{i,j=1}^{n}$, then

$$\delta_{ij} = a_{i1}a_{1j} + a_{i2}a_{2j} + ... + \boxed{a_{im}a_{mj}} + ... + a_{in}a_{nj}. \tag{3.4}$$

From (3.2) *and* (3.3), $\forall i \in \{1,...,n\} \backslash r(A), \forall \ j \in \{1,...,n\} \backslash c(A)$, *we get*

$$a_{im}a_{mj} = 1 \tag{3.5}$$

and from (3.4) *and* (3.5), *we find,* $\forall i \in \{1, ..., n\} \backslash r(A)$, $j \in \{1, ..., n\} \backslash c(A)$,

$$\delta_{ij} \geq 1. \tag{3.6}$$

I.e.,

$$A^k(i,j) \geq 1, \forall i \in \{1, ..., n\} \backslash r(A), \ \forall j \in \{1, ..., n\} \backslash c(A).$$

Thus from (3.1), $k = 2$ *is chose as we want. This complete the proof of Lemma11.* ■

Next, we give $\mathbb{A}_2$ and write it as $\mathbb{A}_2 = \begin{pmatrix} b_{11} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{pmatrix} = \begin{pmatrix} A_{2;1} & A_{2;2} \\ A_{2;3} & A_{2;4} \end{pmatrix}$, where $A_{2;\alpha} \in \mathbf{M}_{2\times 2}(\mathbb{Z})$, $\forall \alpha \in \{1,2,3,4\}$. And we follow the recursive formulae for n-th order transition matrices $\mathbb{A}_n$ from (2.1). Then we prove the main Theorem of this paper.

Theorem 12 *If* $\mathbb{A}_2 \in \mathbf{M}_{4\times 4}(\mathbb{Z})$ *satisfies the following properties*

(1) $\mathbb{A}_2$ *and* $\mathbb{A}_3$ *have safe symbols.*

(2) There exist sequences $\{\beta_k\}_{k=0}^{q}$ *such that for all* $k = 0, ..., q$ *and* $q \geq 2$, *we have*

(a) $\beta_k \in \{1, 4\}$;

(b) there exists $0 \leq m \leq q$, *such that* $\beta_m = \beta_0$ *or* $\beta_{m+1} = \beta_1$;

(c) $s_2 = b_{\beta_0\beta_1}$ *and* $s_3 = b_{\beta_0\beta_1} b_{\beta_1\beta_2} = s_2 b_{\beta_1\beta_2}$;

Then $\mathbb{A}_n$ *is primitive for all* $n \geq 2$.

Proof. We prove it by induction, since $\mathbb{A}_2$ has a safe symbol and $s_2 = b_{\beta_0\beta_1}$, then

$$\mathbb{A}_2(s_2, s_2) = 1, \tag{3.7}$$

$$\mathbb{A}_2(s_2, j) = 1, \ \forall j \in \left\{1...2^2\right\} \backslash c_2, \tag{3.8}$$

and

$$\mathbb{A}_2(i, s_2) = 1, \ \forall i \in \left\{1...2^2\right\} \backslash r_2. \tag{3.9}$$

Therefore

$$S_2 = A_{2;\beta_0}, \tag{3.10}$$

$$R_2 = \{A_{2;\alpha} \mid \alpha \in \{1,2\} \text{ or } \{3,4\}\}, \text{ where } A_{2;\alpha} \text{ has property } C \tag{3.11}$$

and

$$C_2 = \{A_{2;\alpha} \mid \alpha \in \{1,3\} \text{ or } \{2,4\}\}, \text{ where } A_{2;\alpha} \text{ has property } R. \tag{3.12}$$

Since $\mathbb{A}_3$ has a safe symbol and $s_3 = b_{\beta_0\beta_1} b_{\beta_1\beta_2} = s_2 b_{\beta_1\beta_2}$, then

$$\mathbb{A}_3(s_3, s_3) = 1, \tag{3.13}$$

$$\mathbb{A}_3(s_3, j) = 1, \ \forall j \in \{1 ... 2^3\} \backslash c_3, \tag{3.14}$$

and

$$\mathbb{A}_3(i, s_3) = 1, \ \forall i \in \{1 ... 2^3\} \backslash r_3. \tag{3.15}$$

Therefore

$$S_3 = A_{2;\beta_1}, \tag{3.16}$$

$$R_3 = \{A_{2;\alpha} \mid \alpha \in \{1,2\} \text{ or } \{3,4\}\}, \text{ where } A_{2;\alpha} \text{ has property } C \tag{3.17}$$

and

$$C_3 = \{A_{2;\alpha} \mid \alpha \in \{1,3\} \text{ or } \{2,4\}\}, \text{ where } A_{2;\alpha} \text{ has property } R. \tag{3.18}$$

Next, we show that s_4 exists, such that $\mathbb{A}_4(s_4, s_4) = 1$, $\mathbb{A}_4(s_4, j) = 1$, $\forall j \in \{1 ... 2^4\} \backslash c_4$, and $\mathbb{A}_4(i, s_4) = 1$, $\forall i \in \{1 ... 2^4\} \backslash r_4$.

By (2.1), we perform $\mathbb{A}_3$ and $\mathbb{A}_4$ for given $\mathbb{A}_2$ as follows

$$\mathbb{A}_3 = \begin{pmatrix} A_{3;1} & A_{3;2} \\ A_{3;3} & A_{3;4} \end{pmatrix},$$

where

$$\begin{aligned} \mathbb{A}_{3;\alpha} &= \begin{pmatrix} b_{\alpha 1} A_{2;1} & b_{\alpha 2} A_{2;2} \\ b_{\alpha 3} A_{2;3} & b_{\alpha 4} A_{2;4} \end{pmatrix} \\ &= \begin{pmatrix} b_{\alpha 1} b_{11} & b_{\alpha 1} b_{12} & b_{\alpha 2} b_{21} & b_{\alpha 2} b_{22} \\ b_{\alpha 1} b_{13} & b_{\alpha 1} b_{14} & b_{\alpha 2} b_{23} & b_{\alpha 2} b_{24} \\ b_{\alpha 3} b_{31} & b_{\alpha 3} b_{32} & b_{\alpha 4} b_{41} & b_{\alpha 4} b_{42} \\ b_{\alpha 3} b_{33} & b_{\alpha 3} b_{34} & b_{\alpha 4} b_{43} & b_{\alpha 4} b_{44} \end{pmatrix}, \end{aligned}$$

for $\alpha \in \{1, 2, 3, 4\}$, and

$$\mathbb{A}_4 = \begin{pmatrix} \mathbb{A}_{4;1} & \mathbb{A}_{4;2} \\ \mathbb{A}_{4;3} & \mathbb{A}_{4;4} \end{pmatrix},$$

where

$$\mathbb{A}_{4;\alpha} = \begin{pmatrix} b_{\alpha 1}A_{3;1} & b_{\alpha 2}A_{3;2} \\ b_{\alpha 3}A_{3;3} & b_{\alpha 4}A_{3;4} \end{pmatrix}$$

$$= \begin{pmatrix} b_{\alpha 1}b_{11}A_{2;1} & b_{\alpha 1}b_{12}A_{2;2} & b_{\alpha 2}b_{21}A_{2;1} & b_{\alpha 2}b_{22}A_{2;2} \\ b_{\alpha 1}b_{13}A_{2;3} & b_{\alpha 1}b_{14}A_{2;4} & b_{\alpha 2}b_{23}A_{2;3} & b_{\alpha 2}b_{24}A_{2;4} \\ b_{\alpha 3}b_{31}A_{2;1} & b_{\alpha 3}b_{32}A_{2;2} & b_{\alpha 4}b_{41}A_{2;1} & b_{\alpha 4}b_{42}A_{2;2} \\ b_{\alpha 3}b_{33}A_{2;3} & b_{\alpha 3}b_{34}A_{2;4} & b_{\alpha 4}b_{43}A_{2;3} & b_{\alpha 4}b_{44}A_{2;4} \end{pmatrix}, \qquad (3.19)$$

for $\alpha \in \{1, 2, 3, 4\}$.

Next, we use the conditions (2)-(a)~(2)-(b) to take the different s_4 in the follow situations:

(1) If $\beta_1 = \beta_0$, then we take

$$s_4 = b_{\beta_0\beta_0}b_{\beta_0\beta_0}b_{\beta_0\beta_0} = s_3 b_{\beta_0\beta_0}.$$

so

$$S_2 = S_3 = S_4 = A_{2;\beta_0},$$

and

$$\begin{aligned} R_2 &= R_3 = R_4; \\ C_2 &= C_3 = C_4, \end{aligned}$$

(2) If $\beta_0 \neq \beta_1$ and $\beta_1 \neq \beta_2$ $(i.e., \ \beta_2 = \beta_0)$, then we take

$$s_4 = b_{\beta_0\beta_1}b_{\beta_1\beta_0}b_{\beta_0\beta_1} = s_3 b_{\beta_0\beta_1}.$$

so

$$\begin{aligned} S_2 &= A_{2;\beta_0}; \\ S_3 &= A_{2;\beta_1}; \\ S_4 &= A_{2;\beta_0}, \end{aligned}$$

and

$$\begin{aligned} R_2 &= R_4 \neq R_3; \\ C_2 &= C_4 \neq C_3, \end{aligned}$$

(3) If $\beta_0 \neq \beta_1$ and $\beta_2 = \beta_1$, then we take

$$s_4 = b_{\beta_0\beta_1}b_{\beta_1\beta_1}b_{\beta_1\beta_1} = s_3 b_{\beta_1\beta_1}.$$

so

$$\begin{aligned} S_2 &= A_{2;\beta_0}, \\ S_3 &= S_4 = A_{2;\beta_1}, \end{aligned}$$

and

$$\begin{aligned} R_2 &\neq R_3,\ R_3 = R_4; \\ C_2 &\neq C_3,\ C_3 = C_4, \end{aligned}$$

Then from (3.7) and (3.13), we get $\mathbb{A}_4(s_4, s_4) = 1$, from (3.14), (3.19) and (3.11) or (3.17), we get $\mathbb{A}_4(s_4, j) = 1$, $\forall j \in \{1...2^4\} \backslash c_4$, and from (3.15), (3.19) and (3.12) or (3.18), we get $\mathbb{A}_4(i, s_4) = 1$, $\forall i \in \{1...2^4\} \backslash r_4$, i.e., $\mathbb{A}_4$ has at least one safe symbol.

Now, we assume $\mathbb{A}_{n-2}$ and $\mathbb{A}_{n-1}$ have safe symbols and $s_{n-2} = s_{n-3}b_{\beta_0\beta_1}$ and $s_{n-1} = s_{n-3}b_{\beta_0\beta_1}b_{\beta_1\beta_2} = s_{n-2}b_{\beta_1\beta_2}$, in the same fashion of proof of $\mathbb{A}_4$ has at least one safe symbol, since $\mathbb{A}_{n-2}$ has a safe symbol and $s_2 = s_{n-3}b_{\beta_0\beta_1}$, then

$$\mathbb{A}_{n-2}(s_{n-2}, s_{n-2}) = 1, \tag{3.20}$$

$$\mathbb{A}_{n-2}(s_{n-2}, j) = 1,\ \forall j \in \left\{1...2^{n-2}\right\} \backslash c_{n-2}, \tag{3.21}$$

and

$$\mathbb{A}_{n-2}(i, s_{n-2}) = 1,\ \forall i \in \left\{1...2^{n-2}\right\} \backslash r_{n-2}. \tag{3.22}$$

Therefore

$$S_{n-2} = A_{2;\beta_0}, \tag{3.23}$$

$$R_{n-2} = \{A_{2;\alpha} \mid \alpha \in \{1,2\} \text{ or } \{3,4\}\},\ \text{where } A_{2;\alpha} \text{ has property } C \tag{3.24}$$

and

$$C_{n-2} = \{A_{2;\alpha} \mid \alpha \in \{1,3\} \text{ or } \{2,4\}\},\ \text{where } A_{2;\alpha} \text{ has property } R. \tag{3.25}$$

Since $\mathbb{A}_{n-1}$ has a safe symbol and $s_{n-1} = s_{n-3}b_{\beta_0\beta_1}b_{\beta_1\beta_2} = s_{n-2}b_{\beta_1\beta_2}$, then

$$\mathbb{A}_{n-1}(s_{n-1}, s_{n-1}) = 1, \tag{3.26}$$

$$\mathbb{A}_{n-1}(s_{n-1}, j) = 1,\ \forall j \in \left\{1...2^{n-1}\right\} \backslash c_{n-1}, \tag{3.27}$$

and

$$\mathbb{A}_{n-1}(i, s_{n-1}) = 1,\ \forall i \in \left\{1...2^{n-1}\right\} \backslash r_{n-1}. \tag{3.28}$$

Therefore

$$S_{n-1} = A_{2;\beta_1}, \tag{3.29}$$

$$R_{n-1} = \{A_{2;\alpha} \mid \alpha \in \{1,2\} \text{ or } \{3,4\}\}, \text{ where } A_{2;\alpha} \text{ has property } C \tag{3.30}$$

and

$$C_{n-1} = \{A_{2;\alpha} \mid \alpha \in \{1,3\} \text{ or } \{2,4\}\}, \text{ where } A_{2;\alpha} \text{ has property } R. \tag{3.31}$$

Next, we show that s_n exists, such that $\mathbb{A}_n(s_n, s_n) = 1$, $\mathbb{A}_n(s_n, j) = 1$, $\forall j \in \{1...2^n\} \setminus c_n$, and $\mathbb{A}_n(i, s_n) = 1$, $\forall i \in \{1...2^n\} \setminus r_n$.

By (2.1), we perform $\mathbb{A}_{n-1}$ and $\mathbb{A}_n$ for given $\mathbb{A}_{n-2}$ as follows

$$\mathbb{A}_{n-1} = \begin{pmatrix} A_{n-1;1} & A_{n-1;2} \\ A_{n-1;3} & A_{n-1;4} \end{pmatrix},$$

where

$$\begin{aligned} \mathbb{A}_{n-1;\alpha} &= \begin{pmatrix} b_{\alpha 1} A_{n-2;1} & b_{\alpha 2} A_{n-2;2} \\ b_{\alpha 3} A_{n-2;3} & b_{\alpha 4} A_{n-2;4} \end{pmatrix} \\ &= \begin{pmatrix} b_{\alpha 1} b_{11} A_{n-3;1} & b_{\alpha 1} b_{12} A_{n-3;2} & b_{\alpha 2} b_{21} A_{n-3;1} & b_{\alpha 2} b_{22} A_{n-3;2} \\ b_{\alpha 1} b_{13} A_{n-3;3} & b_{\alpha 1} b_{14} A_{n-3;4} & b_{\alpha 2} b_{23} A_{n-3;3} & b_{\alpha 2} b_{24} A_{n-3;4} \\ b_{\alpha 3} b_{31} A_{n-3;1} & b_{\alpha 3} b_{32} A_{n-3;2} & b_{\alpha 4} b_{41} A_{n-3;1} & b_{\alpha 4} b_{42} A_{n-3;2} \\ b_{\alpha 3} b_{33} A_{n-3;3} & b_{\alpha 3} b_{34} A_{n-3;4} & b_{\alpha 4} b_{43} A_{n-3;3} & b_{\alpha 4} b_{44} A_{n-3;4} \end{pmatrix}, \end{aligned}$$

for $\alpha \in \{1,2,3,4\}$, and

$$\mathbb{A}_n = \begin{pmatrix} \mathbb{A}_{n;1} & \mathbb{A}_{n;2} \\ \mathbb{A}_{n;3} & \mathbb{A}_{n;4} \end{pmatrix},$$

where

$$\begin{aligned} \mathbb{A}_{n;\alpha} &= \begin{pmatrix} b_{\alpha 1} A_{n-1;1} & b_{\alpha 2} A_{n-1;2} \\ b_{\alpha 3} A_{n-1;3} & b_{\alpha 4} A_{n-1;4} \end{pmatrix} \\ &= \begin{pmatrix} b_{\alpha 1} b_{11} A_{n-2;1} & b_{\alpha 1} b_{12} A_{n-2;2} & b_{\alpha 2} b_{21} A_{n-2;1} & b_{\alpha 2} b_{22} A_{n-2;2} \\ b_{\alpha 1} b_{13} A_{n-2;3} & b_{\alpha 1} b_{14} A_{n-2;4} & b_{\alpha 2} b_{23} A_{n-2;3} & b_{\alpha 2} b_{24} A_{n-2;4} \\ b_{\alpha 3} b_{31} A_{n-2;1} & b_{\alpha 3} b_{32} A_{n-2;2} & b_{\alpha 4} b_{41} A_{n-2;1} & b_{\alpha 4} b_{42} A_{n-2;2} \\ b_{\alpha 3} b_{33} A_{n-2;3} & b_{\alpha 3} b_{34} A_{n-2;4} & b_{\alpha 4} b_{43} A_{n-2;3} & b_{\alpha 4} b_{44} A_{n-2;4} \end{pmatrix}, \end{aligned} \tag{3.32}$$

for $\alpha \in \{1,2,3,4\}$.

Next, we use the conditions (2)-(a)~(2)-(b) to take the different s_n in the follow situations:

(1) If $\beta_1 = \beta_0$, then we take

$$s_n = s_{n-3} b_{\beta_0\beta_0} b_{\beta_0\beta_0} b_{\beta_0\beta_0} = s_{n-1} b_{\beta_0\beta_0}.$$

so

$$S_{n-2} = S_{n-1} = S_n = A_{2;\beta_0},$$

and

$$\begin{aligned} R_{n-2} &= R_{n-1} = R_n; \\ C_{n-2} &= C_{n-1} = C_n, \end{aligned}$$

(2) If $\beta_0 \neq \beta_1$ and $\beta_1 \neq \beta_2$ $(i.e.,\ \beta_2 = \beta_0)$, then we take

$$s_n = s_{n-3} b_{\beta_0\beta_1} b_{\beta_1\beta_0} b_{\beta_0\beta_1} = s_{n-1} b_{\beta_0\beta_1}.$$

so

$$\begin{aligned} S_{n-2} &= A_{2;\beta_0}; \\ S_{n-1} &= A_{2;\beta_1}; \\ S_n &= A_{2;\beta_0}, \end{aligned}$$

and

$$\begin{aligned} R_{n-2} &= R_n \neq R_{n-1}; \\ C_{n-2} &= C_n \neq C_{n-1}, \end{aligned}$$

(3) If $\beta_0 \neq \beta_1$ and $\beta_2 = \beta_1$, then we take

$$s_n = s_{n-3} b_{\beta_0\beta_1} b_{\beta_1\beta_1} b_{\beta_1\beta_1} = s_{n-1} b_{\beta_1\beta_1}.$$

so

$$\begin{aligned} S_{n-2} &= A_{2;\beta_0}, \\ S_{n-1} &= S_n = A_{2;\beta_1}, \end{aligned}$$

and

$$\begin{aligned} R_{n-2} &\neq R_{n-1},\ R_{n-1} = R_n; \\ C_{n-2} &\neq C_{n-1},\ C_{n-1} = C_n, \end{aligned}$$

Then from (3.20) and (3.26), we get $\mathbb{A}_n(s_n, s_n) = 1$, from (3.27), (3.32) and (3.24) or (3.30), we get $\mathbb{A}_n(s_n, j) = 1$, $\forall j \in \{1...2^n\} \setminus c_n$, and from (3.28), (3.32) and (3.25) or (3.31), we get $\mathbb{A}_n(i, s_n) = 1$, $\forall i \in \{1...2^n\} \setminus r_n$, i.e., $\mathbb{A}_n$ has at least one safe symbol. Therefore, Lemma11 is applied to show $\mathbb{A}_n$ is primitive. This complete the proof of Theorem12. ■

4 Examples for safe symbol existing Case

For the section, we will show some examples in [33] to check primitivity of $\mathbb{A}_n$, $\forall n \geq 2$. In [33], the authors prove the Theorem related to the *extensively weak mixing* property. We will use the conditions of the *extensively weak mixing* property to find H and V. Then we use some results in [33] to construct horizontal transition matrix $\mathbb{H}_2$ and vertical transition matrix $\mathbb{V}_2$. And to find the cases that if $\mathbb{A}_2 = \mathbb{H}_2$ or $\mathbb{A}_2 = \mathbb{V}_2$ can all let $\mathbb{A}_n$ be primitive for all $n \geq 2$.

4.1 Find H and V

We first state the Theorem related to the *extensively weak mixing* property in [33] .

Theorem 13 (Theorem 5.8 of [33]) *Suppose* $\Lambda = \Omega\left(A_1, ..., A_v\right)$ *is a subshift of finite type and each* A_i *is a* $p \times p$ *matrix. Then* Λ *is extensively weak mixing if and only if for all* $\xi \in \Xi\left\{-1, 1\right\}^v$ *there exists* n *satisfying*

$$1 \leq n \leq p^2 - 2p + 2 \tag{4.1}$$

and

$$A_1^{(\xi_1)n} ... A_v^{(\xi_v)n} > 0. \tag{4.2}$$

For simplicity, we only consider two symbols which are given on 2×2 lattice $Z_{2\times 2}$. So we just suppose $\Lambda = \Omega\left(H, V\right)$, $p = 2$ and $\xi = 1$. Since $\xi = 1$, in [33], we know that the matrix subshift $\Lambda = \Omega\left(H, V\right)$ is of finite type if and only if $HV = VH$.

So by the above statement, we can get three conditions below, such that $\Lambda = \Omega\left(H, V\right)$ is extensively weak mixing.

Condition 1 $HV = VH$

Condition 2 $1 \leq n \leq 2$

Condition 3 $H^n V^n > 0$

From the above conditions, we can find eleven cases for H and V.

Case 1 $H = V = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

Case 2 $H = V = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

Case 3 $\quad H = V = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Case 4 $\quad H = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Case 5 $\quad H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

Case 6 $\quad H = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Case 7 $\quad H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, V = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

Case 8 $\quad H = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Case 9 $\quad H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Case 10 $\quad H = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Case 11 $\quad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, V = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Next, we will take these cases of H and V to construct horizontal transition matrix $\mathbb{H}_2$ and vertical transition matrix $\mathbb{V}_2$.

4.2 Construct $\mathbb{H}_2$ and $\mathbb{V}_2$

we consider first some results in [7], which are crucial to the constructs of $\mathbb{H}_2$ and $\mathbb{V}_2$. We begin with 1×2 column pattern h_i,

$$h_i = \begin{pmatrix} u_2 \\ u_1 \end{pmatrix} \text{ or } \begin{array}{|c|}\hline u_2 \\ \hline u_1 \\ \hline \end{array} \tag{4.3}$$

and

$$i = 1 + 2u_1 + u_2. \tag{4.4}$$

A 2×2 pattern $U = (u_{\alpha_1 \alpha_2})$ can now be obtained by a horizontal direct sum of two 1×2 pattern, i.e.,

$$\begin{aligned} h_{i_1 i_2} &\equiv h_{i_1} \oplus h_{i_2} \\ &\equiv \begin{pmatrix} u_{12} & u_{22} \\ u_{11} & u_{21} \end{pmatrix} \text{ or } \begin{array}{|c|c|}\hline u_{12} & u_{22} \\ \hline u_{11} & u_{21} \\ \hline \end{array}, \end{aligned} \tag{4.5}$$

where

$$i_k = 1 + 2u_{k1} + u_{k2}, \quad 1 \le k \le 2. \tag{4.6}$$

Therefore, the complete set of all $16(= 2^{2\times 2})$ 2×2 patterns in $\Sigma_{2\times 2}$ can be listed by a 4×4 matrix $H_2 = (h_{i_1 i_2})$ with 2×2 pattern $h_{i_1 i_2}$ as its entries in

$$\begin{array}{cc}
 & \begin{array}{cccc} \begin{array}{|c|}\hline 0\\\hline 0\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 0\\\hline\end{array} & \begin{array}{|c|}\hline 0\\\hline 1\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array} \end{array} \\
\begin{array}{c} \begin{array}{|c|}\hline 0\\\hline 0\\\hline\end{array} \\ \begin{array}{|c|}\hline 1\\\hline 0\\\hline\end{array} \\ \begin{array}{|c|}\hline 0\\\hline 1\\\hline\end{array} \\ \begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array} \end{array} &
\left(\begin{array}{cccc}
\begin{array}{|c|c|}\hline 0&0\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&0\\\hline 0&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 0&1\\\hline\end{array} \\
\begin{array}{|c|c|}\hline 1&0\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline 0&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 0&1\\\hline\end{array} \\
\begin{array}{|c|c|}\hline 0&0\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&0\\\hline 1&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 1&1\\\hline\end{array} \\
\begin{array}{|c|c|}\hline 1&0\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline 1&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 1&1\\\hline\end{array}
\end{array}\right)
\end{array} \tag{4.7}$$

Similarly, a 2×2 pattern can also be viewed as a vertical direct sum of two 2×1 patterns, i.e.,

$$v_{j_1 j_2} \equiv v_{j_1} \oplus v_{j_2}, \tag{4.8}$$

where

$$v_{j_l} = \begin{pmatrix} u_{1l} & u_{2l} \end{pmatrix} \text{ or } \begin{array}{|c|c|}\hline u_{1l} & u_{2l} \\\hline\end{array}, \tag{4.9}$$

and

$$j_l = 1 + 2u_{1l} + u_{2l}, \quad 1 \le l \le 2. \tag{4.10}$$

A 4×4 matrix $V_2 = (v_{j_1 j_2})$ can also be obtained for $\Sigma_{2\times 2}$, i.e., we have

$$
\begin{array}{cc}
 & \begin{array}{cccc}
\begin{array}{|c|c|}\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{|c|c|}\hline 0&0\\\hline\end{array} \\ \\ \begin{array}{|c|c|}\hline 0&1\\\hline\end{array} \\ \\ \begin{array}{|c|c|}\hline 1&0\\\hline\end{array} \\ \\ \begin{array}{|c|c|}\hline 1&1\\\hline\end{array}
\end{array} &
\left(\begin{array}{cccc}
\begin{array}{|c|c|}\hline 0&0\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 0&0\\\hline\end{array} \\
\begin{array}{|c|c|}\hline 0&0\\\hline 0&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 0&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline 0&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 0&1\\\hline\end{array} \\
\begin{array}{|c|c|}\hline 0&0\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 1&0\\\hline\end{array} \\
\begin{array}{|c|c|}\hline 0&0\\\hline 1&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 1&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline 1&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 1&1\\\hline\end{array}
\end{array}\right)
\end{array}
\tag{4.11}
$$

From above, H_2 can also be represented by $v_{j_1 j_2}$ as

$$
\mathbb{H}_2 = \begin{pmatrix} v_{11} & v_{12} & v_{21} & v_{22} \\ v_{13} & v_{14} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{41} & v_{42} \\ v_{33} & v_{34} & v_{43} & v_{44} \end{pmatrix}. \tag{4.12}
$$

In (4.12), the indices $j_1 j_2$ are arranged by two Z-map successively, as

$$
\begin{pmatrix} 1 & \rightarrow & 2 \\ & \swarrow & \\ 3 & \rightarrow & 4 \end{pmatrix}, \tag{4.13}
$$

i.e., the path from 1 to 4 in (4.13) is Z shaped and is then called a Z-map. More precisely, $\mathbb{H}_2$ can be decomposed by

$$
\mathbb{H}_2 = \begin{pmatrix} V_{2;1} & V_{2;2} \\ V_{2;3} & V_{2;4} \end{pmatrix} \tag{4.14}
$$

and

$$
V_{2;K} = \begin{pmatrix} v_{k1} & v_{k2} \\ v_{k3} & v_{k4} \end{pmatrix}. \tag{4.15}
$$

Where, $\mathbb{H}_2$ is arranged by a Z-map $(V_{2;K})$ in (4.14) and each $V_{2;K}$ is also arranged by a Z-map (v_{kl}) in (4.15). Therefore, the indices of v in (4.12) consist of two Z-map.

Then, we use the above mention to get the value of H_2 and V_2. We first let

$$H = \begin{array}{|c|}\hline 0\\\hline 1\\\hline\end{array}\overset{\begin{array}{|c|c|}\hline 0&1\\\hline\end{array}}{\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}}, \quad V = \begin{array}{|c|}\hline 0\\\hline 1\\\hline\end{array}\overset{\begin{array}{|c|c|}\hline 0&1\\\hline\end{array}}{\begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix}} \tag{4.16}$$

where

$$a_i, b_j \in \{0,1\}, \text{ for } i, j = 1 \text{ to } 4. \tag{4.17}$$

So by (4.3) , (4.9)

$$\begin{aligned} h_1 &= \begin{array}{|c|}\hline 0\\\hline 0\\\hline\end{array} = b_1, \ h_2 = \begin{array}{|c|}\hline 1\\\hline 0\\\hline\end{array} = b_2, \\ h_3 &= \begin{array}{|c|}\hline 0\\\hline 1\\\hline\end{array} = b_3, \ h_4 = \begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array} = b_4 \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} v_1 &= \begin{array}{|c|c|}\hline 0&0\\\hline\end{array} = a_1, \ v_2 = \begin{array}{|c|c|}\hline 0&1\\\hline\end{array} = a_2, \\ v_3 &= \begin{array}{|c|c|}\hline 1&0\\\hline\end{array} = a_3, \ v_4 = \begin{array}{|c|c|}\hline 1&1\\\hline\end{array} = a_4 \end{aligned} \tag{4.19}$$

But by (4.7) and (4.11), if we want to get the value of H_2 and V_2, we must consider the value of H and V at the same time. Therefore,

$$\begin{aligned} H_2 &= \begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{pmatrix} \\ &= \begin{pmatrix} a_1a_1b_1b_1 & a_2a_1b_1b_2 & a_1a_2b_1b_3 & a_2a_2b_1b_4 \\ a_3a_1b_2b_1 & a_4a_1b_2b_2 & a_3a_2b_2b_3 & a_3a_2b_2b_4 \\ a_1a_3b_3b_1 & a_2a_3b_3b_2 & a_1a_4b_3b_3 & a_2a_4b_3b_4 \\ a_3a_3b_4b_1 & a_4a_3b_4b_2 & a_3a_4b_4b_3 & a_4a_4b_4b_4 \end{pmatrix} \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} V_2 &= \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{pmatrix} \\ &= \begin{pmatrix} a_1a_1b_1b_1 & a_2a_1b_1b_2 & a_3a_1b_2b_1 & a_4a_1b_2b_2 \\ a_1a_2b_1b_3 & a_2a_2b_1b_4 & a_3a_2b_2b_3 & a_3a_2b_2b_4 \\ a_1a_3b_3b_1 & a_2a_3b_3b_2 & a_4a_3b_4b_1 & a_4a_3b_4b_2 \\ a_1a_4b_3b_3 & a_2a_4b_3b_4 & a_3a_4b_4b_3 & a_4a_4b_4b_4 \end{pmatrix} \end{aligned} \tag{4.21}$$

From above, we can also find

$$\mathbb{H}_2 = \begin{pmatrix} v_{11} & v_{12} & v_{21} & v_{22} \\ v_{13} & v_{14} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{41} & v_{42} \\ v_{33} & v_{34} & v_{43} & v_{44} \end{pmatrix} = \begin{pmatrix} V_{2;1} & V_{2;2} \\ V_{2;3} & V_{2;4} \end{pmatrix} \tag{4.22}$$

and

$$\mathbb{V}_2 = \begin{pmatrix} h_{11} & h_{12} & h_{21} & h_{22} \\ h_{13} & h_{14} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{41} & h_{42} \\ h_{33} & h_{34} & h_{43} & h_{44} \end{pmatrix} = \begin{pmatrix} H_{2;1} & H_{2;2} \\ H_{2;3} & H_{2;4} \end{pmatrix} \tag{4.23}$$

Therefore we get $\mathbb{V}_2$ for given $\mathbb{H}_2$.

Next, for those eleven cases of H and V in **section4.1**. We also use the introduced method above to get eleven cases below for $\mathbb{H}_2$ and $\mathbb{V}_2$.

Case 1 If $H = V = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

then $\mathbb{H}_2 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\mathbb{V}_2 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Case 2 If $H = V = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

then $\mathbb{H}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$, $\mathbb{V}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$.

Case 3 If $H = V = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

then $\mathbb{H}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, $\mathbb{V}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

Case 4 If $H = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

then $\mathbb{H}_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, $\mathbb{V}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Case 5 If $H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $V = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

then $\mathbb{H}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\mathbb{V}_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Case 6 If $H = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

then $\mathbb{H}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$, $\mathbb{V}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Case 7 If $H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $V = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

then $\mathbb{H}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\mathbb{V}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$.

Case 8 If $H = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

then $\mathbb{H}_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$, $\mathbb{V}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Case 9 If $H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $V = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

then $\mathbb{H}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\mathbb{V}_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$.

Case 10 If $H = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

then $\mathbb{H}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\mathbb{V}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Case 11 If $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $V = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

then $\mathbb{H}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, $\mathbb{V}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Next, we want to check that for these cases, if $\mathbb{A}_2 = \mathbb{H}_2$ or $\mathbb{A}_2 = \mathbb{V}_2$, then $\mathbb{A}_n$ is primitive for all $n \geq 2$.

4.3 Checking all cases in section4.2 can let $\mathbb{A}_n$ be primitive for all $n \geq 2$

We separate these cases into four subsections.

4.3.1 $s_2 = b_{11}$ or b_{44} and one column and one row of $\mathbb{A}$ are all zero

We consider the case1 and case2 in **section4.2**.

Example 14 *Consider* $\mathbb{A}_2 = \begin{pmatrix} \boxed{b_{11}} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{pmatrix} = \mathbb{H}_2 = \begin{pmatrix} \boxed{1} & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

By (2.1), *it is easily checked that*

(1) $\mathbb{A}_2$ *and* $\mathbb{A}_3 = \begin{pmatrix} \boxed{1} & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ *have safe symbols.*

(2) there exists

$$\beta_0 = 1,\ \beta_1 = 1,\ \beta_2 = 1$$

and

$$s_2 = b_{11},\ s_3 = b_{11}b_{11} = s_2 b_{11}$$

such that (2)-(a)~(2)-(c) of Theorem12 hold, then Theorem12 is applied to show that $\mathbb{A}_n$ *is primitive for all* $n \geq 2$.

Example 15 *Consider* $\mathbb{A}_2 = \begin{pmatrix} b_{11} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & \boxed{b_{44}} \end{pmatrix} = \mathbb{H}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & \boxed{1} \end{pmatrix}.$

By (2.1) *, it is easily checked that*

(1) $\mathbb{A}_2$ *and* $\mathbb{A}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & \boxed{1} \end{pmatrix}$ *have safe symbols.*

(2) there exists

$$\beta_0 = 4,\ \beta_1 = 4,\ \beta_2 = 4$$

and

$$s_2 = b_{44},\ s_3 = b_{44}b_{44} = s_2 b_{44}$$

such that (2)-(a)˜(2)-(c) of Theorem12 hold, then Theorem12 is applied to show that $\mathbb{A}_n$ *is primitive for all* $n \geq 2$*.*

4.3.2 $\mathbb{A}$ is full matrix

We consider the case3 in **section4.2**.

Example 16 *Consider* $\mathbb{A}_2 = \begin{pmatrix} \boxed{b_{11}} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{pmatrix} = \mathbb{H}_2 = \begin{pmatrix} \boxed{1} & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$

By (2.1) *, it is easily checked that*

(1) $\mathbb{A}_2$ *and* $\mathbb{A}_3 = \begin{pmatrix} \boxed{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ *have safe symbols.*

(2) there exists

$$\beta_0 = 1,\ \beta_1 = 1,\ \beta_2 = 1$$

and

$$s_2 = b_{11},\ s_3 = b_{11}b_{11} = s_2 b_{11}$$

such that (2)-(a)˜(2)-(c) of Theorem12 hold, then Theorem12 is applied to show that $\mathbb{A}_n$ is primitive for all $n \geq 2$.

4.3.3 $s_2 = b_{11}$ or b_{44} and two column and row of $\mathbb{A}$ are all zero

We consider the case4˜case9 in **section4.2**.

Example 17 *Consider*

$$\begin{aligned}
\mathbb{A}_2 &= \begin{pmatrix} \boxed{b_{11}} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{pmatrix} \\
&= \text{one of } \mathbb{H}_2 \text{ and } \mathbb{V}_2 = \begin{pmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

By (2.1) , *it is easily checked that*

(1) $\mathbb{A}_2$ *and* $\mathbb{A}_3 = \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ *have safe symbols.*

(2) there exists

$$\beta_0 = 1,\ \beta_1 = 1,\ \beta_2 = 1$$

and

$$s_2 = b_{11},\ s_3 = b_{11}b_{11} = s_2 b_{11}$$

such that (2)-(a)˜(2)-(c) of Theorem12 hold, then Theorem12 is applied to show that $\mathbb{A}_n$ is primitive for all $n \geq 2$.

Example 18 *Consider*

$$\mathbb{A}_2 = \begin{pmatrix} b_{11} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & \boxed{b_{44}} \end{pmatrix}$$

$$= \text{one of } \mathbb{H}_2 \text{ and } \mathbb{V}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \boxed{1} \end{pmatrix}.$$

By (2.1), *it is easily checked that*

(1) $\mathbb{A}_2$ *and* $\mathbb{A}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix}$ *have safe symbols.*

(2) there exists

$$\beta_0 = 4, \ \beta_1 = 4, \ \beta_2 = 4$$

and

$$s_2 = b_{44}, \ s_3 = b_{44}b_{44} = s_2 b_{44}$$

such that (2)-(a)~(2)-(c) of Theorem12 hold, then Theorem12 is applied to show that $\mathbb{A}_n$ *is primitive for all* $n \geq 2$.

Example 19 *Consider*

$$\mathbb{A}_2 = \begin{pmatrix} \boxed{b_{11}} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{pmatrix}$$

$$= \text{one of } \mathbb{H}_2 \text{ and } \mathbb{V}_2 = \begin{pmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

By (2.1) *, it is easily checked that*

(1) $\mathbb{A}_2$ *and* $\mathbb{A}_3 = \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ *have safe symbols.*

(2) there exists

$$\beta_0 = 1,\ \beta_1 = 1,\ \beta_2 = 1$$

and

$$s_2 = b_{11},\ s_3 = b_{11}b_{11} = s_2 b_{11}$$

such that (2)-(a)˜(2)-(c) of Theorem12 hold, then Theorem12 is applied to show that $\mathbb{A}_n$ *is primitive for all* $n \geq 2$.

4.3.4 $s_2 = b_{14},\ s_3 = b_{14}b_{44}$ and two column and row of $\mathbb{A}$ are all zero

We consider the case10 and case11 in **section4.2**.

Example 20 *Consider*

$$\begin{aligned} \mathbb{A}_2 &= \begin{pmatrix} b_{11} & b_{12} & b_{21} & b_{22} \\ b_{13} & \boxed{b_{14}} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{pmatrix} \\ &= \text{one of } \mathbb{H}_2 \text{ and } \mathbb{V}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

By (2.1) *, it is easily checked that*

(1) $\mathbb{A}_2$ *and* $\mathbb{A}_3 = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&\boxed{1}&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}$ *have safe symbols.*

(2) there exists

$$\beta_0 = 1,\ \beta_1 = 4,\ \beta_2 = 1$$

and

$$s_2 = b_{14},\ s_3 = b_{14}b_{41} = s_2 b_{41}$$

such that (2)-(a)~(2)-(c) of Theorem12 hold, then Theorem12 is applied to show that $\mathbb{A}_n$ *is primitive for all* $n \geq 2$.

4.4 Conclusion

In the section we describe some Remarks and examples related to the primitivity of $\mathbb{A}_n$.

Remark 21 *For the examples in the* ***section4.3****, we can find all cases for* $\mathbb{H}_2$ *and* $\mathbb{V}_2$ *which satisfying the Theorem5.8 related to the extensively weak mixing property in [33] can all let* $\mathbb{A}_n$ *be primitive for all* $n \geq 2$*. Therefore, we can cover the Theorem5.8 in [33], i.e., the Theorem12 in our paper can be used to show more examples that* $\mathbb{A}_n$ *is primitive therein for all* $n \geq 2$ *than the Theorem5.8 in [33].*

Remark 22 *Observing the examples in the* ***section4.3****, we find*

$$\text{one of } H \text{ and } V \in \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\},$$

In fact, if H *and* V *satisfy one of the follow situations, and using the introduced method to get* $\mathbb{H}_2$ *and* $\mathbb{V}_2$*, then for* $\mathbb{A}_2 =$ *one of* $\mathbb{H}_2$ *and* $\mathbb{V}_2$*, Theorem12 is applied to show that* $\mathbb{A}_n$ *is primitive for all* $n \geq 2$.

(1) one of H *and* $V = E$*, and the other* $\notin \{O\}$;

(2) one of H *and* $V = G$*, and the other* $\in \{U, L, I, T_1, T_2, K_1\}$;

(3) one of H *and* $V = G'$*, and the other* $\in \{U, L, I, T_3, T_4, K_4\}$;

(4) one of H and $V \in \{U, L, I\}$, and the other $\in \{K_1, K_2, K_3, K_4\}$;
(5) one of H and $V \in \{T_1, T_2\}$, and the other $\in \{K_1\}$;
(6) one of H and $V \in \{T_3, T_4\}$, and the other $\in \{K_4\}$;
(7) $H = V$ and $H, V \in \{K_1, K_2, K_3, K_4\}$
, where

$$\begin{aligned} E &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, G = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, G' = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \\ U &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ T_1 &= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, T_3 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, T_4 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \\ K_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, K_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, K_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, K_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Next, we give one example to show the statement given above.

Example 23 (from(1)) *If*

$$H = E = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, V = G = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

then

$$\mathbb{H}_2 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbb{V}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Consider

$$\mathbb{A}_2 = \begin{pmatrix} \boxed{b_{11}} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{pmatrix} = \mathbb{H}_2 = \begin{pmatrix} \boxed{1} & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By (2.1), *it is easily checked that*

$$\text{(1) } \mathbb{A}_2 \text{ and } \mathbb{A}_3 = \begin{pmatrix} \boxed{1} & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{have safe symbols.}$$

(2) there exists

$$\beta_0 = 1, \ \beta_1 = 1, \ \beta_2 = 1$$

and

$$s_2 = b_{11}, \ s_3 = b_{11}b_{11} = s_2 b_{11}$$

such that (2)-(a)~(2)-(c) of Theorem12 hold, then Theorem12 is applied to show that $\mathbb{A}_n$ is primitive for all $n \geq 2$.

Remark 24 *If $\mathbb{A}_2$ is not constructed from H and V, then Theorem12 is also applied to show that $\mathbb{A}_n$ is primitive for all $n \geq 2$.*

we give one example follow.

Example 25 (Simplified Golden Mean) *Consider*

$$\mathbb{A}_2 = \begin{pmatrix} \boxed{b_{11}} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{pmatrix} = \begin{pmatrix} \boxed{1} & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By (2.1), *it is easily checked that*

$$\text{(1) } \mathbb{A}_2 \text{ and } \mathbb{A}_3 = \begin{pmatrix} \boxed{1} & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{have safe symbols.}$$

(2) there exists

$$\beta_0 = 1, \ \beta_1 = 1, \ \beta_2 = 1$$

and

$$s_2 = b_{11},\ \ s_3 = b_{11}b_{11} = s_2 b_{11}$$

such that (2)-(a)˜(2)-(c) of Theorem12 hold, then Theorem12 is applied to show that $\mathbb{A}_n$ *is primitive for all* $n \geq 2$.

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