KMV模型的不同估計方式

學生:施冠宇　　指導教授:李正福 博士
許元春 博士

國立交通大學財務金融研究所

摘　要

這篇論文簡介傳統的信用風險結構模型, 並且提出幾種不同的模型去預測違約的機率。我們採用幾種不同的選擇權評價模型使得公司的資產價格服從不同的機率分配用以預測倒閉的機率。我們希望這些不同的模型假設可以使的倒閉機率的預測更加精確。最後, 我們將做實證測試, 找出能夠對於我們的資料作最佳的詮釋的模型。

關鍵字: 信用風險, Merton 模型, 常數彈性變異模型, 跳躍擴散模型。

Alternative Methods for Estimating KMV Model

Student : Kuan-Yu Shih   Advisors : Dr. Cheng-Few Lee
Dr. Yuan-Chung Sheu

Institute of Finance
National Chiao Tung University

Abstract

In this paper, we introduce the conventional structural credit risk model and propose several different models to approach the default prediction. We apply several different option pricing frameworks which make asset value follows different distribution processes to predict the bankruptcy probability. We hope these distinct setups can predict the bankruptcy probability much accuracy. Finally, we will test these models and compare which one is the best of them.

Keywords: Credit risk, Merton model, CEV model, jump diffusion model.

誌　　謝

轉眼間又過了兩年, 完成了另外一份的碩士論文, 拿到了財務金融的碩士學位。回首這兩年的所學, 感覺還是跟數學、統計、計算脫不了關係, 想學的東西太多, 能力有限與時間有限的結果, 造成自己財務基本功似乎沒有學得很紮實, 學位拿的實在有些心虛。

雖然如此, 還是必須感謝陪伴我度過這兩年的人。首先必須要先感謝我的父母。謝謝他們願意多投資我兩年, 在同年齡的親朋好友都在賺錢或者念博士之際, 他們願意提供資金讓我再念一個國內的碩士。現在也是我該報答他們的時候了。其次必須感謝我指導教授, 李正福老師和許元春老師。感謝李老師不斷的要求我加強財務方面的知識, 並且指導我如何完成一篇好的論文。感謝許老師在數學方面的指導, 讓我因爲當兵而荒廢了兩年的基本數學知識能夠獲得補救。感謝曾經教過我的所有老師, 因爲你們上課的細心指導, 讓我在其他相關領域有更多的了解。感謝博炫學長, 雖然遠在美國, 也常常用 MSN 細心的給予指導。感謝交大財金所的所有同學與學弟妹, 在課業、八卦、歡笑與淚水之中陪伴我度過人生中的一個“兩年”。

最後, 我必須感謝奕 (方方土) 兄在當兵期間的鼓勵, 讓我能夠下定決心在人生的路上多投資兩年在財金碩士學位上。以目前金融改革的大環境來看, 這個投資似乎是對的。相信這兩年的付出將會對我未來的人生有決定性的影響。

謹將此篇論文獻給所有關心我的人, 謝謝大家!

施冠宇 2005.06.30

Contents

List of Tables

List of Figures

1 Introduction

Credit risk is a possibility of a creditor's financial loss occurring due to the contractual counterparty does not meet its debt obligations. In common sense, a bond involving a high amount of credit risk must promise a higher return to the investor than a less credit-risky contract. If we are able to obtain a good relation between the credit risk and bond price, then we will easily to predict the value for both of them.

Black and Scholes [2] are harbingers to discuss the relationship of bond price and its credit risk by option pricing model. Merton [32] further develope the intuition of Black and Scholes and put it into an analytical framework. A large amount of research followed the work of Black, Merton and Scholes.

Merton [32] use "Basic Accounting Equation", firm asset value equal to its liability and stock market value, to define what the bankruptcy event is. The approach assumed that the bankruptcy is occurred when the firm's asset values lower than its liability. When the corporation only issue one zero coupon bond, applying Merton's [32] credit risk model will facilitate to calculate the default probability of the corporate bond.

In addition to Merton model, Black and Cox [1] present an explicit equilibrium model which assumed the bond default at any time before maturity when the asset price touch threshold K. This threshold is exogenous, i.e., it can be different with the bond face value. Furthermore, Longstaff and Schwartz [30] import the dynamic riskless interest rate r, which follows the Vasicek [42] model. The correlation of diffusion term in asset price and interest rate is negative. Zhou [46] follows the assumption of Longstaff and Schwartz [30] and extends their result. He assume the asset value follows the normal jump diffusion model. Chen and Panjer [8] demonstrate that structural model[1] is consistent with reduced-form model[2] in theoretically.

However, because K is exogenously specified, it maybe gets an illogical result, as mentioned by Briys and Varenne [4]. Thus, Briys and Varenne [4] describe a model which assume the bankruptcy threshold is equal to the multiple of discount at the risk-free interest rate up to maturity date of the corporate bond. Hui *et al* [20] propose an extension model which can not only include the Longstaff and Schwartz [30] and Briys and Varenne [4] results, but also fulfill the situation when economical deterioration.

[1]All of the credit risk models which base on "Basic Accounting Equation", such as Merton [32], Black and Cox [1], Longstaff and Schwartz [30], Zhou [46], etc., are called "structural model".

[2]There are three different reduce-form models which try to capture the default risk of bond: recovery of market value (RMV), recovery of treasury (RT) and recovery of face value (RFV). Duffie and Singleton [16] propose RMV model. Jarrow and Turnbull [22] propose RT model. Brenann and Schwartz [3] and Duffee [15] propose RFV model. Chen and Panjer [8] verify that structural model is not only equivalent to RT model, but also converges to RMV model.

All models as mentioned above ignore the leverage ratios effect in credit risk. They assumed the value of the firm is independent of the capital structure of the firm. This is the standard assumption that the Modigliani-Miller Theorem holds. That is, the corporations will not change the credit spread of the old debt when they issue a new one. However, Collin-Dufresne and Goldstein [9] argue that the stationary leverage ratios will increase the credit spreads. Their model is also an extension of Longstaff and Schwartz [30].

Structural model can do very well for fitting the feature of the term structure of credit spread. Unfortunately, these models at least contain two unknown variables in one equation. As pointed out in Jarrow and Turnbull [23], we can not get the model parameters (the assets' expected return and volatility in the case of Merton model) from the market index directly. There are at least three different approaches to overcome this problem in academic literature, as pointed out by Duan *et al* [14]. Jones *et al* [24] and Ronn and Verma [34] (JMR-RV) apply Itô's lemma to make simultaneous equations and solve two unknown variables, asset value and its volatility. Eom *et al* [18] (EHH) propose the second approach. They apply the sum of the market value of equity and total debt as a proxy of firm implied asset value. In their empirical study, Merton model overestimates corporate bond prices substantially. The third approach is proposed by Duan [12]. This approach is based on the maximum likelihood estimation and applying the transformed-data to estimate two unknown variables. Although all of these three different approaches have their reasonable evidence to support their idea, two of the first are contradict with some other theoretical results. The deficiencies of first approach are explained by Duan [12] and Bruche [6]. They argue that the volatility of equity be a constant was unreasonable, especially when the asset value of the firm changes greatly during the estimation period. Besides, because the volatility equation is derived from Itô's lemma, it is redundancy. For the second approach, Wang and Li [45] verify it theoretically conflict with the boundary condition of option pricing model. Also, the Monte Carlo simulation shows that approach is bias the prediction very seriously. Under the simulation, Bruche [6], Wang and Li [45] and Duan *et al* [14] demonstrate the third method is the best approach in three of them.

Although JMR-RV approach has some defects, there are a lot of literatures and textbooks, such as Hull [21] and Schönbucher [38] , introduce this approach only to estimate unknown parameters. Based on this approach, KMV[3] proposed another methodology to find the firm asset value its volatility. Duan *et al* [14] and Lando [27] show that the KMV approach is equivalent to the maximum likelihood estimation approach in the case of Merton model. In theoretically, Duan *et al* [14] shows this method is a kind of EM algorithm, a well-known approach for obtaining maximum likelihood estimates under missing data environment.

[3]KMV is a company which produce several softwares to monitor the credit events. For more detail, please see: http://www.moodyskmv.com/

Under Merton framework, the asset price follows lognormal distribution. However, some empirical studies in stock price, like Tsay [41] and Schroder [36], show that lognormal distribution can not describe the price data very well. It's because the empirical distribution of daily log returns are skewness and excess kurtosis. This mean the empirical distribution of log return is asymmetric and fat tails. If the asset value have a good relation with equity value, we reasonable to doubt the assumption of distribution of asset is good or not. Therefore, in this article, we try to introduce two different asset processes and measure the performance of default forecast under these different assumptions in empirical study.

Cox [10] derives the renowned of Constant Elasticity of Variance (CEV) option pricing model. Cox *et al* [11] use the similar idea to derive the close form of well-known CIR term structure model. The difference between CEV and Black-Scholes models is that the CEV model has elasticity in the variance term of the diffusion model. Schroder [36] show the stock process under CEV model follows noncentral Chi-square distribution. Lee *et al* [29] derive Schroder's result in much detail. Macbeth and Merille [31] and Lee *et al* [28] demonstrate that CEV model is much better than Black-Scholes model. Since the empirical performance of CEV model is perfect in the stock markets, we rationally hope to apply their modelling idea and try to get good results in the credit risk forecasting.

Alternatively, Kou [25] derive the analytical solution of double exponential jump diffusion (JDF) model. It can alleviate the asymmetric and leptokurtic features in some empirical studies of equity price. The model also has better performance in kurtosis than normal jump diffusion model. We will introduce this model and apply it in credit risk prediction.

Although KMV and MLE methods are equivalent in point estimates in Merton model, KMV method cannot work for structural credit risk models that involve any unknown capital structures parameters. Duan *et al* [14] shows this result. Therefore, we apply MLE method, instead of KMV method, to estimate all parameters in CEV and double exponential JDF models.

Alternatively, following Black and Cox [1] approach, Brockman and Turtle [5] proposed barrier option framework to capture the default information. They thought the market value is an down-and-out option of asset value. The firm's debt maybe default if its asset value touch the default barrier before the debt maturity. Although Wang and Choi [44] and Duan *et al* [14] argue the estimating method is not adequate, the model's assumption and idea are still very well.

In empirical study, we choose debt, equity value and interest rate data during 2001 to 2004 from Taiwan Economic Journal. We calculate the default probability of all firms by these three distinct pricing models. In order to discriminate the performance of these models, we adopt the Cumulative Accuracy Profile (CAP), Accuracy Ratios (ARs) and Receiver Operating Characteristic (ROC), which proposed by Sobehart *et al* [40] and Engelmann *et al* [17], to evaluate the results.

The scheme of this article is as follows. In the following section, we describe the mixture of standard structural model framework with three different option pricing methods. Four different estimating approaches are introduced in section 3. We do the Monte Carlo simulation to test the performance of four different estimating methods under Merton model in section 4. Section 5 present the empirical result for these different option pricing models under Taiwanese data. Section 6 conclude the results.

2 European Option Pricing Model Framework

2.1 Merton Model

Merton [32] provide the analytical framework to evaluate the firm's credit risk. The structure of the Merton's model was built on the firm's capital structure. Assume the firm's total asset value is V. Consider a company issues a single liability with promised payoff K at maturing time T. This claim can be interpreted as a zero-coupon bond. For a bond holder, his payoff relies on the relation of V and K. If $V \geq K$, the bond holder will obtain all principal K, as he anticipate. However, if $V < K$, he will only obtain all firm's asset value V, which is less than K. Consequently, the payoff ϕ of the bond holder is

$$\phi = \min(V, K) = K - \max(K - V, 0). \tag{1}$$

The relation of ϕ and V is shown in Figure 1. We can see that when $V_1 < K$, the payoff ϕ is less than K. However, when $V_2 > K$, the payoff ϕ is equal to K.

From Equation (1), we can evaluate the expected value of $\max(K - V, 0)$ to find the risky bond price at time $t \leq T$. Assume the asset value follows a geometric Brownian motion:

$$d \ln V_t = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t. \tag{2}$$

where μ and σ are the expected return and instantaneous standard deviation of the asset value V, respectively. W_t is a Wiener process. Thus, we know

$$V_T = V_t \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}Z\right\}, \tag{3}$$

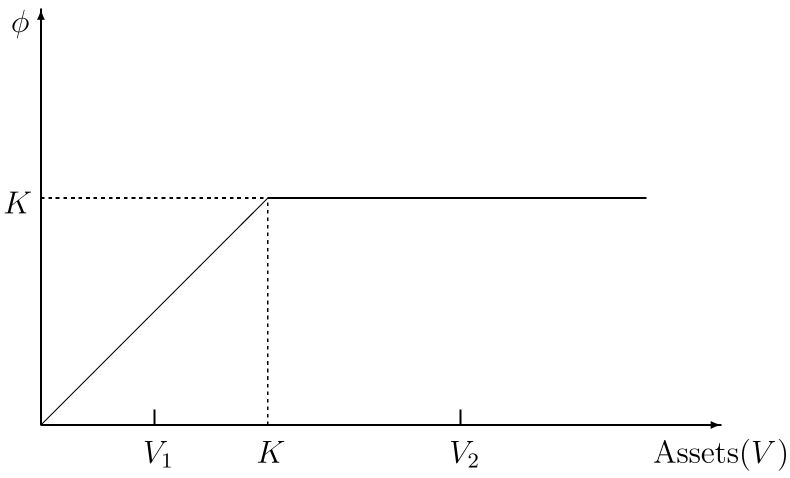

Figure 1: The payoff of the bond holder

where $Z \sim N(0,1)$ and $\tau = T - t$. Under Black-Scholes framework, the price of the firm's risky debt D_t is

$$\begin{aligned} D_t &= Ke^{-r\tau} - Ke^{-r\tau} N\left(-d_t + \sigma\sqrt{\tau}\right) + V_t N(-d_t) \\ &= Ke^{-r\tau} N\left(d_t - \sigma\sqrt{\tau}\right) + V_t N(-d_t), \end{aligned} \tag{4}$$

where $N(\cdot)$ is the cumulate standard normal distribution function and

$$d_t = \frac{\ln(V_t/K) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}.$$

In addition, from "Basic Accounting Equation", we know

$$V_t = S_t + D_t. \tag{5}$$

Employing Equation (5), we can get the formula that the equity price as a call option of asset value,

$$S_t = V_t N(d_t) - Ke^{-r\tau} N(d_t - \sqrt{\tau}). \tag{6}$$

We can conclude the default probability of the risky bond is

$$\begin{aligned} P_{def} = P(V_T \leq K) &= P\left(Z \leq \frac{\ln(K/V_t) - \left(\mu - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) \\ &= P\left(Z \leq -\frac{\ln(V_t/K) + \left(\mu - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) \equiv N(-(d_t - \sigma\sqrt{\tau})). \end{aligned} \tag{7}$$

2.2 CEV Option Pricing Model

2.2.1 CEV Option Pricing Formula

CEV model is first derived by Cox [10]. Schorder [36] and Lee *et al* [29] proof the pricing formula in much detail. We cite the same notations in Lee *et al* [29] and describe some important proof procedure.

The CEV option pricing model assumes that the stock price is governed by the diffusion process

$$dS_t = (\mu - q)S_t dt + \sigma S_t^{\alpha} dW_t, \tag{8}$$

where q is stock dividend rate. Under risk neutral environment, every one can not get any risk premium in the financial market. They only earn riskless interest r. So we set $\mu = r$, a constant interest rate. Let $X_t = S_t^{1/v}$, where $1/v = 2 - 2\alpha$. From Itô's lemma, Equation (8) can be rewrote as

$$dX_t = \left(\frac{1}{v}(r - q)X_t + \frac{1}{2}\frac{1}{v}(\frac{1}{v} - 1)\sigma^2\right) dt + \frac{\sigma}{v}\sqrt{X_t} dW_t. \tag{9}$$

Before we go ahead to get our results, we need the following two useful tools, the Kolmogorov Forward (or Fokker-Planck) Equation and Theorem 1.

Kolmogorov Forward Equation. (Also called the Fokker-Planck equation). *Consider the stochastic differential equation*

$$dX(u) = \beta(u, X(u))du + \gamma(u, X(u))dW(u). \tag{10}$$

where $dW(u)$ *is a Brownian motion. Let* $X(t) = x \geq 0$ *for any initial time* t, $0 \leq t < T$, T *is the termination time for this process,* $X(T) = y$, $X(u) > 0$, $\forall u \in (t, T]$. *Assume* $p(t, T, x, y)$ *be the transition density for the solution to the Equation (10),* $p(t, T, x, y) = 0$ *for* $0 \leq t < T$ *and* $y \leq 0$. *Then* $p(t, T, x, y)$ *satisfies the* Kolomogorov forward equation

$$\frac{\partial}{\partial T}p(t, T, x, y) = -\frac{\partial}{\partial y}(\beta(t, y)p(t, T, x, y)) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(\gamma^2(T, y)p(t, T, x, y)). \tag{11}$$

Proof. Please see Appendix A. □

Theorem 1. *Consider the parabolic equation*

$$\frac{\partial}{\partial t}u = \frac{\partial^2}{\partial x^2}(axu) - \frac{\partial}{\partial x}((bx + h)u), \ \ 0 < x < \infty, \tag{12}$$

where $u = u(x,t)$, and a, b, h are constants, $a > 0$. The explicit form of the fundamental solution of Equation (12) is given by

$$\begin{aligned} u(t,x,x_0) =& \frac{b}{a(e^{bt}-1)} \exp\left\{ \frac{-b(x+x_0e^{bt})}{a(e^{bt}-1)} \right\} \\ & \times \left(\frac{e^{-bt}x}{x_0} \right)^{\frac{(h-a)}{2a}} I_{1-\frac{h}{a}} \left(\frac{2b}{a(1-e^{-bt})} (e^{-bt}xx_0)^{\frac{1}{2}} \right). \end{aligned} \tag{13}$$

where $I_k(x)$ is the modified Bessel function of the first kind of order k and is defined as

$$I_k(x) = \sum_{r=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2r+k}}{r!\Gamma(r+1+k)}. \tag{14}$$

Proof. Please see Feller [19]. □

Remark 1. *$y = I_k(x)$ is the solution of differential equation $x^2y^" + xy^{'} - (x^2+k^2)y = 0$. We can find that $I_{-k}(x)$ is also a solution of this differential equation. Furthermore, $I_k(x) = I_{-k}(x)$.*

Using Kolmogorov forward equation can get the transition density function $p(t,T,x,y)$ of X which follows the equation

$$\frac{\partial}{\partial T}p = -\frac{\partial}{\partial y}\left(\left(\frac{1}{v}(r-q)y + \frac{1}{2}\frac{1}{v}\left(\frac{1}{v}-1\right)\sigma^2 \right) p \right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\frac{\sigma^2}{v^2}yp \right). \tag{15}$$

Applying Theorem 1, let

$$\begin{aligned} &a = \frac{\sigma^2}{2v^2}, \quad b = \frac{1}{v}(r-q), \quad h = \frac{1}{2}\frac{1}{v}(\frac{1}{v}-1)\sigma^2, \\ &x_t = x, \quad x_T = y, \quad \tau = T-t, \end{aligned}$$

the transition probability $p(t,T,x,y)$ is

$$\begin{aligned} p(t,T,x,y) =& \frac{\frac{1}{v}(r-q)}{\frac{\sigma^2}{2v^2}\left(\exp\left(\frac{1}{v}(r-q)\tau\right)-1\right)} \exp\left\{ \frac{-\frac{1}{v}(r-q)(y+xe^{\frac{1}{v}(r-q)\tau})}{\frac{\sigma^2}{2v^2}\left(\exp\left(\frac{1}{v}(r-q)\tau\right)-1\right)} \right\} \\ & \times \left(\frac{ye^{-\frac{1}{v}(r-q)\tau}}{x} \right)^{-\frac{v}{2}} I_v \left(\frac{\frac{2}{v}(r-q)\left(\exp\left(-\frac{1}{v}(r-q)\tau\right)xy\right)^{\frac{1}{2}}}{\frac{\sigma^2}{2v^2}\left(1-\exp\left(-\frac{1}{v}(r-q)\tau\right)\right)} \right). \end{aligned} \tag{16}$$

Let

$$\begin{aligned} k^* &= \frac{2v(r-q)}{\sigma^2\left(\exp\left(\frac{1}{v}(r-q)\tau\right)-1\right)}, \\ x^* &= 2k^*x\exp\left(\frac{1}{v}(r-q)\tau\right) = 2k^*S_t^{1/v}\exp\left(\frac{1}{v}(r-q)\tau\right), \\ y^* &= 2k^*y = 2k^*S_T^{1/v} \Rightarrow dy^* = 2k^*dy, \end{aligned}$$

then

$$
\begin{aligned}
p(t,T,x^*,y^*) &= \frac{1}{2}\left(\frac{x^*}{y^*}\right)^{\frac{v}{2}} \exp\left(-\frac{1}{2}(x^*+y^*)\right) I_v(\sqrt{x^*y^*}) \\
&= \frac{1}{2}\left(\frac{y^*}{x^*}\right)^{-\frac{v}{2}} \exp\left(-\frac{1}{2}(x^*+y^*)\right) I_{-v}(\sqrt{x^*y^*}).
\end{aligned} \tag{17}
$$

We know the pdf of noncentral chisquare distribution with noncentrality λ and degree of freedom v, $\chi_v'^2(\lambda)$, is

$$
p_{\chi_v'^2(\lambda)}(x) = \frac{1}{2}\left(\frac{x}{\lambda}\right)^{(v-2)/4} I_{\frac{1}{2}(v-2)}(\sqrt{\lambda x}) \exp\left(-\frac{1}{2}(\lambda + x)\right).
$$

Thus we can conclude that y^* follows a noncentral chisquare distribution with noncentrality x^* and degree of freedom $2-2v$, $\chi_{2-2v}'^2(x^*)$.

Finally, we can get the option pricing formula under the CEV model is

$$
\begin{aligned}
C_t =& e^{-r\tau}\int_K^\infty p(t,T,S_t,S_T)(S_T-K)dS_T \\
=& e^{-r\tau}\left(\int_K^\infty p(t,T,S_t,S_T)S_T dS_T - K\int_K^\infty p(t,T,S_t,S_T)dS_T\right) \\
=& e^{-r\tau}\left(\int_{2k^*K^{\frac{1}{v}}}^\infty p(t,T,x^*,y^*)\left(\frac{y^*}{2k^*}\right)^v dy^* - K\int_{2k^*K^{\frac{1}{v}}}^\infty p(t,T,x^*,y^*)dy^*\right) \\
=& e^{-r\tau}\int_w^\infty \frac{1}{2}\left(\frac{y^*}{x^*}\right)^{\frac{v}{2}} \exp\left(-\frac{1}{2}(x^*+y^*)\right) I_v\left(\sqrt{x^*y^*}\right)\left(\frac{x^*}{2k^*}\right)^v dy^* \\
& - e^{-r\tau}K\int_w^\infty \frac{1}{2}\left(\frac{y^*}{x^*}\right)^{-\frac{v}{2}} \exp\left(-\frac{1}{2}(x^*+y^*)\right) I_{-v}\left(\sqrt{x^*y^*}\right)dy^* \\
& (\text{Let } w = 2k^*K^{\frac{1}{v}}) \\
=& e^{-q\tau}S_tC_1 - e^{-r\tau}KC_2,
\end{aligned} \tag{18}
$$

where

$$
\begin{aligned}
C_1 &= \int_w^\infty \frac{1}{2}\left(\frac{y^*}{x^*}\right)^{\frac{v}{2}} \exp\left(-\frac{1}{2}(x^*+y^*)\right) I_v\left(\sqrt{x^*y^*}\right)dy^*, \\
C_2 &= \int_w^\infty \frac{1}{2}\left(\frac{y^*}{x^*}\right)^{-\frac{v}{2}} \exp\left(-\frac{1}{2}(x^*+y^*)\right) I_{-v}\left(\sqrt{x^*y^*}\right)dy^*.
\end{aligned}
$$

2.2.2 Greek Letters in CEV Model

The Greek Letters is very useful to test the sensibility of option price versus various parameters, including underlying asset, interest rate, time to maturity, volatility and Delta hedge ratio. The definition of them are shown in Table 1.

Unfortunately, it is too complicate to get the close-form for all Greek letters under CEV model. We have to apply numerical method to find the results.

Delta	Gamma	Theta	Rho	Vega or Lambda
$\frac{\partial C}{\partial S}$	$\frac{\partial^2 C}{\partial S^2}$	$\frac{\partial C}{\partial t}$	$\frac{\partial C}{\partial r}$	$\frac{\partial C}{\partial \sigma}$

Table 1: The Greek letters.

Theorem 2. *Suppose a function $f \in C^2[a,b]$. For any $x_0 \in (a,b)$, consider some $h \neq 0$ such that $x_0 + h \in (a,b)$. The approximation of first derivative of the function f at x_0 is*

$$\begin{aligned} f'(x_0) =& \frac{1}{12h}[f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h)] \\ &+ \frac{h^4}{30} f^{(5)}(\xi), \end{aligned} \tag{19}$$

where $\xi \in (x_0 - h, x_0 + h)$, the error term is of the form $O(h^4)$. The approximation of second derivative is

$$f''(x_0) = \frac{1}{h^2}[f(x_0-h) - 2f(x_0) + f(x_0+h)] - \frac{h^2}{12} f^{(4)}(\xi). \tag{20}$$

Also, $\xi \in (x_0 - h, x_0 + h)$, the error term is of the form $O(h^2)$.

Proof. Please see Burden and Faires [7]. □

Applying Theorem 2 can not only get all numerical results of Table 1, but also do trading strategy to hedge portfolio very easily.

2.2.3 The Default Probability of Risky Bond in CEV Model

Employing CEV model in the credit framework, we derive the relation between the market value and asset value is

$$S_t = V_t C_1 - e^{-r\tau} K C_2, \tag{21}$$

where

$$\begin{aligned} C_1 &= \int_w^\infty \frac{1}{2} \left(\frac{y^*}{x^*}\right)^{\frac{v}{2}} \exp\left(-\frac{1}{2}(x^*+y^*)\right) I_v\left(\sqrt{x^* y^*}\right) dy^*, \\ C_2 &= \int_w^\infty \frac{1}{2} \left(\frac{y^*}{x^*}\right)^{-\frac{v}{2}} \exp\left(-\frac{1}{2}(x^*+y^*)\right) I_{-v}\left(\sqrt{x^* y^*}\right) dy^*, \\ k^* &= \frac{2vr}{\sigma^2\left(\exp\left(\frac{r\tau}{v}\right) - 1\right)}, \quad x^* = 2k^* V_t^{1/v} \exp\left(\frac{r\tau}{v}\right), \\ y^* &= 2k^* V_T^{1/v} \Rightarrow dy^* = \frac{2k^*}{v} V_T^{\frac{1}{v}-1} dV_T, \quad \tau = T - t, \quad w = 2k^* K^{\frac{1}{v}} \end{aligned} \tag{22}$$

Let the return of the asset value is μ. Then the default probability of corporate bond is

$$
\begin{aligned}
P_{def} = P(V_T < K) &= \int_0^K p(t,T,V_t,V_T)dV_T = \int_0^{2k^* K^{\frac{1}{v}}} p(t,T,x^*,y^*)dy^* \\
&= 1 - C_2,
\end{aligned}
\tag{23}
$$

where all riskless interest rate r in the Equation (22) is instead of μ.

2.3 JDF Option Pricing Model

In addition to CEV model in previous section, Kou [25] propose a JDF model to describe the asymmetric leptokurtic and heavy fat tails properties in the return distribution. In this model, the stock price is assumed to follow a Brownian motion plus a compound Poisson process with jump sizes double exponentially distributed. As mentioned by Kou [25], this model has some more excellent properties than other models:

1. The CEV model does not have the leptokurtic feature. The volatility smile in option pricing can not fit very well.
2. Merton [33] propose a jump-diffusion model with normal distribution in jump size. Almost all results are similar with double exponential jump-diffusion models except the analytical path-dependent options. Since Kou's [25] model can handle the "overshoot" problem very well, it can derive the analytical solution of American option, Lookback option and Barrier option.
3. Schoutens [37] propose a theoretical and empirical study in option pricing models based on several different Lévy processes. Because based on the infinitely divisible property in these processes, the empirical study shows that these option pricing models can fit the realize stock price very well. Actually, the double exponential jump-diffusion model is a special case of Lévy process. Although this model is not as well as other distribution in Lévy process pricing model, it is easier to calculate the theoretical price than other distributions.

2.3.1 Theoretical Results in JDF Option Pricing Model

Assume the stock price, S_t, follows the jump-diffusion process, i.e.,

$$
\frac{dS_t}{S_{t-}} = \mu dt + \sigma dW_t + d\left(\sum_{i=1}^{N(t)}(J_i - 1)\right), \tag{24}
$$

where $N(t)$ is a Poisson process with rate λ, and $\{J_i\}$ is a sequence of independent identically distributed (i.i.d.) nonnegative random variables such that $Y = \ln(J)$ has an asymmetric double exponential distribution with the density

$$f_Y(y) = p \cdot \eta_1 \exp(-\eta_1 y) 1_{\{y \geq 0\}} + q \cdot \eta_2 \exp(\eta_2 y) 1_{\{y<0\}}, \tag{25}$$

where $\eta_1 > 1$, $\eta_2 > 0$, $p \geq 0$, $q \geq 0$ and $p + q = 1$. All other notations are same as previously. p and q are the probabilities of upward and downward jumps, i.e.,

$$\ln(J) = Y \stackrel{d}{=} \begin{cases} \xi^+, & \text{with probability } p \\ -\xi^-, & \text{with probability } q \end{cases}, \tag{26}$$

where ξ^+ and ξ^- are exponential random variables with means $1/\eta_1$ and $1/\eta_2$, respectively, and $\stackrel{d}{=}$ means equal in distribution.

Applying Itô's lemma in Equation (24), the process of $\ln S_t$ is

$$d \ln S_t = \left(\mu - \frac{1}{2}\sigma^2 - \lambda\zeta\right) dt + \sigma dW_t + d\left(\sum_{i=1}^{N(t)} Y_i\right), \tag{27}$$

where

$$\zeta = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1, \;\; p + q = 1.$$

Before go ahead to evaluate the option pricing by the double exponential jump-diffusion model, we need to use following results.

Proposition 1. *For every $n \geq 0$, the Hh function is a non increasing function defined by*

$$\begin{aligned} Hh_n(x) &= \int_x^\infty Hh_{n-1}(y)dy = \frac{1}{n!}\int_x^\infty (t-x)^n e^{-t^2/2} dt \geq 0, \\ n &= 0, 1, 2, \ldots \\ Hh_{-1}(x) &= e^{-x^2/2}, \;\; Hh_0(x) = \sqrt{2\pi}N(-x), \end{aligned} \tag{28}$$

where $N(\cdot)$ is a cumulate standard normal distribution function. Hh function can also be written as,

$$\begin{aligned} Hh_n(x) =& 2^{-n/2}\sqrt{\pi}e^{-x^2/2} \times \\ & \left\{ \frac{{}_1F_1(\frac{1}{2}n + \frac{1}{2}, \frac{1}{2}, \frac{1}{2}x^2)}{\sqrt{2}\Gamma(1 + \frac{1}{2}n)} - x\frac{{}_1F_1(\frac{1}{2}n + 1, \frac{3}{2}, \frac{1}{2}x^2)}{\Gamma(\frac{1}{2} + \frac{1}{2}n)} \right\}, \;\; n \geq -1 \end{aligned}$$

and

$$nHh_n(x) = Hh_{n-2}(x) - xHh_{n-1}(x), \;\; n \geq 1.$$

These three equations are equivalent.

Proof. Please see the description in Kou [25] and other properties in Kou and Wang [26]. □

Proposition 2. *Define*

$$I_n(c;\alpha,\beta,\delta) = \int_c^\infty e^{\alpha x} Hh_n(\beta x - \delta)dx, \;\; n \geq 0, \tag{29}$$

1. *If* $\beta > 0$ *and* $\alpha \neq 0$*, then for all* $n \geq -1$*,*

$$\begin{aligned} I_n(c;\alpha,\beta,\delta) = &-\frac{e^{\alpha c}}{\alpha}\sum_{i=0}^{n}\left(\frac{\beta}{\alpha}\right)^{n-i} Hh_i(\beta c - \delta) \\ &+ \left(\frac{\beta}{\alpha}\right)^{n+1}\frac{\sqrt{2\pi}}{\beta}e^{\frac{\alpha\delta}{\beta}+\frac{\alpha^2}{2\beta^2}}N\left(-\beta c + \delta - \frac{\alpha}{\beta}\right). \end{aligned} \tag{30}$$

2. *If* $\beta < 0$ *and* $\alpha < 0$*, then for all* $n \geq -1$*,*

$$\begin{aligned} I_n(c;\alpha,\beta,\delta) = &-\frac{e^{\alpha c}}{\alpha}\sum_{i=0}^{n}\left(\frac{\beta}{\alpha}\right)^{n-i} Hh_i(\beta c - \delta) \\ &+ \left(\frac{\beta}{\alpha}\right)^{n+1}\frac{\sqrt{2\pi}}{\beta}e^{\frac{\alpha\delta}{\beta}+\frac{\alpha^2}{2\beta^2}}N\left(\beta c - \delta + \frac{\alpha}{\beta}\right). \end{aligned} \tag{31}$$

3. *If* $\beta > 0$ *and* $\alpha = 0$*, then for all* $n \geq 0$*,* $I_n(c;\alpha,\beta,\delta) = \frac{1}{\beta}Hh_{n+1}(-\beta c - \delta)$.

4. *If* $\beta \leq 0$ *and* $\alpha \geq 0$*, then for all* $n \geq 0$*,* $I_n(c;\alpha,\beta,\delta) = \infty$.

5. *If* $\beta = 0$ *and* $\alpha < 0$*, then for all* $n \geq 0$*,* $I_n(c;\alpha,\beta,\delta) = \int_c^\infty e^{\alpha x}Hh_n(-\delta)dx = Hh_n(-\delta)e^{\alpha c}/\alpha$.

Proof. Please see Kou [25]. □

Proposition 3. *Suppose* $\{\xi_1, \xi_2, \ldots\}$ *is a sequence of i.i.d. exponential random variables with rate* $\eta > 0$*, and* Z *is a random variable with distribution* $N(0, \sigma^2)$*. Then for every* $n \geq 1$*, we have*

1. *The density function are given by*

$$\begin{aligned} f_{Z+\sum_{i=1}^n \xi_i}(t) &= (\sigma\eta)^n \frac{e^{(\sigma\eta)^2/2}}{\sigma\sqrt{2\pi}}e^{-t\eta}Hh_{n-1}\left(-\frac{t}{\sigma}+\sigma\eta\right), \\ f_{Z-\sum_{i=1}^n \xi_i}(t) &= (\sigma\eta)^n \frac{e^{(\sigma\eta)^2/2}}{\sigma\sqrt{2\pi}}e^{t\eta}Hh_{n-1}\left(\frac{t}{\sigma}+\sigma\eta\right). \end{aligned} \tag{32}$$

2. *The tail probabilities are given by*

$$\begin{aligned} P\left(Z + \sum_{i=1}^n \xi_i \geq x\right) &= \frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}}e^{(\sigma\eta)^2/2}I_{n-1}\left(x; -\eta, -\frac{1}{\sigma}, \sigma\eta\right), \\ P\left(Z - \sum_{i=1}^n \xi_i \geq x\right) &= \frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}}e^{(\sigma\eta)^2/2}I_{n-1}\left(x; \eta, \frac{1}{\sigma}, -\sigma\eta\right). \end{aligned} \tag{33}$$

Proof. Please see Kou [25]. □

Theorem 3. *With $\pi_n := P(N(T) = n)e^{-\lambda T}(\lambda T)^n/n!$ and I_n in Proposition 2, we have*

$$\begin{aligned} P\left(Z(T) \geq a\right) =& \frac{e^{(\sigma\eta_1)^2 T/2}}{\sigma\sqrt{2\pi T}} \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} P_{n,k} \left(\sigma\sqrt{T}\eta_1\right)^k \\ & \times I_{k-1}\left(a - \mu T; -\eta_1, -\frac{1}{\sigma\sqrt{T}}, \sigma\eta_1\sqrt{T}\right) \\ & + \frac{e^{(\sigma\eta_2)^2 T/2}}{\sigma\sqrt{2\pi T}} \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} Q_{n,k} \left(\sigma\sqrt{T}\eta_2\right)^k \\ & \times I_{k-1}\left(a - \mu T; -\eta_2, -\frac{1}{\sigma\sqrt{T}}, \sigma\eta_2\sqrt{T}\right) \\ & + \pi_0 N\left(-\frac{a-\mu T}{\sigma\sqrt{T}}\right) \end{aligned} \tag{34}$$

and

$$\begin{aligned} f_Z(a) =& \frac{e^{(\sigma\eta_1)^2 T/2}}{\sigma\sqrt{2\pi T}} \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} P_{n,k} \left(\sigma\sqrt{T}\eta_1\right)^k e^{-(a-\mu T)\eta_1} \\ & \times Hh_{k-1}\left(-\frac{a-\mu T}{\sigma\sqrt{T}} + \sigma\sqrt{T}\eta_1\right) \\ & + \frac{e^{(\sigma\eta_2)^2 T/2}}{\sigma\sqrt{2\pi T}} \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} Q_{n,k} \left(\sigma\sqrt{T}\eta_2\right)^k e^{(a-\mu T)\eta_2} \\ & \times Hh_{k-1}\left(\frac{a-\mu T}{\sigma\sqrt{T}} + \sigma\sqrt{T}\eta_2\right) \\ & + \pi_0 \varphi\left(\frac{a-\mu T}{\sigma\sqrt{T}}\right), \end{aligned} \tag{35}$$

where

$$\begin{aligned} P_{n,k} &= \sum_{i=k}^{n-1} \binom{n-k-1}{i-k} \binom{n}{i} \left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{i-k} \left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{n-i} p^i q^{n-i}, \\ Q_{n,k} &= \sum_{i=k}^{n-1} \binom{n-k-1}{i-k} \binom{n}{i} \left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{n-i} \left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{i-k} p^{n-i} q^i, \\ Z(T) &= \mu T + \sigma\sqrt{T}Z + \sum_{i=1}^{N(T)} Y_i, \;\; P_{n,n} = p^n, \;\; Q_{n,k} = q^n \end{aligned}$$

and $\varphi(\cdot)$ is a standard normal distribution density function.

Proof. Kou [25] show the result of Equation (34). The proof of Equation (35) is similar with Equation (34). □

Employing Equation (27) and (35) can derive the pdf of stock price, S,

$$f_S(S|S_0) = \frac{f_Z\left(\ln\frac{S}{S_0}\right)}{S}, \tag{36}$$

where μ in Equation (35) is instead by $\mu - \sigma^2/2 - \lambda(p\eta_1/(\eta_1 - 1) + (1-p)\eta_2/(\eta_2 + 1) - 1)$.

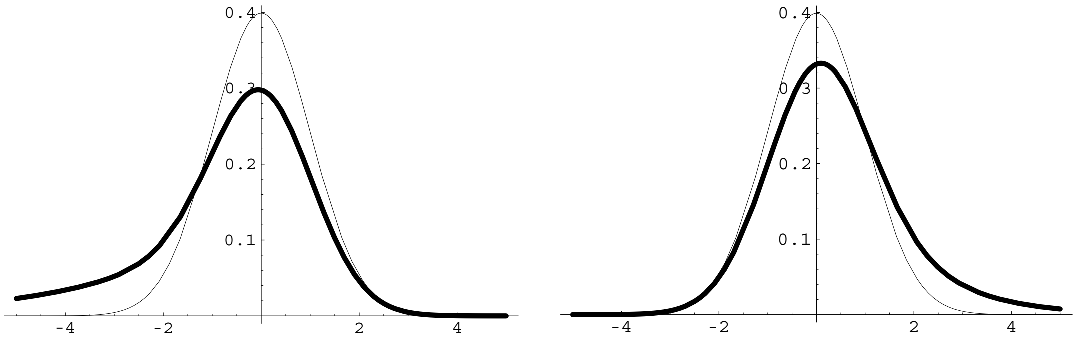

Figure 2: In left panel, the parameters are: $\mu = 0$, $\sigma = 1$, $\eta_1 = 5$, $\eta_2 = 0.5$, $\lambda = 1$ and $p = q = 0.5$. The parameters in right panel is: $\mu = 0$, $\sigma = 1$, $\eta_1 = 1.01$, $\eta_2 = 5$, $\lambda = 1$ and $p = q = 0.5$.

Figure 2 shows the comparing of two different density functions, $f_Z(\cdot)$ in Equation (35) and standard normal distribution. The left graph shows the jump of the return is negative amplitude, and right graph is positive. Because $f_Z(\cdot)$ owns the heavy tail property in both side of distribution, it can fit the return of stock prices very well.

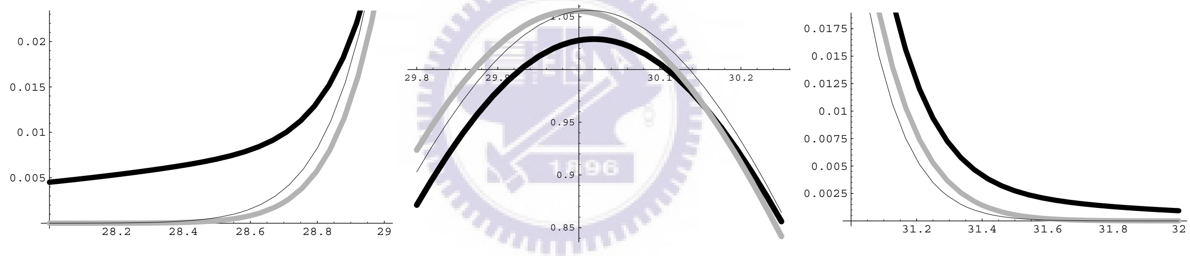

Figure 3: The shape of three different distributions.

Figure 3 shows shape of three different distributions. The parameters are setup as follows: $S_0 = 30$, $r = 1.5\%$, $T = 1/252$, $\sigma = 0.2$, $v = 0.2$, $\eta_1 = 50$, $\eta_2 = 25$, $\lambda = 10$, $p = 0.3$. The black thin line is noncentral chisquare distribution, the gray thick line is lognormal distribution, and the black thick line is $f_S(\cdot)$ in Equation (36).

Let us define

$$\Upsilon(\mu, \sigma, \lambda, p, \eta_1, \eta_2; a, T) := P(Z(T) \geq a). \tag{37}$$

Under the Proposition 1 to Proposition 3 and Theorem 3, Kou [25] showed the option price is

$$\begin{aligned} C_t =& S_t \Upsilon\left(r + \frac{1}{2}\sigma^2 - \lambda\zeta, \sigma, \tilde{\lambda}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2; \ln\left(K/S_t\right), \tau\right) \\ & - Ke^{-r\tau}\Upsilon\left(r - \frac{1}{2}\sigma^2 - \lambda\zeta, \sigma, \lambda, p, \eta_1, \eta_2; \ln\left(K/S_t\right), \tau\right), \end{aligned} \tag{38}$$

where

$$\tilde{p} = \frac{p}{1+\zeta} \cdot \frac{\eta_1}{\eta_1 - 1}, \;\; \tilde{\eta}_1 = \eta_1 - 1, \;\; \tilde{\eta}_2 = \eta_2 + 1,$$
$$\tilde{\lambda} = \lambda(\zeta + 1), \;\; \zeta = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1.$$

2.3.2 The Default Probability of Risky Bond in JDF Model

Assume the firm value process V_t follows the jump diffusion process as in Equation (24), i.e.,

$$\frac{dV_t}{V_{t-}} = \mu dt + \sigma dW_t + d\left(\sum_{i=1}^{N(t)}(J_i - 1)\right), \tag{39}$$

where μ is the asset return, σ is the volatility of the asset value. Notations W_t, $N(t)$ and J_i and properties in Equation (39) are same as before. Employing Equation (38), the market value S_t is

$$\begin{aligned} S_t =& V_t \Upsilon\left(r + \frac{1}{2}\sigma^2 - \lambda\zeta, \sigma, \tilde{\lambda}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2; \ln\left(K/V_t\right), \tau\right) \\ &- Ke^{-r\tau}\Upsilon\left(r - \frac{1}{2}\sigma^2 - \lambda\zeta, \sigma, \lambda, p, \eta_1, \eta_2; \ln\left(K/V_t\right), \tau\right), \end{aligned} \tag{40}$$

where

$$\tilde{p} = \frac{p}{1+\zeta} \cdot \frac{\eta_1}{\eta_1 - 1}, \;\; \tilde{\eta}_1 = \eta_1 - 1, \;\; \tilde{\eta}_2 = \eta_2 + 1,$$
$$\tilde{\lambda} = \lambda(\zeta + 1), \;\; \zeta = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1, \;\; \tau = T - t.$$

The default probability is

$$\begin{aligned} P_{def} &= P(V_T < K) = 1 - P(V_T > K) \\ &= 1 - \Upsilon\left(\mu - \frac{1}{2} - \sigma^2\lambda\zeta, \sigma, \lambda, p, \eta_1, \eta_2; \ln\left(K/V_t\right), \tau\right). \end{aligned} \tag{41}$$

3 Estimating Approaches

As mentioned in introduction, there are at least three different approaches to estimating unknown parameters. We describe much detail in this section.

3.1 JMR-RV Approach

JMR-RV approach is derived from Itô's lemma. From Equation (6), the equity value S_t is an option premium of asset value V_t. Applying Itô's lemma can easy to derive the following equation,

$$dS_t = \left(\frac{\partial S_t}{\partial V_t}\mu_v V_t + \frac{\partial S_t}{\partial t} + \frac{1}{2}\frac{\partial^2 S_t}{\partial V_t^2}\sigma_v^2 V_t^2\right) dt + \left(\frac{\partial S_t}{\partial V_t}\sigma_v V_t\right) dW_t. \tag{42}$$

On the other hand, we also assume the equity value follows geometric Brownian motion,

$$dS_t = \mu_S S_t dt + \sigma_S S_t dW_t. \tag{43}$$

Comparing the Equation (42) and (43), we can get the following equation,

$$\sigma_S = \frac{\partial S_t}{\partial V_t}\frac{V_t}{S_t}\sigma_v. \tag{44}$$

In addition, Equation (6) shows that S_t is a one to one function of V_t, say $S_t = g(V_t; \mu, \sigma)$. The inverse of $g_1(\cdot)$ ($g_1^{-1}(\cdot)$) is exist. Thus, the two unknown variables, V_t and σ_t, can be solved by the simultaneous equations,

$$\begin{cases} S_t & = V_t N(d_t) - Ke^{-r\tau}N(d_t - \sqrt{\tau}) \\ \sigma_S & = \sigma_V \frac{V}{S}\frac{\partial S}{\partial V}. \end{cases} \tag{45}$$

Duan [12] and Bruche [6] argued two deficiencies for this approach. First, set the volatility of equity be a constant was unreasonable, especially when the asset value changes greatly during the estimation period. On the other hand, because the volatility equation is derived from Itô's lemma, it is redundant and can not be used as a separate restriction. We will use Monte Carlo simulation to demonstrate it in section 4.

3.2 KMV Approach

Consider the Equation (45). If $L_t = V_t/S_t$ is small, then $\frac{\partial S}{\partial V}$ is close to 1 and the approximation $\sigma_s = L_t \sigma_v$ works well. Furthermore, if L_t doesn't vary too much over the observation period, then the stock looks like a Brownian motion. Assume the equity value data are observed at $n+1$ equal time interval points. They are denoted by $\{S_0, S_h, S_{2h}, \ldots, S_{nh}\}$, where h is the ratio of the time length between two data over one year. Applying MLE procedure, which described by Duan [14], can estimate these unknown variables:

1. Compute the implied asset value $\hat{V}_t(\hat{\sigma}^{(m)})$ corresponding to the observed equity value S_t, for all $t = 0, h, \ldots, nh$.

2. Compute the implied asset returns $\hat{R}_i^{(m)} = \ln\left(\hat{V}_{ih}(\hat{\sigma}^{(m)})/\hat{V}_{(i-1)h}(\hat{\sigma}^{(m)})\right)$, for all $i = 1, \ldots, n$. and update the asset drift and volatility parameters as follows:

$$\bar{R}^{(m)} = \frac{1}{n}\sum_{k=1}^{n}\hat{R}_k^{(m)}$$

$$\left(\hat{\sigma}^{(m+1)}\right)^2 = \frac{1}{nh}\sum_{k=1}^{n}\left(\hat{R}_k^{(m)} - \bar{R}^{(m)}\right)^2$$

$$\hat{\mu}^{(m+1)} = \frac{1}{h}\bar{R}^{(m)} + \frac{1}{2}\left(\hat{\sigma}^{(m+1)}\right)^2$$

3. If $|\hat{\mu}^{(m+1)} - \hat{\mu}^{(m-1)}| < tol$ and $|\hat{\sigma}^{(m+1)} - \hat{\sigma}^{(m-1)}| < tol$, then stop this procedure. Otherwise, go back to step 1 and repeat it again.

3.3 EHH Approach

In EHH approach, they apply the sum of the market value of equity and total debt as a proxy of firm implied asset value, i.e., $V_{proxy} = K + S$. After get V_{proxy}, applying Equation (44) can estimate the volatility of asset very quickly. Furthermore, asset return μ comes from average monthly change in V. However, Wang and Li [45] show this assumption is unreasonable. Under the option theory, assume the true of asset value is V_{true}, it is easy to fund

$$C(V_{true}, K, T) = S = V_{proxy} - K < C(V_{proxy}, K, T).$$

Because call option function is an increasing function of underlying asset, it implies $V_{proxy} > V_{true}$, overestimate the true value. So this method will get the bias result. Wang and Li [45] also used Monte Carlo simulation to support the result.

3.4 The Transform data MLE Approach

3.4.1 In Merton's Model

Duan [12], Duan [13] and Duan *et al* [14] propose a transform data maximum likelihood estimation to resolve limitation of several unknown variables in one equation.

From Equation (3) and (6), the relation between probability density function of S_t and V_t is

$$\begin{aligned} f(S_t) &= f(g_1^{-1}(S_t)) \left| \frac{\partial g_1^{-1}(S_t)}{\partial S} \right| \\ &= \frac{1}{g_1^{-1}(S_t)\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(\ln g_1^{-1}(S_t) - \mu)^2}{2\sigma^2} \right\} \frac{1}{N(d_t(g_1^{-1}(S_t)))}. \end{aligned} \tag{46}$$

We use the fact that $\partial V_t / \partial S_t = N(d_t)$ in Equation (6).

Under this framework, we obtain the log-likelihood function of S_t is

$$\begin{aligned} L^S(\mu, \sigma; S_0, S_h, S_{2h}, \ldots, S_{nh}) =& L^V(\mu, \sigma; \hat{V}_0(\sigma), \hat{V}_h(\sigma), \hat{V}_{2h}(\sigma), \ldots, \hat{V}_{nh}(\sigma)) \\ & - \sum_{i=1}^{n} \ln(N(\hat{d}_{ih}(\sigma))), \end{aligned} \tag{47}$$

where

$$
\begin{aligned}
L^V(\mu, \sigma; & \hat{V}_0(\sigma), \hat{V}_h(\sigma), \hat{V}_{2h}(\sigma), \ldots, \hat{V}_{nh}(\sigma)) \\
&= -\frac{n}{2}\ln(2\pi\sigma^2 h) - \frac{1}{2}\sum_{i=1}^{n} \frac{\left(\ln\left(\frac{\hat{V}_{ih}(\sigma)}{\hat{V}_{(i-1)h}(\sigma)}\right) - \left(\mu - \frac{\sigma^2}{2}\right)h\right)^2}{\sigma^2 h} - \sum_{i=1}^{n} \ln \hat{V}_{ih}(\sigma), \\
\hat{V}_{ih}(\sigma) &= g_1^{-1}(S_{ih}; \sigma), \\
\hat{d}_{ih}(\sigma) &= \frac{\ln(\hat{V}_{ih}(\sigma)/K) + \left(r + \frac{\sigma^2}{2}\right)(T - ih)}{\sigma\sqrt{T - ih}}.
\end{aligned}
$$

The Algorithm for this procedure is described as follows:

1. Assign an initial value to μ and σ respectively to evaluate the implied asset value $\hat{V}$.
2. Applying Equation (47) to evaluate the MLE of μ and σ.
3. Comparing the absolutely error between MLE and initial value. If it less than a convergence criterion, we stop this procedure. Otherwise, go back to step 1 and repeat this procedure.
4. Using the MLE of σ, $\hat{\sigma}$, to calculate the imply asset value $\hat{V}_{nh}$ and its default probability P_{def}.

3.4.2 In CEV Model

In Macbeth and Merville [31], they propose the following regression, which comes from Equation (8), to estimate unknown parameter v

$$\ln(dS_t - (\mu - q)S_t dt)^2 - \ln dt = 2\ln\sigma + 2\alpha \ln S_t + \ln \chi^2_{(1)},$$

Taking $1/v = 2 - 2\alpha$ to get v. However, S_t is the stock price, a known value. V_t is the firm price, an unknown value. Thus, we can't use this method to estimate unknown parameters.

From Equation (17) and (22), we can obtain the probability density function of random variable V_T is

$$
\begin{aligned}
p(t, T, V_t, V_T) &= p(t, T, x^*, y^*) \left|\frac{dy^*}{dV_T}\right| \\
&= \frac{k^*}{v}\left(e^{r\tau} V_t V_T^{\frac{2}{v}-3}\right)^{\frac{1}{2}} I_v\left(2k^*\left(e^{r\tau} V_t V_T\right)^{\frac{1}{2v}}\right) \\
&\quad \times \exp\left\{-k^*\left(V_t^{\frac{1}{v}} \exp\left(\frac{r\tau}{v}\right) + V_T^{\frac{1}{v}}\right)\right\}.
\end{aligned}
\tag{48}
$$

Obviously, if v tends to infinity, Equation (48) can reduce to the pdf of lognormal distribution. We have to mention that r in the above equations have to be replaced by μ if the market is not complete.

Similar with Equation (47), the log-likelihood of $L^S(\mu, \sigma, v; S_0, S_h, S_{2h}, \ldots, S_{nh})$ is

$$
\begin{aligned}
& L^S(\mu, \sigma, v; S_0, S_h, S_{2h}, \ldots, S_{nh}) \\
=& L^V(\mu, \sigma, v; \hat{V}_0(\sigma, v), \hat{V}_h(\sigma, v), \hat{V}_{2h}(\sigma, v), \ldots, \hat{V}_{nh}(\sigma, v)) - \sum_{i=1}^{n} \ln\left(\left|\frac{\partial S_{ih}}{\partial \hat{V}_{ih}(\sigma, v)}\right|\right) \\
=& n\left(\ln\frac{k^*}{v} + \frac{\mu h}{2}\right) + \frac{1}{2}\sum_{i=0}^{n-1} \ln \hat{V}_{ih}(\sigma, v) + \left(\frac{1}{v} - \frac{3}{2}\right)\sum_{i=1}^{n} \ln \hat{V}_{ih}(\sigma, v) \\
& + \sum_{i=0}^{n-1} \ln\left\{I_v\left(2k^*\left(e^{\mu h}\hat{V}_{ih}(\sigma, v)\hat{V}_{(i+1)h}(\sigma, v)\right)^{\frac{1}{2v}}\right)\right\} \\
& - \sum_{i=0}^{n-1} k^*\left(\hat{V}_{ih}^{\frac{1}{v}}(\sigma, v)\exp\left(\frac{\mu h}{v}\right) + \hat{V}_{(i+1)h}^{\frac{1}{v}}(\sigma.v)\right) - \sum_{i=1}^{n} \ln\left(\left|\frac{\partial S_{ih}}{\partial \hat{V}_{ih}(\sigma, v)}\right|\right)
\end{aligned}
\tag{49}
$$

where $\hat{V}_{ih}(\sigma, v) = g_2^{-1}(S_{ih}; \sigma, v)$ and $k^* = \frac{2\mu v}{\sigma^2(\exp(\frac{\mu h}{v})-1)}$. $\partial\hat{V}_{ih}(\sigma, v)/\partial S_{ih}$ can easily be calculated from Theorem 2.

The computing procedure is the same as mentioned in the previous section.

3.4.3 In JDF Model

Combine the Equation (27) and (35), we know the density function of asset value at time T, V_T, is

$$
\begin{aligned}
& f_Z(\ln(V_T/V_t)) = f\left(\mu - \frac{1}{2}\sigma^2 - \lambda\zeta, \sigma, \eta_1, \eta_2, \lambda, p; \ln(V_T/V_t), \tau\right) \\
=& \frac{e^{(\sigma\eta_1)^2\tau/2}}{\sigma\sqrt{2\pi\tau}}\sum_{n=1}^{\infty}\pi_n\sum_{k=1}^{n}P_{n,k}\left(\sigma\sqrt{\tau}\eta_1\right)^k e^{-d_\tau\eta_1} Hh_{k-1}\left(-\frac{d_\tau}{\sigma\sqrt{\tau}} + \sigma\sqrt{\tau}\eta_1\right) \\
& + \frac{e^{(\sigma\eta_2)^2\tau/2}}{\sigma\sqrt{2\pi\tau}}\sum_{n=1}^{\infty}\pi_n\sum_{k=1}^{n}Q_{n,k}\left(\sigma\sqrt{\tau}\eta_2\right)^k e^{d_\tau\eta_2} Hh_{k-1}\left(\frac{d_\tau}{\sigma\sqrt{\tau}} + \sigma\sqrt{\tau}\eta_2\right) \\
& + \frac{\pi_0}{\sigma\sqrt{\tau}}\varphi\left(\frac{d_\tau}{\sigma\sqrt{\tau}}\right),
\end{aligned}
\tag{50}
$$

where

$$
\begin{aligned}
d_\tau &= \ln(V_T/V_t) - \left(\mu - \frac{1}{2}\sigma^2 - \lambda\zeta\right)\tau, \quad \tau = T - t, \\
\zeta &= \frac{p\eta_1}{\eta_1 - 1} + \frac{(1-p)\eta_2}{\eta_2 + 1} - 1.
\end{aligned}
$$

Also, if $\pi_0 = 1$ and $\pi_n = 0$, $\forall n \geq 1$, the model will reduce to Merton model.

Before we derive the transform-data ML estimation in JDF model, we need to find the delta hedge of this model at first.

Theorem 4. *From Equation (40), the delta hedge of JDF model is*

$$\begin{aligned}\frac{\partial S_t}{\partial V_t} =&\Upsilon\left(r+\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\tilde{\lambda},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2;\ln\left(K/V_t\right),\tau\right)\\ &+f\left(r+\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\tilde{\lambda},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2;\ln\left(K/V_t\right),\tau\right)\\ &-\frac{Ke^{-r\tau}}{V_t}f\left(r-\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\lambda,p,\eta_1,\eta_2;\ln\left(K/V_t\right),\tau\right)\end{aligned} \tag{51}$$

All of the notations in Equation (51) are same as Equation (40).

Proof. Please see Appendix B. □

According to the Equation (51) and the ML Estimation transform data framework, the log-likelihood of $L^S(\mu,\sigma,\eta_1,\eta_2,\lambda,p;S_0,S_h,S_{2h},\ldots,S_{nh})$ is

$$\begin{aligned}&L^S(\mu,\sigma,\eta_1,\eta_2,\lambda,p;S_0,S_h,S_{2h},\ldots,S_{nh})\\ =&L^V(\mu,\sigma,\eta_1,\eta_2,\lambda,p;\hat{V}_0(\sigma,\eta_1,\eta_2,\lambda,p),\hat{V}_h(\sigma,\eta_1,\eta_2,\lambda,p),\ldots,\hat{V}_{nh}(\sigma,\eta_1,\eta_2,\lambda,p))\\ &-\sum_{i=1}^{n}\ln\left(\left|\frac{\partial S_{ih}}{\partial\hat{V}_{ih}(\sigma,\eta_1,\eta_2,\lambda,p)}\right|\right)\\ =&\sum_{i=1}^{n}\ln\left(f\left(\mu-\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\eta_1,\eta_2,\lambda,p;\ln(\hat{V}_{ih}/\hat{V}_{(i-1)h}),h\right)\right)\\ &-\sum_{i=1}^{n}\ln\left(\Upsilon\left(r+\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\tilde{\lambda},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2;\ln\left(K/V_{ih}\right),h\right)\right.\\ &+f\left(r+\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\tilde{\lambda},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2;\ln\left(K/V_{ih}\right),h\right)\\ &\left.-\frac{Ke^{-rh}}{V_t}f\left(r-\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\lambda,p,\eta_1,\eta_2;\ln\left(K/V_{ih}\right),h\right)\right)\end{aligned} \tag{52}$$

4 Monte Carlo Simulation

We do the Monte Carlo simulation to demonstrate which estimating approaches is the best under Merton's model.

From Equation (2), it is easy to derive the following equation:

$$\ln V_t=\ln V_{t-h}+\left(\mu-\frac{\sigma^2}{2}\right)h+\epsilon_t, \tag{53}$$

where

$$\epsilon_t = \sigma\left(W_t - W_{t-h}\right) \sim N(0, \sigma^2 h), \text{ and } Cov(\epsilon_t, \epsilon_{t-i}) = 0, \ \forall i \geq 1.$$

In this simulation, we set $r = 6\%$, $\mu = 0.1$, $\sigma = 0.3$ and initial firm value $V = 10000$. The data which is generated according to Equation (53) have daily (253 days each year) formate. Consider the debt which is maturity after two years has three possible face value, 3000, 5000 and 7000, which represent different leverage level of a company. We simulate one year data for 5000 pathes under each possible debt and assess market value in all different scenarios. Finally, we apply four different approaches to estimate unknown variable, V, and parameters, μ and σ, and comparing the results, which is shown in Table 2.

In this table, we use $\hat{\mu}$ and $\hat{\sigma}$ to denote the estimating results of unknown parameters μ and σ, and $\hat{V}_1$ is the implied asset value at the end of one year. All statistics are come from sample estimation, that is, the results of 5000 pathes estimation results. Although the bias of all approaches are increased when increase K, the best approach is KMV, then MLE, JMR-RV, and the worse approach is EHH. The standard deviation in all approaches are also increased as K is increased.

Although KMV looks better than MLE approach, it can't provide any point estimation information. For MLE approach, we can consider MLEs to be consistent and asymptotically efficient by Cramér-Rao Lower Bound. Duna [12] has shown the results. Besides, because KMV approach can't update other unknown parameters, it also don't suitable to estimate the model which contains other unknown variables. Barrier option pricing model, as proposed by Brockman and Turtle [5], and CEV model are examples.

5 Data and Empirical Study

5.1 Data

We investigate the performance of these models in Taiwanese industry in our empirical study. The data formate is weekly (in case study is daily), and collected from Taiwan Economic Journal (TEJ). We pick all firms which list on Taiwan Stock Exchange Corporation (TSE) during 2001 to 2004. The default event follows Article 49, 50, and 50-1 of "Operating Rules of the Taiwan Stock Exchange Corporation" to define whether the firms are bankruptcy or not during the sample period. In other words, the corporation is called bankruptcy if it has one of these situations, "altered-trading-method", "suspend the trading of such securities", or "Unlisted". We adopt market value and debt value data from two years ago to one year before if the firm has bankruptcy juring our sample period. Otherwise, we adopt all data in 2003 to predict the default probability at the end

K	Statistic	Approach	$\hat{\mu}$	$\hat{\sigma}$	$\hat{V}_1 - V_1$
		True	0.1000	0.3000	0.0000
3000	Mean	JMR-RV	–	0.2975	0.2542
		KMV	0.1008	0.2999	-0.0083
		EHH	0.0395	0.2925	175.0257
		MLE	0.1008	0.2999	-0.0376
3000	Median	JMR-RV	–	0.2991	0.0000
		KMV	0.0949	0.3000	0.0000
		EHH	0.0315	0.2942	174.7085
		MLE	0.0948	0.3000	0.0000
3000	Std	JMR-RV	–	0.0251	1.8459
		KMV	0.2972	0.0134	0.5064
		EHH	0.2920	0.0257	2.0560
		MLE	0.2971	0.0137	0.6147
5000	Mean	JMR-RV	–	0.2917	15.9458
		KMV	0.1009	0.3000	-0.1692
		EHH	0.0297	0.2825	314.3872
		MLE	0.1041	0.3113	-7.1959
5000	Median	JMR-RV	–	0.2967	0.0828
		KMV	0.0958	0.3000	0.0000
		EHH	0.0182	0.2882	293.6056
		MLE	0.0948	0.3101	-0.2892
5000	Std	JMR-RV	–	0.0480	56.2363
		KMV	0.2973	0.0146	10.2210
		EHH	0.2828	0.0499	64.7591
		MLE	0.2964	0.0188	19.7715
7000	Mean	JMR-RV	–	0.2749	119.8667
		KMV	0.1011	0.3002	-0.9099
		EHH	0.0229	0.2556	595.5345
		MLE	0.1127	0.3268	-40.2969
7000	Median	JMR-RV	–	0.2833	12.1483
		KMV	0.0953	0.3000	0.0002
		EHH	-0.036	0.2652	472.2824
		MLE	0.1118	0.3275	-23.9145
7000	Std	JMR-RV	–	0.0770	273.2714
		KMV	0.2973	0.0185	49.5603
		EHH	0.2545	0.0824	305.9758
		MLE	0.2986	0.0241	79.2341

Table 2: Simulation results.

of 2004. If the trading data is less than one year, we omit this firm. The trading day in each year is set 53 weeks (in case study is 253 days). Thus, we use 53 trading data in each firm to predict its default probability after one year. As the result, the sample size is 618. 64 of them have default event juring the four years.

Because the liquidity of the Taiwanese bond market is not very well, we adopt One Year Time Deposits interest rate of Bank of Taiwan for interest rate parameter. Besides, the same as Vassalou and Xing [43], we use the "Debt in One Year" plus half of the "Long-Term Debt" to represent variable K in our model.

5.2 Testing Methodology

We use following three nonparametric methods to test the performance of our model result.

5.2.1 Kolmogorov-Smirnov Test

The Kolmogorov-Smirnov (K-S) test is a very useful tool to judge whether two distributions have common distribution function or not. Test statistic D is the maximum difference between two distribution function. Ross [35] showes how to test whether the empirical distribution function of sample points fit a parametric distribution function.

In addition to one sample test, K-S test also can judge whether two empirical distribution functions can fit with each other. We use this method to test whether our models can predict default event or not. If the model is powerful, the cumulative distribution of model's output in all bankruptcy firms will be different with survival firms. Thus, we divide the sample into two class, A and B. Class A contains all predict result of survival firms. Class B contains all others. The hypothesis test H_0 is the class A and B have same empirical distribution function. The test procedure is shown as follows.

1. Calculate the cumulative bankruptcy probability of class A, $S_d(x)$.
2. Calculate the cumulative survival probability of class B, $S_{nd}(x)$.
3. Find the K-S statistic $D = \max_x |S_d(x) - S_{nd}(x)|$.
4. Set the significant level α. Calculate the P-value $P_F = P(D \leq d)$. If $P_F \leq \alpha$, then we reject H_0. Otherwise, we can't reject our hypothesis.

5.2.2 Cumulative Accuracy Profile

K-S test can measure whether the model can discriminate the default and survival groups. However, it can not compare the power of two different rating systems. Cumulative Accuracy Profile (CAP) is a good method to assess them. Engelmann,*et al* [17] show statistical properties in the method. We only introduce the basic idea and describe how to employ it in our models.

Consider the rating system contains k different scores, $\{s_1, s_2, \ldots, s_k\}$, $s_1 < s_2 < \ldots < s_k$. This system can assign one of them to each debtor. Follows the K-S test, consider the class A and B has distribution S_d and S_{nd}, respectively. S_T is the distribution of total firms. Assume the defaulter has probability p_d^i to earn score value s_i , $\sum_{i=1}^k p_d^i = 1$. Alternatively, the survival firms have probability p_{nd}^i to be assigned score value s_i. Given the default probability π of all debtors, we can suspect the p_T^i is

$$p_t^i = \pi p_d^i + (1-\pi)p_{nd}^i$$

for any debtor has a score value s_i. The cumulative probabilities are defined by

$$CD_d^i = \sum_{j=1}^i p_d^j, \quad i = 1, \ldots k$$

$$CD_{nd}^i = \sum_{j=1}^i p_{nd}^j, \quad i = 1, \ldots k$$

$$CD_t^i = \sum_{j=1}^i p_t^j, \quad i = 1, \ldots k.$$

The CAP curve is obtained by the graph of all points $(CD_t^i, CD_d^i)_{i=0,\ldots,k}$ where the points are connected by a straight line. The concave curve of the rating system means the system is perfect. Otherwise, if the curve is a diagonal line, this means the system assign score randomly. It can't predict the default event very well. We show the graph in Figure 4.

Let the area of perfect model is a_p, and the area of rating system is a_r. We can define the Accuracy Ratio, AR, is

$$AR = \frac{a_r}{a_p},$$

where $0 \leq AR \leq 1$. The higher of the AR is, the better of the rating model.

5.2.3 Receiver Operating Characteristic

Similar with CAP, Engelmann *et al* [17] demonstrate the statistical properties about Receiver Operating Characteristic (ROC). We also introduce some important definitions

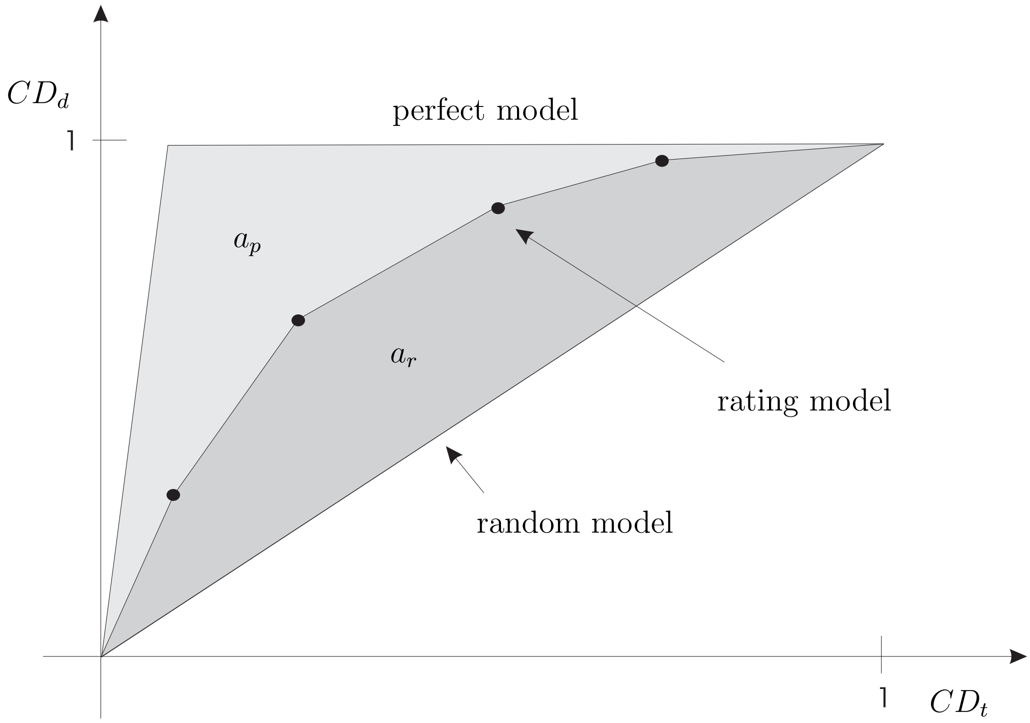

Figure 4: CAP curve.

Rating Score	Default	No Default
Below C	Correct	Wrong
Above C	Wrong	Correct

Table 3: Decision result given the cut-off value C.

and notations at here. Assume someone who assign cut-off value C to classify each debtor into two groups. The debtor whose score less than C means it will default latter, and higher than C if it will non-default. Thus, there are four situation for this prediction, which summarized in Table 3.

If the score of debtor is less than C and the debtor is default subsequently, it means our predict is correct. Otherwise, the rating system make a wrong decision for this debtor (type I error). Alternatively, if the rating score is higher than C and the debtor is survival, then the prediction is correct. Otherwise, also, the decision is wrong (type II error).

Under this assumption, we define the hit rate $HR(C)$ as

$$HR(C) = P(S_d \leq C).$$

The false alarm rate FAR(C) is defined as

$$FAR(C) = P(S_{nd} \leq C)$$

For different cut-off value C, we can compute its $HR(C)$ and $FAR(C)$. The ROC curve is constructed by plotting $HR(C)$ versus $FAR(C)$, for all C. This method is equivalent to connect all points $(CD^i_{nd}, CD^i_D)_{i=0,\ldots,k}$. We show the graph in Figure 5.

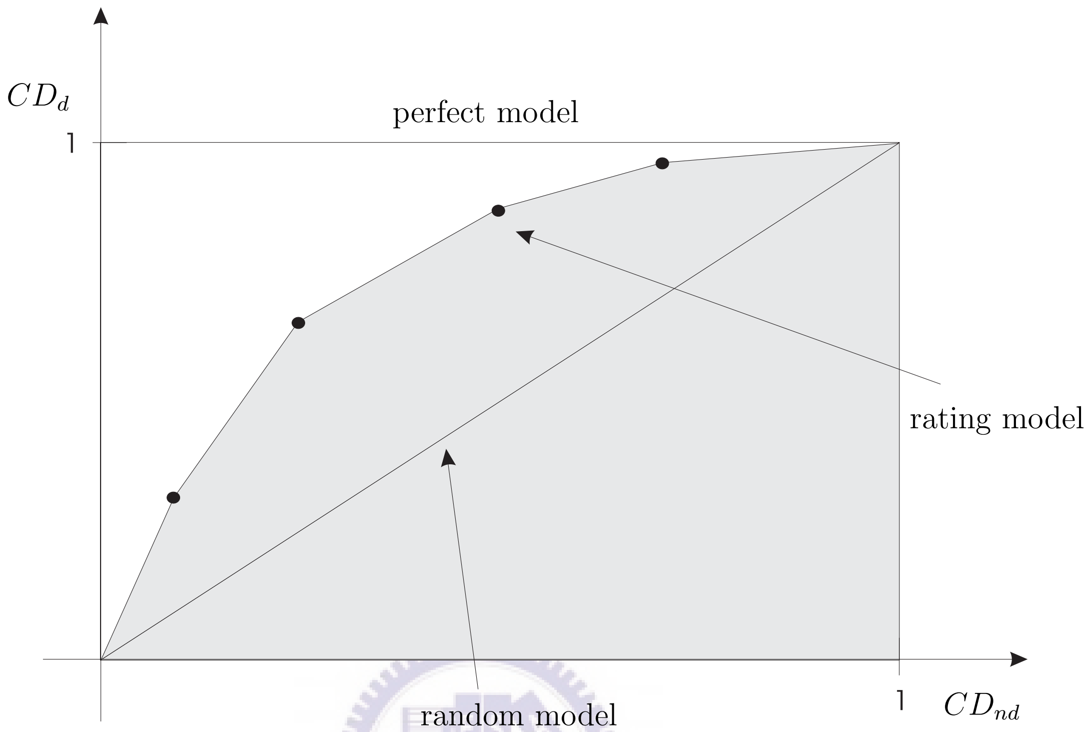

Figure 5: ROC curve.

5.3 Empirical Results

5.3.1 Case Study

We adopt Procomp Informatics Ltd. to do case study. This company is builded in 1990, and the main product is sound card. About ten years ago, Procomp founded the photoelectricity department. It changed the main product to gallium arsenide microchip and IC design. Procomp also do a lot of 3C products, like motherboard, graphic card and sound card. Although Procomp was a famous electronic company in overseas, it didn't sell any product in Taiwan. Procomp planed to become the best microchip company in the world in 2000.

However, in 1999, Procomp CEO do a lot of "fail sale" in order to increase the "revenue". In addition, the CEO took over the firms capital about 500 million NT dollars. The CFO also changed very quickly since 1999. Under these events, we can deduce that there are a lot of manage problems in company.

Because Procomp is listed since 1999/12/18, and default at 2004/06/15, we adopt all trading data exclude one year of default date, i.e., from 1999/12/18 to 2003/06/18. We

i	$S_{i,h}$	$T-ih$	$\hat{V}_{ih}^{KMV}$	$\hat{V}_{ih}^{MLE}$
246	6224400000	1.020	13645366000	13633708000
247	5882400000	1.016	13296957000	13283675000
248	5985000000	1.012	13402714000	13390083000
249	6121800000	1.008	13543182000	13531345000
250	5882400000	1.004	13299525000	13286579000
251	5882400000	1.000	13300377000	13287542000

Table 4: KMV and MLE estimating results for Procomp Informatics company.

Model	Lognormal	CEV
K-S Value	6.449	6.049
P-value	0	0

Table 5: The K-S test result.

are interested in which estimating method can not only give an alarm as soon as possible, but also forecast default probability very well, for one year predict power. The results are shown in following four graphs.

In Figure 6, it is very clear that the default probability is almost zero although the time is close to the end of sample. The maximum default probability in our sample is 0.0442 at date 824. It is also very clear that EHH method in Figure 7 is worse than JMR-RV result.

From Figure 8 and 9, we can find the shape all curves in these two estimating method are very similar. Duan *et al* [14] has showed the equivalent result in theoretically. Our estimating result in Table 4 also support this idea. For all $\hat{\mu}$ and $\hat{\sigma}$ are: $\hat{\mu}^{KMV} = -0.358$, $\hat{\sigma}^{KMV} = 0.3116$, $\hat{\mu}^{MLE} = -0.3535$, $\hat{\sigma}^{MLE} = 0.328$. Because MLE approach can evaluate the standard error to measure the asymptotic efficiency, and the values are $s.e.(\hat{\mu}^{MLE}) = 0.328$, and $s.e.(\hat{\sigma}^{MLE}) = 0.016154375$, we know our estimation is efficiency.

5.3.2 Power of the Rating Systems

We use K-S test to measure the discriminative power between Merton's model and CEV model. The result is shown in Table 5. Because the p-value of two models are booth zero, we can conclude that these models have the power to discriminate two different events very well.

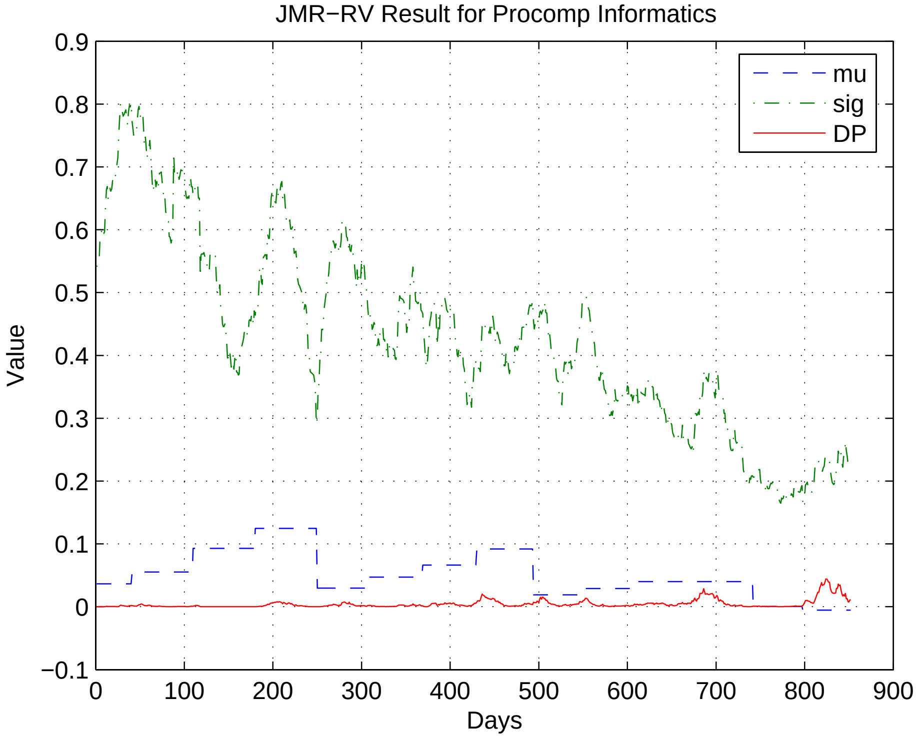

Figure 6: JMR-RV estimating method result.

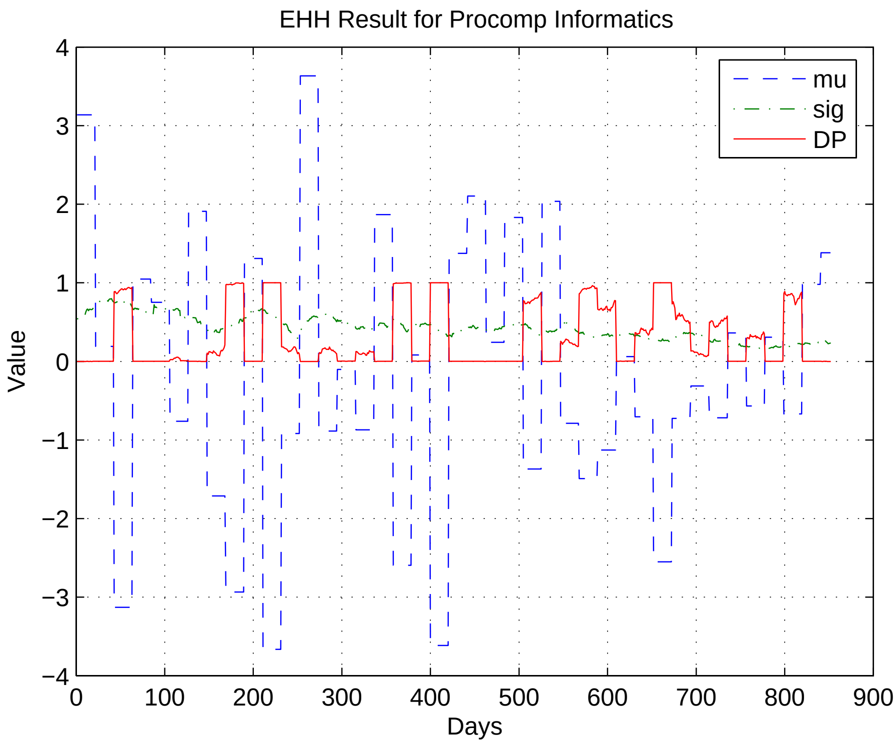

Figure 7: EHH estimating method result.

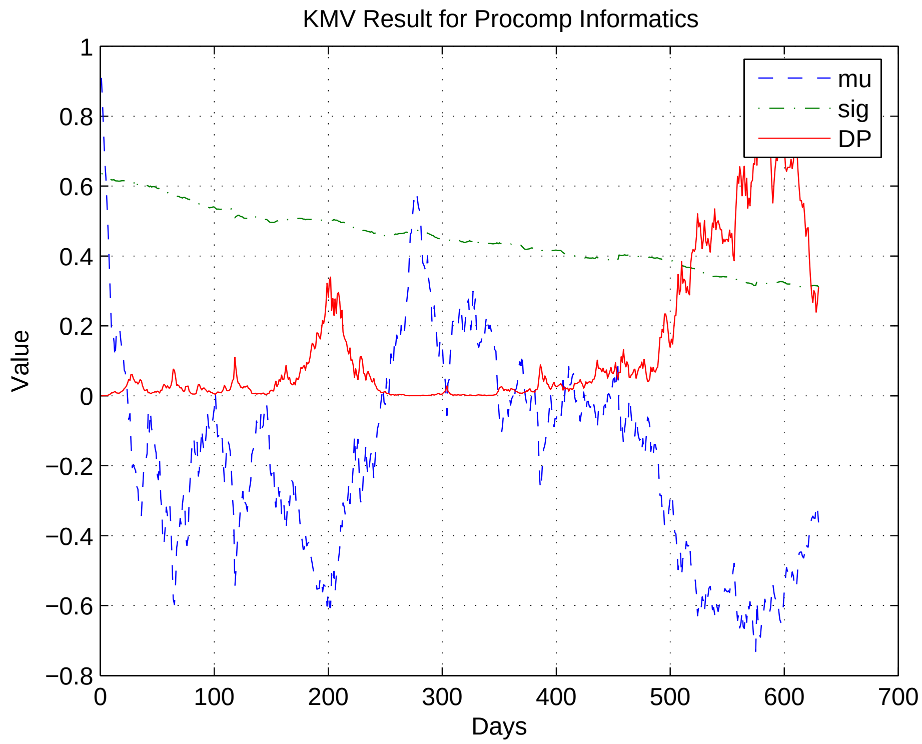

Figure 8: KMV estimating method result.

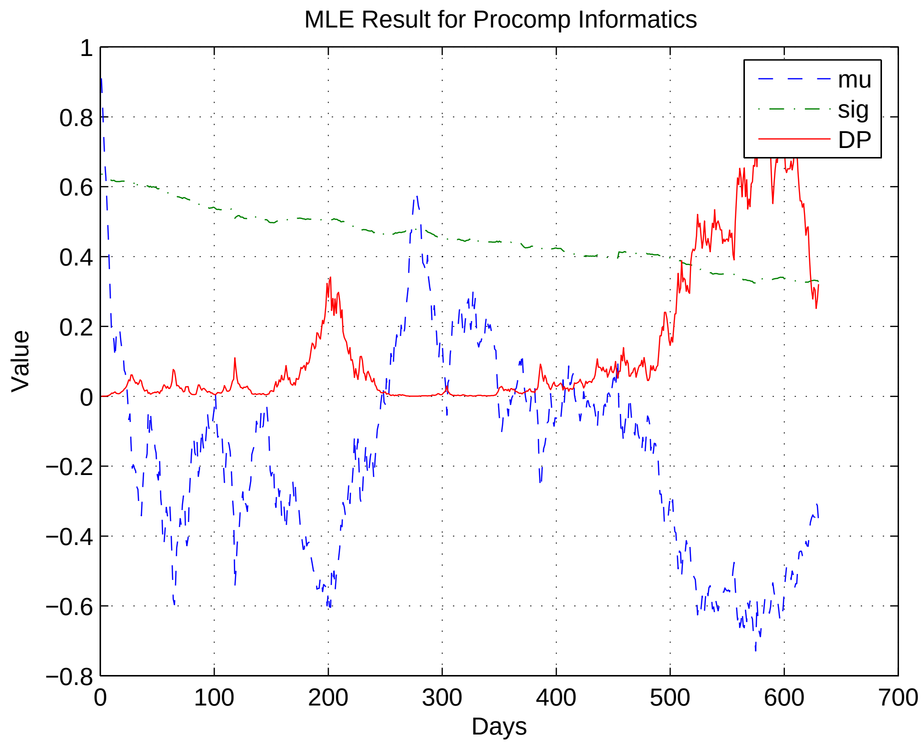

Figure 9: MLE transform data method result.

However, in AR test, we see the AR of Merton model is 0.9359, and CEV model is 0.8896. Merton model is more powerful than CEV. In CAP curve and ROC curve also show the similar results. Thus the CAP, ROC and AR test are consistent.

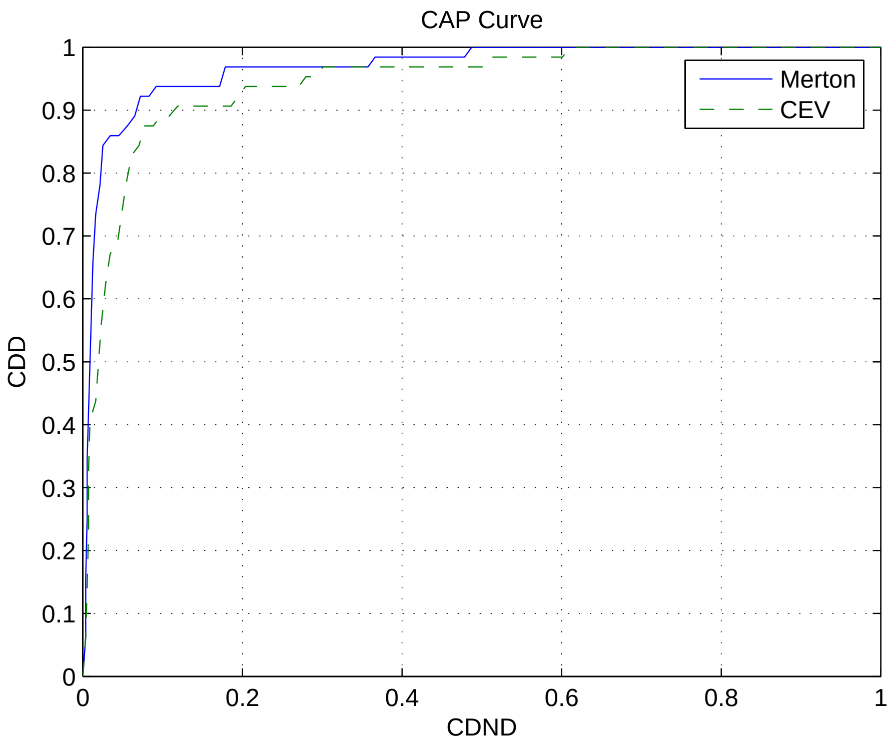

Figure 10: CAP for Merton and CEV models.

6 Conclusions

In this paper, we apply several different firm value assumptions to test which one can fit the Taiwanese company very well. As the empirical results, the Merton model is prefer than CEV. Although the CEV model is more consistent with options price than Merton model, our study can not support this result in credit risk area.

To investigate why the CEV model is under performance, we due to the following reasons:

1. Although some firms has high default probability in CEV model and not default after the end of one year, they usually default before 1.5 year. That is, they still have high probability to default after the end of one year. Tsin Tsin Corp. is a good example. In Merton and CEV model, the default probability at the end of 2004 are 1.8% and 15.12%, respectively. The default event is occurred at 2005/04/22. Thus the CEV model assign a higher default probability than Merton model, even the company still live at the end of 2004.

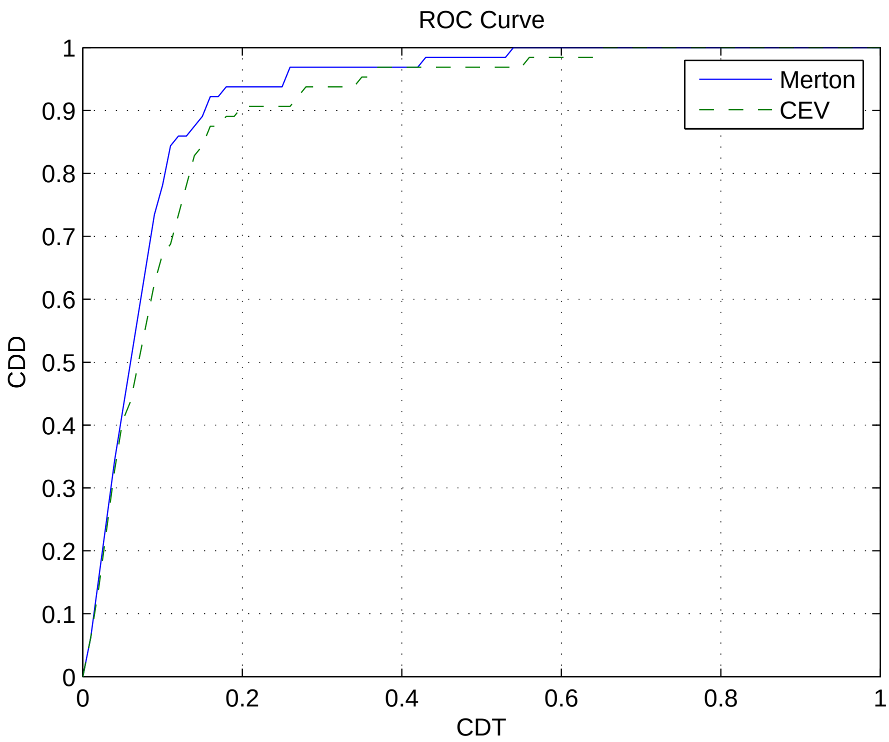

Figure 11: ROC for Merton and CEV models.

2. Some survival firms have higher liability in the third or forth quarter and lower market value in the forth quarter than other time period. The noncentral chisquare distribution can catch this trend. However, the lognormal distribution can not catch very well. Although CEV model do well job in catch trend, the debt of firm actually not default after the end of one year. Edimax Technology Co., Ltd. is a good example. At the begin of 2003, the market value is $9.88E + 08$ dollars, and the debt value is $6.02E + 08$ dollars. However, at the end of 2003, the market value is $5.44E + 08$ dollars, and the debt value is $8.26E + 08$ dollars. That is, the market value is decrease and debt value is increase largely. The forecasting default probability at the end of 2004 for Merton and CEV model are 8.22% and 25.41%, respectively. From fundamental option pricing theorem and the numerical result, we find the CEV model is more reasonable than Merton model.

3. Since maximum likelihood estimator of parameters in noncentral chisquare distribution is very hard to derive, even in numerical method, the computing procedure for some firms maybe escape the calculating loop although the numerical results are not converge.

We should discuss these problems much detail in future.

Although the power of CEV model is less than Merton model, our empirical study shows not all firm value data fit lognormal distribution. In CEV model, if v is very small, the distribution will be different with lognormal distribution. Table 6 shows all v value for all firms in our empirical study. Obviously, not all of v higher than 20. Some of them are less than one. It means not all firm value fit lognormal distribution. CEV model is much better than Merton model to fit firm value distribution.

Alternatively, although we propose the double exponential JDF model, we don't provide any empirical result in this paper. Because the model has five unknown parameters, they are too much to get a converge results in numerical estimation. We will find a good initial value to estimate them and assess the default probability under this model in future.

Ticker	v	Ticker	v	Ticker	v	Ticker	v
1101	15.807916	1307	4.7945983	1445	89.782542	1516	11.265695
1102	19.680323	1308	0.53807518	1446	10.297676	1517	0.49888758
1103	0.41322393	1309	0.50127878	1447	15.398132	1519	22.876671
1104	0.43906694	1310	0.55008135	1449	14.084247	1520	280.84972
1107	20.159997	1311	2.8376321	1450	0.32742237	1521	0.30815183
1108	12.703216	1312	14.057446	1451	78.09191	1522	92.136818
1109	233.90125	1313	0.65877008	1452	59.613859	1523	0.43036493
1110	25.658117	1314	16.307827	1454	102.94189	1524	11.416662
1201	14.655319	1315	48.271545	1455	33.645969	1525	3.0659876
1204	0.31349137	1316	16.244498	1456	13.523349	1526	13.957526
1207	7.0250939	1319	28.784252	1457	14.05625	1527	208.10771
1210	0.24654111	1321	17.889003	1458	12.120736	1528	8.0030019
1212	7.4403538	1323	0.11523948	1459	12.55042	1529	17.480487
1213	141.34409	1324	10.692452	1460	59.463652	1530	0.60429701
1215	0.3032653	1325	48.935439	1462	21.916174	1531	0.22097637
1216	9.6055657	1326	7.4478008	1463	267.08961	1532	8.80989
1217	96.228783	1402	14.081433	1464	27.06358	1533	88.70068
1218	8.5663791	1407	14.784967	1465	0.56118483	1534	10.574197
1219	0.26316806	1408	14.119778	1466	11.780401	1535	9.9592787
1220	61.149794	1409	19.152906	1467	0.46815663	1536	0.27043096
1221	9.1753656	1410	393.7002	1468	172.19052	1537	0.38398095
1224	0.31428558	1413	18.233843	1469	0.3973995	1538	66.360339
1227	0.2161895	1414	14.496862	1470	80.751554	1539	0.43050166
1228	18.841664	1416	12.728027	1471	27.073876	1540	78.991506
1229	15.792868	1417	5.26513	1472	14.911103	4526	2.1118375
1231	47.982197	1418	0.50154198	1473	253.66	4532	38.031503
1232	0.65312152	1419	29.709297	1474	0.22752312	1601	13.723861
1233	11.856904	1422	0.41967498	1475	0.33842562	1602	10.381605
1234	113.68638	1423	31.104663	1476	24.148386	1603	8.4313653
1235	110.88612	1431	21.976953	4414	18.616855	1605	18.957858
1236	0.36218072	1432	7.8466616	1503	9.4046014	1606	10.734243
8722	15.546874	1434	9.9843801	1504	8.72542	1608	35.79069
1301	9.1276637	1438	40.287379	1507	14.953109	1609	9.4603649
1303	15.10464	1439	101.95875	1512	0.40479641	1611	0.2800399
1304	19.915046	1440	15.781695	1513	8.1046387	1612	84.169884
1305	16.996511	1443	10.410289	1514	48.5553	1613	15.376802
1306	0.82269705	1444	18.354448	1515	11.288385	1614	0.4419928

Continue

Continue

Ticker	v	Ticker	v	Ticker	v	Ticker	v
1615	7.3258568	1809	14.785111	2106	15.541137	2333	16.710656
1616	33.778899	1810	14.607316	2107	9.7329192	2335	0.40469672
1617	202.10495	1902	N.A.	2108	159.8126	2336	0.14766339
1618	88.999011	1903	1.3666143	2109	104.38995	2337	31.025494
1701	17.875625	1904	14.560068	2201	0.83704421	2338	106.30455
1702	0.62850101	1905	142.67932	2204	34.67092	2340	15.652741
1704	13.14391	1906	8.0514403	2206	10.34686	2341	11.551822
1708	119.49717	1907	18.639282	2207	14.905526	2342	20.334256
1709	1.0423463	1909	28.444255	1435	40.716727	2343	18.605525
1710	0.49711569	2002	10.052166	1437	0.38710347	2344	20.463139
1711	67.548202	2006	11.902454	1453	0.53566289	2345	12.31367
1712	0.26716336	2007	13.384177	1604	16.643085	2347	1.4094506
1713	2.5783918	2008	9.2549662	2301	7.943008	2348	12.376806
1714	29.556493	2009	35.176454	2302	0.44720803	2349	16.336839
1715	20.76901	2010	2.3459327	2303	19.495424	2350	13.664259
1716	134.00546	2012	8.1295176	2304	32.308407	2351	0.47783137
1717	32.346586	2013	12.055172	2305	0.45093949	2352	21.751811
1718	15.07238	2014	19.447409	2308	3.2409358	2353	71.493388
1720	154.34074	2015	0.51429858	2311	6.6939709	2355	1.9545645
1721	35.148805	2017	18.319816	2312	15.957428	2356	14.976199
1723	0.2630486	2022	0.55888406	2313	14.527227	2357	44.655452
1724	160.97977	2023	12.514533	2314	13.957428	2358	28.995508
1725	50.418267	2024	17.078684	2315	0.36623855	2359	14.345877
1726	3.4642013	2025	23.888428	2316	0.57067206	2360	0.19184421
1727	0.31947767	2027	16.581368	2317	14.680037	2361	0.80205754
1729	209.11625	2029	16.027227	2318	30.031286	2362	1.2675568
1730	0.54400627	2030	0.47048353	2321	17.98297	2363	18.875253
1731	0.13453901	2031	0.38239927	2323	34.258143	2364	13.204818
1732	209.8608	2032	0.58040902	2324	27.401684	2365	6.964435
1733	1.6160904	2033	38.559837	2325	8.2101807	2366	0.36560929
1734	101.00931	2034	0.65156081	2326	19.261034	2367	15.553783
1735	49.074978	2101	N.A.	2327	26.37557	2368	7.1882225
1802	275.06774	2102	0.2840711	2328	108.01729	2369	0.6519674
1805	18.192573	2103	15.702141	2329	11.379737	2370	34.821777
1806	12.617012	2104	17.981393	2330	4.6285402	2371	10.539285
1807	38.457594	2105	0.76933408	2332	0.43945417	2373	17.206298

Continue

Continue

Ticker	v	Ticker	v	Ticker	v	Ticker	v
2374	0.53412698	2413	12.300155	2450	85.618348	2486	48.441875
2375	76.832005	2414	12.163123	2451	0.5788923	2487	5.6409931
2376	104.23045	2415	215.97714	2452	59.580263	2488	29.977097
2377	17.02533	2416	16.009897	2453	0.70487603	2489	5.5334689
2378	22.216958	2417	3.7466813	2454	127.11192	2490	15.767782
2379	17.539154	2418	13.617424	2455	15.775537	2491	15.943891
2380	147.94358	2419	13.29592	2456	46.039658	2492	0.54516226
2381	17.84502	2420	132.86067	2457	0.63737184	2493	12.595111
2382	19.778629	2421	12.711949	2458	5.2947103	2494	28.08291
2383	32.800496	2422	37.070435	2459	67.344754	2495	602.16418
2384	0.50368739	2423	101.07127	2460	0.45331745	2496	44.488324
2385	19.162086	2424	61.363318	2461	20.178857	2497	162.97763
2387	5.7443424	2425	12.333916	2462	0.42485808	2498	1.7673094
2388	28.134861	2426	0.29526333	2463	102.53624	2499	0.14204479
2389	18.249108	2427	48.997709	2464	0.30888446	2544	0.48634809
2390	36.503427	2428	86.939416	2465	0.35944628	3001	18.43568
2391	2.2394213	2429	0.33050079	2466	15.202508	3002	0.4802825
2392	0.22738523	2430	0.16641713	2467	57.159233	3003	121.53535
2393	106.87882	2431	70.434028	2468	13.184931	3004	9.6769648
2394	4.7994809	2432	55.613369	2469	6.9060712	3005	11.943903
2395	0.45255792	2433	14.031004	2470	26.387472	3006	0.48900396
2396	48.068192	2434	99.863424	2471	182.1732	3007	1.2179026
2397	164.88793	2435	7.3452326	2472	0.67689126	3008	634.53644
2398	18.32525	2436	0.30867767	2473	109.3682	3009	17.442787
2399	0.50567283	2437	428.17313	2474	275.91083	3010	0.47966825
2401	0.63488814	2438	0.47933237	2475	15.923493	3011	0.43804661
2402	74.66147	2439	137.4578	2476	43.841853	3012	22.710451
2403	0.46292925	2440	12.024255	2477	0.3028424	3013	0.34718506
2404	28.91038	2441	1.364366	2478	2.6016339	3014	0.47923401
2405	49.413363	2442	10.004294	2479	12.884793	3015	30.486869
2406	0.79037101	2443	16.828574	2480	0.43820239	3016	52.714724
2407	5.9941829	2444	0.38711787	2481	0.48558484	3017	21.388432
2408	10.276401	2446	65.270968	2482	404.60882	3018	12.304879
2409	10.034583	2447	0.27491427	2483	489.10742	3019	7.0133003
2411	198.31909	2448	54.693446	2484	63.513002	3020	0.33295334
2412	14.390006	2449	35.597758	2485	0.65364061	3021	18.390914

Continue

Continue

Ticker	v	Ticker	v	Ticker	v	Ticker	v
3022	1.2709064	3059	63.662679	2501	164.74904	2603	18.539024
3023	0.36044549	4096	0.45525316	2504	11.59083	2605	48.012532
3024	0.25636792	5023	117.90618	2506	10.545541	2606	0.70541945
3025	0.52874198	5305	0.46766162	2509	0.27831908	2607	66.812608
3026	9.9655783	5434	121.80133	2511	11.177028	2608	9.5410088
3027	0.21173826	5469	75.688434	2512	21.953106	2609	0.46106138
3028	0.35958859	5471	0.45005766	2514	39.539587	2610	7.3617178
3029	0.23250006	5484	12.872097	2515	11.044165	2611	41.338155
3030	1.9513724	6112	0.35748337	2516	14.256062	2612	165.59671
3031	1.38953	6115	264.45211	2517	14.844087	2613	0.39389514
3032	7.5558311	6116	14.787256	2518	16.481707	2614	7.6613194
3033	8.9844111	6117	8.6051653	2520	15.701434	2615	3.8040159
3034	11.432979	6119	0.42542013	2523	7.6804306	2616	0.33545315
3035	0.56492946	6128	0.17306457	2524	16.785254	2617	284.6432
3036	13.731599	6131	160.19386	2525	16.27133	2618	8.6113553
3037	29.061584	6132	109.00478	2526	8.632467	5607	0.36310734
3038	86.488286	6133	63.044942	2528	11.896091	5608	42.98233
3039	13.004953	6136	0.42187884	2530	19.450004	2701	339.43832
3040	0.50855452	6139	65.220856	2533	25.788547	2702	26.01889
3041	0.48489083	6141	236.3014	2534	13.309915	2704	6.022627
3042	71.725393	6142	52.134964	2535	0.33675743	2705	0.61685644
3043	286.25738	6145	0.65731051	2536	3.7179111	2706	200.77887
3044	0.49183953	6165	11.469234	2537	0.63004606	2707	3.6998207
3045	57.847536	6166	150.35205	2538	12.83936	2901	249.68028
3046	18.807294	6168	0.27937297	2539	18.099372	2902	13.379019
3047	0.28033771	6172	0.5174026	2540	13.552301	2903	21.056324
3048	5.4114185	6189	2.5771611	2542	19.243972	2905	29.292632
3049	16.833259	6192	46.60068	2543	13.940892	2906	6.7282715
3050	0.53367606	6196	105.90507	2545	0.44508709	2908	19.937711
3051	35.581793	6197	437.06246	2546	10.905093	2910	18.189326
3052	27.083219	6202	0.90953255	2547	16.589215	2911	11.582979
3053	11.208225	6206	60.67007	2548	19.687774	2912	60.137393
3054	24.120323	6209	877.46921	5515	0.64754991	2913	14.199292
3055	199.33973	8008	92.974526	5525	19.889303	2915	0.22026195
3057	184.86788	9912	0.49926319	5534	4.4202737	9801	10.018894
3058	0.37125725	1436	15.726472	2601	0.63867515	2904	15.598134

Continue

Continue

Ticker	v	Ticker	v	Ticker	v	Ticker	v
6201	407.33788	9914	0.357053	9927	0.40131129	9937	8.8818871
8926	25.38709	9915	15.193508	9928	134.02736	9938	0.43673316
9902	14.202463	9917	116.01749	9929	7.8869169	9939	17.855105
9904	0.4356872	9918	5.2167601	9930	107.33184	9940	440.19738
9905	297.46936	9919	0.23416048	9931	0.21035339	9941	3.1766857
9906	218.56611	9921	0.91152524	9933	0.3723244	9942	0.46424946
9907	10.016128	9922	19.155413	9934	36.164242	9943	21.339835
9908	128.01831	9924	78.737605	9935	17.55793	9944	152.6052
9910	135.63028	9925	117.46192	9936	20.16259	9945	12.789164
9911	9.8365288	9926	178.99729				

Table 6: The v value of all firms. N.A. means the numerical procedure does not converge.

Appendices

A The Proof of Kolmogorov Forward Equation

The prove of this equation is an exercise in Shreve [39]. We just follow the hint and show more detail.

Let b be a positive constant and let $h_b(y)$ be a function with continuous first and second derivatives such that $h_b(x) = 0$ for all $x \leq 0$, $h_b^{'}(x) = 0$ for all $x \geq b$, and $h_b(b) = h_b^{'}(b) = 0$. Let $X(u)$ be the solution to the stochastic differential equation with initial condition $X(t) = x \in (0, b)$. Under these assumptions, we can use Itô's lemma on $h_b(X(u))$ to get the following result.

$$\begin{aligned} dh_b(X(u)) =& h_b^{'}(X(u))dX(u) + \frac{h_b^{"}(X(u))}{2}(dX(u))^2 \\ =& h_b^{'}(X(u))(\beta(u, X(u))du + \gamma((u), X(u))dW(u)) \\ &+ \frac{h_b^{"}(X(u))}{2}(\gamma^2(u, X(u)))du. \end{aligned} \tag{54}$$

Let $0 \leq t \leq T$ be given, and integrate Equation (54) from t to T. Take expectations on both sides and use the fact that $X(u)$ has density $p(t, u, x, y)$ in the y-variable to obtain

$$\begin{aligned} &\int_0^b h_b(y)p(t, T, x, y)dy - h_b(x) = \mathbb{E}\left(\int_t^T dh_b(X(u))\right) \\ =& \mathbb{E}(h_b(X(T)) - h_b(X(t))) \\ =& \mathbb{E}\left(\int_t^T \beta(u, X(u))h_b^{'}(X(u))du + \int_t^T \gamma((u), X(u))h_b^{'}(X(u))dW(u) \right. \\ &\left. + \frac{1}{2}\int_t^T \gamma^2(u, X(u))h_b^{"}(X(u))du\right) \\ =& \int_0^b \int_t^T \beta(u, y)p(t, u, x, y)h_b^{'}(y)dudy + \frac{1}{2}\int_0^b \int_t^T \gamma^2(u, y)p(t, u, x, y)h_b^{"}(y)dudy \\ =& \int_t^T \int_0^b \beta(u, y)p(t, u, x, y)h_b^{'}(y)dydu + \frac{1}{2}\int_t^T \int_0^b \gamma^2(u, y)p(t, u, x, y)h_b^{"}(y)dydu. \end{aligned} \tag{55}$$

Integral the integrals $\int_0^b \cdots dy$ on the right-hand side of Equation (55) by parts to obtain

$$\begin{aligned} \int_0^b h_b(y)p(t, T, x, y)dy =& h_b(x) - \int_t^T \int_0^b \frac{\partial}{\partial y}[\beta(u, y)p(t, u, x, y)h_b(y)]dydu \\ &+ \frac{1}{2}\int_t^T \int_0^b \frac{\partial^2}{\partial y^2}[\gamma^2(u, y)p(t, u, x, y)]h_b(y)dydu. \end{aligned} \tag{56}$$

Differentiate Equation (56) with respect to T to obtain

$$\begin{aligned}\int_0^b h_b(y)\left[\frac{\partial}{\partial T}p(t,T,x,y)+\frac{\partial}{\partial y}(\beta(T,y)p(t,T,x,y))\right.\\ \left.-\frac{1}{2}\frac{\partial^2}{\partial y^2}(\gamma^2(T,y)p(t,T,x,y))\right]dy=0.\end{aligned} \tag{57}$$

Finally, let

$$g(y)=\frac{\partial}{\partial T}p(t,T,x,y)+\frac{\partial}{\partial y}(\beta(T,y)p(t,T,x,y))-\frac{1}{2}\frac{\partial^2}{\partial y^2}(\gamma^2(T,y)p(t,T,x,y)). \tag{58}$$

Consider $0 \le y_1 \le y_2 \le b$. Since $h_b(y)$ is any second derivative continuous funcion which satisfies some conditions as above, we can find $g(y)=0$, for all $y \in (y_1, y_2)$. This can conclude that if $g(y)$ is a continuous function, then $g(y)=0$ for every $y>0$, and hence $p(t,T,x,y)$ satisfies Equation (11).

B The Proof of Theorem 4

From Equation (37) and (40), we know

$$\begin{aligned}&\frac{\partial}{\partial V_t}\Upsilon\left(r-\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\lambda,p,\eta_1,\eta_2;\ln\left(K/V_t\right),\tau\right)\\
=&\frac{\partial}{\partial V_t}P\left(Z(\tau)>\ln(K/V_t)\right)\\
=&\frac{\partial}{\partial V_t}\sum_{n=0}^{\infty}\pi_n P\left(\sigma\sqrt{\tau}Z+\sum_{k=1}^{n}Y_k>\ln(K/V_t)-\left(r-\frac{1}{2}\sigma^2-\lambda\zeta\right)\tau\right)\\
=&\frac{\partial}{\partial a}\sum_{n=0}^{\infty}\pi_n P\left(\sigma\sqrt{\tau}Z+\sum_{k=1}^{n}Y_k>a\right)\cdot\left(-\frac{1}{V_t}\right)\\
=&\frac{1}{V_t}\sum_{n=0}^{\infty}\pi_n\frac{\partial}{\partial a}P\left(\sigma\sqrt{\tau}Z+\sum_{k=1}^{n}Y_k<a\right)\\
=&\frac{1}{V_t}f\left(r-\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\lambda,p,\eta_1,\eta_2;\ln\left(K/V_t\right),\tau\right)\end{aligned}$$

where

$$Z(\tau)=\left(r-\frac{1}{2}\sigma^2-\lambda\zeta\right)\tau+\sigma\sqrt{\tau}Z+\sum_{i=1}^{N(\tau)}Y_i,$$

$$Z\sim N(0,1),\ \ a=\ln(K/V_t)-\left(r-\frac{1}{2}\sigma^2-\lambda\zeta\right)\tau$$

The result of $\frac{\partial}{\partial V_t}\Upsilon\left(r+\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\tilde{\lambda},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2;\ln\left(K/V_t\right),\tau\right)$ is similar. Thus,

$$
\begin{aligned}
\frac{\partial S_t}{\partial V_t} =& \Upsilon\left(r+\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\tilde{\lambda},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2;\ln\left(K/V_t\right),\tau\right)\\
&+ f\left(r+\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\tilde{\lambda},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2;\ln\left(K/V_t\right),\tau\right)\\
&- \frac{Ke^{-r\tau}}{V_t} f\left(r-\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\lambda,p,\eta_1,\eta_2;\ln\left(K/V_t\right),\tau\right),
\end{aligned}
$$

which complete the proof.

References

[1] Black, F., and J. C. Cox, "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions", Journal of Finance, 31, pp. 351-367, 1976.

[2] Black, F., and M. Scholes, "The Pricing of Options and Corporate Liabilities", Journal of Political Economy, 81, pp. 637-659, 1973.

[3] Brenann, M., and E. Schwartz, "Analyzing Convertible Bonds", Journal of Financial and Quantitative Analysis, 15, pp. 907-929, 1980.

[4] Briys, E., and F. de varenne, "Valuing Risky Fixed Rate Debt: An Extension", Journal of Financial and Quantitative Analysis, 32, pp. 239-248, 1997.

[5] Brockman, P., and H. Turtle, "A Barrier Option Framework for Corporate Security Valuation", Journal of Financial Economics, 67, pp. 511-529, 2003.

[6] Bruche, M., "Estimating Structural Bond Pricing Models via Simulated Maximum Likelihood," London School of Economics Working Paper, 2005.

[7] Burden, R. L., and J. Douglas Faires, Numerical Analysis - 7th Edition, Brooks/Cole Publishing Company, 2001.

[8] Chen, C. C., and H. Panjer, "Unifying Discrete Structural Models and Reduced-Form Models in Credit Risk Using a Jump-Diffusion Process", Insurance: Mathematics and Economics, 33, pp. 357-380, 2003.

[9] Collin-Dufresne, P., and R. S. Goldstein, "Do Credit Spreads Reflect Stationary Leverage Ratios?", Journal of Finance, 56, pp. 1929-1957, 2001.

[10] Cox, J. C., "Notes on Option Pricing I: Constant Elasticity of Variance Diffusion", Standford University Working Paper. Also, "The Constant Elasticity of Variance Option Pricing Model", Journal of Portfolio Management, Special Issue, pp. 15-17, 1996.

[11] Cox, J. C., J. E. Ingersoll, and S. A. Ross, "A Theory of the Term Structure of Interest Rates," Econometrica, 53, pp. 363-384, 1985.

[12] Duan, J. C, "Maximum Likelihood Estimation Using Pricing Data of the Derivative Contract", Mathematical Finance, 4, pp. 155-167, 1994.

[13] Duan, J. C, "Correction: Maximum Likelihood Estimation Using Pricing Data of the Derivative Contract", Mathematical Finance, 10, pp. 461-462, 2000.

[14] Duan, J. C., G. Gauthier, and J. G. Simonato, "On the Equivalence of the KMV and Maximum Likelihood Methods for Structural Credit Risk Models", University of Toronto Working Paper, 2004.

[15] Duffee, G., "The Relation between Treasury Yields and Corporate Bond Yield Spreads", Journal of finance, 53, pp. 2225-2242, 1998.

[16] Duffie, D., and K. Singleton, "Modelling Term Structures of Defaultable Bonds", The Review of Financial Studies, 12, pp. 687-720, 1999.

[17] Engelmann, B., E. Hayden, and D. Tasche, "Measuring the Discriminative Power of Rating systems," Deutsche Bundesbank Working Paper, 2003.

[18] Eom, Y. H., J. Helwege, and J. Z. Huang, "Structural Models of Corporate Bond Pricing: An Empirical Analysis", The Review of Financial Studies, 17, pp. 499-544, 2004.

[19] Feller, W, "Two Singular Diffusion Problems", Annuals of Mathematics, 54, pp. 173-182, 1951.

[20] Hui, C. H., C. F. Lo, and S. W. Tsang, "Pricing Corporate Bonds with Dynamic Default Barriers", Journal of Risk, 5, pp. 17-37, 2003.

[21] Hull, J. Options Futures and Other Derivative Securities, Prentice Hall, New York, 2003.

[22] Jarrow, R., and S. Turnbull, "Pricing Derivatives on Financial Securities subject to Credit Risk", *Journal of Finance*, 50, pp. 53-86, 1995.

[23] Jarrow, R., and S. Turnbull, "The Intersection of Market and Credit Risk", Journal of Banking and Finance, 24, pp. 271-299, 2000.

[24] Jones, E. P., S. Mason, and E. Rosenfeld., "Contingent Claims Analysis of Corporate Capital Structures: An Empirical Investigation", Journal of Finance, 39, pp. 611-627, 1984.

[25] Kou, S. G., "A Jump-Diffusion Model for Option Pricing", Management Science, 48, pp. 1086-1011, 2002.

[26] Kou, S. G. and H. Wang, "First Passage Times of A Jump Diffusion Process", Advanced Applied Probability, 35, pp. 504-531, 2003.

[27] Lando, D., Credit Risk Modeling: Theory and Applications, Princeton University Press, Princeton, New Jersey, 2004.

[28] Lee, C. F., T. P. Wu, and R. R. Chen, "The Constant Elasticity of Variance Models: New Evidence from S&P500 Index Options", Review of Pacific Basin Financial Markets and Policies, 7, pp. 176-190, 2004.

[29] Lee, C. F., Jack C. Lee, Y. L. Hsu and T. I. Lin, "Constant Elasticity of Variance (CEV) Option Pricing Model : Integration and Detailed Derivations", Working Paper, 2004.

[30] Longstaff, F. A., and E. S. Schwartz, "A simple Approach to Valuing Risky Fixed and Floating Rate Debt", Journal of Finance, 50, pp. 789-819, 1995.

[31] Macbeth, J. D., and L, J, Merville, "Test of the Black-Scholes and Cox Call Option Valuation Models", Journal of Finance, 35, pp. 285-301, 1980.

[32] Merton, R. C., "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates", Journal of Finance, 28, pp. 449-470, 1974.

[33] Merton, R. C., "Option Pricing When Underlying Stock Returns are Discontinuous", Journal of Financial Economics, 3, pp. 125-144, 1976.

[34] Ronn, E. I., and A. K. Verma, "Pricing Risk-Adjusted Deposit Insurance: An Option-Based Model," Journal of Finance, 41, pp. 871-895, 1986.

[35] Ross, S. M., Simulation, Academic Press, USA, 2002.

[36] Schorder, M., "Computing the Constant Elasticity of Variance Option Pricing Formula," Journal of Finance, 44, pp. 211-219, 1989.

[37] Schoutens, W., Lévy Process in Finance: Pricing Financial Derivatives, John Wiley & Sons, Inc, New York, NY, 2003.

[38] Schönbucher, P. J., Credit Derivatives Pricing Models: Models, Pricing and Implementation, John Wiley & Sons Ltd., England, 2003.

[39] Shreve, S. E., Stochastic Calculus for Finance Vol2: Continuous-Time Models, Springer-Verlag Berlin Heidelberg, New York, 2004.

[40] Sobehart, J. R., S. C. Keenan, and R. M. Stein, "Benchmarking Quantitative Default Risk Models: A validation Methodology," Moodys Investors Services, 2000.

[41] Tsay, R. S., Analysis of Financial Time Series, John Wiley & Sons, Inc, New York, 2002.

[42] Vasicek, O., "An Equilibrium Characterization of The Term Structure", Journal of Financial Economics, 5, pp. 177-188, 1977.

[43] Vassalou, M., and Y. Xing, "Default Risk in Equity Returns", Journal of Finance, 59, pp. 831-868, 2004.

[44] Wang, H. Y., and T. W. Choi, "The Impact of Default Barrier on the Market Value of Firm's Asset", The Chinese University of Hong Kong Working Paper, 2004.

[45] Wang, H. Y., and K. L. Li, "On Bias of Testing Merton's Model", The Chinese University of Hong Kong Working Paper, 2004.

[46] Zhou, C., "The Term Structure of Credit Spreads with Jump Risk", Journal of Banking & Finance, 25, pp. 2015-2040, 2001.