國立交通大學

應用數學系

碩士論文

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Topics on Linear Elliptic Equations

研究生 ： 姚文雯

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摘　要

研究線性橢圓偏微分方程(線性橢圓 PDEs)。首先，給一些實用的例子，同時將二階線性偏微分方程式作一分類。接下來，運用幾個古典方法解線性橢圓偏微分方程，並且將該方程式的解以各種形式表示。

當我們運用傅立葉轉換解整個或半平面的偏微分方程時，需要利用逆傅立葉轉換導出該偏微分方程的解，此時被積分函數中常出現平方根的形式，在複數平面上它是多值函數。為了讓逆傅立葉轉換導出的解是正確的，我們結合複數平面上的黎曼曲面，藉由適當的代數建構出平方根在該曲面上是單值，並且完成逆轉換的解析解與數值解。最後藉由例子來說明整個計劃。

Topics on Linear Elliptic Equations

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Abstract

We study the linear elliptic partial differential equations (linear elliptic PDEs). First, we give some practical examples and show that they are governed by such type of the equations. Next, we apply several classical methods to solve the linear elliptic PDEs with the solutions being expressed in various forms. We then identify those solutions.

When we apply Fourier transformations to the whole- and half-line PDEs, it is necessary to perform the inverse Fourier transformations to derive the PDE solutions, and it is quite often that those integrals involve the square root operator which is multi-valued in the complex plane. In order to perform the inverse transformations correctly, we develop the Riemann surfaces from the complex plane with the proper algebraic structures to assure that the square root is now a single-valued function on the surfaces, and we are able to accomplish the inverse transformations analytically and numerically. Some examples are given to illustrate the entire scheme.

誌　謝

在交通大學兩年的研究所生涯中，承蒙恩師 李榮耀教授在論文研究上的悉心指導與建議，始能順利完成，在此向恩師致上最誠摯的感謝。

論文撰寫過程中特別感謝佳樺、美如所給予的鼎力協助。感謝父母親的養育與栽培、老公的鼓勵及家人的配合，你們所給予的支持是我完成論文最大的原動力。一路行來，點滴在心，雖然這不是一部完美的論文，但這部論文的完成，要感謝的真的人很多，無法一一列出，僅以此文表達我的誠摯謝意。

Contents

Ⅰ. Introduction

Many important scientific and engineering problems fall into the field of second-order partial differential equation. We want to recognize the distinguish for second-order partial differential equation.

The distinction as to *Hyperbolic* 、*Parabolic* 、or *Elliptic* for second-order partial differential equation depends on the coefficients of second-derivative term. we can write any such general linear partial differential equation of second order in two variables reads，

$$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y + F(x,y)u = 0$$

where $(x,y) \in \Omega$ (Ω is domain).

Depending on the value $B^2 - 4AC$ ，we classify the equation as

$$\begin{aligned} &Hyperbolic \Rightarrow \text{if } B^2 - 4AC > 0 \ , \\ &Parabolic \Rightarrow \text{if } B^2 - 4AC = 0 \ , \\ &Elliptic \Rightarrow \text{if } B^2 - 4AC < 0 \ . \end{aligned}$$

For example，the *Wave* equation $u_{xx} - u_{yy} = 0$ is of *Hyperbolic* type，and the *Heat* partial differential equation $u_{xx} - u_t = 0$ is parabolic，while *Laplace′s* equation $u_{xx} + u_{yy} = 0$ is *Elliptic* .

Elliptic partial differential equation has many applications in engineering，physics and material science，for example resistance and capacitance extraction in electronic circuit，state decomposition in microwave tube，Navier-Stokes equation in incompressible fluid and device simulation of semiconductor，membrane displacement，torsion and so on .

There is a question，why are most physical problems related to elliptic equation？Since *Elliptic* equation has a term 〝*Laplacian operator*〞，it describe diffusion phenomenon，like heat diffusion、dynamic diffusion etc.

Now consider the steady potential flow in two-dimensional incompressible fluid. First，we define correlation proper noun. In general，the two-dimensional flow is a flow in which the velocity component depends on only two space variables. An example is a plane flow，in which the velocity component depends on two spatial coordinates，x and y，but not z. An incompressible flow exists if the density of each fluid particle remains relatively constant as it moves through the flow field，that is $\frac{d\rho}{dt}=0$，and for an incompressible flow，the differential equation of mass conservation is $\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}=0$ in three-dimensional. The velocity at a given point in space does not vary with time，that is $\frac{\partial u}{\partial t}=0$.We call that is the steady flow. The flow is irrotational we call the potential flow. In this we discuss $xy-plane$，that implies $w_z=0$，we have $\frac{\partial u_y}{\partial x}=\frac{\partial u_x}{\partial y}$.

Let $u(x,y)$ be the velocity of the point (x,y) on $xy-plane$. Then we have the differential of mass conservation of incompressible flow in $xy-plane$.

$$\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}=0 \quad . \qquad (1\text{-}1)$$

This equation is satisfied identically if a function $\psi(x,y)$ is defined such that becomes

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)=0 \quad . \qquad (1\text{-}2)$$

Comparison of (1-1) and (1-2) shows that this new function ψ must be defined such that

$$u_x=\frac{\partial \psi}{\partial y} \quad \text{and} \quad u_y=-\frac{\partial \psi}{\partial x} \quad . \qquad (1\text{-}3)$$

Since this flow is irrotational，we put (1-3) into the $\frac{\partial u_y}{\partial x}=\frac{\partial u_x}{\partial y}$.

We get

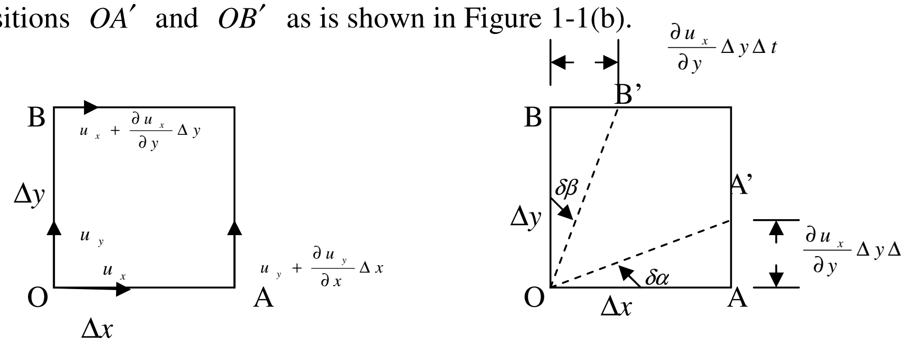
$$-\frac{\partial^2\psi}{\partial x^2}=\frac{\partial^2\psi}{\partial y^2}$$

$$\Rightarrow \quad \frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$$

$$\Rightarrow \quad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi=\nabla^2\psi=0 \quad . \qquad (1\text{-}4)$$

The operator $\nabla^2=\left(\frac{\partial}{\partial x^2}+\frac{\partial}{\partial y^2}\right)$ is called the *Laplacian*，and the equation (1-4) is called *Laplace's* equation in two dimensional. The inviscid、incompressible、irrotational flow fields are governed by *Laplace's* equation. This type of flow is commonly called a potential flow，and the function ψ is called potential function.

In below, we illustrate the angular motion in the $xy-plane$. The velocity variation that causes rotation and angular deformation is illustrated in Figure 1-1(a). In a short time interval Δt the line segments OA and OB will rotate through the angles $\delta\alpha$ and $\delta\beta$ to the new positions OA' and OB' as is shown in Figure 1-1(b).



(a) (b)

Figure1-1. Angular motion and deformation of a fluid element

The angular velocity of line OA, W_{OA} is

$$W_{OA} = \lim_{\Delta t \to 0} \frac{\delta\alpha}{\Delta t} \quad .$$

For small angles, we have

$$\delta\alpha \quad \approx \quad \tan\delta\alpha = \frac{\overline{AA'}}{\Delta x} = \frac{\frac{\partial u_y}{\partial x}\Delta x \Delta t}{\Delta x} = \frac{\partial u_y}{\partial x}\Delta t \quad .$$

So that
$$W_{OA} = \lim_{\Delta t \to 0}\left[\frac{\frac{\partial u_u}{\partial x}\Delta t}{\Delta t}\right] = \frac{\partial u_y}{\partial x} \, .$$

Note that, if $\frac{\partial u_y}{\partial x}$ is positive, W_{OA} will be counterclockwise.

Similarly, the angular velocity of line OB, W_{OB} is

$$W_{OB} = \lim_{\Delta t \to 0} \frac{\delta\beta}{\Delta t} \quad ,$$

and

$$\delta\beta \quad \approx \quad \tan\delta\beta = \frac{\overline{BB'}}{\Delta y} = \frac{\frac{\partial u_x}{\partial y}\Delta y \Delta t}{\Delta y} = \frac{\partial u_x}{\partial y}\Delta t \quad ,$$

so that
$$W_{OB} = \lim_{\Delta t \to 0}\left[\frac{\frac{\partial u_x}{\partial y}\Delta t}{\Delta t}\right] = \frac{\partial u_x}{\partial y} \quad .$$

Note that, if $\frac{\partial u_x}{\partial y}$ is positive, W_{OB} will be clockwise.

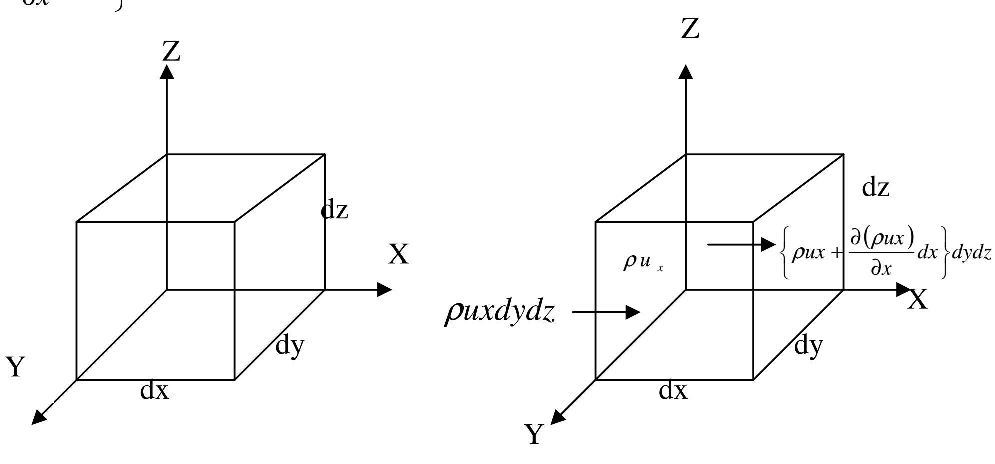
The rotation，W_Z，of the element about the $Z-axis$ is defined as the average of the abgular velocities W_{OA} and W_{OB} of the two mutually perpendicular lines OA and OB. Thus，if counterclockwise rotation is considered to be positive，it follows that

$$W_Z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \quad .$$

Since we derive in $xy-plane$，that implies $W_Z = 0$.
So we get

$$\frac{\partial u_y}{\partial x} = \frac{\partial u_x}{\partial y} \quad .$$

We will take as our control volume the small，stationary cubical element shown in Figure 1-2(a). At the center of the element the fluid density is ρ and the velocity has component u_x、u_y and u_z. The rate of mass flow through the surface of the element can be obtained by considering the flow in each of the coordinate directions separately. For example，in Figure 1-2(b) flow in the $x-direction$ is depicted. Let ρu_x represent the x component of the mass rate of flow per unit area at the center of the element，the rate at which mass is crossing the left side of the element are obtained as $\rho u_x dydz$ and the rate at which mass is crossing the right side of the element are obtained as $\left\{\rho u_x + \frac{\partial(\rho u_x)}{\partial x}dx\right\}dydz$.



(a) (b)

Figure 1-2. A differential element for the development of conservation of mass

When these two expressions are combined，the net rate of mass flowing from the element through the two surfaces can be expressed as ：

Net rate of mass outflow in $x-direction$ $= \left\{\rho u_x + \frac{\partial(\rho u_x)}{\partial x}dx\right\}dydz - \rho u_x dydz$

$$= \frac{\partial(\rho u_x)}{\partial x}dxdydz \quad .$$

For simplicity，only flow in the $x-direction$ has been considered in Figure 1-2(b)， in general，there will also be flow in the y and $z-direction$. An analysis similar to the one used for flow in the $x-direction$ shown that

Net rate of mass outflow in $y-direction$ $= \left\{\rho u_y + \frac{\partial(\rho u_y)}{\partial y}dy\right\}dxdz - \rho u_y dxdz$

$$= \frac{\partial(\rho u_y)}{\partial y}dxdydz \quad ,$$

and

Net rate of mass outflow in $z-direction$ $= \left\{\rho u_z + \frac{\partial(\rho u_z)}{\partial z}dz\right\}dxdy - \rho u_z dxdy$

$$= \frac{\partial(\rho u_z)}{\partial z}dxdydz \quad .$$

Since we derive the incompressible flow， *i.e.* $\frac{\partial \rho}{\partial t} = 0$ and ρ is constant.

Thus by the conservation of mass，we have

$$\text{Net rate of mass outflow} = \frac{\partial(\rho u_x)}{\partial x}dxdydz + \frac{\partial(\rho u_y)}{\partial y}dxdydz + \frac{\partial(\rho u_z)}{\partial z}dxdydz = 0$$

$$\Rightarrow \frac{\partial(\rho u_x)}{\partial x} + \frac{\partial(\rho u_y)}{\partial y} + \frac{\partial(\rho u_z)}{\partial z} = 0 \quad .$$

Since ρ is constant.

Therefore $$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0 \quad .$$

As previously mentioned，this equation is also commonly referred to as the continuity equation.

In below，we consider the a situation that is typical，in which the temperatures is a function of the coordinates of position of the point in equation.

A piece of metal is 12in.×3in.×6ft. There feet of the slab is kept inside a furnace but half of the slab protrudes (see Figure 1-3). In order to decrease heat losses to the air，the protruding half is covered with a 1-in.thickness of insulation. If the furnace is maintained at $950^0 F$ ，will at points of the metal reach a temperature of $800^0 F$ or higher，in spite of heat loss through the insulation？ Such a question might arise in heat-treating the slab when the only furnace available to heat the metal is too small to contain the whole slab.

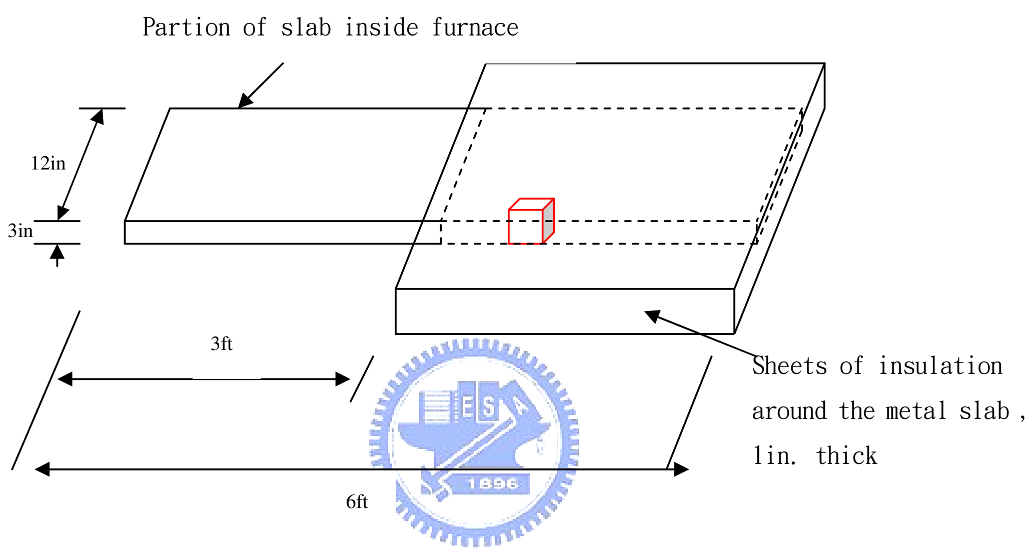


Figure 1-3. A piece of metal is 12in.×3in.×6ft

We derive the relationship for temperature u as a function of space variables for the equilibrium temperature distribution by the metal piece protruding from the furnace. In this consider the two spatial coordinates，that is derive the relationship for temperature u as a function of space variables x and y for the equilibrium temperature distribution on a flat plate.

First，ideal supposition. One：consider only the case where the temperatures do not change with time. Second：assume that heat flows only in the x and y-directions and not in the perpendicular direction (If the plate is very thin，or if the upper and lower surfaces are both well insulated，the physical situation will agree with our assumption) . Three：assume that no heat is being generated at points in the plate. (see Figure 1-4)

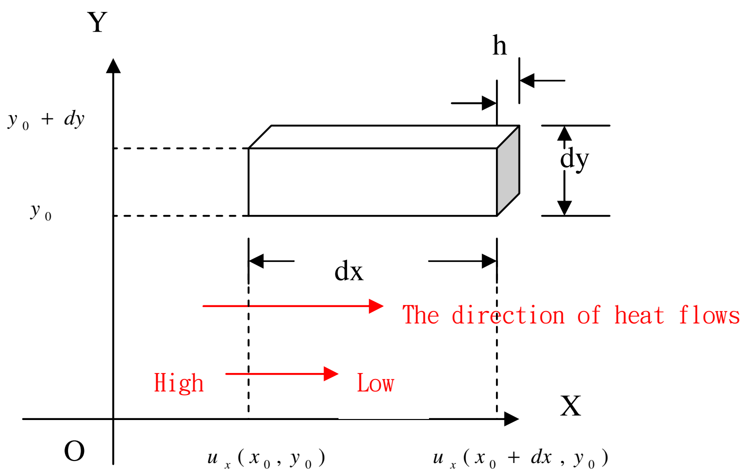


Figure 1-4. The plate which is thin and small

Let h be the thickness of the plate. Heat flows at a rate proportional to the cross-sectional area，to the temperature rate of change (u_x or u_y)，and to the thermal conductivity k，which we will assume constant at all points. The flow of heat is from high to low temperature，of course，meaning opposite to the direction of increasing temperature rate of change. We use a minus sign in the equation to account for this：

In the x-directions，the rate of heat flow into element at $x = x_0$ is $-k(hdy)u_x$.

The rate of change at $x_0 + dx$ is the rate of change at x_0 plus the increment in the rate of change over the distance dx：

The rate of change at $x_0 + dx$： $u_x + u_{xx}dx$.

Rate of heat flow out of element at $x = x_0 + dx$： $-khdy[u_x + u_{xx}dx]$.

Net rate of heat into element in x-directions： $-k(hdy)[u_x - (u_x + u_{xx}dx)] = kh(dxdy)u_{xx}$.

Similarly，in the y-directions we have the Net rate of heat into element in y-directions：

$$-khdx[u_y - (u_y + u_{yy}dy)] = kh(dxdy)u_{yy} .$$

The total heat flowing into the elemental by conduction is the sum of these net flows in the x and y-directions. If there is equilibrium as to temperature distribution，that is steady-state，the total rate of heat flow into the element plus heat generated must be zero.

Hence

$$kh(dxdy)(u_{xx}+u_{yy})+Qh(dxdy)=0$$

where Q is the rate of heat generation per unit area and Q will often be a function of x and y.

By above assume second，we have $Q=0$，
and

$$kh(dxdy)(u_{xx}+u_{yy})=0$$

$$\Rightarrow \quad u_{xx}+u_{yy}=\nabla^2 u=0 \quad . \qquad (1\text{-}5)$$

The operator $\nabla^2=(\frac{\partial}{\partial x^2}+\frac{\partial}{\partial y^2})$ is called the *Laplacian*，and the equation (1-5) is called *Laplace′s* equation in two dimensional. *Laplace′s* equation arises in steady-state heat conduction problems involving homogeneous solids. For three dimensional heat flow problems，we would have，analogously，

$$\nabla^2 u=(\frac{\partial}{\partial x^2}+\frac{\partial}{\partial y^2}+\frac{\partial}{\partial z^2})u=0 \quad .$$

Consider that heat is being generated at points in the plate. Assume this removal rate to be a function of the location of the element in the $xy-plane$，$f(x,y)$，we would have，with Q equal to the rate of heat generation per unit area，

$$kh(dxdy)(u_{xx}+u_{yy})+Q(x,y)h(dxdy)=0$$

$$\Rightarrow \quad kh(dxdy)(\nabla^2 u)=-Q(x,y)h(dxdy)$$
$$\Rightarrow \quad k(\nabla^2 u)=-Q(x,y)$$
$$\Rightarrow \quad \nabla^2 u=-\frac{1}{k}Q(x,y)=f(x,y).$$

This equation is called *Poisson′s* equation (non homogeneous).

A typical steady-state heat flow problem is the following： A thin steel plate is a 10×20 an rectangular. If one of the 10-cm edges in held at $100^0 C$ and the other three edge are held at $0^0 C$，what are the steady-state temperatures at interior points？ For steel，$k=0.16\ cal/\sec\bullet cm^2\bullet {}^0C/cm$.

Math model：Find $u(x, y)$ such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ,$$
$$u(x,0) = 100 \quad ,$$
$$u(x,20) = 0 \quad ,$$
$$u(0, y) = 0 \quad ,$$
$$u(10, y) = 0 \quad .$$

In this statement of the problem，we imagine one corner of the plate at the origin，with boundary conditions as sketched in Figure 1-5.

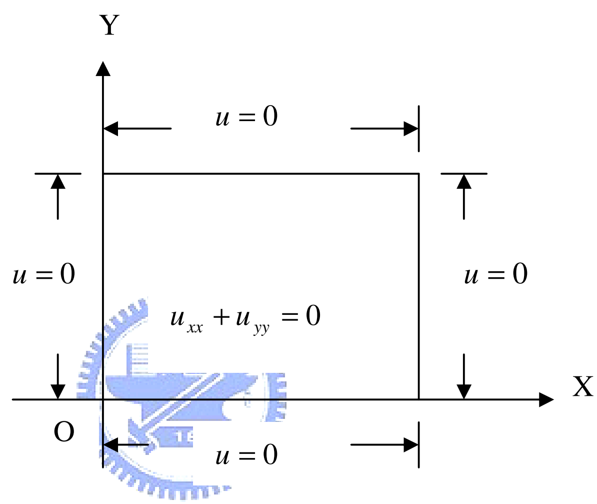


Figure 1-5. *Laplace′s* equation for a rectangular domain

Because the field of application of *Laplace′s* equation and *Poisson′s* equation do not involve time，initial conditions are not prescribed for the solution of equation. Rather，we find that it is proper to simply prescribed a single boundary condition. Such problems are them call simply boundary value problems (*BVPs*).

The basic example of an elliptic partial differential equation is *Laplace′s* equation，$i.e. \nabla^2 u = 0$ in Ω (that is domain) in $n-$dimensional Euclidean space，other examples of elliptic partial differential equations include the nonhomogeneous *Poisson′s* equation，$i.e. \nabla^2 u = f(x, y)$ in Ω (that is domain). These two equations include most of the physical applications of elliptic partial differential equation.

Elliptic partial differential equation may have non-constant coefficients and be non-linear. Despite this variety，the elliptic equations have a well-developed theory. In this paper，we discuss the *linear Elliptic* partial differential equation.

By above math model, we know in two dimensions, *Laplace′s* equation has the rectangular coordinate representation:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \text{for} \quad 0<x<a \quad \text{and} \quad 0<y<b \ ,$$

$$u(x,0)=f(x) \ ,$$

$$u(x,b)=0 \ ,$$

$$u(o,y)=0 \ ,$$

$$u(a,y)=0 \ .$$

In rectangular domain, we imagine one corner of the plate at the origin, with boundary conditions as sketched in Figure 1-6.

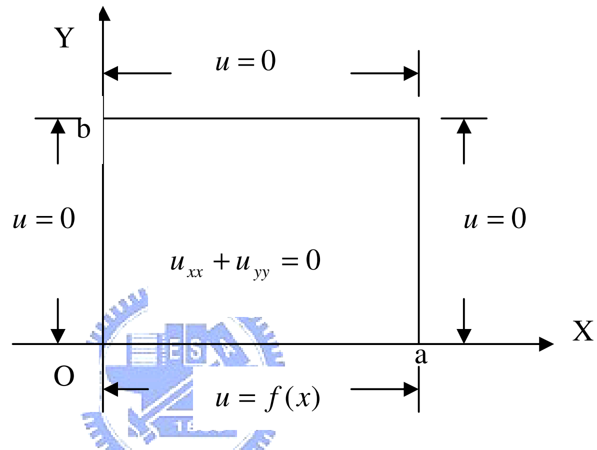
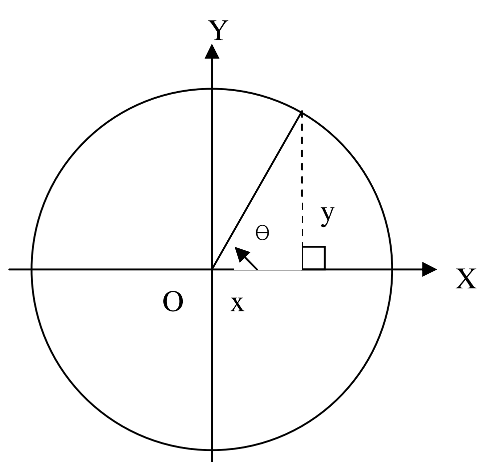


Figure 1-6. *Laplace′s* equation for a rectangular domain

Many two dimensional problems involving *Laplace′s* equation are in region that lend themselves to a polar description in terms of r and θ, rather than rectangular coordinates x and y. This means that we need an expression for the *Laplacian* in terms of polar coordinates.

Let us consider in the unit circle $x^2+y^2<1$ with its values given on the boundary $x^2+y^2=1$. It is natural to introduce the poor coordinates transformation.



$$(x,y) \ \rightarrow \ (r,\theta)$$

$$r=\sqrt{x^2+y^2}$$

Setting $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ and $\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\frac{y}{x} = \tan^{-1}(yx^{-1})$

We want to $(x,y) - PDE \Rightarrow (r,\theta) - PDE$

$$u_x = u_r \cdot r_x + u_\theta \cdot \theta_x = u_r \cdot \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(2x) + u_\theta \cdot \frac{-yx^{-2}}{1+(\frac{y}{x})^2} = u_r \cdot (x^2+y^2)^{-\frac{1}{2}}(x) + u_\theta \cdot (\frac{-y}{x^2+y^2})$$

$$u_{xx} = [u_{rr} \cdot r_x + u_{r\theta} \cdot \theta_x](x^2+y^2)^{-\frac{1}{2}}(x) + u_r[\frac{-1}{2}(x^2+y^2)^{-\frac{3}{2}}(2x)(x) + (x^2+y^2)^{-\frac{1}{2}}]$$
$$+[u_{\theta\theta} \cdot \theta_x + u_{\theta r} \cdot r_x](\frac{-y}{x^2+y^2}) + u_\theta[(-y)(-1)(x^2+y^2)^{-2}(2x)]$$

$$= [u_{rr} \cdot (x^2+y^2)^{-\frac{1}{2}}(x) + u_{r\theta} \cdot (\frac{-y}{x^2+y^2})](x^2+y^2)^{-\frac{1}{2}}(x) + u_r[(-x^2)(x^2+y^2)^{-\frac{3}{2}} + (x^2+y^2)^{-\frac{1}{2}}]$$

$$+[u_{\theta\theta} \cdot (\frac{-y}{x^2+y^2}) + u_{\theta r} \cdot (x^2+y^2)^{-\frac{1}{2}}(x)](\frac{-y}{x^2+y^2}) + u_\theta[(2xy)(x^2+y^2)^{-2}]$$

$$u_y = u_r \cdot r_y + u_\theta \cdot \theta_y = u_r \cdot \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(2y) + u_\theta \cdot \frac{x^{-1}}{1+(\frac{y}{x})^2} = u_r \cdot (x^2+y^2)^{-\frac{1}{2}}(y) + u_\theta \cdot (\frac{x}{x^2+y^2})$$

$$u_{yy} = [u_{rr} \cdot r_y + u_{r\theta} \cdot \theta_y](x^2+y^2)^{-\frac{1}{2}}(y) + u_r[\frac{-1}{2}(x^2+y^2)^{-\frac{3}{2}}(2y)(y) + (x^2+y^2)^{-\frac{1}{2}}]$$
$$+[u_{\theta\theta} \cdot \theta_y + u_{\theta r} \cdot r_y](\frac{x}{x^2+y^2}) + u_\theta[(-2xy)(x^2+y^2)^{-2}]$$

$$= [u_{rr} \cdot (x^2+y^2)^{-\frac{1}{2}}(y) + u_{r\theta} \cdot (\frac{x}{x^2+y^2})](x^2+y^2)^{-\frac{1}{2}}(y) + u_r[(-y^2)(x^2+y^2)^{-\frac{3}{2}} + (x^2+y^2)^{-\frac{1}{2}}]$$

$$+[u_{\theta\theta} \cdot (\frac{x}{x^2+y^2}) + u_{\theta r} \cdot (x^2+y^2)^{-\frac{1}{2}}(y)](\frac{x}{x^2+y^2}) + u_\theta[(-2xy)(x^2+y^2)^{-2}]$$

Hence

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=u_{rr}(x^2+y^2)^{-1}x^2+u_{r\theta}(-xy)(x^2+y^2)^{-\frac{3}{2}}+u_r[(-x^2)(x^2+y^2)^{\frac{-3}{2}}+(x^2+y^2)^{-\frac{1}{2}}]$$

$$+u_{rr}(x^2+y^2)^{-1}y^2+u_{r\theta}(xy)(x^2+y^2)^{-\frac{3}{2}}+u_r[(-y^2)(x^2+y^2)^{-\frac{3}{2}}+(x^2+y^2)^{-\frac{1}{2}}]$$

$$+u_{\theta\theta}(x^2+y^2)^{-2}y^2+u_{\theta r}(-xy)(x^2+y^2)^{-\frac{3}{2}}+u_\theta[(2xy)(x^2+y^2)^{-2}]$$

$$+u_{\theta\theta}(x^2+y^2)^{-2}x^2+u_{\theta r}(xy)(x^2+y^2)^{-\frac{3}{2}}+u_\theta[(-2xy)(x^2+y^2)^{-2}]$$

$$=u_{rr}+u_r[-(x^2+y^2)^{-\frac{3}{2}}(x^2+y^2)+2(x^2+y^2)^{-\frac{1}{2}}]+u_{\theta\theta}(x^2+y^2)^{-1}$$

$$=u_{rr}+u_r(-r^{-3}\cdot r^2+2r^{-1})+u_{\theta\theta}\cdot r^{-2}$$

$$=u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$$

Therefore，a computation shows that *Laplace′s* equation in polar coordinates is

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0 \quad \text{for } 0<r<1 \text{ and } -\pi<\theta<\pi \ ,$$

$$u(r,\theta)=f(\theta) \ .$$

In circular domain，with boundary conditions as sketched in Figure 1-7.

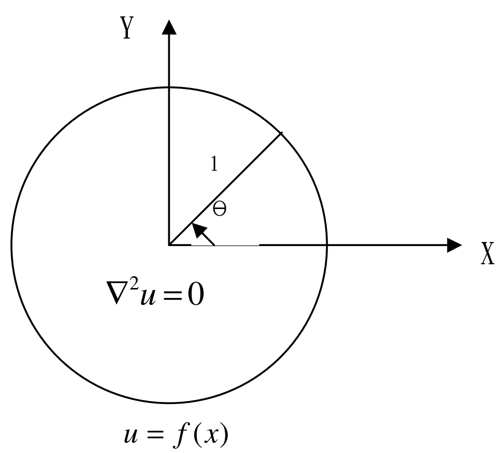


Figure 1-7. *Laplace′s* equation for a circular domain

Laplace's equation，also called the potential equation，the concept of a potential function seems to have been first used by Daniel Bernoulli (1700 ~ 1782)，son of the more famous Jean Bernoulli，in 〝Hydrodynamica〞 in 1738，and Euler wrote *Laplace's* equation in 1752 ，from the continuity equation for incompressible fliulds. The real progress was made by two of the three *L's* ，Adrien-Marie Legendre (1752 ~ 1833) and Pierre-Simon Laplace (1749 ~ 1827) . (The other *L* was Lagrange.) Legendre looked at the gravitational attraction of spheroids in 1785 and developed the Legendre polynomials as part of this work. Laplace used expansions in spherical functions to solve the equation since named after him，and both mathematicians continued their work into the 1790_s .

Ⅱ. The methods of solving Elliptic PDE

In this chapter，we considers various aspects of the solution of boundary value problems for second-order *linear elliptic* prtial differential equations in two variables.

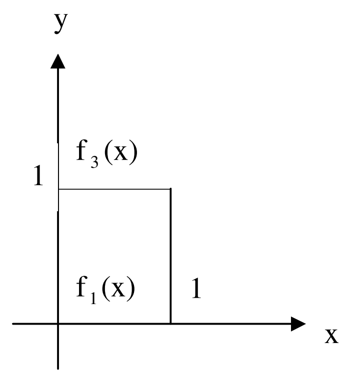
Ⅱ-1 Separation of variables to construct solution of system of *Laplace's* equation

Ⅱ-1.1 The domain is a rectangular

Consider $u_{xx} + u_{yy} = 0$ for $0 < x < \pi$，$0 < y < \pi$.

To solve
$$u(x,0) = f_1(x),$$
$$u(x,\pi) = f_3(x),$$
$$u(0,y) = f_2(y),$$
$$u(\pi,y) = f_4(y),$$

graph:



where f_1, f_2, f_3, f_4 are given functions .

Ansatz $u(x,y) = X(x)Y(y)$.

Since $u_{xx} = X''(x)Y(y)$ and $u_{yy} = X(x)Y''(y)$.

Put it in the above equation，we have

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

$$\Rightarrow \frac{X''(x)Y(y) + X(x)Y''(y)}{X(x)Y(y)} = 0$$

$$\Rightarrow \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

$$\Rightarrow \frac{X''}{X}(x) = -\frac{Y''}{Y}(y)$$

$$\Rightarrow \frac{d}{dx}\left(\frac{X''}{X}(x)\right) = -\frac{d}{dx}\left(\frac{Y''}{Y}(y)\right)$$

$$\Rightarrow \begin{cases} \dfrac{X''}{X}(x) = -\lambda \\ \dfrac{Y''}{Y}(y) = \lambda \end{cases}，\lambda \text{ is any constant .}$$

Thus $u = X(x)Y(y)$ is a solution of *Laplace's* equation if and only if $X(x)$ and $Y(y)$ satisfy the two ordinary differential equations .

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ Y''(y) - \lambda Y(y) = 0 \end{cases} \text{ for some constant } \lambda \ . \qquad (2\text{-}1)$$

For each value of λ each of the above second order equations has two linearly independent solutions.

Consider $\begin{cases} \lambda > 0 \\ \lambda = 0 \\ \lambda < 0 \end{cases}$, then we get the two linearly independent solutions

① For each $\lambda > 0$, we have

$$u(x, y) = X(x)Y(y) \Rightarrow \text{ linear combination of } \{ e^{\pm\sqrt{\lambda}y} \cos\sqrt{\lambda}x \text{ 、 } e^{\pm\sqrt{\lambda}y} \sin\sqrt{\lambda}x \}_{\lambda>0} \ .$$

② For each $\lambda = 0$, we have

$$u(x, y) = X(x)Y(y) \Rightarrow \text{ linear combination of } 1 \text{ 、 } x \text{ and } y \Rightarrow \{ 1 \text{ 、 } x \text{ 、 } y \text{ 、 } xy \}.$$

③ For each $\lambda < 0$, we have

$$u(x, y) = X(x)Y(y) \Rightarrow \text{ linear combination of } \{ e^{\pm\sqrt{-\lambda}x} \cos\sqrt{-\lambda}y \text{ 、 } e^{\pm\sqrt{-\lambda}x} \sin\sqrt{\lambda}y \}_{\lambda<0} \ .$$

Since we are dealing with a linear problem , the solution can be found as the sum of the solution of

$$u_{xx} + u_{yy} = 0 \text{ and } \begin{cases} 0 < x < \pi \\ 0 < y < \pi \end{cases} , \qquad (2\text{-}2)$$

$$u(x,0) = f_1(x),$$

$$u(x,\pi) = 0 \ ,$$

$$u(0, y) = 0,$$

$$u(\pi, y) = 0,$$

and three other boundary value problems , in each of which $u = 0$ except on one edge. It is therefore sufficient to solve problems of this kind.

Since we wish to have $u=0$ for $x=0$ and $x=\pi$, we only consider those solutions of the equation (2-1) which satisfy these conditions. We must have

$$\begin{cases} X''+\lambda X=0 \quad , \quad 0<x<\pi \\ X(0)=X(\pi)=0 \end{cases} \text{ and } \begin{cases} Y''-\lambda Y=0 \quad , \quad 0<y<\pi \\ Y(\pi)=0 \end{cases} .$$

Consider $X(x)$ and

$$\begin{cases} X''+\lambda X=0 \quad , \quad 0<x<\pi \\ X(0)=X(\pi)=0 \end{cases} .$$

This homogeneous problem always has the trivial solution $X\equiv 0$, but this is of no use to us. We are interested in case to find the non-trivial solution of $X(x)$. So we must check $\lambda>0$ 、 $\lambda=0$ and $\lambda<0$

① Let $\lambda>0 \Rightarrow X(x)=\sin\sqrt{\lambda}x$ or $\cos\sqrt{\lambda}x$.

The general solution of the equation is $X(x)=a\sin\sqrt{\lambda}x+b\cos\sqrt{\lambda}x$, where a,b are to be determined to satisfy $X(0)=X(\pi)=0$.

So we have

$$X(0)=a\cdot 0+b\cdot 1=0 \Rightarrow b=0 .$$

$$X(x)=a\sin\sqrt{\lambda}x \Rightarrow X(\pi)=a\sin\sqrt{\lambda}\pi=0 ;$$

$$\text{either } \begin{cases} a=0 \Rightarrow X(x)\equiv 0 \;\; (trivial-solution) \\ \sin\sqrt{\lambda}\pi=0 \Rightarrow \sqrt{\lambda}=n \;\; , \;\; n=1,2,.. \end{cases} ,$$

we have $\lambda=n^2=\lambda_n$. In this $X_n(x)=\sin\sqrt{\lambda_n}x$ are solutions.

Take $X_n(x)$ satisfies $\begin{cases} X''+\lambda_n X=0 \quad , \quad 0<x<\pi \\ X(0)=X(\pi)=0 \end{cases}$.

② Let $\lambda=0 \Rightarrow X(x)=a\cdot 1+b\cdot x$

$$X(0)=a=0 \text{ and } X(\pi)=b\cdot 1=b=0$$

$$\Rightarrow X(x)\equiv 0 \;\; (trivial-solution)$$

③ Let $\lambda < 0 \Rightarrow X(x) = e^{\sqrt{-\lambda}x}$ or $e^{-\sqrt{-\lambda}x}$.

The general solution form $X(x) = ae^{\sqrt{-\lambda}x} + be^{-\sqrt{-\lambda}x}$,

and $X(0) = a + b = 0$ 、 $X(\pi) = ae^{\sqrt{-\lambda}} + be^{-\sqrt{-\lambda}} = 0$.

Since $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$,

this implies $\sinh\sqrt{-\lambda}x = \dfrac{e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}}{2}$ and $\cosh\sqrt{-\lambda}x = \dfrac{e^{\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x}}{2}$.

So the general solution is $X(x) = A\sinh\sqrt{-\lambda}x + B\cosh\sqrt{-\lambda}x$,

and $X(0) = A\sinh 0 + B\cosh 0 = 0 \Rightarrow B = 0$.

$\Rightarrow X(x) = A\sinh\sqrt{-\lambda}x$

and $X(\pi) = A\sinh\sqrt{-\lambda} = A\cdot\dfrac{e^{\sqrt{-\lambda}} - e^{-\sqrt{-\lambda}}}{2} = 0 \Rightarrow A = 0$.

We get the solution is $X(x) \equiv 0$ $(trivial - solution)$

Finally , for $\lambda = \lambda_n = n^2$ with $n = 1,2,3,\ldots\ldots$

The system $\begin{cases} X''(x) + \lambda_n X(x) = 0, 0 < x < \pi \\ X(0) = 0 \\ X(\pi) = 0 \end{cases}$ have non-trivial solution .

We have $X_n(x) = \sin nx$ and it not zero .

Now we have $\begin{cases} X''(x) + \lambda\ X(x) = 0, 0 < x < 1 \\ X(0) = 0 \\ X(1) = 0 \end{cases}$ and ,

the eigenvalues $\{\ \lambda_n = n^2\ \}_{n=1}^{\infty}$ and the eigenfunction $\{X_n = \sin nx\ \}_{n=1}^{\infty}$.

In below , we consider $Y(y)$ system

Notice $\begin{cases} Y'' - \lambda Y = 0 \quad , \quad 0 < y < \pi \\ Y(\pi) = 0 \end{cases}$ must have non-trivial solution.

For each λ_n : $Y''(y) - n^2 Y(y) = 0$.

We have the linear independent solution of $Y(y)$ equation are

$$Y(y) = e^{\sqrt{\lambda}y} \text{、} e^{-\sqrt{\lambda}y}(\lambda > 0) \Rightarrow Y^*(y) = \sinh\sqrt{\lambda}y \text{ or } \cosh\sqrt{\lambda}y \ .$$

Combination the above solution form ，

we get $Y(y) = ae^{\sqrt{\lambda}y} + be^{-\sqrt{\lambda}y}$ and $Y(\pi) = ae^{\sqrt{\lambda}\pi} + be^{-\sqrt{\lambda}\pi} = 0$

$$\Rightarrow \quad a = e^{-\sqrt{\lambda}\pi} \text{ and } b = -e^{\sqrt{\lambda}\pi} \ ,$$

so $Y(y) = ae^{\sqrt{\lambda}y} + be^{-\sqrt{\lambda}y} = e^{-\sqrt{\lambda}\pi}e^{\sqrt{\lambda}y} + (-e^{\sqrt{\lambda}\pi})e^{-\sqrt{\lambda}y}$

$$= e^{\sqrt{\lambda}\pi(y-1)} - e^{-\sqrt{\lambda}\pi(y-1)}$$

$$= 2\left[\frac{e^{\sqrt{\lambda}\pi(y-1)} - e^{-\sqrt{\lambda}\pi(y-1)}}{2}\right] = 2\sinh[\sqrt{\lambda}(y-\pi)] = -2\sinh[\sqrt{\lambda}(\pi - y)] \ .$$

For each $\lambda_n = n^2 \Rightarrow X_n(x) = \sin nx$ and $Y_n(y) = \sinh(n)(\pi - y)$

We have constructed the particular solutions

$$u_n(x, y) = X_n(x)Y_n(y) = \sin nx \cdot \sinh(n)(\pi - y) \ ,$$

which satisfy all the homogeneous conditions of the problem (2-2). The same true of any finite linear combination. We attempt to represent the solution u of (2-2) as an infinite series in the functions u_n:

$$u(x, y) = \sum_{n=1}^{\infty} c_n \cdot \sin nx \cdot \sinh(n)(\pi - y) \ . \qquad (2\text{-}3)$$

We need to determine the coefficients c_n in such a way that $u(x,0) = f_1(x)$ ， $f_1(x)$ is given function. We must then still check to see whether the convergence of the series is sufficiently good to ensure the satisfaction of the differential equation and the homogeneous boundary conditions.

We put $y = 0$ in each term of the series to obtain the condition

$$f_1(x) = \sum_{n=1}^{\infty} c_n \cdot \sin nx \cdot \sinh(n\pi) \ .$$

If we let

$$b_n = c_n \cdot \sinh(n\pi) \quad ,$$

our problem is to determine $b_1, b_2, \ldots$ in such a way that for a given function $f_1(x)$

$$f_1(x) = \sum_{n=1}^{\infty} b_n \cdot \sin nx \quad .$$

The expansion of an arbitrary function in a series of eigenfunctions is called a Fourier series. The particular case where the eigenfunctions are all sines is called a Fourier sines series. Now we derived the problem (2-2) solution is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \cdot \sin nx \cdot \sinh(n)(\pi - y) \quad ,$$

with $u(x,0) = f_1(x) = \sum_{n=1}^{\infty} b_n \cdot \sin nx$ where $b_n = c_n \cdot \sinh(n\pi)$.

In below，we give a example to illustrate above statement.

Example 2-1 : (Using <u>Separation of variables</u> to solve *Laplace's* equation)

Solve $u_{xx} + u_{yy} = 0$ for $0 < x < \pi$ and $0 < y < \pi$,

$u(\pi, y) = u(x,\pi) = u(0, y) = 0$,

$u(x,0) = x^2(\pi - x)$.

<u>*Solution*</u> :

By equation (2-3)，we have

$$u(x,y) = \sum_{n=1}^{\infty} b_n \cdot \frac{\sinh n(\pi - y)}{\sinh(n\pi)} \sin(nx) \quad \text{and} \quad u(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx) = x^2(\pi - x) \quad .$$

$$\Rightarrow \quad b_n = \frac{< x^2(\pi - x), \sin(nx) >}{< \sin(nx), \sin(nx) >} = \frac{2}{\pi}\int_0^{\pi} x^2(\pi - x)\sin(nx)dx \, .$$

Ⅱ-1.2 The domain is a circular

We consider a solution u of *Laplace's* equation in the unit circle $x^2+y^2<1$ with its values given on the boundary $x^2+y^2=1$. It is natural to introduce the polar coordinates $r=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}\frac{y}{x}$. A computation shows that *Laplace's* equation in these coordinates is

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0 \quad . \qquad (2\text{-}4)$$

We seek a solution $u(r,\theta)$ of this equation for $r<1$ which is continuous for $r\le 1$ and satisfies

$$u(1,\theta)=f(\theta) \quad . \qquad (2\text{-}5)$$

The function $f(\theta)$ is a given continuously differentiable function which is periodic of period 2π. The solution $u(r,\theta)$ must also be periodic of period 2π in θ.

We apply separation of variables to *Laplace's* equation by seeking solutions of the form $R(r)\theta(\theta)$.

Substituting，we have

$$R''(r)\theta(\theta)+\frac{1}{r}R'(r)\theta(\theta)+\frac{1}{r^2}R(r)\theta''(\theta)=0$$

$$\Rightarrow \quad r^2R''(r)\theta(\theta)+rR'(r)\theta(\theta)+R(r)\theta''(\theta)=0$$

$$\Rightarrow \quad r^2\frac{R''(r)\theta(\theta)}{R(r)\theta(\theta)}+r\frac{R'(r)\theta(\theta)}{R(r)\theta(\theta)}+\frac{R(r)\theta''(\theta)}{R(r)\theta(\theta)}=0$$

$$\Rightarrow \quad r^2\frac{R''(r)}{R(r)}+r\frac{R'(r)}{R(r)}+\frac{\theta''(\theta)}{\theta(\theta)}=0$$

$$\Rightarrow \quad r^2\frac{R''(r)}{R(r)}+r\frac{R'(r)}{R(r)}=-\frac{\theta''(\theta)}{\theta(\theta)}=\lambda \quad \text{where } \lambda \text{ is a constant}$$

$$\Rightarrow \quad \begin{cases} \theta''(\theta)+\lambda\theta(\theta)=0 \\ r^2R''(r)+rR'(r)-\lambda R(r)=0 \end{cases} ,$$

consider the eigenvalue equation for θ. We are interested in functions which are periodic of period 2π. We consider in the interval $(-\pi,\pi)$，and pose the boundary conditions

$$\theta(-\pi)-\theta(\pi)=0 \quad ,$$
$$\theta'(-\pi)-\theta'(\pi)=0 \quad .$$

It is easy to see that has solutions of period 2π if and only if $\lambda = n^2$ with $n = 0,1,2,...$ ，corresponding to these eigenvalues n^2 we have the eigenfunctions $\cos(n\theta)$ and $\sin(n\theta)$. Phere are two eigenfunctions corresponding to each eigenvalue except $\lambda = 0$. The eigenvalues $\lambda = n^2$ with $n = 0,1,2,...$ ，are said to be double eigenvalues.

We turn now to the equation for $R(r)$ ，for $n = 0$ this has the general solution $a + b\log_r$ and for $n = 1,2,...$ ，the general solution is $ar^n + br^{-n}$. The equation is to be satisfied on the interval $0 < r < 1$. In place of a boundary condition at $r = 0$ we simply impose the condition that $R(r)$ be finite there.

We are left with the product solutions $r^n \sin(n\theta)$ and $r^n \cos(n\theta)$. We seek to solve the problem (2-4) and (2-5) by a series

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n r^n \cos n\theta + b_n r^n \sin n\theta) . \qquad (2\text{-}6)$$

Putting $r = 1$ ，we see that the coefficients a_n and b_n are to be chosen so that

$$f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta) \quad ,$$

which is a full Fourier series .

Hence，we deduce that

$$\begin{cases} a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(\phi)\cos n\phi d\phi & for \quad n = 0,1,2,... \\ b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(\phi)\sin n\phi d\phi & for \quad n = 1,2,3,... \end{cases} .$$

We examine the function

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta) \quad .$$

If $c = \frac{1}{\pi}\int_{-\pi}^{\pi}|f(\theta)|d\theta$ ，so that $|a_n| \le c$ and $|b_n| \le c$ ，we find that the series for u and its first and second partial derivatives are dominated by the series $\sum 2cn^2 r^{n-2}$. This series converges uniformly for $r \le r_0$ for any $r_0 < 1$. It follows that u is twice continuously differentiable for $r<1$，and its derivatives may be formed by term-by term differentiation of its series. Then

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=\sum_{n=1}^{\infty}r^n(a_n\cos n\theta+b_n\sin n\theta)[n(n-1)+n-n^2]=0 \quad ,$$

so that $u(r,\theta)$ is harmonic，that is it ，satisfies *Laplace's* equation.

In below，we give a example to illustrate above statement.

Example 2-2 : (Using Separation of variables to solve *Laplace's* equation)

Solve $\quad u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0 \quad$ for $\quad r<1$,

$u(1,\theta)=\sin^3\theta$.

Solution :

By equation (2-6)，we have

$$u(r,\theta)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}r^n(a_n\cos n\theta+b_n\sin n\theta) \quad ,$$

and

$$u(1,\theta)=\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)=\sin^3\theta .$$

$$\Rightarrow a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\sin^3\phi\cos n\phi d\phi=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{8}[3\sin(n+1)\phi-3\sin(n-1)\phi-\sin(n+3)\phi+\sin(n-3)\phi]d\phi=0 \quad .$$

And $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\sin^3\phi\sin n\phi d\phi=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{8}[3\cos(n-1)\phi-3\cos(n+1)\phi-\cos(n-3)\phi+\cos(n+3)\phi]d\phi$

$$=\begin{cases}\frac{3}{4} & , \quad n=1\\ \frac{-1}{4} & , \quad n=3\\ 0 & , \quad otherwise\end{cases} \quad .$$

Finally，we get $u(r,\theta)=\frac{3}{4}r\sin\theta-\frac{1}{4}r^3\sin 3\theta$.

Ⅱ-2 **Finite Fourier transform to construct solution of system of** *Laplace's* **equation**

We shall now treat the corresponding non-homogeneous problem

$$u_{xx} + u_{yy} = F(x, y) \quad \text{for} \quad 0 < x < \pi \quad \text{and} \quad 0 < y < 1 \ , \qquad (2\text{-}7)$$

$$u(x,1) = u(0, y) = u(\pi, y) = 0 \ ,$$
$$u(x,0) = 0 \ ,$$

by expanding the solution in a Fourier series in terms of the same set of functions.

To solve the above non-homogeneous problem，we expand the solution in a Fourier sine series for each fixed y ：

$$u(x, y) \sim \sum_{n=1}^{\infty} b_n(y) \sin nx \ .$$

The set of sine coefficients

$$b_n(y) = \frac{2}{\pi}\int_0^{\pi} u(x, y)\sin nx dx \ ,$$

which is a function of the integer n and y，determines $u(x, y)$ uniquely. It is called the finite sine transform of $u(x, y)$.

If $\frac{\partial^2 u}{\partial x^2}$ is continuous，its finite sine transform is given by

$$\frac{2}{\pi}\int_0^{\pi} u_{xx}(x, y)\sin nx dx = \frac{2}{\pi}[u_x(x, y)\sin nx \ \Big|_0^{\pi} - \int_0^{\pi} u_x(x, y)\cdot n\cos nx dx]$$

$$= \frac{-2}{\pi}\cdot n^2 \int_0^{\pi} u(x, y)\sin nx dx$$

$$= (-n^2) b_n(y) \ ,$$

because $u(0, y) = u(\pi, y) = 0$. Differentiating u with respect to x twice corresponds to the simpler operation of multiplying its finite sine transform by $(-n^2)$.

If $\frac{\partial^2 u}{\partial y^2}$ is continuous，we can interchange integration and differentiation to show that

$$\frac{2}{\pi}\int_0^{\pi} u_{yy}(x, y)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} \frac{\partial^2}{\partial y^2} u(x, y)\sin nx dx$$

$$= \frac{d}{dy}(\frac{2}{\pi}\int_0^{\pi} u(x, y)\sin nx dx)$$

$$= b_n''(y) \ .$$

Taking the finite sine transform of both sides of (2-7) therefore leads to the equation

$$u_{xx} + u_{yy} = F(x, y)$$

$$\Rightarrow \frac{2}{\pi}\int_0^{\pi} u_{xx}(x, y)\sin nx dx + \frac{2}{\pi}\int_0^{\pi} u_{yy}(x, y)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} F(x, y)\sin nx dx$$

$$\Rightarrow (-n^2)b_n(y) + \frac{d^2}{dy^2}b_n(y) = B_n(y) \quad \text{for} \quad n = 1,2,3,...$$

$$\Rightarrow b_n''(y) - n^2 b_n(y) = B_n(y) \quad .$$

The condition $u(x,0) = 0$ means that

$$b_n(0) = 0 \quad .$$

Taking sine transform has reduced the problem (2-7) for a partial differential to the problem for an ordinary differential equation，that is

$$\begin{cases} b_n''(y) - n^2 b_n(y) = B_n(y) \\ b_n(0) = 0 \end{cases} .$$

Solving this by a method，we can use ***Green's function*** to solves it，and the solution has Fourier sine series form. By ***Schwarz's inequality*** for sums and ***Parseval's equation***，we have proved the series $\sum b_n(y)\sin nx$ converges uniformly for $0 \le x \le \pi$, $0 \le y \le 1$.

Under this condition，we get

$$u(x, y) = \sum_{n=1}^{\infty} b_n(y)\sin nx \quad .$$

In below，we give a example to illustrate above statement.

Example 2-3 : (Using <u>Finite Fourier Transform</u> to solve *Laplace's* equation)

Solve $u_{xx}+u_{yy}=y(1-y)\sin^3 x$ for $0<x<\pi$, $0<y<1$,

$u(x,0)=u(x,1)=u(0,y)=u(\pi,y)=0$.

<u>*Solution*</u> :

Let $u(x,y)=X(x)Y(y)$

$$\Rightarrow \frac{X''(x)}{X(x)}-\frac{Y''(y)}{Y(y)}=-\lambda$$

$$\Rightarrow \begin{cases} X''+\lambda X=0 \quad , \quad 0<x<\pi \\ X(0)=X(\pi)=0 \end{cases} \quad \text{and} \quad \begin{cases} Y''-\lambda Y=0 \quad , \quad 0<y<1 \\ Y(1)=0 \end{cases} .$$

When $\lambda>0$, we have $X(x)=a\sin\sqrt{\lambda}x+b\cos\sqrt{\lambda}x$.

And $X(0)=b\cdot 1=0 \Rightarrow b=0$,

and $X(\pi)=a\sin\sqrt{\lambda}\pi=0 \Rightarrow a=0(trivial)$ or $\sin\sqrt{\lambda}\pi=0 \Rightarrow \lambda=n^2$ with $n=1,2,..$.

So we get $X(x)=\sin(nx)$ with $n=1,2,..$.

For $\lambda=0$ and $\lambda<0 \Rightarrow$ trivial solution .

Since $\lambda=n^2$ with $n=1,2,..$.

We have $Y(y)=Ae^{ny}+Be^{-ny}$ and $Y(1)=Ae^{n}+Be^{-n}=0 \Rightarrow A=-Be^{-2n}$.

So $Y(y)=Ae^{ny}+Be^{-ny}=-Be^{n(y-2)}+Be^{-ny}=\frac{-B}{e}(e^{n(y-2)}+e^{n(1-y)})=\sinh n(y-1)$ with $n=1,2,..$.

Hence $u(x,y)=\sum_{n=1}^{\infty}b_n\sinh n(y-1)\sin nx$.

<u>Ansatz</u> $u(x,y)=\sum_{n=1}^{\infty}b_n(y)\sin nx$ where $b_n(y)=\frac{2}{\pi}\int_0^{\pi}u(x,y)\sin nx dx$.

We have $\frac{2}{\pi}\int_0^{\pi}u_{xx}(x,y)\sin nx dx=\frac{2}{\pi}[u_x(x,y)\sin nx \,\Big|_0^{\pi} - \int_0^{\pi}u_x(x,y)\cdot n\cos nx dx]$

$$=\frac{-2}{\pi}\cdot n^2\int_0^{\pi}u(x,y)\sin nx dx = (-n^2)b_n(y) ,$$

and $\frac{2}{\pi}\int_0^{\pi}u_{yy}(x,y)\sin nx dx=\frac{2}{\pi}\int_0^{\pi}\frac{\partial^2}{\partial y^2}u(x,y)\sin nx dx=\frac{d}{dy}(\frac{2}{\pi}\int_0^{\pi}u(x,y)\sin nx dx) = b_n''(y)$.

Given $u_{xx} + u_{yy} = y(1-y)\sin^3 x$

$$\Rightarrow \frac{2}{\pi}\int_0^\pi u_{xx}\sin nxdx + \frac{2}{\pi}\int_0^\pi u_{yy}\sin nxdx = \frac{2}{\pi}\int_0^\pi y(1-y)\sin^3 x\cdot\sin nxdx$$

$$\Rightarrow (-n^2)b_n(y) + b_n''(y) = \frac{2}{\pi}y(1-y)\int_0^\pi \sin^3 x\cdot\sin nxdx \quad \text{for} \quad n = 1,2,\ldots$$

$$\Rightarrow b_n''(y) - n^2 b_n(y) = \frac{2}{\pi}y(1-y)\int_0^\pi \sin^3 x\sin nxdx \ .$$

In this case， $\begin{cases} u(x,o) = 0 \Rightarrow b_n(0) = 0 \\ u(x,1) = 0 \Rightarrow b_n(1) = 0 \end{cases}$,

Since

$$\int_0^\pi \sin^3 x\cdot\sin nxdx = \int_0^\pi \frac{1}{2}\sin^2 x[\cos(n-1)x - \cos(n+1)x]dx$$

$$= \int_0^\pi \frac{1}{4}\sin x[\sin nx - \sin(n-2)x - \sin(n+2)x + \sin nx]dx$$

$$= \frac{1}{8}\int_0^\pi [3\cos(n-1)x - 3\cos(n+1)x - \cos(n-3)x + \cos(n+3)x]dx$$

$$= \begin{cases} \frac{3\pi}{8} &, \ n=1 \\ \frac{-\pi}{8} &, \ n=3 \\ 0 &, \ otherwise \end{cases} ,$$

and $\begin{cases} \frac{3\pi}{8}\cdot\frac{2}{\pi}y(1-y) = \frac{3}{4}y(1-y) &, \ n=1 \\ \frac{-\pi}{8}\cdot\frac{2}{\pi}y(1-y) = \frac{-1}{4}y(1-y) &, \ n=3 \end{cases}$.

Now we use *Green′s function* to solves it，

and we have $\begin{cases} p(y) = 1 \\ q(y) = -n^2 \end{cases}$ and $\begin{cases} \alpha = 0 \\ \beta = 1 \end{cases}$, about v_1, v_2 satisfy above equation $v'' - n^2 v = 0$.

Let $v = e^{ry}$ $\Rightarrow$ $v_1(x) = e^{ny}$ and $v_2(x) = e^{-ny}$

$\Rightarrow$ $v_1'(x) = ne^{ny}$ and $v_2'(x) = -ne^{-ny}$.

We have $k = p(x)[v_1'(x)v_2(x) - v_2'(x)v_1(x)] = 1 \cdot [ne^{ny} \cdot e^{-ny} + ne^{-ny} \cdot e^{ny}] = 2n$,

and $D = v_1(\alpha)v_2(\beta) - v_1(\beta)v_2(\alpha) = e^{-n} - e^{n}$.

When $\xi \le x$, we have

$$G(x,\xi) = \frac{1}{kD}[v_1(\xi)v_2(\alpha) - v_1(\alpha)v_2(\xi)][v_1(x)v_2(\beta) - v_1(\beta)v_2(x)]$$

$$= \frac{1}{2n(e^{-n} - e^{n})}[e^{n\xi} - e^{-n\xi}][e^{ny} \cdot e^{-n} - e^{n} \cdot e^{-ny}]$$

$$= \frac{1}{2n(e^{n} - e^{-n})}[e^{n\xi} - e^{-n\xi}][e^{n(1-y)} - e^{-n(1-y)}] \ .$$

When $\xi \ge x$, we have

$$G(x,\xi) = \frac{1}{kD}[v_1(x)v_2(\alpha) - v_1(\alpha)v_2(x)][v_1(\xi)v_2(\beta) - v_1(\beta)v_2(\xi)]$$

$$= \frac{1}{2n(e^{n} - e^{-n})}[e^{ny} - e^{-ny}][e^{n(1-\xi)} - e^{-n(1-\xi)}] \ .$$

So

$$\int_0^y (e^{n\xi} - e^{-n\xi})\xi(1-\xi)d\xi = \int_0^y \xi e^{n\xi} - \xi e^{-n\xi} - \xi^2 e^{n\xi} + \xi^2 e^{-n\xi} d\xi$$

$$= (\frac{y}{n} - \frac{1}{n^2} - \frac{y^2}{n} + \frac{2y}{n^2} - \frac{2}{n^3})e^{ny} + (\frac{y}{n} + \frac{1}{n^2} - \frac{y^2}{n} - \frac{2y}{n^2} - \frac{2}{n^3})e^{-ny} + \frac{4}{n^3} \ ,$$

and

$$\int_y^1 (e^{n(1-\xi)} - e^{-n(1-\xi)})\xi(1-\xi)d\xi = (\frac{y}{n} + \frac{1}{n^2} - \frac{y^2}{n} - \frac{2y}{n^2} - \frac{2}{n^3})e^{n(1-y)} + (\frac{y}{n} - \frac{1}{n^2} - \frac{y^2}{n} + \frac{2y}{n^2} - \frac{2}{n^3})e^{-n(1-y)} + \frac{4}{n^3} \ .$$

Hence

$$\frac{1}{2n(e^{n} - e^{-n})}\{[e^{n(1-y)} - e^{-n(1-y)}]\int_0^y (e^{n\xi} - e^{-n\xi})\xi(1-\xi)d\xi + [e^{ny} - e^{-ny}]\int_y^1 (e^{n(1-\xi)} - e^{-n(1-\xi)})\xi(1-\xi)d\xi\}$$

$$= \frac{1}{n^2}y(1-y) - \frac{2}{n^4} + \frac{2}{n^4} \cdot \frac{\cosh n(y - \frac{1}{2})}{\cosh \frac{n}{2}} \ .$$

$$\Rightarrow \quad b_n(y) = \int_0^y G(y,\xi) f(\xi) d\xi + \int_y^1 G(y,\xi) f(\xi) d\xi \; = \begin{cases} \dfrac{3}{4} & , \quad n=1 \\ \dfrac{-1}{4} & , \quad n=3 \\ 0 & , \quad otherwise \end{cases} \quad .$$

Therefore，the solution is

$$u(x,y) = \frac{3}{4}\{y(1-y) - 2 + 2 \cdot \frac{\cosh(y-\frac{1}{2})}{\cosh\frac{1}{2}}\}\sin x - \frac{1}{4}\{\frac{y}{9}(1-y) - \frac{2}{81} + \frac{2}{81} \cdot \frac{\cosh 3(y-\frac{1}{2})}{\cosh\frac{3}{2}}\}\sin 3x \quad .$$

II-3 <u>Fourier Transform to construct solution of system of</u> *Laplace's* <u>equation</u>

Just as problems on the finite intervals lead to Fourier series，problems on the whole line $(-\infty, \infty)$ lead to Fourier transform. To understand this relationship，consider a function $f(x)$ defined on the interval $(-l, l)$. Its Fourier series，In complex notation，is

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} \quad ,$$

where the coefficients are

$$c_n = \frac{1}{2l} \int_{-l}^{l} f(y) e^{\frac{-in\pi y}{l}} dy \quad .$$

The coefficients c_n define the function $f(x)$ uniquely in the interval $(-l, l)$.
The Fourier integral comes from letting $l \to \infty$. However，this limit is one of the trickiest in all mathematics because the interval grows simultaneously as the terms change. If we write $k = \dfrac{n\pi}{l}$，and substitute the coefficients into the series，we get

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} [\int_{l}^{l} f(y) e^{-iky} dy] e^{ikx} \frac{\pi}{l} \quad .$$

As $l \to \infty$，the interval expands to the whole line and the points k get closer toghter. In the limit we should expect k to become a continuous variable，and the sum to become an integral. The distance between two successive $k's$ is $\Delta k = \dfrac{\pi}{l}$，which we may think of as becoming dk in the limit. Therefore，we expect the result

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}f(y)e^{-iky}dy\right]e^{ikx}dk \quad . \qquad (2\text{-}8)$$

Another way to state the above identity (2-8) is

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(w)e^{ikx}\frac{dw}{2\pi} \quad \text{where} \quad F(w)=\int_{-\infty}^{\infty}f(x)e^{iwx}dx \quad .$$

Let $$\hat{f}(w)=\int_{-\infty}^{\infty}f(x)e^{iwx}dx \quad , \qquad (2\text{-}9)$$

then $$f(x)=\frac{1}{2\pi}\lim_{L\to\infty}\int_{-L}^{L}\hat{f}(w)e^{-iwx}dw \quad .$$

If the integral in (2-9) converges，it is called the Fourier transform of $f(x)$. It is sometimes denoted by $F[f]$. The integral converges if $\int_{-\infty}^{\infty}|f(x)|dx$ does.

The Fourier transform of $f(x)$ is

$$F[f](w)=\hat{f}(w)=\int_{-\infty}^{\infty}f(x)e^{iwx}dx \quad ,$$

and the inverse Fourier transform is

$$f(x)=F^{-1}[f]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(w)e^{-iwx}dw \quad .$$

For functions of two variables，say $u(x,y)$，and we define

$$F[u](w,y)\equiv\hat{u}(w,y)=\int_{-\infty}^{\infty}u(x,y)e^{iwx}dx \quad .$$

A basic property of the Fourier transform is that the k th derivative $u^{(k)}$ with $k=1,2,...$ transforms to an algebraic expression，that is

$$F[u^{k}](w,y)=(-iw)^{k}\,\hat{u}(w,y) \quad ,$$

confirming our comment that derivatives are transformed to multiplication. This formula is easily proved by integration by parts.

One of the many important formulae which is used in this field is given in the convolution theorem. The convolution $f * g$ of two functions f and g is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du = \int_{-\infty}^{\infty} f(x-u)g(u)du \quad .$$

Now

$$F[f * g] = \int_{-\infty}^{\infty} e^{iwx} \int_{-\infty}^{\infty} f(u)g(x-u)dudx$$

$$= \int_{-\infty}^{\infty} f(u)\int_{-\infty}^{\infty} g(x-u)e^{iwx}dxdu \quad .$$

After applying this change of variables in above equation，we deduce the convolution theorem which states that

$$F[f * g] = \int_{-\infty}^{\infty} f(x)\int_{-\infty}^{\infty} g(v)e^{iw(x+v)}dvdx$$

$$= \int_{-\infty}^{\infty} f(x)e^{iwx}dx \cdot \int_{-\infty}^{\infty} g(v)e^{iwv}dv = F[f]F[g] = \hat{f} \cdot \hat{g} \quad ,$$

and

$$F^{-1}[\hat{f} \cdot \hat{g}] = f * g = \int_{-\infty}^{\infty} f(u)g(x-u)du \quad .$$

This is useful relationship in solving differential equations.
Following is a table of some important basic properties of transforms

	$f(x)$	$\hat{f}(w)$
1	f'	$iw\hat{f}$
2	$xf(x)$	$i\hat{f}'$
3	$f(x-a)$	$e^{-iaw}\hat{f}$
4	$e^{iax}f(x)$	$\hat{f}(w-a)$
5	$af(x)+bg(x)$	$a\hat{f}+b\hat{b}$
6	$f(ax)$	$\frac{1}{a}F[\frac{w}{a}]$

Table 2-1. Basic properties of transforms

In below，we give a example to illustrate above statement.

Example 2-4 : (Using Fourier Transform to solve *Laplace's* equation)

Consider $u_{xx} + u_{yy} = 0$ in the half plane $y \geq 0$ subject to the boundary condition

$u(x,0) = \delta(x)$ with $x \in R$ and the condition $u(x,y) \to 0$ as $\sqrt{x^2+y^2} \to \infty$.

Solution :

Using Fourier transform with respect to x ,

$$F[u(x,y)] = \hat{u}(w,y) = \int_{-\infty}^{\infty} u(x,y)e^{iwx}dx \quad ,$$

and $$F[\frac{\partial^2 u}{\partial y^2}] = \hat{u}_{yy} \quad , \quad F[\frac{\partial^2 u}{\partial x^2}] = (-iw)^2 \hat{u} \quad .$$

Which implies $\hat{u}$ satisfies the ODE

$$\hat{u}_{yy} - w^2 \hat{u} = 0 \quad \text{for} \quad y > 0 \quad , \quad F[(w,0)] = 1 .$$

The solutions of the ODE are $e^{\pm wy}$. We must reject a positive exponent since $\hat{u}$ would grow exponentially as $|w| \to \infty$ and would not have Fourier transform.

So $\hat{u}(w,y) = e^{-|w|y}$. Therefore ,

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-y|w|}e^{iwx}dw \quad , \quad w \in R \quad \text{and} \quad y \geq 0 \quad .$$

This improper integral clearly converges for $y > 0$. It is split into to parts and integrated directly as

$$u(x,y) = \frac{1}{2\pi(ix-y)}e^{iwx-wy} \Big|_{0}^{\infty} + \frac{1}{2\pi(ix+y)}e^{iwx+wy} \Big|_{-\infty}^{0}$$

$$= \frac{1}{2\pi}(\frac{1}{y-ix} + \frac{1}{y+ix}) \quad .$$

$$= \frac{y}{\pi(x^2+y^2)} \quad .$$

Ⅱ-4 Finite Difference to construct solution of system of *Laplace′s* equation

One scheme for solving all kinds of partial differential equations is to replace the derivatives by difference quotients , converting the equation to a difference equation. We then write the difference equation corresponding to each point at the intersections of a gridwork that subdivides the region of interest at which the function values are unknown. Solving these equations simultaneously gives values for the function at each node that approximate the true values. We begin with the two-dimensional case.

Let $h = \Delta x =$ equal spacing of gridwork in the $x - direction$, see Figure 2-1. We assume that the function $f(x)$ has a continuous fourth derivative. By Taylor series ,

$$f(x_n + h) = f(x_n) + f'(x_n)h + \frac{f''(x_n)}{2}h^2 + \frac{f'''(x_n)}{6}h^3 + \frac{f^{(IV)}(x_n)}{24}h^4 \ ,$$
$$x_n < \xi_1 < x_n + h \ ,$$

$$f(x_n - h) = f(x_n) - f'(x_n)h + \frac{f''(x_n)}{2}h^2 - \frac{f'''(x_n)}{6}h^3 + \frac{f^{(IV)}(x_n)}{24}h^4 \ ,$$
$$x_n - h < \xi_2 < x_n \ .$$

It follows that

$$\frac{f(x_n + h) - 2f(x_n) + f(x_n - h)}{h^2} = f''(x_n) + \frac{f^{(IV)}(\xi)}{12}h^4 \quad \text{where} \quad x_n - h < \xi < x_n + h \ .$$

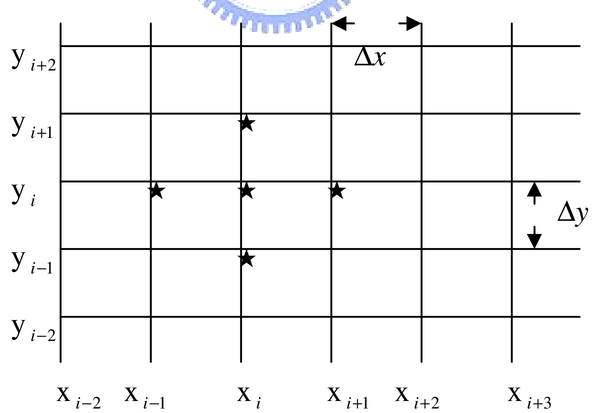


Figure 2-1. Taking five interior points

A subscript notation is convenient :

$$\frac{f_{n+1} - 2f_n + f_{n-1}}{h^2} = f_n'' + O(h^2) \ .$$

In above equation，the subscripts on f indicate the $x-values$ at which it is evaluated. The order relation $O(h^2)$ signifies that the error approaches proportionality to h^2 as $h \to 0$.

Similarly，the first derivative is approximated，

$$\frac{f(x_n+h)-f(x_n-h)}{2h} = f'(x_n) + \frac{f^{(III)}(\xi)}{6}h^2 \quad \text{where } x_n - h < \xi < x_n + h .$$

$$\Rightarrow \qquad \frac{f_{n+1}-f_{n-1}}{2h} = f_n' + O(h^2) \ .$$

When f is a function of both x and y，we get the second partial derivative with respect to x，$\partial^2 u / \partial x^2$，by holding y constant and evaluating the function at three points where x equals x_n，$x_n + h$ and $x_n - h$. The partial derivative $\partial^2 u / \partial y^2$ is similarly computed，holding x constant. We require that fourth derivatives with respect to both variables exist.

To solve the $Laplace's$ equation on a region in the $xy-plane$，we subdivide the region with equispaced lines parallel to the $x-$ and $y-axis$. Consider a portion of the region near (x_i, y_j). We wish to approximate

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \qquad \text{in } D,$$

$$u = f \qquad \text{in } C,$$

in a bounded domain D with boundary C.

Replacing the $Laplace's$ equation by the finite difference equation，we get

$$\nabla^2 v(x_i, y_j) = \frac{v(x_{i+1}, y_j) - 2v(x_i, y_j) + v(x_{i-1}, y_j)}{(\Delta x)^2} + \frac{v(x_i, y_{j+1}) - 2v(x_i, y_j) + v(x_i, y_{j-1})}{(\Delta y)^2} = 0 \ .$$

It is convenient to let double subscripts on u indicate the $x-$ and $y-values$:

$$\nabla^2 v_{i,j} = \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{(\Delta x)^2} + \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{(\Delta y)^2} = 0 \ .$$

We call the points $(i+1, j)$、$(i-1, j)$、$(i, j+1)$ and $(i, j-1)$ the nearest neighbors of the mesh point (i, j). If (i, j) and all its nearest neighbors lie in $D+C$ ，we call (i, j) an interior point.

It is common to take $\Delta x = \Delta y = h$ ，resulting in considerable simplification，so that

$$\nabla^2 v_{i,j} = \frac{1}{h^2}[v_{i+1,j} + v_{i-1,j} + v_{i,j+1} + v_{i,j-1} - 4v_{i,j}] = 0 \quad . \qquad (2\text{-}9)$$

Note that five points are involved in the relationship of equation (2-9)，points to the right、left、above and below the central point (x_i, y_j). The approximation has $O(h^2)$ error，provided that u is sufficiently smooth. This formula is referred to as the five-point star formula.

The system we get in this way has exactly one solution. To prove this，suppose that there were two solutions，$\{u_{i,j}\}$ and $\{v_{i,j}\}$ of (2-9) in D with identical boundary values. Their difference $\{u_{i,j} - v_{i,j}\}$ also satisfies (2-9) in D but with zero boundary values. By the maximum principle，$u_{i,j} - v_{i,j} \le 0$ ，hence $u_{i,j} = v_{i,j}$. So there is at most one solution.

Now，if we define the error function $w = u - v$.

The boundary value problem for u is therefore properly posed. As $h \to 0$ the error $w = u - v$ approaches zero. That is，v converges to u.

In below，we give a example to illustrate above statement.

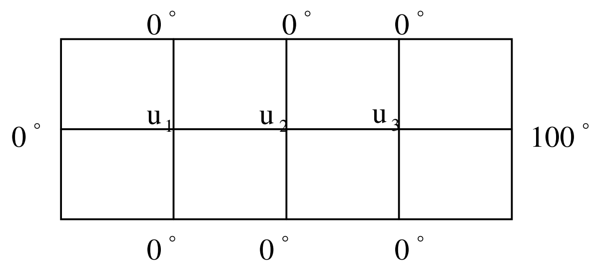
Example 2-5 : (Using <u>Finite Difference</u> to solve *Laplace's* equation)

Find $u(x, y)$ such that

$$u_{xx} + u_{yy} = 0 \quad ,$$

$$u(x,o) = u(x,10) = u(o,y) = 0 \quad ,$$
$$u(20,y) = 100 \quad .$$

<u>*Solution* :</u>



We replace the differential equation by a difference equation:

$$\frac{1}{h^2}[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}] = 0$$

$$\Rightarrow \quad u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0$$

$$\Rightarrow \quad 4u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j-1} - u_{i,j+1} = 0 \quad .$$

Suppose we choose $h = 5$ ，the system of equations is

$$\frac{1}{5^2}(0 + 0 + u_2 + 0 - 4u_1) = 0 \quad ,$$

$$\frac{1}{5^2}(u_1 + 0 + u_3 + 0 - 4u_2) = 0 \quad ,$$

$$\frac{1}{5^2}(u_2 + 0 + 100 + 0 - 4u_3) = 0 \quad .$$

We can write equations as matrix form and usung 〝Metlab〞 to solve.

The solution to the set of equations is easy when there are only three of them :

$$u_1 = 1.786 \quad 、 \quad u_2 = 7.143 \quad 、 \quad u_3 = 26.786 \quad .$$

Ⅲ. <u>The limit of the methods of solving Ellpitic PDE</u>

In this chapter，we want to analysis the limit of four methods of solving *Laplace′s equation* in chapter Ⅱ.

1、The limit of Separation of variables

The standard technique for solving PDE_s on bounded (rectangular) domains is called separation of variables. The idea is to assume that the unknown function $u = u(x, y)$ in an initial boundary value problem can be written as a product of a funvtion of x and a function of y，that is，$u(x, y) = X(x)Y(y)$. Thus，the variables separate. If the method is to be successful，when this product is substituted into the PDE，the PDE separates into two ODE_s，one for $X(x)$ and one for $Y(y)$. Therefore，we are left with an ODE boundary value problem for $X(x)$ and an ODE for $Y(y)$. When we solve for $X(x)$ and $Y(y)$，we will have a product solution $u(x, y)$ of the PDE that satisfies the boundary conditions.

Whether or not the method of separation of variables can be applied to a particular problem depends not only on the differential equation but also on the shape of the boundary and on the form of the boundary conditions.

Three things are needed to apply the method to a problem in two variables x and y:

(a) The differential operator L must be separable. For example，this elliptic equation

$u_{xx} + u_{xy} + u_{yy} = 0$，it can not use Separation of variables to find solution.

(b) All initial and boundary conditions must be on lines $x \equiv cons\tan t$ and $y \equiv cons\tan t$.

(c) The linear operators defining the boundary conditions at $x \equiv cons\tan t$ must involve no partial derivatives of u with respect to y，and their coefficients must be independent of y. Those at $y \equiv cons\tan t$ must involve no partial derivatives of u with respect to x，and their coefficients must be independent of x.

That the method of separation of variables can only be applied to a special class of problems.

2、Finite Fourier transform

To solve the nonhomogenous problem，we expand the solution in a Fourier sine series. The Finite Fourier transforms，are simply Fourier coefficients. Whenever a homogeneous problem can be solved by separation of variables in the form of a Fourier series，the Finite

Fourier transform reduces the partial differential equation to an infinite system of ordinary differential equations. These equations can then be solved by the methods of one-sided *Green's function* or *Green's function*. The Finite Fourier transform is using half - space domain.

3、Fourier transform

The Fourier transforms are first encountered in elementary differential equations courses as a technique for solving linear，constant-coefficient ordinary differential equations; Fourier transforms convert an ODE into an algebra problem. The ideas easily extend to PDE_s，where the operation of Fourier transformation converts PDE_s into ODE_s. Thus the Fourier transforms is useful as a computational tool in solving differential equations. In PDE_s the Fourier transform is usually applied to the spatial variable when it varies over whole line. That is，the Fourier transform is using whole space domain.

4、Finite Difference

The finite difference method is using the domain of rechangular domain or irregular shape. This methos solution form is discrete solution and it is the approximate solution (value). All we need to do is to continue to make h smaller. However，this procedure runs into severe difficulties. It is apparent that the number of equations increases inordinately fast. With $h=1.25$，we would have 105 discrete interior points; with $h=0.625$，we have 465 discrete interior points and so on. Storing a matrix with 105 rows and 105 columns would require 105^2 of computer memory. Few computer systems allow us such a generous partition，and overlaying memory space from disk storage would be extremely time-consuming. Along with memory requirements，we worry about execution times.

Compared with four methods :

(a) The homogeneous problem can be solved by Separation of variables、Fourier Transform、Finite Difference. But to solve the nonhomogeneous problem，we can use Finite Fourier Transform.

(b) Separation of variables、Finite Fourier Transform and Fourier Transform reduces the partial differential equation to ordinary differential equations and facilitates us to solve.

(c) The solution caused by Separation of variables、Finite Fourier Transform or Fourier

Transform is continuous，whereas the solution caused by Finite Difference is discrete type and it is the approximate solution.

(d) The Separation of variables method can be applied to rectangle domain; the Finite Fourier Transform method can be applied to half-space domain; the Fourier Transform method can be applied to whole space domain (whole line); the Finite Difference methid can be applied to rectangular domain or irregular shape domain.

Ⅳ. ***Integral evaluations on three-sheeted Riemann surface of genus N***

We know that there are some differential equations whose solution space is in the

Riemann surface. In this chapter，we want to compute the integrals $\int_{\gamma}\frac{1}{f(z)}dz$ ，where γ is in the Riemann surface of algebraic curve $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$. We will develop an algorithm such that we can compute the integrals $\int_{\gamma}\frac{1}{\sqrt{\prod_{j=1}^{n}(z-z_j)}}dz$ by 〝Mathematica〞.

Before computing integrals，it is necessary to discuss the Riemann surface of $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$.

Ⅳ-1 Fundamental introduction

For simplicity, we take $f(z)=\sqrt{z}$ to define a single-value and analytic function on the Riemann surface.

Now we let $z\in C$, and use polar form for z. That is,

$$z = re^{i\theta} \qquad (4\text{-}1)$$

$$= re^{i(\theta+2\pi)}. \qquad (4\text{-}2)$$

Then by (4-1)

$$\sqrt{z} = r^{\frac{1}{2}}e^{i\frac{\theta}{2}},$$

and by (4-2)

$$\sqrt{z} = r^{\frac{1}{2}}e^{i(\frac{\theta+2\pi}{2})} = r^{\frac{1}{2}}e^{i(\frac{\theta}{2}+\pi)} = -r^{\frac{1}{2}}e^{i\frac{\theta}{2}}.$$

Therefore $f(z)=\sqrt{z}$ is a multi-valued function at each $z\in C$ and is not analytic on C.

How to make $f(z)=\sqrt{z}$ to be a single-valued and analytic at every point on C?

Consider two cuts from 0 to $-\infty$ ($i.e.$ the negative real axis).

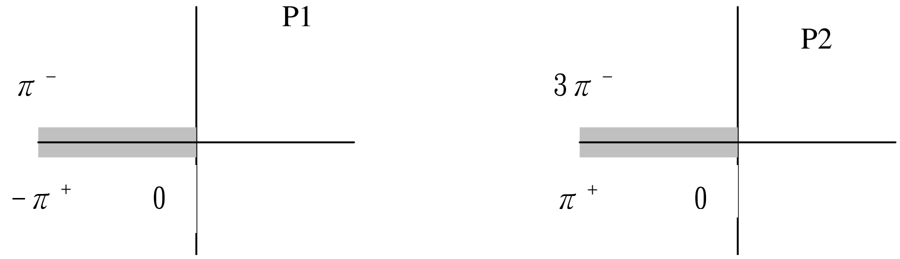
Let

$$P_1 = \{C \setminus (-\infty,0] \mid \theta_1 = \arg z \in [-\pi^+, \pi^-)\ \} \ ,$$

and

$$P_2 = \{C \setminus (-\infty,0] \mid \theta_2 = \arg z \in [\pi^+, 3\pi^-)\ \} \ ,$$

as Figure 4-1 shows.



(a) (b)

Figure 4-1. Cut from 0 to $-\infty$ on P1 and P2

Define

$$f_1(z) = \sqrt{z} \ , \ z \in P_1 \ ,$$

$$f_2(z) = \sqrt{z} \ , \ z \in P_2 \ ,$$

then $f_1(z) = \sqrt{z} = |z|^{\frac{1}{2}} e^{i\frac{\theta_1}{2}}$ is single-valued at each $z \in P_1$ and analytic on P_1.

$$f_2(z) = \sqrt{z} = |z|^{\frac{1}{2}} e^{i\frac{\theta_2}{2}} = |z|^{\frac{1}{2}} e^{i\frac{\theta_1 + 2\pi}{2}} = |z|^{\frac{1}{2}} e^{i\frac{\theta_1}{2}} e^{i\pi} = -|z|^{\frac{1}{2}} e^{i\frac{\theta_1}{2}} = -f_1(z) \, .$$

is also single-valued at each $z \in P_2$ and analytic on P_2.

Let

$$D_1 = \{\ (-\infty,0] \mid \arg z = \pi\ \} \ ,$$

as Figure 4-2 shows.

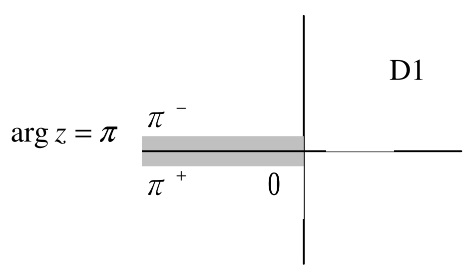


Figure 4-2. Cut from 0 to $-\infty$ on D1

If $z \in P_1$ and $\arg z$ tends to π^-, then $\sqrt{z} = |z|^{\frac{1}{2}} e^{i\frac{\arg z}{2}} \approx |z|^{\frac{1}{2}} e^{i\frac{\pi}{2}} = i|z|^{\frac{1}{2}}$.

If $z \in P_2$ and $\arg z$ tends to π^+ , then $\sqrt{z} = |z|^{\frac{1}{2}} e^{i\frac{\arg z}{2}} \approx |z|^{\frac{1}{2}} e^{i\frac{\pi}{2}} = i|z|^{\frac{1}{2}}$.

So , $\sqrt{z}$ is continuous cross the cut $(-\infty, 0]$ for $z \in D_1$.

We define

$$f_3(z) = \sqrt{z} \text{ , } z \in D_1 \text{ ,}$$

then

$$f_3(z) = \sqrt{z} = |z|^{\frac{1}{2}} e^{i\frac{\pi}{2}} = i|z|^{\frac{1}{2}} \text{ for } z \in D_1 \text{ and analytic on } D_1.$$

Let

$$D_2 = \{ \ (-\infty, 0] \mid \arg z = 3\pi \ \} \text{ ,}$$

as Figure 4-3 shows.

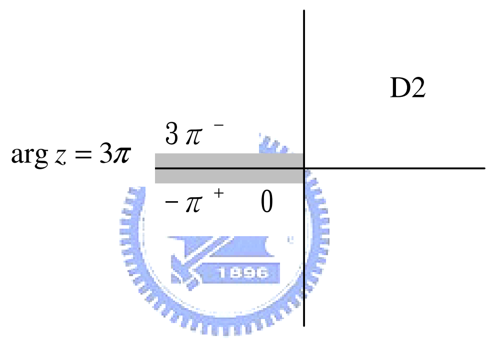


Figure 4-3 . Cut from 0 to $-\infty$ on D2

If $z \in P_2$ and $\arg z$ tends to $3\pi^-$, then $\sqrt{z} = |z|^{\frac{1}{2}} e^{i\frac{\arg z}{2}} \approx |z|^{\frac{1}{2}} e^{i\frac{3\pi}{2}} = -i|z|^{\frac{1}{2}}$.

If $z \in P_1$ and $\arg z$ tends to $-\pi^+$, then $\sqrt{z} = |z|^{\frac{1}{2}} e^{i\frac{\arg z}{2}} \approx |z|^{\frac{1}{2}} e^{i(-\frac{\pi}{2})} = -i|z|^{\frac{1}{2}}$.

So , $\sqrt{z}$ is continuous cross the cut $(-\infty, 0]$ for $z \in D_2$.

We define

$$f_4(z) = \sqrt{z} \text{ , } z \in D_2 \text{ ,}$$

then

$$f_4(z) = -i|z|^{\frac{1}{2}} = -f_3(z) \text{ for } z \in D_2 \text{ and analytic on } D_2 .$$

According the discuss above , we can construct a single-valued function for $\sqrt{z}$.

We have the conclusion as the following:

Let $R_2 = P_1 \cup P_2 \cup (-\infty, 0]$ and a function $F : R_2 \to C$，define

$$F(z) = \begin{cases} f_1(z) & , \quad z \in P_1 \\ f_2(z) & , \quad z \in P_2 \\ f_3(z) & , \quad z \in D_1 \\ f_4(z) & , \quad z \in D_2 \end{cases} ,$$

then $F(z)$ is single-valued and analytic at every point $z \in R_2$.

Note that $f_1(z) = -f_2(z)$ and $f_3(z) = -f_4(z)$.

Moreover，$F(z)$ is defined on a Riemann surface R_2 which is a generalization of the complex plane to a surface of more than one sheet such that a multi-valued function has only one value corresponding to each point on the surface.

Ⅳ-2 Riemann surface of the algebraic curve $f(z) = \sqrt{\prod_{j=1}^{n}(z - z_j)}$ with $z_j \in R$

Consider $f(z) = \sqrt{\prod_{j=1}^{n}(z - z_j)}$，$z_j \in R$ and $z_1 > z_2 > z_3 > \ldots > z_n$ with n distance branch points.

Ⅳ-2.1 The horizontal cut structure of $f(z)$

Since $f(z)$ is a two-valued function，in order to define a single-valued and analytic function，therefore we need branch cuts. But how can we construct branch cuts？

In this paper，we by face the left direction to do cut explained. For convenience，let $n = 4$ and $n = 5$ to see what is going on？

First，we check if there is any cut，for $n = 4$ and $z_1 = 1$、$z_2 = 2$、$z_3 = 3$ and $z_4 = 4$，as Figure 4-4 shows.

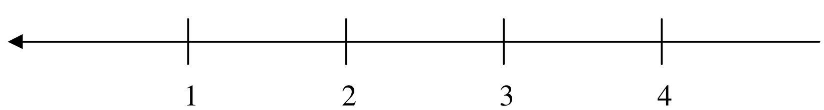


Figure 4-4. The branch points are $z_1 = 1$、$z_2 = 2$、$z_3 = 3$ and $z_4 = 4$

Put point -1 and $-1 \in (-\infty, 1)$，then we have

$$\arg(-1-1)=\arg(-2)=\begin{cases}-\pi\\ \pi\end{cases},$$

$$\arg(-1-2)=\arg(-3)=\begin{cases}-\pi\\ \pi\end{cases},$$

$$\arg(-1-3)=\arg(-4)=\begin{cases}-\pi\\ \pi\end{cases},$$

$$\arg(-1-4)=\arg(-5)=\begin{cases}-\pi\\ \pi\end{cases},$$

taking $-\pi$: $\sqrt{-2}\cdot\sqrt{-3}\cdot\sqrt{-4}\cdot\sqrt{-5}=|2|^{\frac{1}{2}}|3|^{\frac{1}{2}}|4|^{\frac{1}{2}}|5|^{\frac{1}{2}}e^{i(\frac{-2\pi}{2})}=-|120|^{\frac{1}{2}}$, (4-3)

taking π : $\sqrt{-2}\cdot\sqrt{-3}\cdot\sqrt{-4}\cdot\sqrt{-5}=|2|^{\frac{1}{2}}|3|^{\frac{1}{2}}|4|^{\frac{1}{2}}|5|^{\frac{1}{2}}e^{i(\frac{2\pi}{2})}=-|120|^{\frac{1}{2}}$. (4-4)

Since (4-3) = (4-4) .

So，there is no cut in $(-\infty,1)$.

Put point $\frac{3}{2}$ and $\frac{3}{2}\in(1,2)$ ，then we have

$$\arg(\frac{3}{2}-1)=\arg(\frac{1}{2})=0 ,$$

$$\arg(\frac{3}{2}-2)=\arg(-\frac{1}{2})=\begin{cases}-\pi\\ \pi\end{cases},$$

$$\arg(\frac{3}{2}-3)=\arg(-\frac{3}{2})=\begin{cases}-\pi\\ \pi\end{cases},$$

$$\arg(\frac{3}{2}-4)=\arg(-\frac{5}{2})=\begin{cases}-\pi\\ \pi\end{cases},$$

taking $-\pi$: $\sqrt{\frac{1}{2}}\cdot\sqrt{-\frac{1}{2}}\cdot\sqrt{-\frac{3}{2}}\cdot\sqrt{-\frac{5}{2}}=\left|\frac{1}{2}\right|^{\frac{1}{2}}\left|\frac{1}{2}\right|^{\frac{1}{2}}\left|\frac{3}{2}\right|^{\frac{1}{2}}\left|\frac{5}{2}\right|^{\frac{1}{2}}e^{i(\frac{-3\pi}{2})}=i\left|\frac{15}{16}\right|^{\frac{1}{2}}$, (4-5)

taking π : $\sqrt{\frac{1}{2}}\cdot\sqrt{-\frac{1}{2}}\cdot\sqrt{-\frac{3}{2}}\cdot\sqrt{-\frac{5}{2}}=\left|\frac{1}{2}\right|^{\frac{1}{2}}\left|\frac{1}{2}\right|^{\frac{1}{2}}\left|\frac{3}{2}\right|^{\frac{1}{2}}\left|\frac{5}{2}\right|^{\frac{1}{2}}e^{i(\frac{3\pi}{2})}=-i\left|\frac{15}{16}\right|^{\frac{1}{2}}$. (4-6)

Since (4-5) $\neq$ (4-6) .

So，there is a cut in $(1,2)$.

Put point $\frac{5}{2}$ and $\frac{5}{2} \in (2,3)$ ，then we have

$$\arg(\frac{5}{2}-1) = \arg(\frac{3}{2}) = 0 \quad ,$$

$$\arg(\frac{5}{2}-2) = \arg(\frac{1}{2}) = 0 \quad ,$$

$$\arg(\frac{5}{2}-3) = \arg(-\frac{1}{2}) = \begin{cases} -\pi \\ \pi \end{cases} \quad ,$$

$$\arg(\frac{5}{2}-4) = \arg(-\frac{3}{2}) = \begin{cases} -\pi \\ \pi \end{cases} \quad ,$$

taking $-\pi$: $\sqrt{\frac{3}{2}} \cdot \sqrt{\frac{1}{2}} \cdot \sqrt{-\frac{1}{2}} \cdot \sqrt{-\frac{3}{2}} = \left|\frac{3}{2}\right|^{\frac{1}{2}} \left|\frac{1}{2}\right|^{\frac{1}{2}} \left|\frac{1}{2}\right|^{\frac{1}{2}} \left|\frac{3}{2}\right|^{\frac{1}{2}} e^{i(-\pi)} = -\left|\frac{9}{16}\right|^{\frac{1}{2}}$, (4-7)

taking π : $\sqrt{\frac{3}{2}} \cdot \sqrt{\frac{1}{2}} \cdot \sqrt{-\frac{1}{2}} \cdot \sqrt{-\frac{3}{2}} = \left|\frac{3}{2}\right|^{\frac{1}{2}} \left|\frac{1}{2}\right|^{\frac{1}{2}} \left|\frac{1}{2}\right|^{\frac{1}{2}} \left|\frac{3}{2}\right|^{\frac{1}{2}} e^{i(\pi)} = -\left|\frac{9}{16}\right|^{\frac{1}{2}}$. (4-8)

Since (4-7) = (4-8) .
So，there is no cut in (2,3) .

Put point $\frac{7}{2}$ and $\frac{7}{2} \in (3,4)$ ，then we have

$$\arg(\frac{7}{2}-1) = \arg(\frac{5}{2}) = 0 \quad ,$$

$$\arg(\frac{7}{2}-2) = \arg(\frac{3}{2}) = 0 \quad ,$$

$$\arg(\frac{7}{2}-3) = \arg(\frac{1}{2}) = 0 \quad ,$$

$$\arg(\frac{7}{2}-4) = \arg(-\frac{1}{2}) = \begin{cases} -\pi \\ \pi \end{cases} \quad ,$$

taking $-\pi$: $\sqrt{\frac{5}{2}} \cdot \sqrt{\frac{3}{2}} \cdot \sqrt{\frac{1}{2}} \cdot \sqrt{-\frac{1}{2}} = \left|\frac{5}{2}\right|^{\frac{1}{2}} \left|\frac{3}{2}\right|^{\frac{1}{2}} \left|\frac{1}{2}\right|^{\frac{1}{2}} \left|\frac{1}{2}\right|^{\frac{1}{2}} e^{i(-\frac{\pi}{2})} = -i\left|\frac{15}{16}\right|^{\frac{1}{2}}$, (4-9)

taking π : $\sqrt{\frac{5}{2}} \cdot \sqrt{\frac{3}{2}} \cdot \sqrt{\frac{1}{2}} \cdot \sqrt{-\frac{1}{2}} = \left|\frac{5}{2}\right|^{\frac{1}{2}} \left|\frac{3}{2}\right|^{\frac{1}{2}} \left|\frac{1}{2}\right|^{\frac{1}{2}} \left|\frac{1}{2}\right|^{\frac{1}{2}} e^{i(\frac{\pi}{2})} = i\left|\frac{15}{16}\right|^{\frac{1}{2}}$. (4-10)

Since (4-9) $\neq$ (4-10) .
So，there is a cut in (3,4) .

Hence we have the branch cuts in [1,2] and [3,4]. As Figure 4-5 shows.

Figure 4-5. The cut structure for $n=4$ branch points in horizontal

But we can use another easier way to get branch cut，as Figure 4-6 shows.

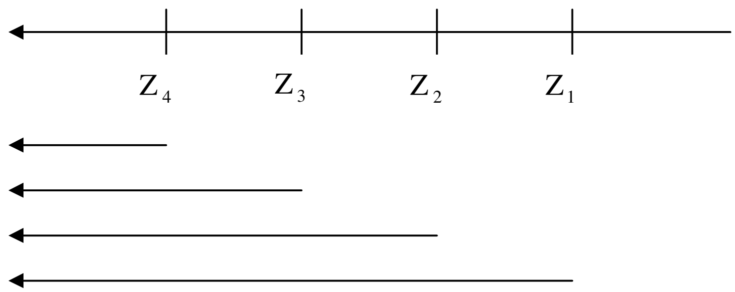


Figure 4-6. The cut appears at $z < z_j$ for each z_j

When crossing the cut even times in each line section，it will not change sign. When crossing the cut odd times in each line section will change sign，this implies the line section will form a branch cut. Hence we have the branch cuts in $[z_4, z_3]$ and $[z_2, z_1]$. The cut structure is showed in Figure 4-7.

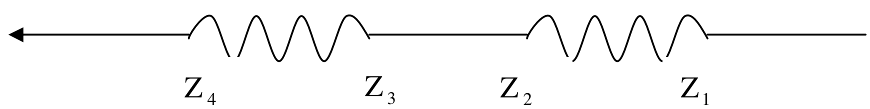


Figure 4-7. The cut structure for four branch points in horizontal

For $n=5$，as Figure 4-8 shows. (in a easier way to illustrate)

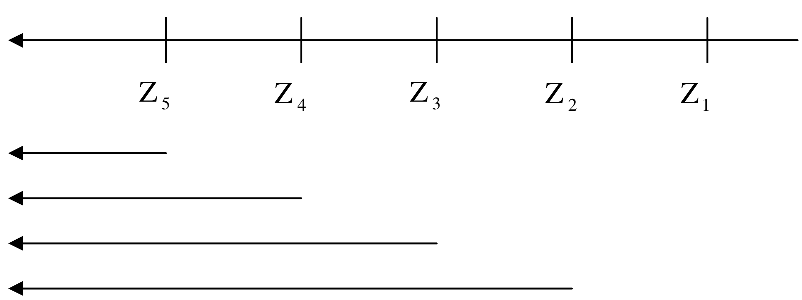


Figure 4-8. The cut appears at $z < z_j$ for each z_j

We have the branch cuts in $(-\infty, z_5]$, $[z_4, z_3]$ and $[z_2, z_1]$. The cut structure is showed in Figure 4-9.

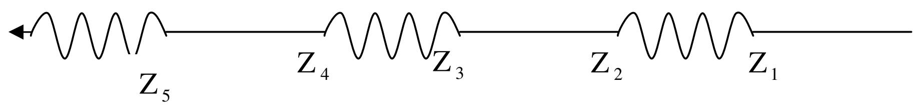


Figure 4-9. The cut structure for five branch points in horizontal

Ⅳ-2.2 The algebraic and geometric structure for Riemann surface of $f(z)$

For simplicity , we use $n=4$ to discuss the structure for Riemann surface of $f(z)=\sqrt{\prod_{j=1}^{4}(z-z_j)}$ in horizontal cut.

(i) Algebraic structure

As Figure 4-10 shows , $[z_4, z_3]$ 、 $[z_2, z_1]$ represent the cuts in this Riemann surface and 〝+〞, 〝–〞 are defined as following (depend on countclockwise — initial edge denote by + , terminus edge denote by –) :

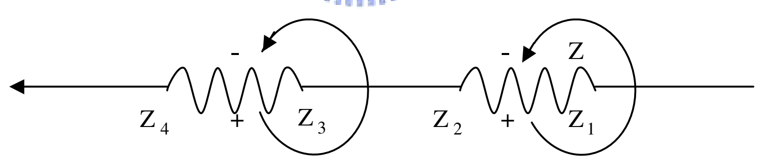


Figure 4-10. The algebraic structure for four branch points in horizontal

As we know , a curve crosses the cut from the sheet to another sheet , so the argument will increase 2π . We can defined the argument of + edge is $-\pi^{+}$ and the argument of – edge is π^{-}; or the argument of + edge is π^{+} and the argument of – edge is $3\pi^{-}$.

Case one : If $z \in I^{+}$ (+ edge of sheet Ⅰ)

As the Figure 4-10 shows , $z \in [z_2, z_1]$

Since $z - z_j > 0 \Rightarrow \arg(z - z_j) = 0$ for $j = 2,3,4$,

$z - z_j < 0 \Rightarrow \arg(z - z_j) = -\pi$ for $j = 1$.

Then $f(z)=\sqrt{\prod_{j=1}^{4}(z-z_j)}$

$$=\prod_{j=1}^{4}\sqrt{z-z_j}$$

$$=\left|z-z_1\right|^{\frac{1}{2}}e^{i(-\frac{\pi}{2})}\cdot\prod_{j=2}^{4}\left|z-z_j\right|^{\frac{1}{2}}e^{i\cdot 0}$$

$$=e^{i(-\frac{\pi}{2})}\cdot\prod_{j=1}^{4}\left|z-z_j\right|^{\frac{1}{2}}$$

$$=(-i)\cdot\prod_{j=1}^{4}\left|z-z_j\right|^{\frac{1}{2}} .$$

Case two : If $z\in I^{-}$ (– edge of sheet Ⅰ)

As the Figure 4-10 shows， $z\in[z_2,z_1]$

Since $z-z_j>0 \Rightarrow \arg(z-z_j)=0$ for $j=2,3,4$,

$z-z_j<0 \Rightarrow \arg(z-z_j)=\pi$ for $j=1$.

Then $f(z)=\sqrt{\prod_{j=1}^{4}(z-z_j)}$

$$=\prod_{j=1}^{4}\sqrt{z-z_j}$$

$$=\left|z-z_1\right|^{\frac{1}{2}}e^{i(\frac{\pi}{2})}\cdot\prod_{j=2}^{4}\left|z-z_j\right|^{\frac{1}{2}}e^{i\cdot 0}$$

$$=e^{i(\frac{\pi}{2})}\cdot\prod_{j=1}^{4}\left|z-z_j\right|^{\frac{1}{2}}$$

$$=(i)\cdot\prod_{j=1}^{4}\left|z-z_j\right|^{\frac{1}{2}} .$$

Note that $f(z)\ \Big|_{I^-} = -f(z)\ \Big|_{I^+}$.

(ii) Geometric structure

After knowing the algebraic structure，we will discuss about how to construct a geometric structure for Riemann surface of $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$. According to algebraic structure for Riemann surface，we know that

if n is even，then the branch cuts are $[z_n, z_{n-1}]$、$[z_{n-2}, z_{n-3}]$…… and $[z_2, z_1]$，implies we have $\frac{n}{2}-1$ holes，and

if n is odd，then the branch cuts are $(-\infty, z_n]$、$[z_{n-1}, z_{n-2}]$…… and $[z_2, z_1]$，implies we have $\frac{n-1}{2}$ holes.

And we obtain one sheet with two edges in each cut by taken of counterclockwise which labeled the edge of lower- cut with + and the edge of upper- cut with –. Since there are two surface，one is，say sheet Ⅰ with $\arg f(z)\in[-\pi,\pi)$; another is，say sheet Ⅱ with $\arg f(z)\in[\pi,3\pi)$.

By definition，the – edge of sheet Ⅰ is joined to the + edge of sheet Ⅱ，and the + edge of sheet Ⅰ is joined to the – edge of sheet Ⅱ. Whenever crossing the cut，we pass from one sheet to the other sheet and the value is continuous which from our construction.

Note that $f(z)\mid_{II} = -f(z)\mid_{I}$ and for $f(z)$，supra – half – ball represents sheet Ⅰ，and infra – half – ball represents sheet Ⅱ.

We take $n=4$ to discuss the geometric structure for Riemann surface of $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$ in horizontal cuts. Show as Figure 4-11.

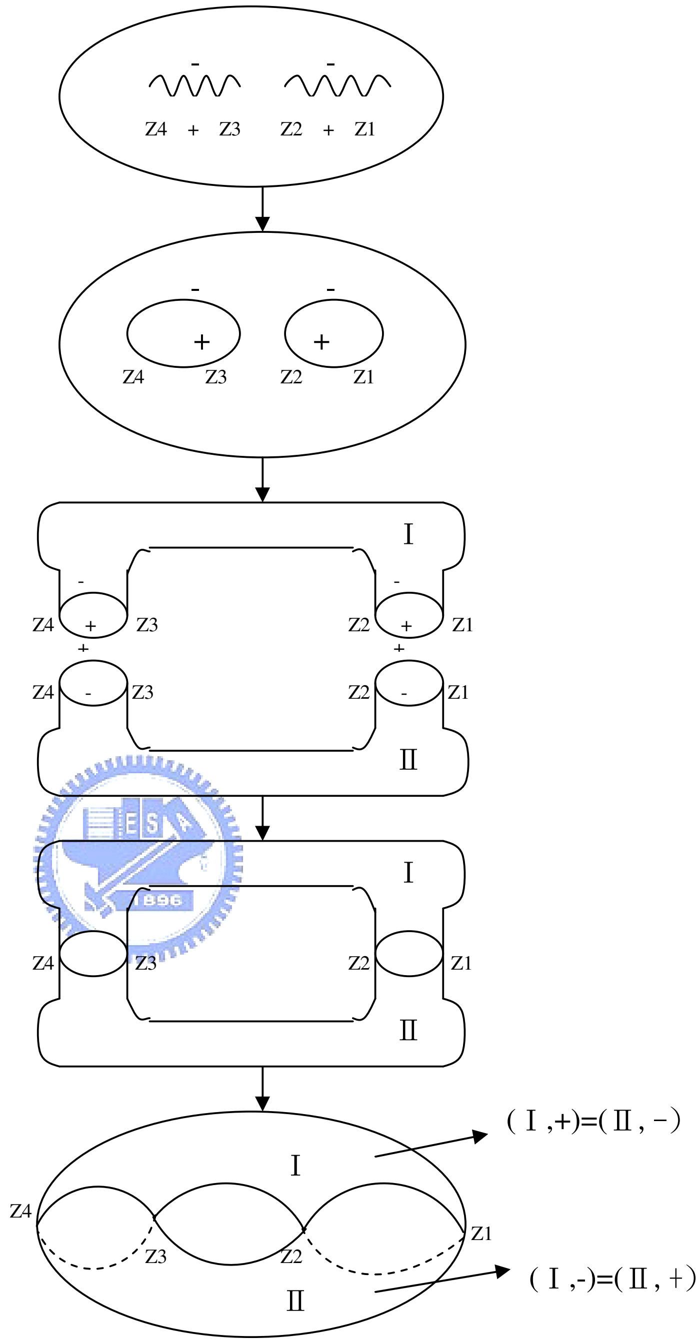


Figure 4-11. The geometric structure for Riemann surface with $n = 4$ in horizontal cut

(iii) Algebraic structure *v.s* Geometric structure

We also use $n = 4$ to discuss. Before talking about the relation between algebraic structure and geometric structure， we need to denote something as the following :

(a) If the curve is drawn by solid line :

In algebraic structure，it means the curve is in sheet Ⅰ;

In geometric structure，it means the curve is in the overhead Riemann surface.

(b) If the curve is drawn by dash line :

In algebraic structure，it means the curve is in sheet Ⅱ;

In geometric structure，it means the curve is in the ventral Riemann surface.

We give some example to show that the curve in algebraic structure and its corresponding in geometric structure in Figure 4-12 to Figure 4-13.

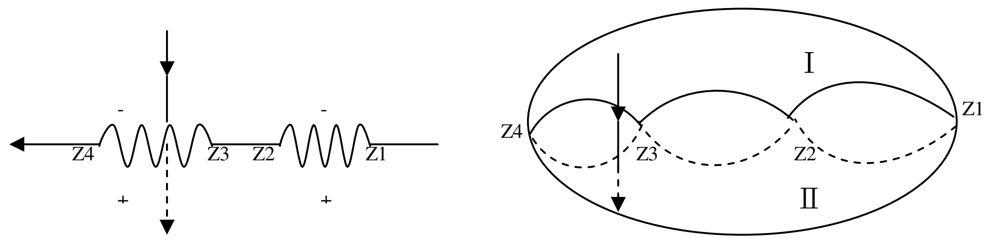


Figure 4-12. The rule in algebraic structure and geometric structure

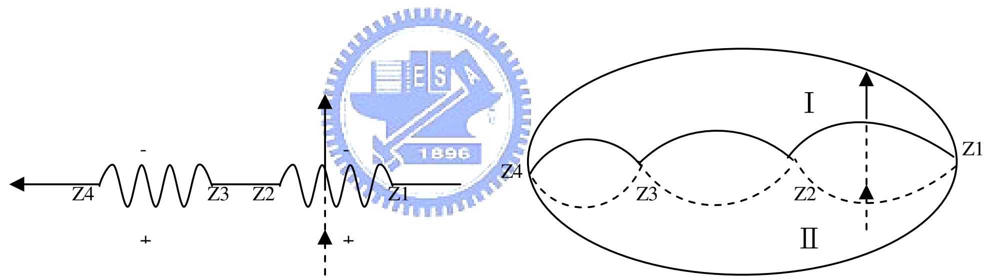


Figure 4-13. The rule in algebraic structure and geometric structure

Ⅳ-3 **Riemann surface of the algebraic curve** $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$ **with** $z_j \in C$

In this section，we discuss the vertical cut structure. We will present two styles of vertical cuts.

In vertical cut structure，we define that $(z, f(z))$ belong to sheet Ⅰ if and only if $\arg\prod_{j=1}^{n}(z-z_j)\in[-\frac{3\pi}{2},\frac{\pi}{2})$，*i.e.* $\arg(z-z_j)\in[-\frac{3\pi}{2},\frac{\pi}{2})$ for each j ; $(z, f(z))$ belong to sheet Ⅱ if and only if $\arg\prod_{j=1}^{n}(z-z_j)\in[\frac{\pi}{2},\frac{5\pi}{2})$，*i.e.* $\arg(z-z_j)\in[\frac{\pi}{2},\frac{5\pi}{2})$ for each j.

Ⅳ-3.1 **The vertical cut structure of** $f(z)$

We consider $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$ with $z_j \in C$ for $j=1,2,3,\cdots,n$ and we by face the up direction to do cut explained. If n is even and $z_{\frac{n}{2}-k+1}=\overline{z_{\frac{n}{2}+k}}$, $k=1,2,\cdots,\frac{n}{2}$, $z_1, z_2, \cdots, z_n$ represent the n branch points and $\overline{z_1z_2}, \overline{z_3z_4}, \cdots, \overline{z_{n-1}z_n}$ represent the cuts showed in Figure 4-14.

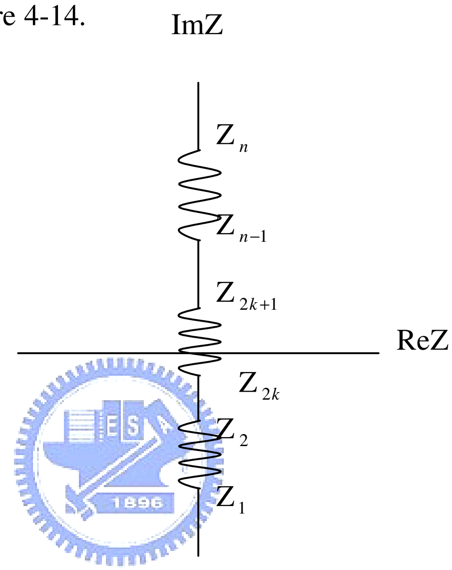


Figure 4-14. The vertical cut structure

About vertical cut structure analysis methid is the same as horizontal cut structure.

First，we check if there is any cut，for $n=2$ and $z_1=i$ 、 $z_2=2i$ ， as Figure 4-15 shows.

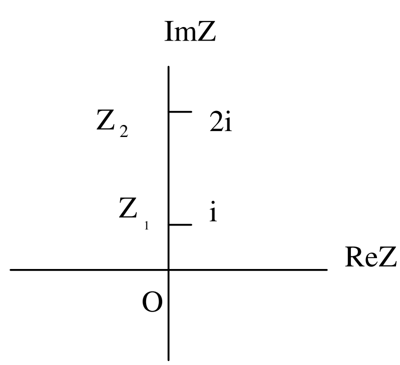


Figure 4-15. The branch points are $z_1=i$ and $z_2=2i$

Put point $3i$ and $3i \in (\infty, 2i)$, then we have

$$\arg(3i - i) = \arg(2i) = \begin{cases} -\frac{3\pi}{2} \\ \frac{\pi}{2} \end{cases} \quad \text{and} \quad \arg(3i - 2i) = \arg(i) = \begin{cases} -\frac{3\pi}{2} \\ \frac{\pi}{2} \end{cases},$$

taking $-\frac{3\pi}{2}$: $\sqrt{2i} \cdot \sqrt{i} = length \times e^{i(\frac{-3\pi}{2})} = (i) \times length$, (4-11)

taking $\frac{\pi}{2}$: $\sqrt{2i} \cdot \sqrt{i} = length \times e^{i(\frac{\pi}{2})} = (i) \times length$. (4-12)

Since (4-11) = (4-12) .

So, there is no cut in $(\infty, 2i)$.

Put point $\frac{3i}{2}$ and $\frac{3i}{2} \in (i, 2i)$, then we have

$$\arg(\frac{3i}{2} - i) = \arg(\frac{i}{2}) = \begin{cases} -\frac{3\pi}{2} \\ \frac{\pi}{2} \end{cases} \quad \text{and} \quad \arg(\frac{3i}{2} - 2i) = \arg(-\frac{i}{2}) = -\frac{\pi}{2},$$

taking $-\frac{3\pi}{2}$: $\sqrt{\frac{i}{2}} \cdot \sqrt{-\frac{i}{2}} = length \times e^{i(-\pi)} = -length$, (4-13)

taking $\frac{\pi}{2}$: $\sqrt{\frac{i}{2}} \cdot \sqrt{-\frac{i}{2}} = length \times e^{i(0)} = length$. (4-14)

Since (4-13) $\neq$ (4-14) .

So, there is a cut in $(i, 2i)$.

Hence we have the branch cuts in $[i, 2i]$. As Figure 4-16 shows.

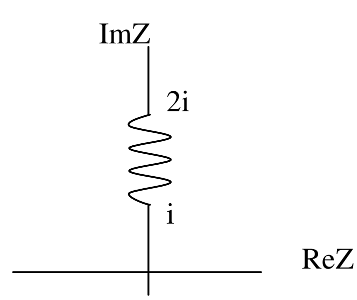


Figure 4-16. The cut structure for $n = 2$ branch points in vertical

But we can use easier way to get branch cut，in this we take $n=4$ and $z_1 = i$ 、 $z_2 = 2i$ 、 $z_3 = 3i$ and $z_4 = 4i$，that is $z_1 < z_2 < z_3 < \ldots < z_n$，as Figure 4-17 shows.

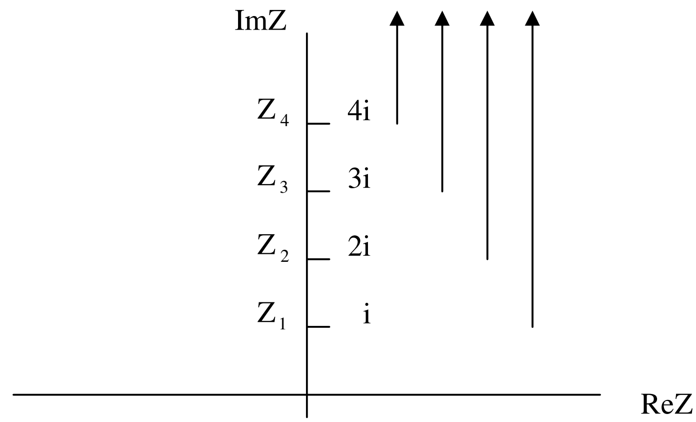


Figure 4-17. The cut appears at $z < z_j$ for each z_j

When crossing the cut even times in each line section，it will not change sign. When crossing the cut odd times in each line section will change sign，this implies the line section will form a branch cut. Hence we have the branch cuts in $[z_4, z_3]$ and $[z_2, z_1]$. The cut structure is showed in Figure 4-18.

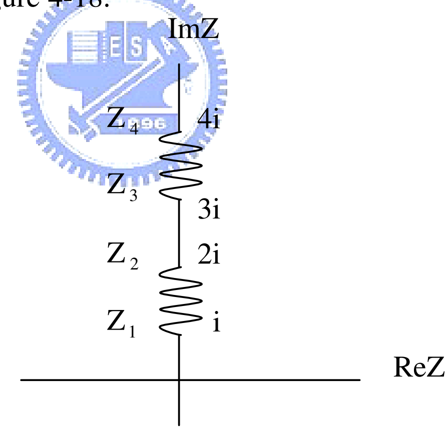


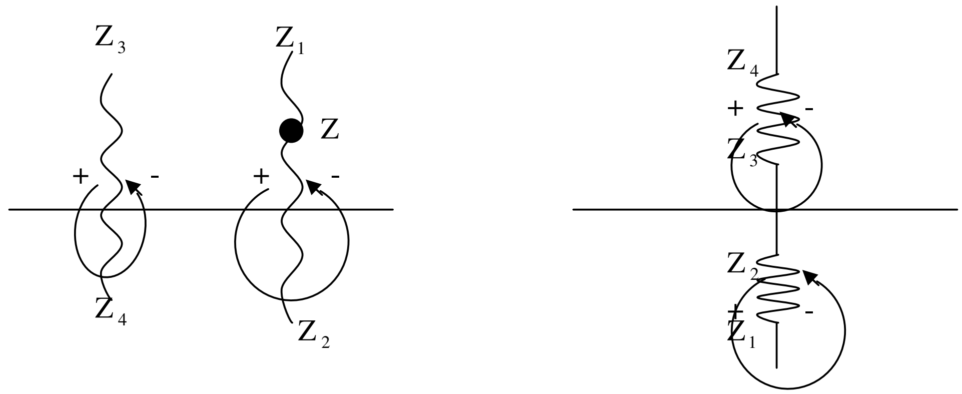
Figure 4-18. The cut structure for four branch points in vertical

Ⅳ-3.2 The algebraic and geometric structure for Riemann surface of $f(z)$

For simplicity，we use $n=4$ to discuss the structure for Riemann surface of $f(z) = \sqrt{\prod_{j=1}^{4}(z - z_j)}$ in vertical cut. In the cut structure，we still depond on the countclockwise to take〝+〞、〝−〞 sign. That is the right hand side of each cut represents the + edge and the left hand side represents the – edge. The definition of solid – line and dash – line are the same as horizontal cut case.

(i) Algebraic structure

As Figure 4-19 shows, $[z_4, z_3]$ and $[z_2, z_1]$ represent the cuts in Riemann surface.



(a) (b)

Figure 4-19. The algebraic structure for four branch points in vertical

Case one : If $z \in I^{+}$ (+ edge of sheet I)

As the Figure 4-19 (a) shows, $z \in [z_2, z_1]$

Since $\arg(z - z_1) = -\dfrac{\pi}{2}$ and $\arg(z - z_2) = -\dfrac{3\pi}{2}$,

$\arg(z - z_j) \in (-\pi, \dfrac{\pi}{2})$ for $j = 3,4$.

Then $f(z) = \sqrt{\prod_{j=1}^{4}(z - z_j)} = \prod_{j=1}^{4}\sqrt{z - z_j}$

$$= \left|z - z_2\right|^{\frac{1}{2}} e^{i(-\frac{3\pi}{4})} \cdot \prod_{j=1,3,4}\left|z - z_j\right|^{\frac{1}{2}} e^{i\frac{\arg(z - z_j)}{2}}$$

$$= (-\frac{\sqrt{2}}{2} i)\left|z - z_2\right|^{\frac{1}{2}} \cdot \prod_{j=1,3,4}\left|z - z_j\right|^{\frac{1}{2}} e^{i\frac{\arg(z - z_j)}{2}} .$$

Case two : If $z \in I^{-}$ (− edge of sheet I)

As the Figure 4-19 (a) shows, $z \in [z_2, z_1]$

Since $\arg(z - z_1) = -\dfrac{\pi}{2}$ and $\arg(z - z_2) = \dfrac{\pi}{2}$,

$\arg(z - z_j) \in (-\pi, \dfrac{\pi}{2})$ for $j = 3,4$.

Then $f(z)=\sqrt{\prod_{j=1}^{4}(z-z_j)}=\prod_{j=1}^{4}\sqrt{z-z_j}$

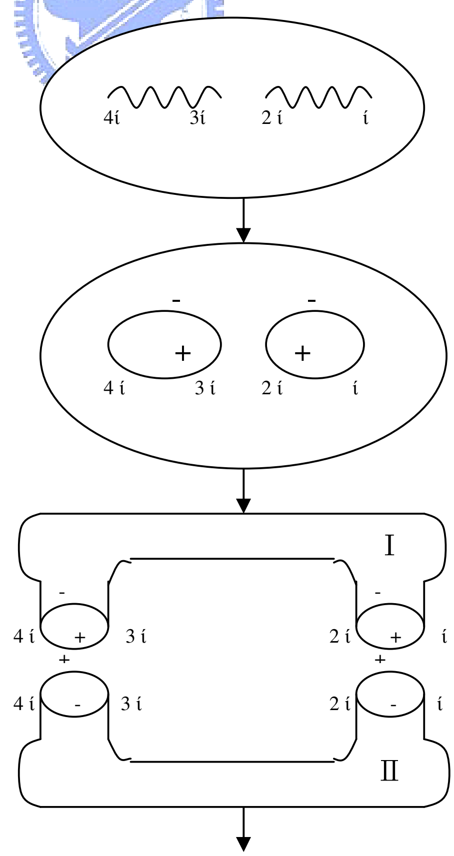
$$=\left|z-z_2\right|^{\frac{1}{2}}e^{i(\frac{\pi}{4})}\cdot\prod_{j=1,3,4}\left|z-z_j\right|^{\frac{1}{2}}e^{i\frac{\arg(z-z_j)}{2}}$$

$$=(\frac{\sqrt{2}}{2}i)\left|z-z_2\right|^{\frac{1}{2}}\cdot\prod_{j=1,3,4}\left|z-z_j\right|^{\frac{1}{2}}e^{i\frac{\arg(z-z_j)}{2}}\ .$$

Note that $f(z)\mid_{I^-}=-f(z)\mid_{I^+}$.

(ii) Geometric structure

The construct a geometric structure for Riemann surface of $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$ is the same as horizontal cuts.By above example and illustration，we discusses the geometric structure foe Riemann surface in vertical cuts. Show as Figure 4-20.



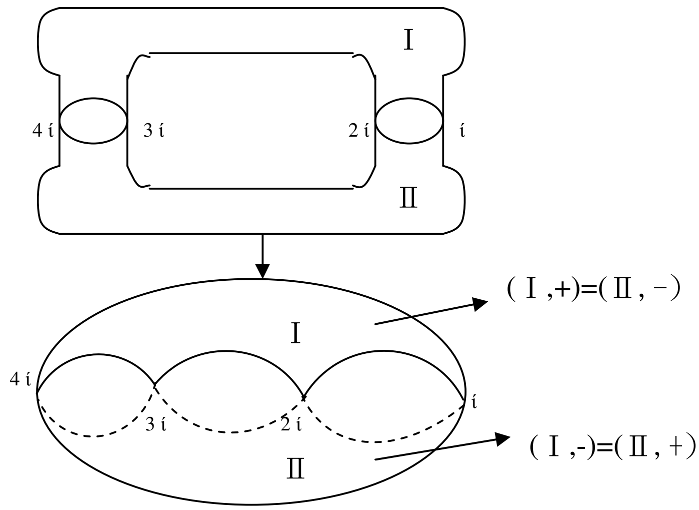


Figure 4-20. The geometric structure for Riemann surface with $n=4$ in vertical cuts

IV-4 The integrals over a,b cycles for the horizontal cuts and vertical cuts

We want to evaluate $\oint_a \frac{1}{f(z)}dz$ and $\oint_b \frac{1}{f(z)}dz$ for n branch points where a, b represent the a, b cycles over the Riemann surface of $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$ with $z_j \in C$, and develop an algorithm such that the integrals can be easily computed.

IV-4.1 The a, b cycles over the Riemann surface of $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$

(i) In horizontal cut :

Let $z_1, z_2, \cdots, z_n$ are the n branch points in $x-axis$ with $z_j \in C$, then $f(z)=\sqrt{\prod_{j=1}^{n}(z-z_j)}$ forms a $N-holes$ Riemann surface where $N \in Z^+ \cup \{0\}$ and

$$\begin{cases} N=\dfrac{n-1}{2} & for \quad n \quad odd \\ N=\dfrac{n-2}{2} & foe \quad n \quad even \end{cases} .$$

So there are N a, b cycles. The Figure 4-21 represents the a, b cycles in the Riemann surface for n is even and the Figure 4-22 is the case for n is odd.

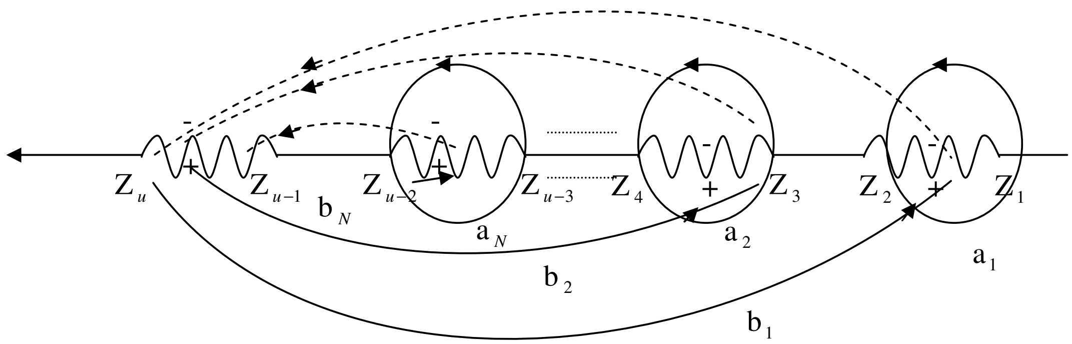


Figure 4-21. a, b cycles for horizontal cuts of even branch points

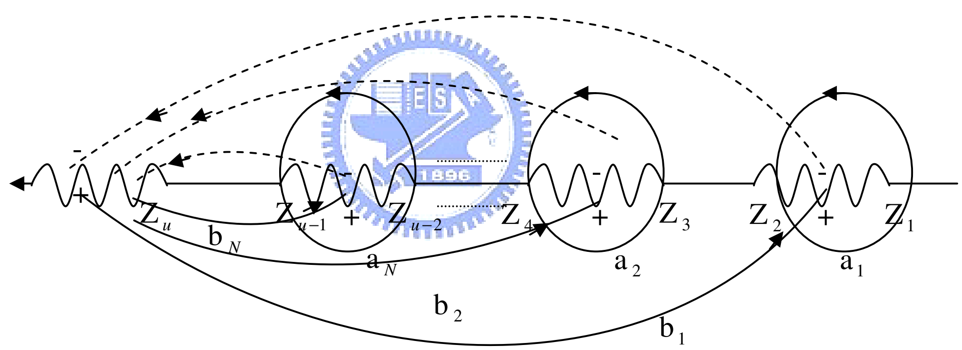


Figure 4-22. a, b cycles for horizontal cuts of odd branch points

(ii) In vertical cut :

Let $z_1, z_2, \cdots, z_n \in C$ are the n branch points where n is even and $z_{2k} = \bar{z}_{2k-1}$, $k = 1, 2, \cdots, \frac{n}{2}$. There are $\frac{n-2}{2}$ a, b cycles in the Riemann surface showed in Figure 4-23. For a_k cycle, it encloses the cut $\overline{z_{2k-1} z_{2k}}$, b_k cycle is passed through the cut $\overline{z_{2k-1} z_{2k}}$ from one sheet to the other.

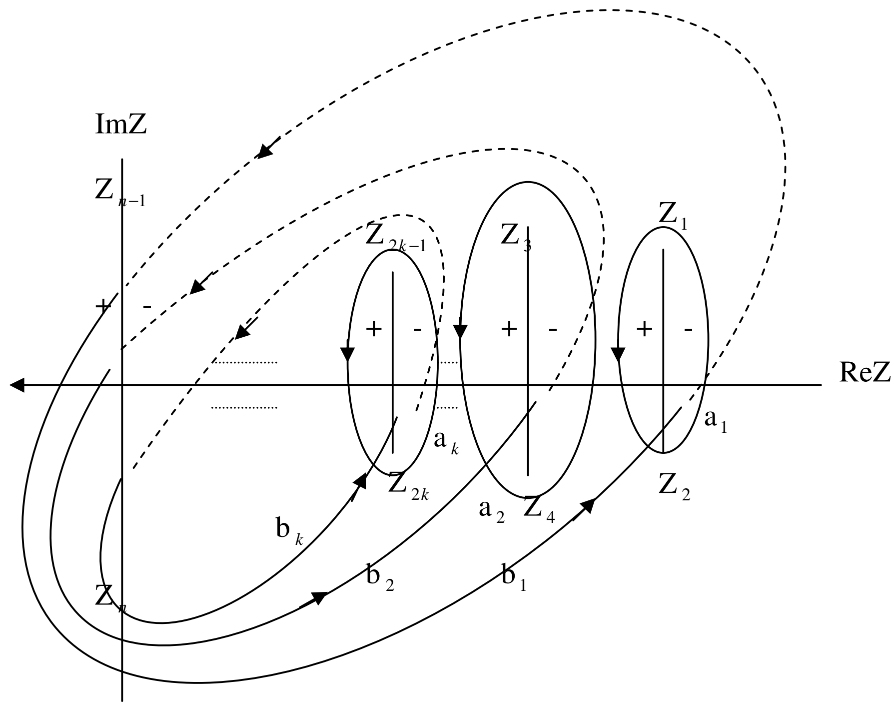


Figure 4-23. a, b cycles for vertical cuts

Let $z_1, z_2, \cdots, z_n \in C$ are the n branch points where n is even and $z_{2k} = \bar{z}_{2k-1}$, $k = 1,2,\cdots,\frac{n}{2}$. There are $\frac{n-2}{2}$ a, b cycles in the Riemann surface showed in Figure 4-24.

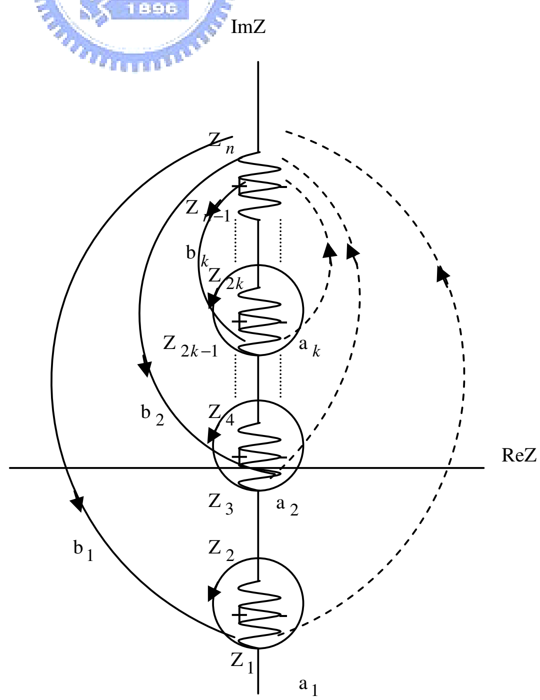


Figure 4-24. a, b cycles for vertical cuts

Ⅳ-4.2 About 〝 Mathematica 〞 and How to modify

All programs in this paper are run by Mathematica . But we can not compute directly，before computing we need to give some adjustments. Since Mathematica reads argument of any complex number in $(-\pi,\pi]$ only，then it just gives right answer in sheet Ⅰ in horizontal cuts (expect at the argument $-\pi$).

If $\arg(z-z_j)\notin(-\pi,\pi]$ ， Mathematica will change the argument into $(-\pi,\pi]$ automatically，this will make some error in our calculation. In order to get the correct values for the argument not belong to $(-\pi,\pi]$ ，we should modify the function before computing. In horizontal cut structure，Mathematica gives correct values in sheet Ⅰ，we base on $f(z)\mid_{II} = -f(z)\mid_{I}$ to the values in sheet Ⅱ.

In vertical cuts ， Mathematica does not give correct value in sheet Ⅰ. If $\arg(z-z_j)\in[-\frac{3\pi}{2},\frac{\pi}{2}]$ for some j，then Mathematica will regards as $\arg(z-z_j)\in[\frac{\pi}{2},\pi]$. This implies we need to modify before computing，so we will have the correct results. The same as in horizontal cut，the values in sheet Ⅱ is from $f(z)\mid_{II} = -f(z)\mid_{I}$.

By above illustration，we get vertical cut structure . Now，we want to know how to compute the path integral in vertical cut ? Note that the vertical cut $angle\in(-\frac{3\pi}{2},\frac{\pi}{2})$ and the $angle$ in Mathematica is $(-\pi,\pi]$. So，we know when the $angle\in(-\frac{3\pi}{2},-\pi)\in II$ ，it need to modify by Mathematica. Therefore，we can get the method to compute the path integral in vertical cut . First，we use circle 、 rectangle or closed path to cover the a , b cycles .Then taking every branch points are the coordinate plane zero point，drawing a coordinate plane，then may divide into the plane to four parts. Since we have several branch points，so we will to partition of several parts in the circle 、 rectangle or closed path. In below，by vector analysis ; if the $angle\in II$ (Second quadrant)，that implies the path need to modify .

Note that，if the path is not to modify and in sheet Ⅰ，then by Mathematica to compute，we use $+M$ sign to express it . If the path is need to modify and in sheet Ⅰ，then by Mathematica to compute，we use $-M$ sign to express it .

Ⅳ-4.3 An application for the integrals over a , b cycles

In this section，we give two examples is with horizontal cut and vertical cut.

Example 4-1:

Let $n=6$，and $z_1=4$、$z_2=3$、$z_3=2$、$z_4=1$、$z_5=-1$ and $z_6=-2$ are six branch points form a horizontal cut as Figure 4-25 shows ; and form a 2- hole Riemann surface.

If $f(z)=\prod_{j=1}^{6}(z-z_j)^{\frac{1}{2}}$，then $\oint_r \frac{1}{f(z)}dz$ where $r=a,b$ cycles ?

We use 〝Mathematica〞 to compute the integral.

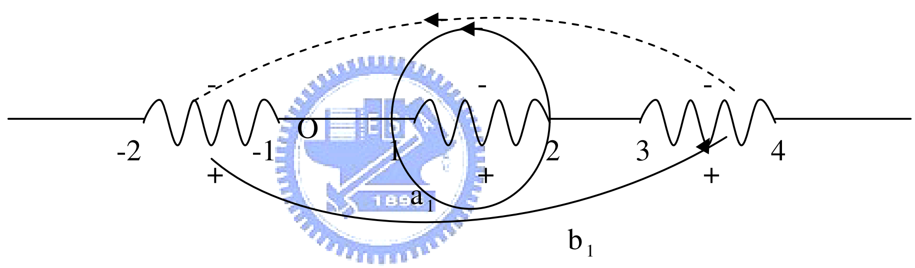


Figure 4-25. a_1, b_1 cycles for six branch points in horizontal cut

(i) For the equivalent path a_1^* : since $\arg(z-z_j)=-\pi$ is not the valid range in Mathematica，$f(z)$ need to multiple a scalar $e^{-i\pi}=-1$. As Figure 4-26 shows.

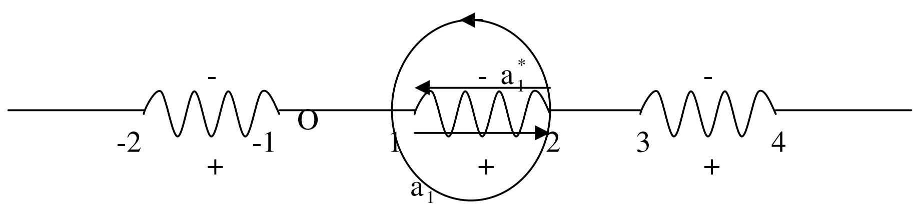


Figure 4-26. a_1 cycle and equivalent path a_1^*

Branch points			$\forall z \in +$ edge of sheet Ⅰ of a_1^* Interval $(1,2)$ angle	value	$\forall z \in -$ edge of sheet Ⅰ of a_1^* Interval $(2,1)$ angle	value
$z-z_1$	$\Leftrightarrow$	$z-4$	$-\pi$	$-M$	π	$+M$
$z-z_2$	$\Leftrightarrow$	$z-3$	$-\pi$	$-M$	π	$+M$
$z-z_3$	$\Leftrightarrow$	$z-2$	$-\pi$	$-M$	π	$+M$
$z-z_4$	$\Leftrightarrow$	$z-1$	0	$+M$	0	$+M$
$z-z_5$	$\Leftrightarrow$	$z+1$	0	$+M$	0	$+M$
$z-z_6$	$\Leftrightarrow$	$z+2$	0	$+M$	0	$+M$
Sheet Ⅰ or sheet Ⅱ			Sheet Ⅰ	$+M$	Sheet Ⅰ	$+M$
Total				$-M$		$+M$

By 〝Mathematica〞，

$$-\int_1^2 \frac{1}{\sqrt{z+1}\sqrt{z+2}\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}}dz+\int_2^1 \frac{1}{\sqrt{z+1}\sqrt{z+2}\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}}dz$$

$$=-2\int_1^2 \frac{1}{\sqrt{z+1}\sqrt{z+2}\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}}dz = 3.3819\times 10^{-49}-1.13022i \quad .$$

Therefore，the integral over a_1 cycle is

$$\oint_{a_1}\frac{1}{f(z)}dz=\oint_{a_1^*}\frac{1}{f(z)}dz=3.3819\times 10^{-49}-1.13022i \quad .$$

(ii) For the equivalent path b_1^* : since the interval $(-1,1)$ and $(2,3)$ are not have cut，so solid line is in sheet Ⅰ and implies + sign ; dash line is in sheet Ⅱ and implies – sign ; now，we illustration the interval $(1,2)$ and it is a cut. As Figure 4-27 shows.

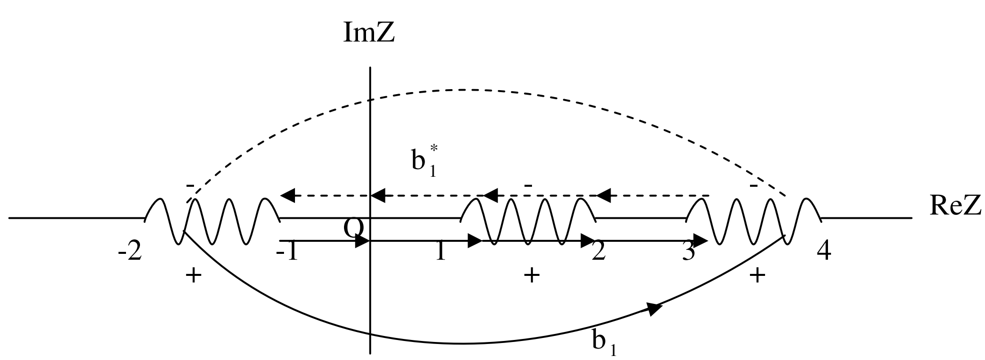


Figure 4-27. b_1 cycle and equivalent path b_1^*

$\forall z \in +$ edge of sheet Ⅰ of b_2^* $\forall z \in -$ edge of sheet Ⅰ of b_2^*

Branch points			Interval (1,2) angle	Interval (1,2) value	Interval (2,1) angle	Interval (2,1) value
$z-z_1$	$\Leftrightarrow$	$z-4$	$-\pi$	$-M$	π	$+M$
$z-z_2$	$\Leftrightarrow$	$z-3$	$-\pi$	$-M$	π	$+M$
$z-z_3$	$\Leftrightarrow$	$z-2$	$-\pi$	$-M$	π	$+M$
$z-z_4$	$\Leftrightarrow$	$z-1$	0	$+M$	0	$+M$
$z-z_5$	$\Leftrightarrow$	$z+1$	0	$+M$	0	$+M$
$z-z_6$	$\Leftrightarrow$	$z+2$	0	$+M$	0	$+M$
Sheet Ⅰ or sheet Ⅱ			Sheet Ⅰ	$+M$	Sheet Ⅱ	$-M$
Total				$-M$		$-M$

By 〝Mathematica〞，

$$\int_{-1}^{0}\frac{1}{\sqrt{z+1}\sqrt{z+2}\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}}dz+\int_{2}^{3}\frac{1}{\sqrt{z+1}\sqrt{z+2}\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}}dz$$

$$-\int_{3}^{2}\frac{1}{\sqrt{z+1}\sqrt{z+2}\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}}dz-\int_{1}^{0}\frac{1}{\sqrt{z+1}\sqrt{z+2}\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}}dz$$

$$+\int_{0}^{1}\frac{1}{\sqrt{z+1}\sqrt{z+2}\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}}dz-\int_{0}^{-1}\frac{1}{\sqrt{z+1}\sqrt{z+2}\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}}dz$$

$$-\int_{1}^{2}\frac{1}{\sqrt{z+1}\sqrt{z+2}\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}}dz-\int_{2}^{1}\frac{1}{\sqrt{z+1}\sqrt{z+2}\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}\sqrt{z-4}}dz$$

$$=-0.0760776+3.77621\times10^{-49}i\ .$$

Therefore，the integral over b_2 cycle is

$$\oint_{b_1}\frac{1}{f(z)}dz=\oint_{b_1^*}\frac{1}{f(z)}=-0.0760776+3.77621\times10^{-49}i\ .$$

Example 4-2 :

Let $n=6$，and $z_1=1+2i$、$z_2=1$、$z_3=3i$、$z_4=i$、$z_5=-1+3i$ and $z_6=-1+i$ are six branch points form a vertical cut as Figure 4-28 shows ; and form a 2- hole Riemann surface.

If $f(z)=\prod_{j=1}^{6}(z-z_j)^{\frac{1}{2}}$，then $\oint_r\frac{1}{f(z)}dz$ where $r=a,b$ cycles ?

Note that，in vertical cut，we use 〝Mathematica〞to compute the integral，we must modify the equation first. That is the $angle\in[-\frac{3\pi}{2},-\pi)\in III$，the $f(z)$ need to multiple a scalar $e^{-i\pi}=-1$.

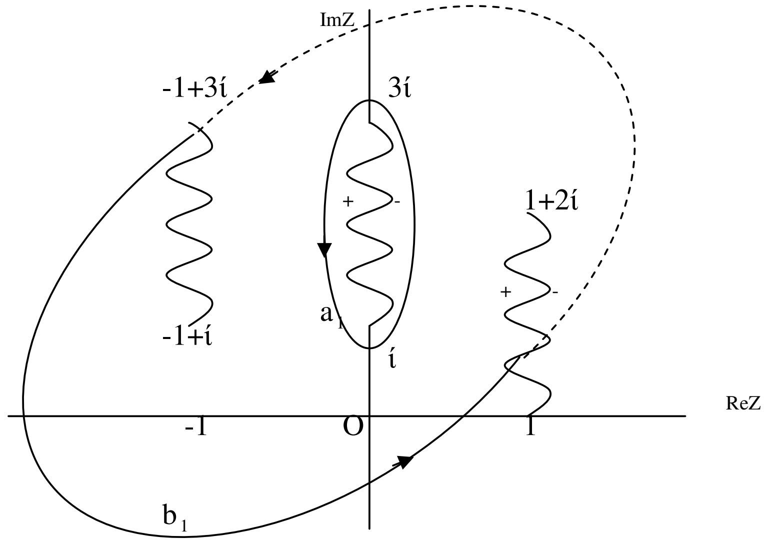


Figure 4-28. a_1, b_1 cycles for six branch points in vertical cut

(i) For the equivalent path a_1^* : as Figure 4-29 shows .

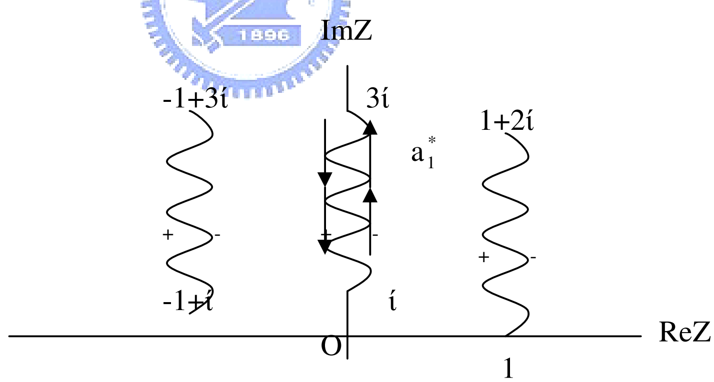


Figure 4-29. Equivalent path a_1^*

Since $\arg(z-z_j)\in(-\frac{3\pi}{2},-\pi)\in II$ for $j=1,2,3,4,5,6$，$f(z)$ need to multiple a scalar $e^{-i\pi}=-1$.

Branch points	$\forall z\in +$ edge of sheet Ⅰ of a_1^*				$\forall z\in -$ edge of sheet Ⅰ of a_1^*			
	Interval $(3i,2i)$		Interval $(2i,i)$		Interval $(2i,3i)$		Interval $(i,2i)$	
	angle	value	angle	value	angle	value	angle	value

$z-z_1$	$\Leftrightarrow$	$z-(1+2i)$	II	$-M$	III	$+M$	II	$-M$	III	$+M$
$z-z_2$	$\Leftrightarrow$	$z-1$	II	$-M$	II	$-M$	II	$-M$	II	$-M$
$z-z_3$	$\Leftrightarrow$	$z-3i$	$-\frac{\pi}{2}$	$+M$	$-\frac{\pi}{2}$	$+M$	$-\frac{\pi}{2}$	$+M$	$-\frac{\pi}{2}$	$+M$
$z-z_4$	$\Leftrightarrow$	$z-i$	$-\frac{3\pi}{2}$	$-M$	$-\frac{3\pi}{2}$	$-M$	$\frac{\pi}{2}$	$+M$	$\frac{\pi}{2}$	$+M$
$z-z_5$	$\Leftrightarrow$	$z-(-1+3i)$	IV	$+M$	IV	$+M$	IV	$+M$	IV	$+M$
$z-z_6$	$\Leftrightarrow$	$z-(-1+i)$	I	$+M$	I	$+M$	I	$+M$	I	$+M$
Sheet Ⅰ or sheet Ⅱ			sheetI	$+M$	sheetI	$+M$	sheetI	$+M$	sheetI	$+M$
Total				$-M$		$+M$		$+M$		$-M$

By 〝Mathematica〞,

$$\oint_{a_1^*}\frac{1}{f(z)}dz=-\int_{3i}^{2i}\frac{1}{f(z)}dz+\int_{2i}^{i}\frac{1}{f(z)}dz-\int_{i}^{2i}\frac{1}{f(z)}dz+\int_{2i}^{3i}\frac{1}{f(z)}dz$$

$$=-2\int_{i}^{2i}\frac{1}{f(z)}dz+2\int_{2i}^{3i}\frac{1}{f(z)}dz \quad =1.38321-2.33762i \quad .$$

Therefore, the integral over a_1 cycle is

$$\oint_{a_1}\frac{1}{f(z)}dz=\oint_{a_1^*}\frac{1}{f(z)}dz=1.38321-2.33762i \quad .$$

(ii) For the equivalent path b_1^* : as Figure 4-30 shows .

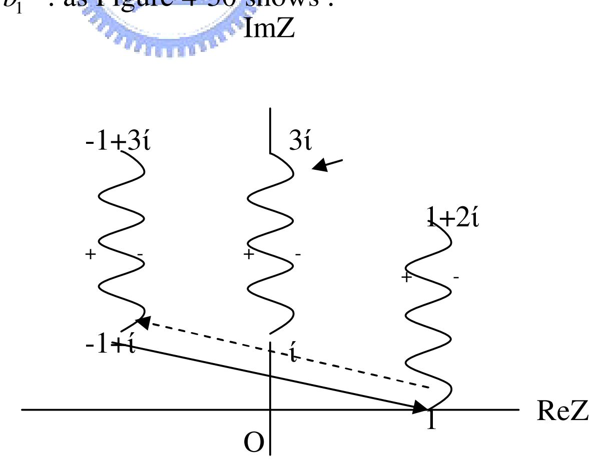


Figure 4-30. Equivalent path b_1^*

Since $\arg(z-z_j)\in(-\frac{3\pi}{2},-\pi)\in II$ for $j=1,2,3,4,5,6$, $f(z)$ need to multiple a scalar $e^{-i\pi}=-1$.

Branch points	$\forall z\in +$ edge of sheet Ⅰ of b_2^* Interval $(-1+i,1)$ angle	value	$\forall z\in -$ edge of sheet Ⅰ of b_2^* Interval $(1,-1+i)$ angle	value
$z-z_1$	III	$+M$	III	$+M$
$z-z_2$	II	$-M$	II	$-M$

$z-z_3$	III	$+M$	III	$+M$
$z-z_4$	III	$+M$	III	$+M$
$z-z_5$	IV	$+M$	IV	$+M$
$z-z_6$	IV	$+M$	IV	$+M$
Sheet Ⅰ or sheet Ⅱ	Sheet Ⅰ	$+M$	Sheet Ⅱ	$-M$
Total		$-M$		$+M$

By 〝Mathematica〞，

$$\oint_{b_1^*}\frac{1}{f(z)}dz=-\int_{-1+i}^{1}\frac{1}{f(z)}dz+\int_{1}^{-1+i}\frac{1}{f(z)}dz=2\int_{1}^{-1+i}\frac{1}{f(z)}dz=0.590344-1.16143i \quad .$$

Therefore，the integral over b_1 cycle is

$$\oint_{b_1}\frac{1}{f(z)}dz=\oint_{b_1^*}\frac{1}{f(z)}dz=0.590344-1.16143i \quad .$$

Ⅳ-5 <u>An application for Riemann integrals</u>

Consider $u_{xx}+u_{yy}=0$ in the half plane $y\geq 0$ subject to the boundary condition $u(x,0)=\sqrt{x^2+1}$ ，with $x\in R$ and the condition $u(x,y)\to 0$ as $\sqrt{x^2+y^2}\to\infty$.

First，we using <u>Fourier transform</u> with respect to x

$$F[u(x,y)]=\hat{u}(w,y)=\int_{-\infty}^{\infty}u(x,y)e^{iwx}dx \quad ,$$

and $$F[\frac{\partial^2 u}{\partial y^2}]=\hat{u}_{yy} \quad , \quad F[\frac{\partial^2 u}{\partial x^2}]=(-iw)^2\hat{u}=-w^2\hat{u} \quad .$$

Which implies $\hat{u}$ satisfies the ODE $\hat{u}_{yy}-w^2\hat{u}=0$,

with the solution of the ODE are $\hat{u}(w,y)=Ae^{wy}+Be^{-wy}$ ，$w\in R$ and $y\geq 0$.

The boundary conditions give

$$\hat{u}(w,0)=F[f]=\hat{f}=A+B \quad ,$$

and $\hat{u}(w,y)\to 0$ as $y\to\infty$.

If $\begin{cases} w>0 \;\Rightarrow\; A=0, B=\hat{f} \\ w<0 \;\Rightarrow\; B=0, A=\hat{f} \end{cases}$, which gives $\hat{u}(x,y)=\hat{f}\,e^{-|w|y}$, $w\in R$ and $y\geq 0$.

By the exponential form of Fourier transform，we have the formula is

$$F^{-1}[e^{-|w|y}] = \frac{1}{\pi} \cdot \frac{y}{x^2 + y^2} \quad .$$

So the convolution theorem yields

$$u(x, y) = \int_{-\infty}^{\infty} f(x-s) \frac{1}{\pi} \frac{y}{s^2 + y^2} ds$$

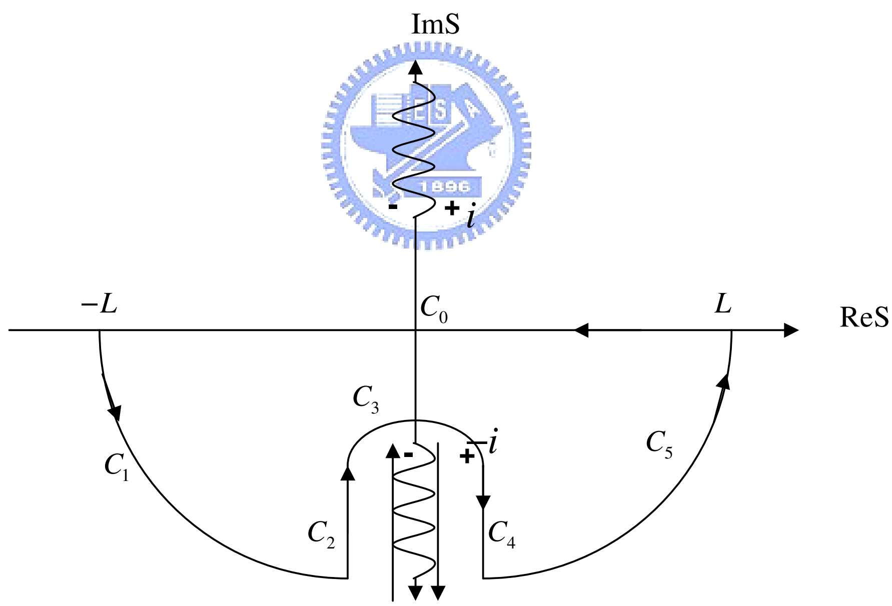
$$= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(x-s)}{s^2 + y^2} ds \quad = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(s)}{(s-x)^2 + y^2} ds \quad .$$

Given boundary conditions is

$$u(x,0) = f(x) = \sqrt{x^2 + 1} \quad \Rightarrow \quad u(s,0) = f(s) = \sqrt{s^2 + 1} \quad .$$

So $$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\sqrt{s^2 + 1}}{(s-x)^2 + y^2} ds \quad .$$

Since $\sqrt{s^2 + 1} = \sqrt{s+i} \cdot \sqrt{s-i}$ have two branch points $\pm i$.



We choose close contone C such that $\dfrac{\sqrt{s^2+1}}{(s-x)^2+y^2}$ is analytic.

That is $\displaystyle\int_C \frac{\sqrt{s^2+1}}{(s-x)^2+y^2} ds = 0 \quad \Rightarrow \quad \sum_{k=0}^{5} \int_k \frac{\sqrt{s^2+1}}{(s-x)^2+y^2} ds = 0$

Since $\displaystyle\int_{C_1} \frac{\sqrt{s^2+1}}{(s-x)^2+y^2} ds$ and $\displaystyle\int_{C_5} \frac{\sqrt{s^2+1}}{(s-x)^2+y^2} ds \to 0$ as $L \to \infty$

$$\int_{C_3} \frac{\sqrt{s^2+1}}{(s-x)^2+y^2}\,ds \to 0 \quad \text{as} \quad \varepsilon \to 0\,.$$

And $\sqrt{s^2+1} = \sqrt{s+i}\sqrt{s-i} = |s+i|^{\frac{1}{2}} e^{\frac{i}{2}[\arg(s+i)]} |s-i|^{\frac{1}{2}} e^{\frac{i}{2}[\arg(s-i)]} = |s^2+1|^{\frac{1}{2}} e^{\frac{i}{2}[\arg(s+i)+\arg(s-i)]}$

For C_2: let $s = -0 + ia$ and a from －L to $-(1+\varepsilon)$

then $ds = ida$ and

$$-0+ia+i = -0+i(a+1) \Rightarrow \arg(-0+ia+i) \in \frac{3\pi}{2}$$

$$-0+ia-i = -0+i(a-1) \Rightarrow \arg(-0+ia-i) \in \frac{-\pi}{2}$$

$$\int_{C_2} \frac{\sqrt{s^2+1}}{(s-x)^2+y^2}\,ds = \frac{y}{\pi}\int_{-\infty}^{-i} \frac{\sqrt{s^2+1}}{(s-x)^2+y^2}\,ds$$

$$= \frac{y}{\pi}\int_{-\infty}^{-1} \frac{1}{(ia-x)^2+y^2} |(ia)^2+1|^{\frac{1}{2}}\, i \cdot ida$$

$$= \frac{-y}{\pi}\int_{-\infty}^{-1} \frac{1}{(ia-x)^2+y^2} (a^2-1)^{\frac{1}{2}}\, da$$

For C_4: let $s = 0 + ia$ and a from $-(1+\varepsilon)$ to -L

then $ds = ida$ and

$$0+ia+i = 0+i(a+1) \Rightarrow \arg(0+ia+i) \in \frac{-\pi}{2}$$

$$0+ia-i = 0+i(a-1) \Rightarrow \arg(0+ia-i) \in \frac{-\pi}{2}$$

$$\int_{C_4} \frac{\sqrt{s^2+1}}{(s-x)^2+y^2}\,ds = \frac{y}{\pi}\int_{-i}^{-\infty} \frac{\sqrt{s^2+1}}{(s-x)^2+y^2}\,ds$$

$$= \frac{y}{\pi}\int_{-1}^{-\infty} \frac{1}{(ia-x)^2+y^2} |(ia)^2+1|^{\frac{1}{2}} (-i) \cdot ida$$

$$= \frac{-y}{\pi}\int_{-\infty}^{-1} \frac{1}{(ia-x)^2+y^2} (a^2-1)^{\frac{1}{2}}\, da$$

Therefore， $$u(x,y) = \frac{-2y}{\pi}\int_{-\infty}^{-1} \frac{1}{(ia-x)^2+y^2} (a^2-1)^{\frac{1}{2}}\, da \qquad \text{(analytic solution)}$$

$$u(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\sqrt{s^2+1}}{(s-x)^2+y^2}\,ds=\frac{y}{\pi}\Big[-\int_{-\infty}^{-i}\frac{\sqrt{s^2+1}}{(s-x)^2+y^2}\,ds+\int_{-i}^{-\infty}\frac{\sqrt{s^2+1}}{(s-x)^2+y^2}\,ds\Big]$$

$$=-\frac{2y}{\pi}\int_{-\infty}^{-i}\frac{\sqrt{s^2+1}}{(s-x)^2+y^2}\,ds \qquad \text{(Mathematica)}$$

Now，fixed $y-value$ and input $x-value$ into above $u(x,y)$ equations,

	analytic solution	Mathematica
(2,1)	13470.8+0.0898203i	13470.8+0.0898203i
(3,1)	13471.1+0.0469379i	13471.1+0.0469379i
(4,1)	13471.3+0.0282633i	13471.3+0.0282633i
(10,1)	13471.9+0.00491445i	13471.9+0.00491445i
(20,1)	13472.3+0.00124455i	13472.3+0.00124455i

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