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碩 士 論 文

二維細胞類神經網路之一般模版：
全域花樣

Two-Dimensional CNN with General Template：
Global Patterns

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指導教授：林松山　教授

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摘　　　要

此篇論文是研究如何將 3×3 的花樣擴展為 4×4 的花樣，進而利用置換矩陣去研究花樣生成的問題。從禁止集合的觀點來看，如果 c_k 是 3×3 中不允許的花樣，則可以找到 4×4 中不允許的花樣是 $F(c_k)\equiv F_1(c_k)\cup F_2(c_k)\cup F_3(c_k)\cup F_4(c_k)$，意指收集 c_k 分別落在 4×4 花樣中第一、二、三及第四像限位置的所有 4×4 的花樣，我們也可找到其置換矩陣。因此，在二維細胞類神經網路之一般模版中，我們知道每一區可允許的局部花樣，且利用置換矩陣的遞迴公式，可找出全域的花樣。此外，可採用連結算子去估計空間熵的下界。

Two-Dimensional CNN with General Template：Global Patterns

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ABSTRACT

In this paper, we investigate how to extend 3×3 patterns to 4×4 patterns and then we can use transition matrices created by Ban and Lin to study pattern generation problems. From the viewpoint of forbidden sets, if c_k is a forbidden pattern in $\Sigma_{3\times3}$, then we can find the forbidden set in $\Sigma_{4\times4}$ is $F(c_k) \equiv F_1(c_k) \cup F_2(c_k) \cup F_3(c_k) \cup F_4(c_k)$, it means collecting all 4×4 patterns which c_k located in the first, second, third, and the fourth quadrant of 4×4 patterns respectively, then we also can find the transition matrix. Therefore, in two-dimensional CNN with general template, we have knew the admissible local patterns, then we can find global patterns by the recursive formula of transition matrix, Furthermore, we can use connecting operators to estimate a lower bound of spatial entropy.

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目 錄

1 Introduction

Many systems have been studied as models for spatial pattern formation in biology,chemistry, engineering and physics.In Lattice Dynamical Systems(LDS), especially Cellular Neural Networks(CNNs) which has been proposed by Chua and Young, the set of global stationary solutions(global patterns) has received considerable attention in recent years.The CNNs without input terms are of the form

$$\frac{dx_{i,j}}{dt} = -x_{i,j} + \sum_{|k|\le 1, |\ell|\le 1} a_{k,l} f(x_{i+k,j+\ell}) + z, \ (i,j) \in \mathbb{Z}^2, \tag{1.1}$$

$$x_{i,j}(0) = x^0_{i,j} \tag{1.2}$$

Here the nonlinearity f is a piecewise-linear function of the form

$$f(x) = \frac{1}{2}(|x+1| - |x-1|). \tag{1.3}$$

The numbers $a_{k,\ell}, |k| \le 1, |\ell| \le 1, k, \ell \in \mathbb{Z}$, are arranged in a 3×3 matrix form, which is called a space-invariant A-template

$$A = \begin{bmatrix} a_{-1,1} & a_{0,1} & a_{1,1} \\ a_{-1,0} & a_{0,0} & a_{1,0} \\ a_{-1,-1} & a_{0,-1} & a_{1,-1} \end{bmatrix} \tag{1.4}$$

The quantities $x_{i,j}$ denote the state of a cell $C_{i,j}$. If $x_{i,j} > 1$(resp. $x_{i,j} < -1$), then its corresponding cell $C_{i,j}$ is called a positively (resp. negatively) saturated cell and the state is called + (resp. -) state. A situation in which $|x_{i,j}| = 1$ (resp. $|x_{i,j}| < 1$) is called a transitional or marginal state (resp. or linear state). The output of a cell $C_{i,j}$ defined as $y_{i,j} = f(x_{i,j})$, equals 1,-1 and $x_{i,j}$ when $x_{i,j}$ is a +,- and linear, respectively. The quantity z is called threshold or bias term which is related to an independent voltage source in an electrical circuit.

Lattices play an important role in many scientific models, such as in modeling underlying spatial structures. Notable examples include models arising from biology,chemical reaction and phase transitions,image processing and pattern recognition, and material science.

As generally known, stationary solutions $\bar{x} = (\bar{x}_{i,j})$ of (1.1) are essential for understanding CNN systems; their outputs $\bar{y} = (f(\bar{x}_{i,j}))$ are called patterns. Juang and Lin [12]used "building block" technique to study the patterns generation and obtained lower bounds of the spatial entropy for CNN with square-cross or diagonal-cross templates. For CNN with general templates, Hsu et al [11] investigated the generation of admissible local patterns and obtained the basic set for any parameter, i.e., the first step in studying the patterns generation problem. Later, Ban and Lin [14] constructed the "ordering matrix" $\mathbf{X}_2$ for $\Sigma_{2\ell\times2\ell}$ to study the patterns generation and obtained recursion formulas for $\mathbf{X}_n$ for $\Sigma_{2\ell\times n\ell}$ where $\ell > 1$ is a fixed positive integer and $n \leq 2$. The recursive formulas for $\mathbf{X}_n$ imply the recursive formula for the associated transition matrices $\mathbb{H}_n(\mathcal{B})$ of $\Sigma_{2\ell\times n\ell}(\mathcal{B})$.

Motivated by the transition matrices in [1], this work introduce how to find global patterns for each general template A and threshold z. First, we extension 3×3 admissible local patterns to 4×4 admissible local patterns, when given a basic set for any parameter.The method is considering the forbidden set, means that if c_k is a forbidden pattern in $\Sigma_{3\times3}$, we define $F_1(c_k), F_2(c_k), F_3(c_k)$ and $F_4(c_K)$ be sets collecting 4×4 patterns with c_k located at left-down, left-up, right-down and right-up respectively,then we can know that the forbidden patterns in $\Sigma_{4\times4}$ is $F(c_k) \equiv F_1(c_k) \cup F_2(c_k) \cup F_3(c_k) \cup F_4(c_k)$. Therefore, if $\mathcal{B}^c$ is a forbidden set of local patterns in $\Sigma_{3\times3}$, then $\bigcup\limits_{c_k\in\mathcal{B}^c} F(c_k)$ is the set with all forbidden local patterns in $\Sigma_{4\times4}$. Then, we can find out the transition matrix is $\circ(H_2(c_k)),\ c_k \in \mathcal{B}^c$, and using the method of Ban and Lin, global patterns can be found, and spatial entropy is known: $h(\mathcal{B}) = \lim\limits_{n\to\infty} \dfrac{log\rho(\mathbb{H}_n)}{4n}$, the detail is in section 3.

The rest of this paper is organized as follows. Section 2 describes that the parameters space can be divided into finitely many subregions such that in each region (1.1) has the same mosaic patterns. Section 3 addresses the ordering matrix and transition matrix of patterns in $\Sigma_{4\times4}$, and the rule putting 3×3 patterns into 4×4 patterns such that transition matrices can be found in $\Sigma_{4\times4}$. Section 4 presents the application of connecting operators to estimate a lower bound of spatial entropy and take an example to compare with Juang and Lin's "building block" method.

2 Partition of the Parameters Space

In this section,according to [11],we partition the parameters space

$$\mathcal{P}^{10} = \{(A, z) : A \text{ is a } 3\times 3 \text{ real matrix and } z \in \mathbb{R}^1\} \quad (2.1)$$

into finite subregions such that each region has the same mosaic pattern.

For a given mosaic solution $\bar{x}$, the state at cell $C_{i,j}$ is $+$, i.e. $\bar{x}_{i,j} > 1$, if and only if

$$(a-1)+z+\sum_{|k|,|\ell|\leq 1,(k,\ell)\neq(0,0)} a_{k,\ell}\bar{y}_{i+k,j+\ell} > 0, \quad (2.2)$$

where $a = a_{0,0}$. Similarly, if the state at cell $C_{i,j}$ is $-$, i.e. $\bar{x}_{i,j} < -1$, if and only if

$$(a-1)-z-\sum_{|k|,|\ell|\leq 1,(k,\ell)\neq(0,0)} a_{k,\ell}\bar{y}_{i+k,j+\ell} > 0. \quad (2.3)$$

2.1 General template

For a given general template, like (1.4)

denote

$$\alpha = (a_{1,1}, a_{1,0}, a_{1,-1}, a_{0,1}, a_{0,-1}, a_{-1,1}, a_{-1,0}, a_{-1,-1}),$$

and

$$V = (y_{i+1,j+1}, y_{i+1,j}, y_{i+1,j-1}, y_{i,j+1}, y_{i,j-1}, y_{i-1,j+1}, y_{i-1,j}, y_{i-1,j-1}).$$

Vector α represents the surrounding template of A without center and vector V represents the surrounding outputs at cell $C_{i,j}$. Therefore, V is called a local pattern associated with (A, z). we have two sets of parallel straight lines$\{\ell_j^+\}_{j=1}^N$ and $\{\ell_j^-\}_{j=1}^N$ in $(z, a-1)$ plane, here

$$\mathcal{L}_j^+(z, a) = (a-1) + z + \alpha \cdot V_j,$$

$$\ell_j^+ = \{(z, a) : \mathcal{L}_j^+(z, a) = 0\},$$

$$\mathcal{L}_j^-(z, a) = (a-1) - z - \alpha \cdot V_j,$$

and

$$\ell_j^- = \{(z, a) : \mathcal{L}_j^-(z, a) = 0\},$$

for $j = 1, \cdots, N$. In general,it may happen that $\alpha \cdot V_i = \alpha \cdot V_i$ for some $i \neq j$, and in this case $\ell_i^+ = \ell_j^+$ and $\ell_i^- = \ell_j^-$.

If the values in template A are all different, there are 2^8 lines for both ℓ_j^+ and ℓ_j^-, and denote the region

$$[m, k] = \{(z, a-1) : \mathcal{L}_m^+(z, a) > 0 > \mathcal{L}_{m+1}^+(z, a) \ and \ \mathcal{L}_n^-(z, a) < 0 < \mathcal{L}_{n+1}^-(z, a)\}$$

which is bounded byℓ_m^+, ℓ_{m+1}^+ and ℓ_n^-, ℓ_{n+1}^- as in Fig. 1.

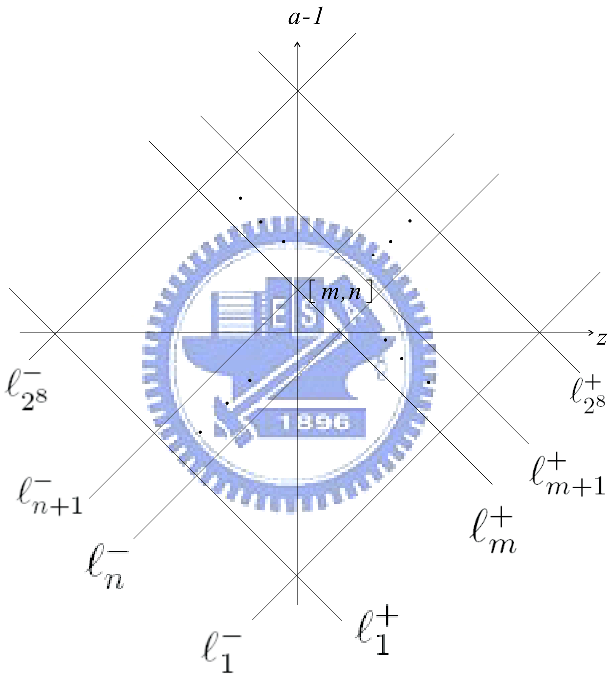


Fig. 1. Partition of $(z, a-1)$ plane.

2.2 Square-cross template

If template A is square-cross,e.g.

$$A = A^+ \equiv \begin{bmatrix} 0 & b & 0 \\ b & a & b \\ 0 & b & 0 \end{bmatrix},$$

the $5+5$ lines are denoted by

$$\begin{aligned}
&\ell_1^+ : (a-1)+z+4b=0, &&\ell_1^- : (a-1)-z+4b=0,\\
&\ell_2^+ : (a-1)+z+2b=0, &&\ell_2^- : (a-1)-z+2b=0,\\
&\ell_3^+ : (a-1)+z=0, &&\ell_3^- : (a-1)-z=0,\\
&\ell_4^+ : (a-1)+z-2b=0, &&\ell_4^- : (a-1)-z-2b=0,\\
&\ell_5^+ : (a-1)+z-4b=0, &&\ell_5^- : (a-1)-z-4b=0.
\end{aligned}$$

Then the$(z, a-1)$ plane can be partition up to 36 subregions. For example, when $b > 0$, we have the bifurcation diagram as in Fig.2, details can be found in [12].

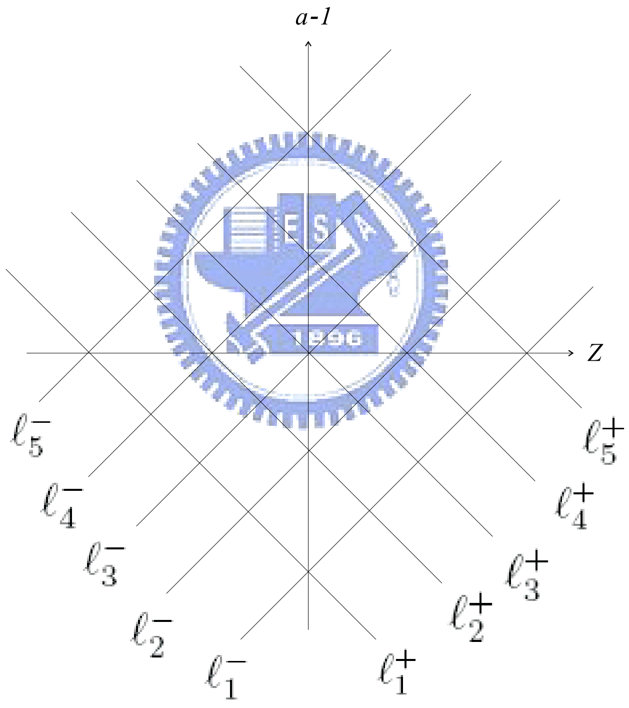


Fig. 2. Partition of $(z, a-1)$ plane with $b > 0$.

2.3 Diagonal-cross template

If template A is diagonal-cross, e.g.,

$$A = A^{\times} \equiv \begin{bmatrix} c & 0 & c \\ 0 & a & 0 \\ c & 0 & c \end{bmatrix}$$

the $5+5$ lines are denoted by

$$\begin{aligned}
&\ell_1^+ : z+(a-1)+4c=0, && \ell_1^- : z-(a-1)+4c=0,\\
&\ell_2^+ : z+(a-1)+2c=0, && \ell_2^- : z-(a-1)+2c=0,\\
&\ell_2^+ : z+(a-1)=0, && \ell_3^- : z-(a-1)=0,\\
&\ell_4^+ : z+(a-1)-2c=0, && \ell_4^- : z-(a-1)-2c=0,\\
&\ell_5^+ : z+(a-1)-4c=0, && \ell_5^- : z-(a-1)-4c=0.
\end{aligned}$$

Then the$(z, a-1)$ plane can be partition up to 36 subregions like Fig.2.

2.4 Double-cross template

If template A is the form

$$A = \begin{bmatrix} c & b & c \\ b & a & b \\ c & b & c \end{bmatrix}$$

we denote the 50 lines by

$$\begin{aligned}
&\ell_1^+ : (a-1)+z+4b+4c=0 && \ell_1^- : (a-1)-z-4b-4c=0\\
&\ell_2^+ : (a-1)+z+4b+2c=0 && \ell_2^- : (a-1)-z-4b-2c=0\\
&\ell_3^+ : (a-1)+z+4b=0 && \ell_3^- : (a-1)-z-4b=0\\
&\ell_4^+ : (a-1)+z+4b-2c=0 && \ell_4^- : (a-1)-z-4b+2c=0\\
&\ell_5^+ : (a-1)+z+4b-4c=0 && \ell_5^- : (a-1)-z-4b+4c=0\\
&\ell_6^+ : (a-1)+z+2b+4c=0 && \ell_6^- : (a-1)-z-2b-4c=0\\
&\ell_7^+ : (a-1)+z+2b+2c=0 && \ell_7^- : (a-1)-z-2b-2c=0\\
&\ell_8^+ : (a-1)+z+2b=0 && \ell_8^- : (a-1)-z-2b=0\\
&\ell_9^+ : (a-1)+z+2b-2c=0 && \ell_9^- : (a-1)-z-2b+2c=0\\
&\ell_{10}^+ : (a-1)+z+2b-4c=0 && \ell_{10}^- : (a-1)-z-2b+4c=0\\
&\ell_{11}^+ : (a-1)+z+4c=0 && \ell_{11}^- : (a-1)-z-4c=0\\
&\ell_{12}^+ : (a-1)+z+2c=0 && \ell_{12}^- : (a-1)-z-2c=0\\
&\ell_{13}^+ : (a-1)+z=0 && \ell_{13}^- : (a-1)-z=0
\end{aligned}$$

$$
\begin{array}{ll}
\ell_{14}^{+} : (a-1)+z-2c=0 & \ell_{14}^{-} : (a-1)-z+2c=0 \\
\ell_{15}^{+} : (a-1)+z-4c=0 & \ell_{15}^{-} : (a-1)-z+4c=0 \\
\ell_{16}^{+} : (a-1)+z-2b+4c=0 & \ell_{16}^{-} : (a-1)-z+2b-4c=0 \\
\ell_{17}^{+} : (a-1)+z-2b+2c=0 & \ell_{17}^{-} : (a-1)-z+2b-2c=0 \\
\ell_{18}^{+} : (a-1)+z-2b=0 & \ell_{18}^{-} : (a-1)-z+2b=0 \\
\ell_{19}^{+} : (a-1)+z-2b-2c=0 & \ell_{19}^{-} : (a-1)-z+2b+2c=0 \\
\ell_{20}^{+} : (a-1)+z-2b-4c=0 & \ell_{20}^{-} : (a-1)-z+2b+4c=0 \\
\ell_{21}^{+} : (a-1)+z-4b+4c=0 & \ell_{21}^{-} : (a-1)-z+4b-4c=0 \\
\ell_{22}^{+} : (a-1)+z-4b+2c=0 & \ell_{22}^{-} : (a-1)-z+4b-2c=0 \\
\ell_{23}^{+} : (a-1)+z-4b=0 & \ell_{23}^{-} : (a-1)-z+4b=0 \\
\ell_{24}^{+} : (a-1)+z-4b-2c=0 & \ell_{24}^{-} : (a-1)-z+4b+2c=0 \\
\ell_{25}^{+} : (a-1)+z-4b-4c=0 & \ell_{25}^{-} : (a-1)-z+4b+4c=0
\end{array}
$$

Then in each region of (b, c) plane as Fig. 5, the $(z, a-1)$ plane can be partitioned up to 26×26 subregions. For example, when $b > 4c > 0$, we have the bifurcation diagram as in Fig.6.

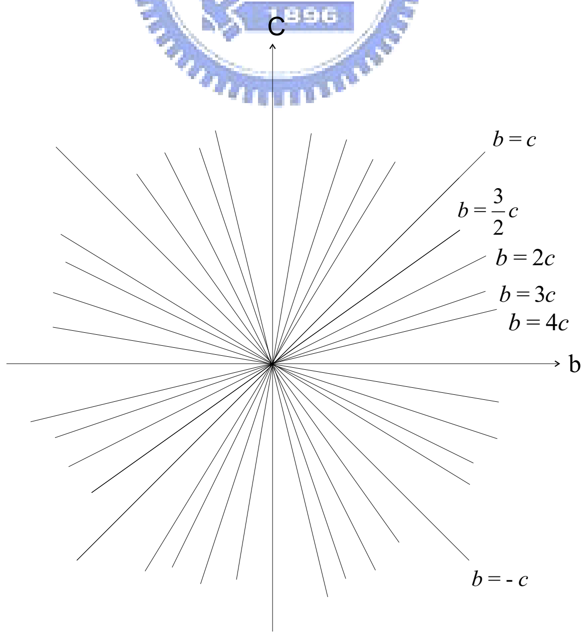


Fig. 5. Partition of (b, c) plane.

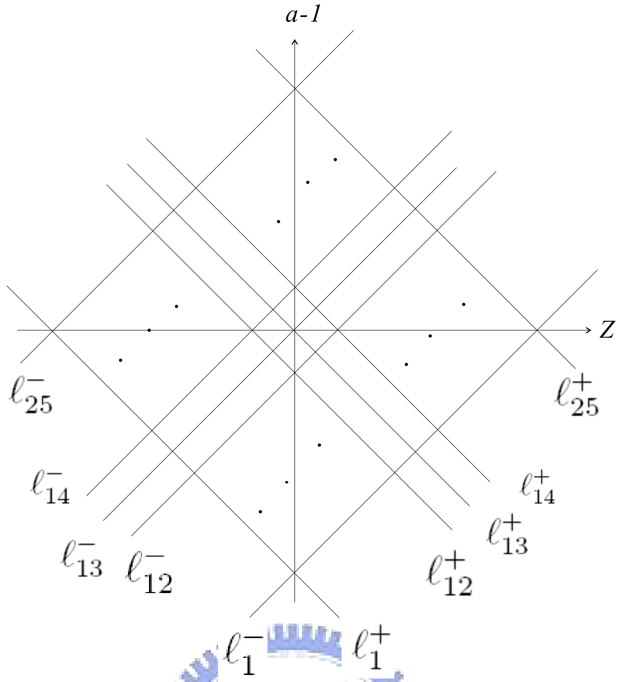


Fig. 6. Partition of $(z, a-1)$ plane with $b > 4c > 0$.

3 Ordering Matrices and Transition Matrices

This section describes two dimensional patterns generation, and the rule that 3×3 patterns putted into 4×4 patterns, then we can use pattern generation method to study the spatial entropy of CNN with general template,square-cross template or diagonal-cross template.

3.1 Ordering matrices for $\Sigma_{4\times4}$

For $2 \times 2n$ pattern $U = (u_k), 1 \le k \le 2n$ in $\Sigma_{2\times2n}$,U is assigned the number

$$i = 1+\sum_{k=1}^{2n} u_k 2^{2n-k} \tag{3.1}$$

As denoted by the $2 \times 2n$ column pattern $x_{2n;i}$,

$$x_{2n;i} = \begin{array}{|c|c|} \hline u_{2n-2} & u_{2n} \\ \hline u_{2n-3} & u_{2n-1} \\ \hline \vdots & \vdots \\ \hline u_2 & u_4 \\ \hline u_1 & u_3 \\ \hline \end{array} \tag{3.2}$$

In particular, when $n = 2$, as denoted by$x_i = x_{4;i}$,

$$i = 1 + \sum_{k=1}^{8} 2^{8-k} u_k \tag{3.3}$$

and

$$x_i = \begin{array}{|c|c|} \hline u_6 & u_8 \\ \hline u_5 & u_7 \\ \hline u_2 & u_4 \\ \hline u_1 & u_3 \\ \hline \end{array} \tag{3.4}$$

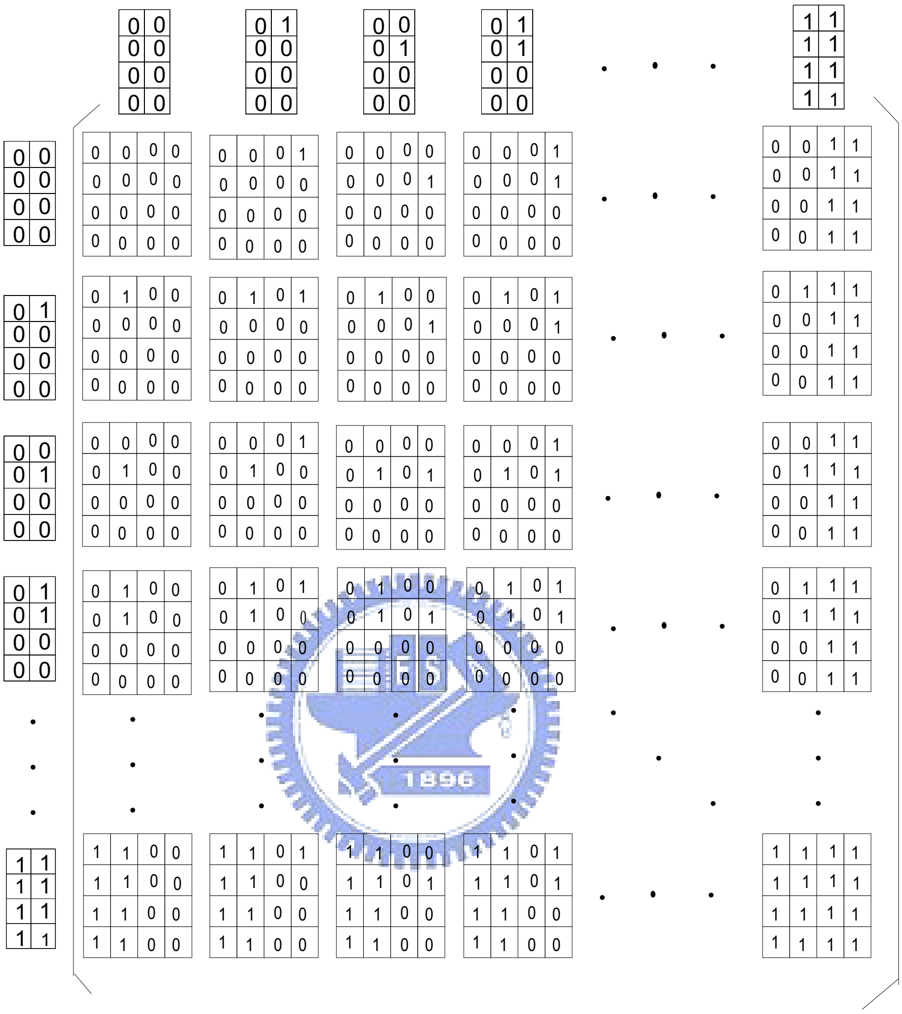
A 4×4 pattern can now be obtained by a horizontal direct sum of two 2×4 patterns, i.e,

$$\begin{aligned} x_{i_1,i_2} &\equiv x_{i_1} \oplus x_{i_2} \\ &\equiv \begin{array}{|c|c|c|c|} \hline u_{16} & u_{18} & u_{26} & u_{28} \\ \hline u_{15} & u_{17} & u_{25} & u_{27} \\ \hline u_{12} & u_{14} & u_{22} & u_{24} \\ \hline u_{11} & u_{13} & u_{21} & u_{23} \\ \hline \end{array} \end{aligned} \tag{3.5}$$

where

$$i_k = 1 + \sum_{j=1}^{8} 2^{8-j} u_{kj} \ , \qquad 1 \le k \le 2. \tag{3.6}$$

Therefore, the complete set of all 2^{16} patterns in $\Sigma_{4\times 4}$ can be listed by a 256×256 matrix $\mathbf{X_2} = [x_{i_1,i_2}]$with 4×4 pattern x_{i_1,i_2} as its entries in



(3.7)

and

$$\chi(x_{i_1,i_2}) = 256(i_1 - 1) + i_2 \tag{3.8}$$

i.e, we are counting local patterns in $\Sigma_{4\times 4}$ by going through each row successively in Table (3.7). Correspondingly, $\mathbf{X_2}$ can be referred to as an ordering matrix for $\Sigma_{4\times 4}$. Similarly, a 4×4 pattern can also be viewed as a vertical direct sum of two 4×2 patterns, i.e,

$$y_{i_1,i_2} = y_{j_1} \oplus y_{j_2}, \tag{3.9}$$

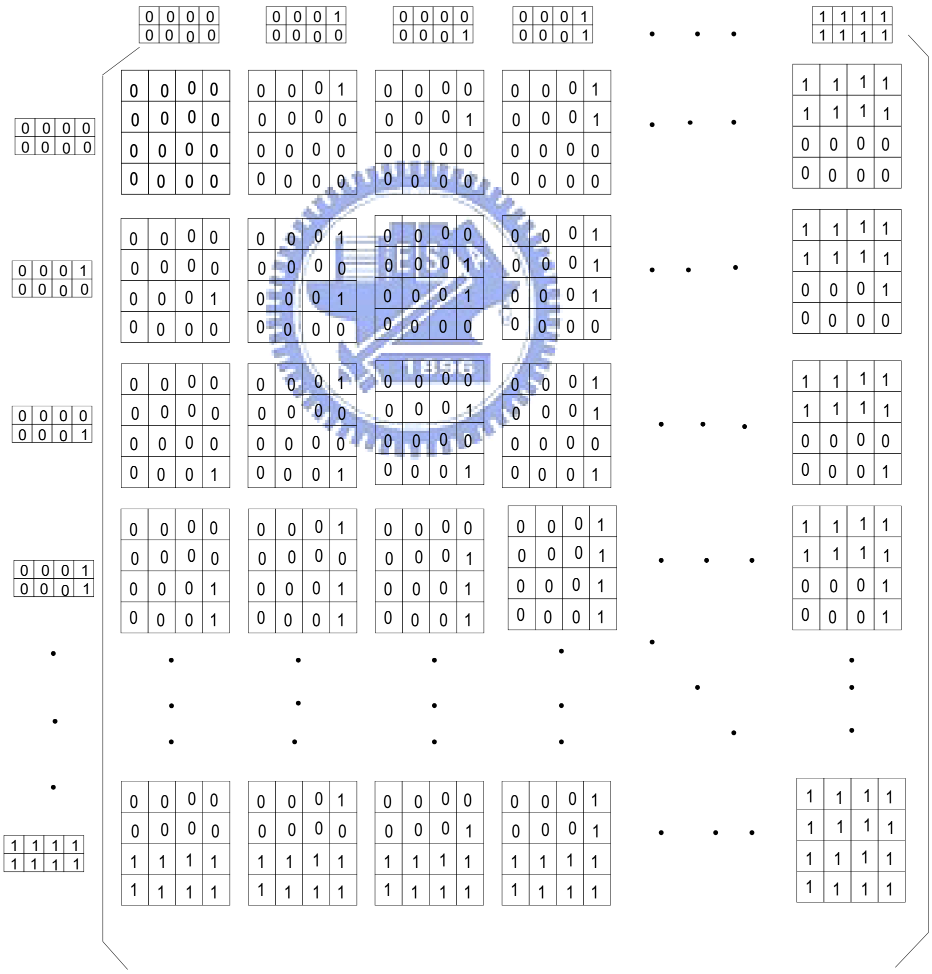
where

$$y_{j_l} = \begin{array}{|c|c|c|c|} \hline u_{2l} & u_{4l} & u_{6l} & u_{8l} \\ \hline u_{1l} & u_{3l} & u_{5l} & u_{7l} \\ \hline \end{array} \tag{3.10}$$

and

$$j_l = 1 + \sum_{k=1}^{8} 2^{8-k} u_{kl}, 1 \leq l \leq 2. \tag{3.11}$$

A 4×4 matrix $\mathbf{Y}_2 = [y_{j_1,j_2}]$ can be obtained for $\Sigma_{4\times4}$. i.e, we have

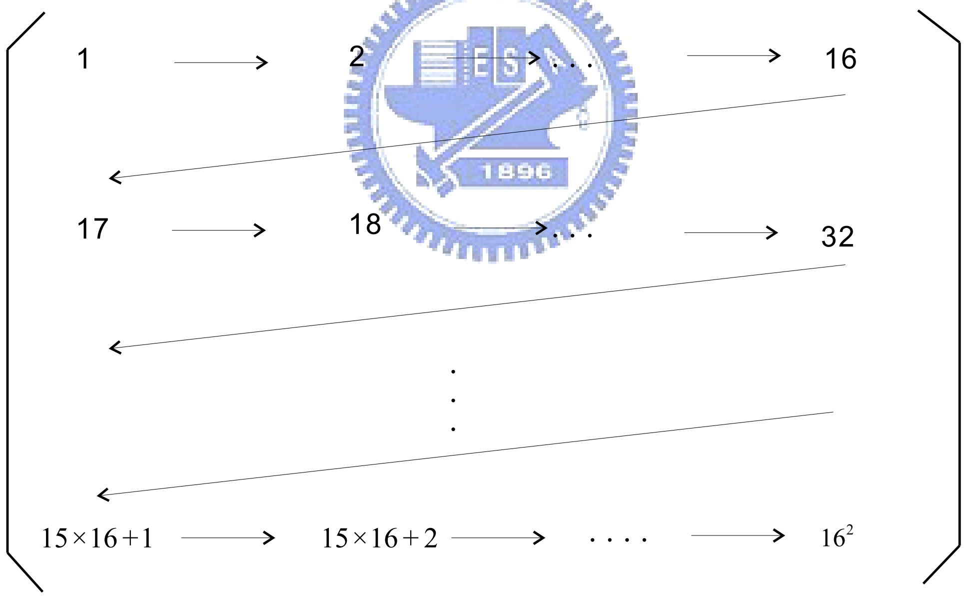


(3.12)

According to [14],the relation between $\mathbf{X}_2$ and $\mathbf{Y}_2$ can be explored. we can discover that $\mathbf{X}_2$ can be represented by y_{j_1,j_2} as

$$\mathbf{X}_2 = \begin{bmatrix} y_{1,1} & \cdots & y_{1,16} & \cdots & y_{16,1} & \cdots & y_{16,16} \\ y_{1,17} & \cdots & y_{1,32} & \cdots & y_{16,17} & \cdots & y_{16,32} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ y_{1,15\cdot16+1} & \cdots & y_{1,16^2} & \cdots & y_{16,15\cdot16+1} & \cdots & y_{16,16^2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ y_{15\cdot16+1,1} & \cdots & y_{15\cdot16+1,16} & \cdots & y_{16^2,1} & \cdots & y_{16^2,16} \\ y_{15\cdot16+1,17} & \cdots & y_{15\cdot16+1,32} & \cdots & y_{16^2,17} & \cdots & y_{16^2,32} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ y_{15\cdot16+1,15\cdot16+1} & \cdots & y_{15\cdot16+1,16^2} & \cdots & y_{16^2,15\cdot16+1} & \cdots & y_{16^2,16^2} \end{bmatrix} \tag{3.13}$$

In (3.13), the indices j_1, j_2 are arranged by two Z-maps successively, as



(3.14)

More precisely, $\mathbf{X}_2$ can be decomposed by

$$\mathbf{X}_2 = \begin{bmatrix} Y_{2;1} & y_{2;2} & \cdots & Y_{2;16} \\ Y_{2;17} & Y_{2;18} & \cdots & Y_{2;32} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{2;15\cdot16+1} & y_{2;15\cdot16+2} & \cdots & Y_{2;16^2} \end{bmatrix}_{16\times16} \tag{3.15}$$

and

$$Y_{2;k} = \begin{bmatrix} y_{k,1} & y_{k,2} & \cdots & y_{k,16} \\ y_{k,17} & y_{k,18} & \cdots & y_{k,32} \\ \vdots & \vdots & \ddots & \vdots \\ y_{k,15\cdot 16+1} & y_{k,15\cdot 16+2} & \cdots & y_{k,16^2} \end{bmatrix}_{16\times 16}. \tag{3.16}$$

where $\mathbf{X}_2$ is arranged by a Z-map ($Y_{2;k}$) in (3.15) and each $Y_{2;k}$ is also arranged by a Z-map (y_{kl}) in (3.16).Therefor, the indices of y in (3.13) consist of two Z-maps.

Now, we can state recursion formulas for higher ordering matrix $\mathbf{X}_n = [x_{n;i_1 i_2}]_{16^n\times 16^n}$ as follows.

Proposition 3.1 ([14])

For any $n \geq 2, \Sigma_{4\times 2n} = \{y_{j_1 j_2\cdots j_n}\}$, where $y_{j_1 j_2\cdots j_n} \equiv y_{j_1 j_2} \hat{\oplus} y_{j_2 j_3} \hat{\oplus} \cdots \hat{\oplus} y_{j_{n-1} j_n}$, $1 \leq j_k \leq 16^2$ and $1 \leq k \leq n$. Furthermore, the ordering matrix $\mathbf{X}_n$ can be decomposed by n Z-maps successively as

$$\mathbf{X}_n = \begin{bmatrix} Y_{n;1} & Y_{n;2} & \cdots & Y_{n;16} \\ Y_{n;17} & Y_{n;18} & \cdots & Y_{n;32} \\ \vdots & \vdots & \cdots & \vdots \\ Y_{n;15\cdot 16+1} & Y_{15\cdot 16+2} & \cdots & Y_{n;16^2} \end{bmatrix},$$

$$Y_{n;j_1\cdots j_k} = \begin{bmatrix} Y_{n;j_1,\cdots,j_k,1} & Y_{n;j_1,\cdots,j_k,2} & \cdots & Y_{n;j_1,\cdots,j_k,16} \\ Y_{n;j_1,\cdots,j_k,17} & Y_{n;j_1,\cdots,j_k,18} & \cdots & Y_{n;j_1,\cdots j_k,32} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n;j_1,\cdots,j_k,15\cdot 16+1} & Y_{n;j_1,\cdots j_k,15\cdot 16+2} & \cdots & Y_{n;j_1,\cdots j_k,16^2} \end{bmatrix},$$

for$1 \leq k \leq n-2$, and

$$Y_{n;j_1\cdots j_{n-1}} = \begin{bmatrix} y_{j_1,\cdots,j_{n-1},1} & y_{j_1,\cdots,j_{n-1},2} & \cdots & y_{j_1,\cdots,j_{n-1},16} \\ y_{j_1,\cdots,j_{n-1},17} & y_{j_1,\cdots,j_{n-1},18} & \cdots & y_{j_1\cdots,j_{n-1},32} \\ \vdots & \vdots & \ddots & \vdots \\ y_{i_1,\cdots,j_{n-1},15\cdot 16+1} & y_{j_1,\cdots,y_{n-1},15\cdot 16+2} & \cdots & y_{j_1,\cdots,j_{n-1},16^2} \end{bmatrix}.$$

3.1.1 General 3×3 patterns

Definition 3.2 $\Sigma_{3\times3}$ is the set of all 3×3 patterns with two different symbols,i.e,

$$\Sigma_{3\times3} = \left\{ \begin{array}{|c|c|c|} \hline \ell_{-1,1} & \ell_{0,1} & \ell_{1,1} \\ \hline \ell_{-1,0} & \ell_{0,0} & \ell_{1,0} \\ \hline \ell_{-1,-1} & \ell_{0,-1} & \ell_{1,-1} \\ \hline \end{array} : \ell_{i,j} \in \{0,1\} \right\}$$

For $c_i \in \Sigma_{3\times3}, c_i$ is assigned the number

$$i = 1 + \ell_{1,1} + 2\ell_{1,0} + 2^2\ell_{1,-1} + 2^3\ell_{0,1} + 2^4\ell_{0,0} + 2^5\ell_{0,-1} + 2^6\ell_{-1,1} + 2^7\ell_{-1,0} + 2^8\ell_{-1,-1}.$$

Definition 3.3 Given $c_k \in \Sigma_{3\times3}$,

(i) c_k is located in the first quadrant(right-up) of the 4×4 pattern in (3.5)

Let

$$u_{25} = \ell_{0,0}, u_{14} = \ell_{-1,-1}, u_{17} = \ell_{-1,0}, u_{18} = \ell_{-1,1},$$

$$u_{22} = \ell_{0,-1}, u_{24} = \ell_{1,-1}, u_{26} = \ell_{0,1}, u_{27} = \ell_{1,0}, u_{28} = \ell_{1,1}.$$

and let

$$k_1 = \{1+\ell_{-1,1}+2\ell_{-1,0}+2^2u_{16}+2^3u_{15}+2^4\ell_{-1,-1}+2^5u_{13}+2^6u_{12}+2^7u_{11} : u_{1j} \in \{0,1\}\}$$

and

$$k_2 = \{1+\ell_{1,1}+2\ell_{1,0}+2^2\ell_{0,1}+2^3\ell_{0,0}+2^4\ell_{1,-1}+2^5u_{23}+2^6\ell_{0,-1}+2^7u_{21} : u_{2j} \in \{0,1\}\}$$

Then we define

$$F_1(c_k) \equiv \{x_{i_1,i_2} : i_1 \in k_1, i_2 \in k_2\}$$

be the set of all patterns with index i_1 satisfy k_1 and i_2 satisfy k_2 in $\Sigma_{4\times4}$.

(ii) c_k is located in the second quadrant(left-up) of the 4×4 pattern in (3.5)

Let

$$u_{17} = \ell_{0,0}, u_{12} = \ell_{-1,-1}, u_{14} = \ell_{0,-1}, u_{15} = \ell_{-1,0}, u_{16} = \ell_{-1,1}, u_{18} = \ell_{0,1},$$

$$u_{22} = \ell_{1,-1}, u_{25} = \ell_{1,0}, u_{26} = \ell_{1,1}.$$

and let

$$k_1 = \{1+\ell_{0,1}+2\ell_{0,0}+2^2\ell_{-1,1}+2^3\ell_{-1,0}+2^4\ell_{0,-1}+2^5u_{13}+2^6\ell_{-1,-1}+2^7u_{11} : u_{1j} \in \{0,1\}\}$$

and

$$k_2 = \{1+u_{28}+2u_{27}+2^2\ell_{1,1}+2^3\ell_{1,0}+2^4u_{24}+2^5u_{23}+2^6\ell_{1,-1}+2^7u_{21} : u_{2j} \in \{0,1\}\}$$

Then we define

$$F_2(c_k) \equiv \{x_{i_1,i_2} : i_1 \in k_1, i_2 \in k_2\}$$

be the set of all patterns with index i_1 satisfying k_1 and i_2 satisfying k_2 in $\Sigma_{4\times 4}$.

(iii) c_k is located in the third quadrant(left-down) of the 4×4 pattern in (3.5)
Let

$$u_{14} = \ell_{0,0}, u_{11} = \ell_{-1,-1}, u_{12} = \ell_{-1,0}, u_{13} = \ell_{0,-1}, u_{15} = \ell_{-1,1}, u_{17} = \ell_{0,1},$$

$$u_{21} = \ell_{-1,-1}, u_{22} = \ell_{1,0}, u_{25} = \ell_{1,1},$$

and let

$$k_1 = \{1+u_{18}+2\ell_{0,1}+2^2u_{16}+2^3\ell_{-1,1}+2^4\ell_{0,0}+2^5\ell_{0,-1}+2^6\ell_{-1,0}+2^7u_{-1,-1} : u_{1j} \in \{0,1\}\}$$

and

$$k_2 = \{1+u_{28}+2u_{27}+2^2u_{26}+2^3\ell_{1,1}+2^4u_{24}+2^5u_{23}+2^6\ell_{1,0}+2^7\ell_{1,-1} : u_{2j} \in \{0,1\}\}$$

Then we define

$$F_3(c_k) \equiv \{x_{i_1,i_2} : i_1 \in k_1, i_2 \in k_2\}$$

be the set of all patterns with index i_1 satisfying k_1 and i_2 satisfying k_2 in $\Sigma_{4\times 4}$.

(iv) c_k is located in the fourth quadrant(right-down) of the 4×4 pattern in (3.5)

Let

$$u_{22} = \ell_{0,0}, u_{13} = \ell_{-1,-1}, u_{14} = \ell_{-1,0}, u_{17} = \ell_{-1,1},$$

$$u_{21} = \ell_{0,-1}, u_{23} = \ell_{1,-1}, u_{24} = \ell_{1,0}, u_{24} = \ell_{1,0}, u_{25} = \ell_{0,1}, u_{27} = \ell_{1,1}.$$

and let

$$k_1 = \{1 + u_{18} + 2\ell_{-1,1} + 2^2 u_{16} + 2^3 u_{15} + 2^4 \ell_{-1,0} + 2^5 \ell_{-1,-1} + 2^6 u_{12} + 2^7 u_{11} : u_{1j} \in \{0,1\}\}$$

and

$$k_2 = \{1 + u_{28} + 2\ell_{1,1} + 2^2 u_{26} + 2^3 \ell_{0,1} + 2^4 \ell_{1,0} + 2^5 \ell_{1,-1} + 2^6 \ell 0, 0 + 2^7 \ell_{0,-1} : u_{2j} \in \{0,1\}\}$$

Then we define

$$F_4(c_k) \equiv \{x_{i_1,i_2} : i_1 \in k_1, i_2 \in k_2\}$$

be the set of all patterns with index i_1 satisfying k_1 and i_2 satisfying k_2 in $\Sigma_{4\times4}$.

Definition 3.4 we define $F(c_k)$ be the set collecting all patterns in $F_1(c_k), F_2(c_k), F_3(c_k)$ and $F_4(c_k)$, i.e,

$$F(c_k) \equiv F_1(c_k) \cup F_2(c_k) \cup F_3(c_k) \cup F_4(c_k)$$

Now,given an admissible basic set $\mathcal{B} \in \Sigma_{3\times3}$,we know the forbidden basic set $\mathcal{B}^c = \Sigma_{3\times3} - \mathcal{B}$ in $\Sigma_{3\times3}$,and we have the following theorem

Theorem 3.5 If $\mathcal{B}$ is an admissible basic set in $\Sigma_{3\times3}$, then the forbidden basic set in $\Sigma_{4\times4}$ is

$$\mathcal{F} \equiv \bigcup_{c_k \in \mathcal{B}^c} F(c_k)$$

Proof. By definition 3.4, $F(c_k)$ is the set that collect all 4×4 patterns which 3×3 pattern c_k is located in it. Because $c_k \in \mathcal{B}^c$, i.e c_k is a forbidden pattern in $\Sigma_{3\times3}$, patterns in $F(c_k)$ are forbidden in $\Sigma_{4\times4}$. Therefore, $\bigcup_{c_k \in \mathcal{B}^c} F(c_k)$ is the set collecting all 4×4 patterns which are forbidden in $\Sigma_{4\times4}$.

3.2 Transition Matrices

3.2.1 Transition matrices for $\Sigma_{4\times 4}$. Given a forbidden basic set $\mathcal{F} \subset \Sigma_{4\times 4}$, horizontal and vertical transition matrices $\mathbb{H}_2$ and $\mathbb{V}_2$ can be define by

$$\mathbb{H}_2 = [h_{i_1 i_2}] \quad \text{and} \quad \mathbb{V}_2 = [v_{j_1 j_2}],$$

two 256×256 matrices with entries either 0 or 1 according to following rules:

$$\begin{cases} h_{i_1 i_2} = 0 & if \quad x_{i_1 i_2} \in \mathcal{F} \\ h_{i_1 i_2} = 1 & if \quad x_{i_1 i_2} \in \Sigma_{4\times 4} - \mathcal{F} \end{cases}$$

and

$$\begin{cases} v_{j_1 j_2} = 0 & if \quad y_{j_1 j_2} \in \mathcal{F} \\ v_{j_1 j_2} = 1 & if \quad y_{j_1 j_2} \in \Sigma_{4\times 4} - \mathcal{F} \end{cases}$$

Obviously, $h_{i_1 i_2} = v_{j_1 j_2}$, where (i_1, i_2) and (j_1, j_2) are related according to 3.1. Here, $\mathbb{H}_2$ is also called the transition matrix for $\mathcal{B}$, and can be defined by

$$\mathbb{H}_2 \equiv \mathbb{H}_2(\mathcal{B}) = \begin{bmatrix} v_{1,1} & \cdots & v_{1,16} & \cdots & v_{16,1} & \cdots & v_{16,16} \\ v_{1,17} & \cdots & v_{1,32} & \cdots & v_{16,17} & \cdots & v_{16,32} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ v_{1,15\cdot 16+1} & \cdots & v_{1,16^2} & \cdots & v_{16,15\cdot 16+1} & \cdots & v_{16,16^2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ v_{15\cdot 16+1,1} & \cdots & v_{15\cdot 16+1,16} & \cdots & v_{16^2,1} & \cdots & v_{16^2,16} \\ v_{15\cdot 16+1,17} & \cdots & v_{15\cdot 16+1,32} & \cdots & v_{16^2,17} & \cdots & v_{16^2,32} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ v_{15\cdot 16+1,15\cdot 16+1} & \cdots & v_{15\cdot 16+1,16^2} & \cdots & v_{16^2,15\cdot 16+1} & \cdots & v_{16^2,16^2} \end{bmatrix}. \quad (3.17)$$

According to [14], transition matrix has following proposition:

proposition 3.6 ([14])

Let $\mathbb{H}_2$ be a transition matrix given by (3.17). Then for higher order transition matrices $\mathbb{H}_n, n \geq 3$, we have the following three equivalent expressions

(I) $\mathbb{H}_n$ can be decomposed into n successive 16×16 matrices as follows:

$$\mathbb{H}_n = \begin{bmatrix} H_{n;1} & \cdots & H_{n;16} \\ H_{n;17} & \cdots & H_{n;32} \\ \vdots & \ddots & \vdots \\ H_{n;15\cdot16+1} & \cdots & H_{n;16^2} \end{bmatrix}$$

$$H_{n;j_1\cdots j_k} = \begin{bmatrix} H_{n;j_1,\cdots,j_k,1} & \cdots & H_{n;j_1,\cdots,j_k16} \\ H_{n;j_1,\cdots,j_k,32} & \cdots & H_{n;j_1,\cdots,j_k32} \\ \vdots & \ddots & \vdots \\ H_{n;j_1,\cdots,j_k15\cdot16+1} & \cdots & H_{m;j_1,\cdots,j_k16^2} \end{bmatrix}$$

for $1 \leq k \leq n-2$ and

$$H_{n;j_1\cdots j_{n-1}} = \begin{bmatrix} v_{j_1,\cdots j_{n-1},1} & \cdots & v_{j_1,\cdots,j_{n-1},16} \\ v_{j_1,\cdots,j_{n-1},17} & \cdots & v_{j_1,\cdots,j_{n-1},32} \\ \vdots & \ddots & \vdots \\ v_{j_1,\cdots,j_{n-1},15\cdot16+1} & \cdots & v_{j_1,\cdots,j_{n-1},16^2} \end{bmatrix}$$

Furthermore,

$$H_{n;k} = \begin{bmatrix} v_{k,1}H_{n-1;1} & \cdots & v_{k,16}H_{n-1;16} \\ v_{k,17}H_{n-1;17} & \cdots & v_{k,32}H_{n-1;32} \\ \vdots & \ddots & \vdots \\ v_{k,15\cdot16+1}H_{n-1;15\cdot16+1} & \cdots & v_{k,16^2}H_{n-1;16^2} \end{bmatrix}$$

(II) Starting from

$$\mathbb{H}_2 = \begin{bmatrix} H_1 & \cdots & H_{16} \\ H_{17} & \cdots & H_{32} \\ \cdots & \ddots & \cdots \\ H_{15\cdot16+1} & \cdots & H_{16^2} \end{bmatrix}$$

with

$$H_k = \begin{bmatrix} h_{k,1} & \cdots & h_{k,16} \\ h_{k,17} & \cdots & h_{k,32} \\ \vdots & \ddots & \vdots \\ h_{k,15\cdot16+1} & \cdots & h_{k,16^2} \end{bmatrix},$$

$\mathbb{H}_n$ can be obtained from $\mathbb{H}_{n-1}$ by replacing H_k by $H_k \circ \mathbb{H}_2$.

(III)

$$\mathbb{H}_n = (\mathbb{H}_{n-1})_{16^{n-1}\times 16^{n-1}} \circ (E_{16^{n-2}} \otimes \mathbb{H}_2),$$

where E_{16^k} is the $16^k \times 16^k$ matrix with 1 as its entries.

Proposition 3.7 ([14])

Given a basic set $\mathcal{B} \subset \Sigma_{4\times 4}$. Suppose ρ_n be the largest eigenvalue of the associated transition matrix $\mathbb{H}_n$. Then the spatial entropy

$$h(\mathcal{B}) = \lim_{n\to\infty} \frac{log\rho_n}{4n}$$

3.2.2 Transition matrices for general 3×3 patterns.

Now, we would like to find transition matrices in $\Sigma_{3\times 3}$.Given a pattern $c_k \in \Sigma_{3\times 3}$, we define

$$H_2(c_k) = [h_{i_1 i_2}] \text{ and } V_2(c_k) = [v_{j_1 j_2}],$$

two 256×256 matrices with entries either 0 or 1 according to following rules:

$$\begin{cases} h_{i_1 i_2} = 0 & if \quad x_{i_1 i_2} \in F(c_k) \\ h_{i_1 i_2} = 1 & if \quad x_{i_1 i_2} \in \Sigma_{4\times 4} - F(c_k) \end{cases}$$

and

$$\begin{cases} v_{j_1 j_2} = 0 & if \quad y_{j_1 j_2} \in F(c_k) \\ v_{j_1 j_2} = 1 & if \quad y_{j_1 j_2} \in \Sigma_{4\times 4} - F(c_k) \end{cases}$$

Obviously, $h_{i_1 i_2} = v_{j_1 j_2}$, where (i_1, i_2) and (j_1, j_2) are related according to 3.1

Consider an admissible basic set $\mathcal{B} \subset \Sigma_{3\times 3}$, then we have the following theorem

Theorem 3.8 If $\mathcal{B}$ is an admissible basic set in $\Sigma_{3\times 3}$, then the transition matrix is

$$\mathbb{H}_2 = \underset{c_k \in \mathcal{B}^c}{\circ} (H_2(c_k))$$

here the Hadamard product of $A \circ B$ means that for any two $n \times n$ matrices $A = (a_{i,j})$ and $B = (b_{i,j})$, $A \circ B = (a_{i,j} \cdot b_{i,j})$.

Proof. By the definition of transition matrix, $\mathbb{H}_2 = [h_{i_1,i_2}]$ is a 256×256 matrix with entries

$$\begin{cases} h_{i_1,i_2} = 1 \quad if \quad x_{i_1,i_2} \in \mathcal{F}^c \\ h_{i_1,i_2} = 0 \quad if \quad x_{i_1,i_2} \in \mathcal{F} \end{cases} \text{where } \mathcal{F} = \bigcup_{c_k \in \mathcal{B}^c} F(c_k),$$

and by definition of $H_2(c_k)$, it is easy to know $\mathbb{H}_2 = \circ(H_2(c_k)), c_k \in \mathcal{B}^c$.

If we have found the transition matrix of the admissible basic set $\mathcal{B} \subset \Sigma_{3\times3}$, we can use proposition 3.6 to find $\mathbb{H}_n$, and proposition 3.7 to know the spatial entropy.

4 Connecting Operators and Computation of Spatial Entropy

4.1 Connecting operators

According to [15],the ordering matrix $\mathbf{X}_{m,n}$ allows the elementary patterns to be tracked during the reduction from $\mathbb{H}^m_{n+1}$ to $\mathbb{H}^m_n$, then they found connecting operators to estimate of spatial entropy.

Definition 4.1 For $m \geq 2$, define

$$\mathbb{C}_m = \begin{bmatrix} C_{m;1,1} & \cdots & C_{m;1,256} \\ C_{m;2,1} & \cdots & C_{m;2,256} \\ \vdots & \ddots & \vdots \\ C_{m;256,1} & \cdots & C_{256,256} \end{bmatrix}$$

$$
= \begin{bmatrix}
S_{m;1,1} & \cdots & S_{m;1,16} & \cdots & S_{m;16,1} & \cdots & S_{m;16,16} \\
S_{m;1,17} & \cdots & S_{m;1,32} & \cdots & S_{m;16,17} & \cdots & S_{m;16,32} \\
\vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
S_{m;1,15\cdot 16+1} & \cdots & S_{m;1,16^2} & \cdots & S_{m;16,15\cdot 16+1} & \cdots & S_{m;16,16^2} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\
S_{m;15\cdot 16+1,1} & \cdots & S_{m;15\cdot 16+1,16} & \cdots & S_{m;16^2,1} & \cdots & S_{m;16^2,16} \\
\vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
S_{m;15\cdot 16+1,15\cdot 16+1} & \cdots & S_{m;15\cdot 16,16^2} & \cdots & S_{m;16^2,15\cdot 16+1} & \cdots & S_{m,16^2,16^2}
\end{bmatrix}, \qquad (4.1)
$$

where

$$
C_{m;\alpha,\beta} = \left(\begin{bmatrix} a_{\alpha,1} & \cdots & a_{\alpha,16} \\ \vdots & \ddots & \vdots \\ a_{\alpha,15\cdot 16+1} & \cdots & a_{\alpha,16^2} \end{bmatrix} \circ \left(\otimes \begin{bmatrix} V_{2;1} & \cdots & V_{2;16} \\ \vdots & \ddots & \vdots \\ V_{2;15\cdot 16+1} & \cdots & V_{2;16^2} \end{bmatrix}^{m-2} \right)\right)
$$

$$
\circ \left(E_{4^{m-2}\times 4^{m-2}} \otimes \begin{bmatrix} a_{1,\beta} & \cdots & a_{16,\beta} \\ \vdots & \ddots & \vdots \\ a_{15\cdot 16+1,\beta} & \cdots & a_{16^2,\beta} \end{bmatrix} \right) \qquad (4.2)
$$

and $\mathbb{C}_{m+1}$ can be found from $\mathbb{C}_m$ by a recursive formula,

Proposition 4.2 ([15])

For any $m \geq 2$ and $1 \leq \alpha, \beta \leq 256$,

$$
C_{m+1;\alpha\beta} = \begin{bmatrix}
a_{\alpha,1}C_{m;1,\beta} & \cdots & a_{\alpha,16}C_{m;16,\beta} \\
a_{\alpha,17}C_{m;17,\beta} & \cdots & a_{\alpha,32}C_{m;32,\beta} \\
\vdots & \ddots & \vdots \\
a_{\alpha,15\cdot 16+1}C_{m;15\cdot 16+1,\beta} & \cdots & a_{\alpha,16^2}C_{m;16^2,\beta}
\end{bmatrix}
$$

Proposition 4.3 ([15])

For any $m \geq 2$, let $S_{m;\alpha\beta}$ be given as in (4.1) and (4.2). Then

$$
X_{m,n+1;\alpha;\beta} = S_{m;\alpha\beta}X_{m,n;\beta},
$$

or equivalently, the recursive formula holds,

$$H^{(k)}_{m,n+1;\alpha} = \begin{bmatrix} \sum_{l=1}^{16^{m-1}} (S_{m;\alpha 1})_{kl} H^{(l)}_{m,n;1} & \cdots & \sum_{l=1}^{16^{m-1}} (S_{m;\alpha 16})_{kl} H^{(l)}_{m,n;16} \\ \sum_{l=1}^{16^{m-1}} (S_{m;\alpha 17})_{kl} H^{(l)}_{m,n;17} & \cdots & \sum_{l=1}^{16^{m-1}} (S_{m;\alpha 32})_{kl} H^{(l)}_{m,n;32} \\ \vdots & \ddots & \cdots \\ \sum_{l=1}^{16^{m-1}} (S_{m;\alpha 15\cdots 16+1})_{kl} H^{(l)}_{m,n;15\cdot 16+1} & \cdots & \sum_{l=1}^{16^{m-1}} (S_{m;\alpha 16^2})_{kl} H^{(l)}_{m,n;16^2} \end{bmatrix}$$

The connecting operator $\mathbb{C}_m$ is employed to estimate the lower bound of entropy, and in particular, to verify the positivity of entropy.

Proposition 4.4 ([15])

Let $\beta_1\beta_2\cdots\beta_k\beta_1$ be diagonal cycle. Then for any $m \geq 2$,

$$h(\mathbb{H}_2) \geq \frac{1}{4mk} log\rho(S_{m;\beta_1\beta_2} S_{m;\beta_2\beta_3} \cdots S_{m;\beta_k\beta_1})$$

4.2 Computation of spatial entropy

Now, we use the property of "connecting operators" to estimate a lower bound of spatial entropy. We **Example 4.4** Consider CNN with square-cross template A,in region $[4,4]_{b>0}$,the forbidden local patterns inΣ_+ are

$$F = \left\{ \begin{array}{ccc} & - & \\ - & + & - \\ & - & \end{array} , \begin{array}{ccc} & + & \\ + & - & + \\ & + & \end{array} \right\},$$

using above method, we can find $\mathbb{H}_2$,and find

$$S_{2;1,1} = C_{2;1,1} = \begin{bmatrix} 1&1&0&1&1&1&0&1&0&0&1&1&1&1&1&1 \\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 1&1&0&1&1&1&0&1&0&0&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&0&0&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&0&1&1&1&0&1&0&0&1&1&1&1&1&1 \\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 1&1&0&1&1&1&0&1&0&0&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&0&0&1&1&1&1&1&1 \\ 1&1&0&1&1&1&0&1&1&1&1&1&1&1&1&1 \\ 0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&0&1&1&1&0&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{bmatrix},$$

the largest eigenvalue of $C_{2;1,1}$ is 13.0153, so a lower bound of spatial entropy is$\frac{log13}{8}$.

We also can compute a lower bound of spatial entropy on the other regions by the same method.the result is followings:

$$h([m,n]) \geq \begin{cases} \frac{log16}{8} & if[m,n] = [5,4],[4,5] \\ \frac{log12}{8} & if[m,n] = [5,3],[3,5] \\ \frac{log13}{8} & if[m,n] = [4,4] \\ \frac{log9}{8} & if[m,n] = [4,3],[3,4] \\ \frac{log4}{8} & if[m,n] = [5,2],[4,2],[2,4],[2,5],[3,3] \\ \frac{log2}{8} & if[m,n] = [3,2],[2,3] \end{cases}$$

and

$$h([5,5]) = log2.$$

See[12],they used "building block" technique to study the patterns generation and obtain lower bounds of the spatial entropy,

$$h[m,n] \geq \begin{cases} log2 & if[m,n]=[5,5] \\ \frac{log10}{4} & if\beta=4 \\ \frac{log4}{4} & if\beta=3 \\ \frac{log4}{9} & if\beta=2,\alpha=5 \\ \frac{log4}{12} & if\beta=2,\alpha=4 \\ \frac{log2}{16} & if\beta=2,\alpha=3 \end{cases}$$

where $\alpha = max\{m,n\}, \beta = min\{m,n\}$

In this example, we compare the estimation of spatial entropy with two different methods, when we take $m = 2$ to find $C_{2;1,1}$ and compute the maximum eigenvalue, we discover the lower bound of spatial entropy is better than the results with "building block" method in subregions $[5,2], [4,2], [3,2], [2,3], [2,4]$ and$[2,5]$, but the others are not, since we use $m = 2$ which is not big enough. We believe that if we take $m \geq 6$ to compute, the lower bound could be better than [12], and for general template, we can compute a lower bound of spatial entropy with pattern generation method.

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