國立交通大學

應用數學系

碩　士　論　文

2維有限型的子移位之原始性質

The primitive property of subshift of finite type in 2-dimensional lattice

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指導教授：林松山　教授

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2維有限型的子移位之原始性質

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摘要

在這篇論文中，討論n階置換矩陣$\mathbb{A}_n$的原始性質。而這些主題與2維有限型的移位之混合性質有關。

我們的目的是給定2階置換矩陣$\mathbb{A}_2$的某些必備條件，進而証得矩陣$\mathbb{A}_n$的原始性質。

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ABSTRACT

In this paper, the primitivity of n-th order transition matrices $\mathbb{A}_n$ defined on $\mathbb{Z}_{2\times n}$ are studied, this topics related to the mixing property of 2- dimensional shift of finite type.

Our purpose is to give some necessary conditions for $\mathbb{A}_2$ to guarantee the primitivity of $\mathbb{A}_n$.

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目 錄

The primitive property of subshift of finite type in 2-dimensional lattice

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Abstract

In this paper, the primitivity of n-th order transition matrices $\mathbb{A}_n$ defined on $\mathbb{Z}_{2\times n}$ are studied,this topics related to the mixing property of 2-dimensional shift of finite type.

Our purpose is to give some necessary conditions for $\mathbb{A}_2$ to guarantee the primitivity of $\mathbb{A}_n$.

1 Introduction

Many systems have been studied as models for spatial pattern formation in biology, chemistry, engineering and physics. Lattices play important roles in modeling underlying spatial structures. We mention some works arising in biology ([8],[9],[22],[23],[24],[28],[29],[30]), chemical reaction and phase transitions ([7],[13],[14],[15],[16],[25],[34]), image processing and pattern recognition ([12],[13],[14],[17],[18],[19],[20],[26],[33]), as well as materials science ([11],[21],[27]). In Lattice Dynamical Systems (LDS), especially Cellular Neural Networks(CNN), the complexity of the set of all global patterns has received considerable attention in recent years ([1],[2],[5],[10]). One of the interesting problem comes from the statistic mechanism is d-demensional shift of finite type, state as follows, given a list of patterns with shape $F \in \mathbb{Z}^d$, consider the set

$$X = X_{\mathcal{L}} = \{x \in \mathcal{A}^{\mathbb{Z}^d} | \text{for all } n \in \mathbb{Z}^d, \text{and } \sigma^n(x)|F \in \mathcal{L}\}, \tag{1.1}$$

where $\mathcal{A}$ is a finite set, we call it symbol, and without loss of generality, F is d-dimensional cube, i.e., $F = \{(n_1, ..., n_d)|1 \leq n_k \leq k, \forall k = 1, ..., d\}$, many invariants related to the shift of finite will discussed likewise in [32], e.g., the topological entropy, measure-theoretical entropy, variational principle, mixing property, and extension problem. Unfortunately, unlike the the one dimensional case, it is extremely difficulty to compute and check those invariants, for example, only a very few example of entropy of 2-dimensional shift of finite type can be computed explicitly,also for mixing property. In this paper we start to study the mixing property of d-dimension shift of finite type, and we focus on d=2. In [3], the authors construct a finite approximation scheme of higher dimensional shift of finite type, and call it the series of transition matrices in multi-dimensional lattice model in $\mathbb{Z}^2$, we are going to use the structure of such transition matrices to study the mixing property of higher dimensional shift of finite type.

We first recall some results in [3], which are crucial in this study. For simplicity, we only consider two symbols which are given on 2×2 lattice $\mathbb{Z}_{2\times 2}$. We begin with a consideration of given horizontal transition matrix

$$H_2 = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix}, \tag{1.2}$$

which is related to a set of admissible local patterns on $\mathbb{Z}_{2\times 2}$, and

$$h_{ij} \in \{0, 1\} \text{ for } 1 \leq i, j \leq 4. \tag{1.3}$$

The associated vertical transition matrix V_2 is defined by

$$V_2 = \begin{bmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{bmatrix}. \tag{1.4}$$

In 2-dimensional shift of finite type, one can immediate construct the H_2 according to the list of pattern with shape $F = \{(n_1, n_2)|1 \leq n_i \leq 2, \forall i = 1, 2\}$. In [3], H_2 and V_2 possess the following property to each other

$$\mathbb{H}_2 = \begin{bmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{bmatrix} = \begin{bmatrix} \mathrm{H}_{2;1} & \mathrm{H}_{2;2} \\ \mathrm{H}_{2;3} & \mathrm{H}_{2;4} \end{bmatrix}, \tag{1.5}$$

and

$$\mathbb{V}_2 = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix} = \begin{bmatrix} \mathrm{V}_{2;1} & \mathrm{V}_{2;2} \\ \mathrm{V}_{2;3} & \mathrm{V}_{2;4} \end{bmatrix}. \tag{1.6}$$

The recursive formula for n-th order horizontal transition matrices $\mathrm{H_n}$ defined on $\mathbb{Z}_{2\times n}$ has been obtained [3] by the following procedure:

$$\mathbb{H}_{k+1} = \begin{bmatrix} v_{11}\mathrm{H}_{k;1} & v_{12}\mathrm{H}_{k;2} & v_{21}\mathrm{H}_{k;1} & v_{22}\mathrm{H}_{k;2} \\ v_{13}\mathrm{H}_{k;3} & v_{14}\mathrm{H}_{k;4} & v_{23}\mathrm{H}_{k;3} & v_{24}\mathrm{H}_{k;4} \\ v_{31}\mathrm{H}_{k;1} & v_{32}\mathrm{H}_{k;2} & v_{41}\mathrm{H}_{k;1} & v_{42}\mathrm{H}_{k;2} \\ v_{33}\mathrm{H}_{k;3} & v_{34}\mathrm{H}_{k;4} & v_{43}\mathrm{H}_{k;3} & v_{44}\mathrm{H}_{k;4} \end{bmatrix}, \tag{1.7}$$

whenever

$$\mathbb{H}_k = \begin{bmatrix} \mathrm{H}_{k;1} & \mathrm{H}_{k;2} \\ \mathrm{H}_{k;3} & \mathrm{H}_{k;4} \end{bmatrix}, \tag{1.8}$$

for $k \geq 2$. The number of all admissible patterns defined on $\mathbb{Z}_{m\times n}$ which can be generated from $\mathbb{H}_2$ is now defined by

$$\begin{aligned} \Gamma_{m\times n}(\mathrm{H}_2) &= |\mathrm{H}_n^{m-1}| \\ &= \text{the summation of all entries in } \mathrm{H}_n^{m-1}. \end{aligned}$$

The quantitative properties of $\mathbb{H}_n$ for $n \geq 2$ are interesting problem in matrix theory and combinatorial dynamics, the most important one is the primitive property, in matrix analysis, the primitivity of a nonnegative matrix will guarantee the positivity of the maximal eigenvalue of a given matrix, and according to the discussion above, if some $\mathbb{H}_2$ is induced from some of 2-dimensional shift of finite type, then primitivity of $\mathbb{H}_2$ demonstrate the shift is mixing. And some interesting dynamics will appear therein, for example, the periodic orbits is dense, and there exists a unique measure of maximal entropy. Thus, it give rise to the study the primitivity of $\mathbb{H}_n$, $\forall n \geq 2$.

The difficulties of this study is that the size of $\mathbb{H}_n$ grows exponentially, i.e., $\mathbb{H}_n \in M_{2^n\times 2^n}$, then it is of nature and interesting to ask that which kind of sufficient conditions will guarantee the primitivity for $\mathbb{H}_n$. [4] and [31] have given some results. To overcome this problem, the powerful tool Σ, Σ', Γ, Γ' will be introduced, thus we obtain some checkable conditions of $\mathbb{H}_2$ to guarantee the primitivity for $\mathbb{H}_n, \forall n \geq 2$.

The paper is organized as follows, Section 2 introduce some definitions, the main results and proof will presented in section 3. Furthermore, the results in section 3 can be generalized to p-symbols and it will be introduced in section 4.

2 Preliminaries

As mentioned in the introduction, horizontal transition matrix $\mathbb{H}_2$ and vertical transition matrix $\mathbb{V}_2$ are related to each other. However, in application, usually it is better working on one matrix then the other. Therefore, we use $\mathbb{A}_2$ and $\mathbb{B}_2$ to replace $\mathbb{H}_2$ and $\mathbb{V}_2$ throughout this paper, i.e., if $\mathbb{A}_2 = \mathbb{H}_2$ then $\mathbb{B}_2$=$\mathbb{V}_2$ and if $\mathbb{A}_2$=$\mathbb{V}_2$ then $\mathbb{B}_2 = \mathbb{H}_2$. Therefore, for simplicity, only $\mathbb{A}_2$ is stated herein.

Definition 2.1. A matrix $\mathrm{A} \in M_{n\times n}(\mathbb{Z})$ is called non-compressible if no columns and rows of A are all zero.

Definition 2.2. A matrix $\mathrm{A} \in M_{n\times n}(\mathbb{Z})$ has property C, if no columns of A are all zero; and has property R, if no rows of A are all zero.

Next, $R(\alpha)$, $\widetilde{R}(\alpha)$, $C(\alpha)$, $\widetilde{C}(\alpha)$, Σ, Σ', Γ, Γ' are introduced, these concepts are defined in, and is crucial for our study. We follow the notation from [3] to denote the recursive formulae for n-th order transition matrices $\mathbb{A}_n$ defined on $\mathbb{Z}_{2\times n}$ *or* $\mathbb{Z}_{n\times 2}$, by

$$\mathbb{A}_n = (\mathbb{A}_{n-1})_{2^{n-1}\times 2^{n-1}} \circ \left(E_{2^{n-2}\times 2^{n-2}} \otimes \begin{pmatrix} \mathrm{A}_{2;1} & \mathrm{A}_{2;2} \\ \mathrm{A}_{2;3} & \mathrm{A}_{2;4} \end{pmatrix} \right)_{2^{n-1}\times 2^{n-1}}, \quad (2.1)$$

for $n > 2$, where

$$\mathbb{A}_2 = \begin{bmatrix} \mathrm{A}_{2;1} & \mathrm{A}_{2;2} \\ \mathrm{A}_{2;3} & \mathrm{A}_{2;4} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{bmatrix} \quad (2.2)$$

and $A_{2;\alpha} \in M_{2\times 2}(\mathbb{Z}),\ \forall\ \alpha \in \{1,2,3,4\}$

Definition 2.3. From (2.2), we define

$$C(\alpha) = \{C_{\alpha;j} | j \in \mathcal{U}\}, \text{ where } C_{\alpha;j} = \min\{\beta | b_{\alpha\beta} = 1,\ \beta = j + 2 \cdot (k-1)\},$$
$$R(\alpha) = \{R_{\alpha;j} | j \in \mathcal{U}\}, \text{ where } R_{\alpha;j} = \min\{\beta | b_{\alpha\beta} = 1,\ \beta = k + 2 \cdot (j-1)\},$$
$$\widetilde{C}(\alpha) = \{\widetilde{C}_{\alpha;j} | j \in \mathcal{U}\}, \text{ where } \widetilde{C}_{\alpha;j} = \max\{\beta | b_{\alpha\beta} = 1,\ \beta = j + 2 \cdot (k-1)\},$$
$$\widetilde{R}(\alpha) = \{\widetilde{R}_{\alpha;j} | j \in \mathcal{U}\}, \text{ where } \widetilde{R}_{\alpha;j} = \max\{\beta | b_{\alpha\beta} = 1,\ \beta = k + 2 \cdot (j-1)\},$$

where where $k \in \mathcal{U}$, and $\mathcal{U} = \{1, 2\}$.

Definition 2.4. A sequence $\{\alpha_k\}_{k=1}^m$ is called an eventually periodic sequence if there exists $1 \leq n < m$ such that $a_m = a_n$ and $a_p \neq a_q, \text{if} 1 \leq p \neq q < m$.

Example 2.5. Let sequence $\{\alpha_k\}_{k=1}^4 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \{2, 3, 1, 3\}$. Since there exists 2 such that $\alpha_4 = \alpha_2$, and $\alpha_p \neq \alpha_q, \text{if} 1 \leq p \neq q < 4$ then we call $\{\alpha_i\}_{i=1}^4$ is an eventually periodic sequence.

Definition 2.6. From definition 2.3, we define
(1) $\Sigma = \Sigma_e \cup \Sigma_0$, where Σ_e=the set of all eventually periodic sequences $\{\alpha_k\}_{k=1}^m$ which satisfy $\alpha_0 = 1,\ \alpha_k \in C(\alpha_{k-1})$. Σ_0 =the set of all sequences $\{\alpha_k\}_{k=1}^m$ which satisfy $\alpha_0 = 1,\ \alpha_k \in C(\alpha_{k-1}),\ 1 \leq k \leq m-1,\ C(\alpha_m) = \emptyset$.
(2) $\Sigma' = \Sigma'_e \cup \Sigma'_0$, where Σ'_e=the set of all eventually periodic sequences $\{\alpha_k\}_{k=1}^m$ which satisfy $\alpha_0 = 1,\ \alpha_k \in R(\alpha_{k-1})$. Σ_0 =the set of all sequences $\{\alpha_k\}_{k=1}^m$ which satisfy $\alpha_0 = 1,\ \alpha_k \in R(\alpha_{k-1}),\ 1 \leq k \leq m-1,\ R(\alpha_m) = \emptyset$.
(3) $\Gamma = \Gamma_e \cup \Gamma_0$, where Γ_e =the set of all eventually periodic sequences $\{\alpha_k\}_{k=1}^m$ which satisfy $\alpha_0 = 1,\ \alpha_k \in \widetilde{C}(\alpha_{k-1})$. Γ_0 =the set of all sequences $\{\alpha_k\}_{k=1}^m$ which satisfy $\alpha_0 = 1,\ \alpha_k \in \widetilde{C}(\alpha_{k-1}),\ 1 \leq k \leq m-1,\ \widetilde{C}(\alpha_m) = \emptyset$.
(4) $\Gamma' = \Gamma'_e \cup \Gamma'_0$, where Γ'_e =the set of all eventually periodic sequences $\{\alpha_k\}_{k=1}^m$ which satisfy $\alpha_0 = 1,\ \alpha_k \in \widetilde{R}(\alpha_{k-1})$. Γ'_0 =the set of all sequences $\{\alpha_k\}_{k=1}^m$ which satisfy $\alpha_0 = 1,\ \alpha_k \in \widetilde{R}(\alpha_{k-1}),\ 1 \leq k \leq m-1,\ \widetilde{R}(\alpha_m) = \emptyset$.

Example 2.7. Let $\mathbb{A}_2 = \begin{bmatrix} b_{11} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.
From Definition 2.3, we have $C(1) = \{C_{1;1}, C_{1;2}\} = \{1, 2\}, ,R(1) = \{R_{1;1}, R_{1;2}\} = \{1, 3\}$, and by Definition 2.4, 2.6, we have
$\Sigma = \Sigma_e \cup \Sigma_0 = \{\{1,1\},\ \{1,2,1\}\} \cup \{\{1,2,4\}\}$=$\{\{1,1\},\ \{1,2,1\},\ \{1,2,4\}\}$ and $\Sigma' = \Sigma'_e = \{\{1,1\}, \{1,3,1\}, \{1,3,3\}\}$.

Example 2.8. Let $\mathbb{A}_2 = \begin{bmatrix} b_{11} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.
From Definition 2.3, we have $\widetilde{C}(3) = \{\widetilde{C}_{3;1}, \widetilde{C}_{3;2}\} = \{3, 4\}$, $\widetilde{R}(1) = \{\widetilde{R}_{1;1}, \widetilde{R}_{1;2}\} = \{2, 4\}$, and by Definition 2.4, 2.6, we have $\Gamma = \Gamma_e \cup \Gamma_0 = \Gamma_e = \{\{4, 4\}, \{4, 3, 3\}, \{4, 3, 4\}\}$ and $\Gamma' = \Gamma'_e = \{\{4, 4\}, \{4, 2, 4\}, \{4, 2, 1, 2\}, \{4, 2, 1, 4\}\}$.

3 Main Theorem (2-Symbols)

Definition 3.1. Let A $\in M_{n\times n}(\mathbb{Z})$ is called primitive if there exists an integer $k \geq 1$ such that $A^k \geq E_{n\times n}$ (full matrix), and let $\tau(A)$ be the minimum number of such k, i.e.,

$$\tau(A) \equiv min\{k : A^k \geq E_{n\times n}\}.$$

In this paper we follow the notation from [6]to denote the multiplication m-times of $\mathbb{A}_n$ i.e., $\mathbb{A}_n^m$, by

$$\mathbb{A}_n^m = \begin{bmatrix} \mathrm{A}_{m,n;1} & \mathrm{A}_{m,n;2} \\ \mathrm{A}_{m,n;3} & \mathrm{A}_{m,n;4} \end{bmatrix}, \tag{3.1}$$

and by matrix multiplication we have

$$A_{m,n;\alpha} = \sum_{k=1}^{2^{m-1}} A_{m,n;\alpha}^{(k)} \text{ where } A_{m,n;\alpha}^{(k)} = A_{n;j_1\cdot j_2} \cdot A_{n;j_2\cdot j_3} \cdot \ldots . A_{n;j_m\cdot j_{m+1}} \tag{3.2}$$

$$k = 1 + \sum_{i=2}^{m} 2^{m-i} \cdot (j_i - 1) \text{ and } \alpha = 2 \cdot (j_1 - 1) + j_{m+1}. \tag{3.3}$$

Lemma 3.2.
(a) If for any sequence $\{\alpha_k\}_{k=1}^{m(k)}$ belongs to Σ, $1 \leq k \leq m(k)$, $A_{2;\alpha_k}$ has property C then for any $n \geq 2$, $1 \leq k \leq m(k)$, $A_{n;\alpha_k}$ has property C.
(b) If for any sequence $\{\beta_k\}_{k=1}^{m(k)}$ belongs to Σ', $1 \leq k \leq m(k)$, $A_{2;\beta_k}$ has property R then for any $n \geq 2$, $1 \leq k \leq m(k)$, $A_{n;\beta_k}$ has property R.

Proof. Firstly, by recursive formulae (2.1), we have
if

$$\mathbb{A}_2 = \begin{bmatrix} b_{11} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} \mathrm{A}_{2;1} & \mathrm{A}_{2;2} \\ \mathrm{A}_{2;3} & \mathrm{A}_{2;4} \end{bmatrix} \tag{3.4}$$

,then

$$\mathbb{A}_{n+1} = \begin{bmatrix} b_{11}\mathrm{A}_{n;1} & b_{12}\mathrm{A}_{n;2} & b_{21}\mathrm{A}_{n;1} & b_{22}\mathrm{A}_{n;2} \\ b_{13}\mathrm{A}_{n;3} & b_{14}\mathrm{A}_{n;4} & b_{23}\mathrm{A}_{n;3} & b_{24}\mathrm{A}_{n;4} \\ b_{31}\mathrm{A}_{n;1} & b_{32}\mathrm{A}_{n;2} & b_{41}\mathrm{A}_{n;1} & b_{42}\mathrm{A}_{n;2} \\ b_{33}\mathrm{A}_{n;3} & b_{34}\mathrm{A}_{n;4} & b_{43}\mathrm{A}_{n;3} & b_{44}\mathrm{A}_{n;4} \end{bmatrix} \tag{3.5}$$

whenever

$$\mathbb{A}_n = \begin{bmatrix} \mathrm{A}_{n;1} & \mathrm{A}_{n;2} \\ \mathrm{A}_{n;3} & \mathrm{A}_{n;4} \end{bmatrix}, \text{ for } n \geq 2 \tag{3.6}$$

or equivalently,

$$\mathrm{A}_{n+1;\alpha} = \begin{bmatrix} b_{\alpha 1}\mathrm{A}_{n;1} & b_{\alpha 2}\mathrm{A}_{n;2} \\ b_{\alpha 3}\mathrm{A}_{n;3} & b_{\alpha 4}\mathrm{A}_{n;4} \end{bmatrix}, \text{ for } \alpha \in \{1,2,3,4\}. \tag{3.7}$$

Next, we prove (a) by induction on n.
When $n = 2$, by condition (a), it is trivial that the result holds for $n = 2$. Now, assume that for any sequence $\{\alpha_k\}_{k=1}^{m(k)}$ belongs to Σ, $1 \leq k \leq m(k)$, $A_{n;\alpha_k}$ has property C ; the goal is to show that it also holds for n+1. Firstly, we let $\{\alpha_j^{(1)}\}_{j=1}^{m}$ be a sequence in Σ and α_i be the $\mathrm{i-th}$ term of $\{\alpha_j^{(1)}\}_{j=1}^{m}$, next we consider the following situations to show $A_{n+1;\alpha_i}$ has property C.
Case 1: $1 \leq i < m$.
By condition (a), because $A_{2;\alpha_i}$ has property C, so $|C(\alpha_i)| = 2$, and we denote it as $C(\alpha_i) = \{p,q\}$ where $p,q \in \{1,2,3,4\}$, i.e., $b_{\alpha_i;p} = 1$, $b_{\alpha_i;q} = 1$. By condition (a), we have for any sequence $\{\alpha_k\}_{k=1}^{m(k)}$ belongs to Σ, $1 \leq k \leq m(k)$, $|C(\alpha_k)| = 2$ and $\Sigma = \Sigma_e$. Therefore, it is trivial that there exists another sequence $\{\alpha_j^{(2)}\}$ which belong to Σ and satisfies the following properties
(a)$\alpha_j^{(1)} = \alpha_j^{(2)}$, where$1 \leq j \leq i$, $\alpha_{i+1}^{(1)} = p$, $\alpha_{i+1}^{(2)} = q$.

(b)$b_{\alpha_i;p} = 1,\ b_{\alpha_i;q} = 1$
(c)$A_{n;p},\ A_{n;q}$ have property C.
Therefore by (3.7) $A_{n+1;\alpha_i}$ has property C.
Case 2: $i = m$.
Since $\{\alpha_j^{(1)}\}_{j=1}^{m}$ is an eventually periodic sequence, i.e., there exists $1 \leq M < n$ such that $\alpha_i = \alpha_n = \alpha_M$. By case 1, we have $A_{n+1;\alpha_M}$ has property C, i.e., $A_{n+1;\alpha_i}$ has property C.

Finally, using the same argument of case 1 and case 2, we obtain for any sequence $\{\alpha_k\}_{k=1}^{m(k)}$ belongs to Σ, $1 \leq k \leq m(k)$, $A_{n+1;\alpha_k}$ has property C.

In the same fashion of proof (a), we also have for any sequence $\{\beta_k\}_{k=1}^{m(k)}$ belongs to Σ', $1 \leq k \leq m(k)$, $A_{n;\beta_k}$ has property R for any $n \geq 2$. This completes the proof of lemma 3.2. □

Lemma 3.3.
Let $E_n \in M_{n\times n}(\mathbb{Z})$ is full matrix, i.e., for all $1 \leq i, j \leq n$, $e_{ij} = 1$.
(1) If $A \in M_{n\times n}(\mathbb{Z})$ has property C, then $E_n \cdot A \geq E_n$.
(2) If $A \in M_{n\times n}(\mathbb{Z})$ has property R, then $A \cdot E_n \geq E_n$.

Proof. (1) Since

$$(E_n \cdot A)_{pq} = E_{n^{(p)}} \cdot A^{(q)}, \tag{3.8}$$

where $E_{n^{(p)}}$ is the $p-th$ row of matrix E_n; $A^{(q)}$ is the $q-th$ column of matrix A and A has property C, E_n is full matrix, so we have $(E_n \cdot A)_{pq} \geq 1$ and this imply $E_n \cdot A \geq E_n$.
(2) Since

$$(A \cdot E_n)_{pq} = A_{(p)} \cdot E_n^{(q)}, \tag{3.9}$$

where $A_{(p)}$ is the $p-th$ row of matrix A; $E_n^{(q)}$ is the $q-th$ column of matrix E_n and A has property R, E_n is full matrix, so we have $(A \cdot E_n)_{pq} \geq 1$ and this imply $A \cdot E_n \geq E_n$. This completes the proof of lemma 3.3. □

Lemma 3.4. Let

$$\mathbb{A}_n = \begin{bmatrix} A_{n;1} & A_{n;2} \\ A_{n;3} & A_{n;4} \end{bmatrix} = \begin{bmatrix} A_{n;11} & A_{n;12} \\ A_{n;21} & A_{n;22} \end{bmatrix},$$

where $A_{n;ij} \in M_{2^{n-1}\times 2^{n-1}}(\mathbb{Z})$, for $i, j \in \{1, 2\}$.
If $\mathbb{A}_n$ satisfies the following properties
(1) There exists an integer k and indices $1 \leq i_0, i_1, ..., i_k \leq n$ such that

(a) $i_0 = i_k = 1$ $(i_0 = i_k = 2)$;

(b) $\prod_{i=1}^{k} A_{n;i_{j-1}i_j}$ is primitive.

(2) $A_{n;11}$ $(A_{n;22})$ is nonzero matrix; $A_{n;12}$ has property C (R) and $A_{n;21}$ has property R(C).

Then $\mathbb{A}_n$ is primitive.

Proof. By the definition of primitive, it suffices to show that there exists $l \in \mathbb{N}$, such that $\mathbb{A}_n^l \geq E_{2^n \times 2^n}$.

Let $m = k+1$, we consider

$$\mathbb{A}_n^m = \begin{bmatrix} \mathrm{A}_{m,n;1} & \mathrm{A}_{m,n;2} \\ \mathrm{A}_{m,n;3} & \mathrm{A}_{m,n;4} \end{bmatrix}, \tag{3.10}$$

by (3.1)(3.2), we have

$$A_{m,n;1} = \sum_{k=1}^{2^{m-1}} A_{m,n;1}^{(k)} \geq A_{n;11} \cdot \prod_{i=1}^{k} A_{n;i_{j-1}i_j} + \prod_{i=1}^{k} A_{n;i_{j-1}i_j} \cdot A_{n;11}, \tag{3.11}$$

$$A_{m,n;2} = \sum_{k=1}^{2^{m-1}} A_{m,n;2}^{(k)} \geq \prod_{i=1}^{k} A_{n;i_{j-1}i_j} \cdot A_{n;12}, \tag{3.12}$$

$$A_{m,n;3} = \sum_{k=1}^{2^{m-1}} A_{m,n;3}^{(k)} \geq A_{n;21} \cdot \prod_{i=1}^{k} A_{n;i_{j-1}i_j}. \tag{3.13}$$

From condition (2), we have $A_{n;11}$ is nonzero matrix, so there exists

$$(A_{n;11})_{kl} \neq 0 \quad \text{for} \quad k, l \in \{1, 2, ..., 2^{n-1}\}. \tag{3.14}$$

Furthermore, from condition (1)(b), (3.11) and (3.14), we have $(A_{m,n;1})_{kj} \geq 1$ and $(A_{m,n;1})_{il} \geq 1$ for all $i, j \in \{1, 2, .., 2^{n-1}\}$. Therefore, $A_{m,n;1}$ has property R and C.

From condition (1) (2), we have $\prod_{i=1}^{k} A_{n;i_{j-1},i_j} \geq E$, $A_{n;12}$ has property C; $A_{n;21}$ has property R, then by (3.12)(3.13) and lemma 3.3, we have $A_{m,n;2} \geq E$ and $A_{m,n;3} \geq E$.

Finally, choose $l = 2m$, we have $\mathbb{A}_n^l = (\mathbb{A}_n^m)^2 \geq E$. This completes the proof of lemma 3.4. □

Next, we give $\mathbb{A}_2$ and write it as $\mathbb{A}_2 = \begin{bmatrix} b_{11} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} \mathrm{A}_{2;1} & \mathrm{A}_{2;2} \\ \mathrm{A}_{2;3} & \mathrm{A}_{2;4} \end{bmatrix} =$ $\begin{bmatrix} \mathrm{A}_{2;11} & \mathrm{A}_{2;12} \\ \mathrm{A}_{2;21} & \mathrm{A}_{2;22} \end{bmatrix}$, where $A_{2;\alpha} \in M_{2\times 2}(\mathbb{Z}), \forall \alpha \in \{1,2,3,4\}$. And, we follow the recursive formulae for n-th order transition matrices $\mathbb{A}_n$ from (2.1). Then we prove the following Theorem.

Theorem 3.5. Given $\mathbb{A}_2 \in M_{4\times 4}(\mathbb{Z})$, where $(A_{2;11})_{1j} = (A_{2;11})_{i1} = 1$ for all $i,j \in \{1,2\}$. If $\mathbb{A}_2$ satisfies the following properties
(a) Every sequence $\{\alpha_k\}_{k=1}^{m(k)}$ in Σ, $A_{2;\alpha_k}$ has property C, $\forall 1 \le k \le m(k)$.
(b) Every sequence $\{\beta_k\}_{k=1}^{m(k)}$ in Σ', $A_{2;\beta_k}$ has property R, $\forall 1 \le k \le m(k)$.
Then $\mathbb{A}_n$ is primitive for all $n \ge 2$.

Proof. Firstly, for matrix multiplication, the indices of $A_{n;\alpha}$ are conveniently expressed as

$$\mathbb{A}_n = \begin{bmatrix} \mathrm{A}_{n;11} & \mathrm{A}_{n;12} \\ \mathrm{A}_{n;21} & \mathrm{A}_{n;22} \end{bmatrix}. \tag{3.15}$$

Clearly, $A_{n;\alpha} = A_{n;j_1 j_2}$, where

$$\alpha = \alpha(j_1, j_2) = 2(j_1 - 1) + j_2. \tag{3.16}$$

Next, we divide this proof into three steps.
Step 1: Since $C(1) = \{1,2\}$, then there exists a sequence $\{(s_k)_{k=1}^{m}\}$ which belongs to Σ with $s_1 = 1, s_2 = 2$. Therefore, by lemma 3.2 and condition (a), $A_{n;1}$ and $A_{n;2}$ have property C for all $n \ge 2$.
Step 2: Since $R(1) = \{1,3\}$, then there exists a sequence $\{(d_l)_{l=1}^{n}\}$ which belongs to Σ' with $d_1 = 1, d_2 = 3$. Therefore, by lemma 3.2 and condition (b), $A_{n;1}$ and $A_{n;3}$ have property R for all $n \ge 2$.
Step 3: The goal is to show that there exists $k(n)$ such that $A_{n;11}^{k(n)} \ge E$ for all $n \ge 2$. This imply there exists an integer $k(n)$ and indices $i_0 = i_1 = \ldots = i_{k(n)} = 1$ such that

$$\prod_{i=1}^{k(n)} A_{n;i_{j-1},i_j} \ge E. \tag{3.17}$$

From (3.7) and (3.16), we have to show that

$$A_{n+1;11}^{k(n+1)} \geq E \tag{3.18}$$

is equivalent to show that

$$\begin{bmatrix} \mathrm{A}_{n;11} & \mathrm{A}_{n;12} \\ \mathrm{A}_{n;21} & b_{14}\mathrm{A}_{n;22} \end{bmatrix}^{k(n+1)} \geq E. \tag{3.19}$$

We prove (3.18) by induction on n.
When n=1, we choose $k(2) = 2$, it is trivial that $A_{2;11}^2 \geq E$.
When n=2, since $A_{2;11}^2 \geq E$, $A_{2;11}$ is nonzero matrix, $A_{2;12}$ has property C, and $A_{2;21}$ has property R, by lemma 3.4 and (3.19) there exists $k(3)$ such that $A_{3;11}^{k(3)} \geq E$. Now, assume that holds for n, the goal is to show that it also holds for n+1.Since $A_{n;11}^{k(n)} \geq E$, $A_{n;11}$ is nonzero matrix, $A_{n;12}$ has property C, and $A_{n;21}$ has property R, by lemma 3.4 and (3.19) there exists $k(n+1)$ such that $A_{n+1;11}^{k(n+1)} \geq E$.

Finally, by step 1, step 2, step 3, and lemma 3.4, we have $\mathbb{A}_n$ is primitive for all $n \geq 2$. This completes the proof of Theorem of 3.5 . □

Example 3.6. Consider $\mathbb{A}_2 = \begin{bmatrix} b_{11} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.
By definition 2.3, 2.4 and 2.6, we have $\Sigma = \{\{1,1\},\{1,2,2\},\{1,2,3,2\},\{1,2,3,3\}\}$ and $\Sigma' = \{\{1,1\},\{1,3,3\},\{1,3,2,2\},\{1,3,2,3\}\}$. From $\mathbb{A}_2$, we get $A_{2;1}, A_{2;2}, A_{2;3}$ have property R and C. It is easily checked that (a) and (b) of Theorem 3.5 hold, then Theorem 3.5 is applied to show that $\mathbb{A}_n$ is primitive for all $n \geq 2$.

Example 3.7. Consider $\mathbb{A}_2 = \begin{bmatrix} b_{11} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.
By definition 2.3, 2.4 and 2.6, we have $\Sigma = \{\{1,1\},\{1,2,1\},\{1,2,2\})\}$ and $\Sigma' = \{\{1,1\},\{1,3,1\},\{1,3,4,1\},\{1,3,4,3\}\}$. From $\mathbb{A}_2$, we get $A_{2;1}, A_{2;2}$ have property C and $A_{2;1}, A_{2;3}$ R. It is easily checked that (a) and (b) of Theorem 3.5 hold, then Theorem 3.5 is applied to show that $\mathbb{A}_n$ is primitive for all $n \geq 2$.

Corollary 3.8. Given $\mathbb{A}_2 \in M_{4\times4}(\mathbb{Z})$, where $(A_{2;22})_{2j} = (A_{2;22})_{i2} = 1$ for all $i, j \in \{1, 2\}$. If $\mathbb{A}_2$ satisfies the following properties
(a)Every sequence $\{\alpha_k\}_{k=1}^{m(k)}$ in Γ, $A_{2;\alpha_k}$ has property C, $\forall\ 1 \leq k \leq n(k)$.
(b)Every sequence $\{\beta_k\}_{k=1}^{m(k)}$ in Γ', $A_{2;\beta_k}$ has property R, $\forall\ 1 \leq k \leq n(k)$.
Then $\mathbb{A}_n$ is primitive for all $n \geq 2$.

Proof. The proof is similar to Theorem 3.5, the details are omitted. □

Example 3.9. Consider $\mathbb{A}_2 = \begin{bmatrix} b_{11} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.
By definition 2.3, 2.4, 2.6, we have
$\Gamma = \{\{4, 4\}, \{4, 3, 4\}, \{4, 3, 1, 3\}, \{4, 3, 1, 4\}\}$ and $\Gamma' = \{\{4, 4\}, \{4, 2, 4\}, \{4, 2, 2\}\}$
From $\mathbb{A}_2$, we get $A_{2;1}, A_{2;3}, A_{2;4}$ have property C and $A_{2;2}, A_{2;4}$ have property R. It is easily checked that (a) and (b) of Corollary 3.8 hold, then Theorem 3.5 is applied to show that $\mathbb{A}_n$ is primitive for all $n \geq 2$.

Next, we will formulate the second main theorem of our study and proof. First, we define $\mathbb{A}_1$ of $\mathbb{A}_2$.

Definition 3.10. Given $\mathbb{A}_2$ and write it as

$$\mathbb{A}_2 = \begin{bmatrix} A_{2;11} & A_{2;12} \\ A_{2;21} & A_{2;22} \end{bmatrix}$$

, where $A_{2;ij} \in M_{2\times2}(\mathbb{Z})$, then $\mathbb{A}_1 \in M_{2\times2}(\mathbb{Z})$ is defined as follows

$$(\mathbb{A}_1)_{ij} = \begin{cases} 1\ , & if\ A_{2;ij} \neq O_{2\times2}. \\ 0\ , & if\ A_{2;ij} = O_{2\times2}. \end{cases}$$

Theorem 3.11. Given $\mathbb{A}_2 \in M_{4\times4}(\mathbb{Z})$, where $A_{2;12}, A_{2;21} \in \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$.

If $\mathbb{A}_2$ satisfies one of the following properties
$(a)\ A_{2;11} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbb{A}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.
$(b)\ A_{2;11} \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbb{A}_1 \in \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} . \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$.
Then $\mathbb{A}_n$ is primitive for all $n \geq 2$.

Proof. We divide this proof into three steps, and we prove these steps by induction on n.

Step 1: The goal is to show $A_{n;1} \neq O_{2^{n-1}\times 2^{n-1}}$ for all $n \geq 2$.

Case 1: If $A_{2;11} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, we prove that $A_{n;i} \neq O_{2^{n-1}\times 2^{n-1}}$ for $i \in \{2^0, .., 2^2\}$ by induction on n.

When n=2, by condition (a), it is trivial that $A_{2;i} \neq O_{2\times 2}$ for $i \in \{2^0, .., 2^2\}$. Suppose $A_{n;i} \neq O_{2^{n-1}\times 2^{n-1}}$ for $i \in \{2^0, .., 2^2\}$, next we need to claim it also holds for n+1. Since

$$A_{2;\alpha} \neq O_{2\times 2}, \text{i.e.,} \begin{bmatrix} b_{\alpha 1} & b_{\alpha 2} \\ b_{\alpha 3} & b_{\alpha 4} \end{bmatrix} \neq O_{2\times 2} \tag{3.20}$$

and

$$A_{n;i} \neq O_{2^{n-1}\times 2^{n-1}} \text{for } i \in \{2^0, .., 2^2\} \tag{3.21}$$

then

$$\mathrm{A}_{n+1;\alpha} = \begin{bmatrix} b_{\alpha 1}\mathrm{A}_{n;1} & b_{\alpha 2}\mathrm{A}_{n;2} \\ b_{\alpha 3}\mathrm{A}_{n;3} & b_{\alpha 4}\mathrm{A}_{n;4} \end{bmatrix} \neq O_{2^n\times 2^n}. \tag{3.22}$$

Case 2: If $A_{2;11} \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then in the same fashion of proof case 1, we have $A_{n;1} \neq O_{2^{n-1}\times 2^{n-1}}$.

Step 2: The goal is to show $A_{n;2}$, $A_{n;3}$ have property R and C.

When n=2, it is trivial that $A_{2;2}$, $A_{2;3}$ have property R and C. Next, suppose $A_{n;2}, A_{n;3}$ has property R and C, then

$$A_{n+1;\alpha} = \begin{bmatrix} A_{n;1} & A_{n;2} \\ A_{n;3} & b_{\alpha 4}A_{n;4} \end{bmatrix} \text{for } \alpha \in \{2, 3\}, \tag{3.23}$$

also have property R and C.

Step 3: The goal is to show that for all $n \geq 2$, there exists an even number $k(n)$, and indices $1 \leq i_0, ..., i_{k(n)} \leq n$ such that $\prod_{i=1}^{k(n)} A_{n;i_{j-1}i_j} \geq E$, where $i_l = 1$ if l is even; $i_l = 2$ if l is odd.

When n=2, we choose $k(2) = 2$, then $\prod_{i=1}^{2} A_{2;i_{j-1}i_j} \geq E$. Now, assume the result holds for n, i.e., there exists an even number $k(n)$, and indices $1 \leq$

$i_0, ..., i_{k(n)} \le n$, such that $\prod_{i=1}^{k(n)} A_{n;i_{j-1}i_j} \ge E$, where $i_l = 1$ if l is even; $i_l = 2$ if l is odd. The goal is to show that it also holds for n+1. From the assumption $\prod_{i=1}^{k(n)} A_{n;i_{j-1}i_j} \ge E$, $i_0 = i_{k(n)} = 1$ and step1, step2; lemma 3.3 is applied to show that there exists $m(n)$ such that

$$A_{n+1;\alpha}^{m(n)} = \begin{bmatrix} A_{n;11} & A_{n;12} \\ A_{n;21} & O \end{bmatrix}^{m(n)} \ge E \text{ for } \alpha \in \{2,3\}. \tag{3.24}$$

We choose $k(n+1) = 2\cdot m(n)$ and indices $1 \le i_0, ..., i_{k(n+1)} \le n$, where $i_l = 1$ if l is even; $i_l = 2$ if l is odd, then $\prod_{i=1}^{k(n+1)} A_{n+1;i_{j-1}i_j} \ge \begin{bmatrix} A_{n;11} & A_{n;12} \\ A_{n;21} & O \end{bmatrix}^{2\cdot m(n)} \ge E$, this imply the result also holds for $n+1$.

Finally, from step 1 step 2 and step 3, lemma 3.3 is applied to show that $\mathbb{A}_n$ is primitive for all $n \ge 2$. This completes the proof of Theorem of 3.8 . □

Example 3.12. Consider $\mathbb{A}_2 = \begin{bmatrix} b_{11} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

From $\mathbb{A}_2$, we have $\mathbb{A}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and it is easily checked that (b) of Theorem 3.10 holds, then Theorem 3.10 is applied to show that $\mathbb{A}_n$ is primitive for all $n \ge 2$.

Corollary 3.13. Given $\mathbb{A}_2 \in M_{4\times 4}(\mathbb{Z})$, where $A_{2;12}, A_{2;21} \in \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$.

If $\mathbb{A}_2$ satisfies one of the following properties

(a) $A_{2;22} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbb{A}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(b) $A_{2;22} \ne \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbb{A}_1 \in \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$.

Then $\mathbb{A}_n$ is primitive for all $n \ge 2$.

Proof. The proof is similar to Theorem 3.11, the details are omitted. □

Example 3.14. Consider $\mathbb{A}_2 = \begin{bmatrix} b_{11} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

From $\mathbb{A}_2$, we have$\mathbb{A}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and it is easily checked that (a) of Corollary 3.13 holds, then Corollary 3.13 is applied to show that $\mathbb{A}_n$ is primitive for all $n \geq 2$.

Remark 3.15. From theorem 3.5, 3.11 and corollary 3.8, 3.13, we can find the marginal states for classes of $\mathbb{A}_2$. To be clearly, we can easily seen that if $A \geq B$ (in the sense that $A \geq O$, all entries of A are nonegative), then B is primitive imply that A is also. Thus we search for all marginal $\mathbb{A}_2$ which is prime, and use comparison to show others are also.

We let

$$E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad G' = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

$$U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$T_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad K_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

marginal states for Theorem 3.5			
$A_{2;1}$	$A_{2;2}$	$A_{2;3}$	$A_{2;4}$
G	J	J	O
G	T_2	T_1	O
G	T_2	I	T_1, I
G	T_4	I	G, I
G	I	T_1	T_2, I
G	I	T_3	G, I
G	I	I	J, I
G	I	J	J, I
G	J	I	J, I
G	G	T_3	T_1, T_3, I, J
G	U	T_3	T_1, T_3, J
G	U	J	T_1, T_3
G	T_4	G	T_2, T_4, I, J
G	T_4	L	T_2, T_4, J
G	L	J	T_2, T_4
G	J	L	T_2, T_4
G	J	U	T_1, T_3

marginal states for Corollary 3.8			
$A_{2;1}$	$A_{2;2}$	$A_{2;3}$	$A_{2;4}$
O	T_3	T_4	G'
O	J	J	G'
I, G'	T_1	I	G'
T_4, I	T_3	I	G'
T_3, I	I	T_4	G'
I, G'	I	T_2	G'
I, J	I	I	G'
I, J	I	J	G'
I, J	J	I	G'
T_1, T_3, I, J	T_1	L	G'
T_1, T_3, I, J	T_1	G'	G'
T_1, T_3, I, J	J	L	G'
J	L	T_2	G'
T_2, T_4, J	U	T_2	G'
T_2, T_4, I, J	G'	T_2	G'
T_2, T_4	J	U	G'

marginal states for Theorem 3.11			
K_1	G	G	O
K_2	G	G	O
K_3	G	G	O
K_4	G	G	K_1, K_2, K_3, K_4

marginal states for Corollary 3.13			
K_1, K_2, K_3, K_4	G'	G'	K_1
O	G'	G'	K_2
O	G'	G'	K_3
O	G'	G'	K_4

4 Main Theorem (P-Symbols)

The results in last two subsections can be generalized to p-symbols.
Next, we follow the notation from [3] to denote the recursive formulae for higher order transition matrices $\mathbb{A}_n$ defined on $\mathbb{Z}_{2l\times 2l}$, by

$$\mathbb{A}_n = (\mathbb{A}_{n-1})_{p^{n-1}\times p^{n-1}} \odot (E_{p^{n-2}} \otimes \mathbb{A}_2), \tag{4.1}$$

$$\mathbb{A}_2 = \begin{bmatrix} A_{2;1} & \cdots & A_{2;p} \\ A_{2;p+1} & \cdots & A_{2;2p} \\ \vdots & \ddots & \vdots \\ A_{2;(p-1)p+1} & \cdots & A_{2;p^2} \end{bmatrix} \tag{4.2}$$

and

$$A_{2;\alpha} = \begin{bmatrix} b_{\alpha;1} & \cdots & b_{\alpha;p} \\ b_{\alpha;p+1} & \cdots & b_{\alpha;2p} \\ \vdots & \cdots & \vdots \\ b_{\alpha;p(p-1)+1} & \cdots & b_{\alpha;p^2} \end{bmatrix} \tag{4.3}$$

for $\alpha \in \{1, 2, ..., p\}, n \geq 2$.

Definition 4.1. From (4.3), we define

$$\begin{aligned} C(\alpha) &= \{C_{\alpha;j} | j \in \mathcal{U}\}, \text{ where } C_{\alpha;j} = \min\{\beta | b_{\alpha\beta} = 1,\ \beta = j + p\cdot(k-1)\}, \\ R(\alpha) &= \{R_{\alpha;j} | j \in \mathcal{U}\}\}, \text{ where } R_{\alpha;j} = \min\{\beta | b_{\alpha\beta} = 1,\ \beta = k + p\cdot(j-1)\}, \\ \widetilde{C}(\alpha) &= \{\widetilde{C}_{\alpha;j} | j \in \mathcal{U}\}, \text{ where } \widetilde{C}_{\alpha;j} = \max\{\beta | b_{\alpha\beta} = 1,\ \beta = j + p\cdot(k-1), \\ \widetilde{R}(\alpha) &= \{\widetilde{R}_{\alpha;j} | j \in \mathcal{U}\}, \text{ where } \widetilde{R}_{\alpha;j} = \max\{\beta | b_{\alpha\beta} = 1,\ \beta = k + p\cdot(j-1)\}, \end{aligned}$$

where $k \in \mathcal{U}$, and $\mathcal{U} = \{1, 2, ..., p\}$.

Definition 4.2. From definition 4.1, we define
(1) $\Sigma = \Sigma_e \cup \Sigma_0$, where Σ_e=the set of all eventually periodic sequences $\{\alpha_k\}_{k=1}^m$ which satisfy $\alpha_0 = 1,\ \alpha_k \in C(\alpha_{k-1})$. Σ_0 =the set of all sequences $\{\alpha_k\}_{k=1}^m$ which satisfy $\alpha_0 = 1,\ \alpha_k \in C(\alpha_{k-1}),\ 1 \leq k \leq m-1,\ C(\alpha_m) = \emptyset$.
(2) $\Sigma' = \Sigma'_e \cup \Sigma'_0$, where Σ'_e=the set of all eventually periodic sequences $\{\alpha_k\}_{k=1}^m$ which satisfy $\alpha_0 = 1,\ \alpha_k \in R(\alpha_{k-1})$. Σ_0 =the set of all sequences $\{\alpha_k\}_{k=1}^m$ which satisfy $\alpha_0 = 1,\ \alpha_k \in R(\alpha_{k-1}),\ 1 \leq k \leq m-1,\ R(\alpha_m) = \emptyset$.
(3) $\Gamma = \Gamma_e \cup \Gamma_0$, where Γ_e =the set of all eventually periodic sequences $\{\alpha_k\}_{k=1}^m$ which satisfy $\alpha_0 = p^2,\ \alpha_k \in \widetilde{C}(\alpha_{k-1})$. Γ_0 =the set of all sequences $\{\alpha_k\}_{k=1}^m$ which satisfy $\alpha_0 = p^2,\ \alpha_k \in \widetilde{C}(\alpha_{k-1}),\ 1 \leq k \leq m-1,\ \widetilde{C}(\alpha_m) = \emptyset$.
(4) $\Gamma' = \Gamma'_e \cup \Gamma'_0$, where Γ'_e =the set of all eventually periodic sequences $\{\alpha_k\}_{k=1}^m$ which satisfy $\alpha_0 = p^2,\ \alpha_k \in \widetilde{R}(\alpha_{k-1})$. Γ'_0 =the set of all sequences $\{\alpha_k\}_{k=1}^m$ which satisfy $\alpha_0 = p^2,\ \alpha_k \in \widetilde{R}(\alpha_{k-1}),\ 1 \leq k \leq m-1,\ \widetilde{R}(\alpha_m) = \emptyset$.

Next, we let $\mathbb{A}_n \in M_{p^n\times p^n}(\mathbb{Z})$, $\mathbb{A}_n = [A_{n;ij}]_{p\times p}$ where $i,j \in \{1,2,...,p\}$, $A_{n;ij} \in M_{p^{n-1}\times p^{n-1})}(\mathbb{Z})$. By matrix multiplication we denote $\mathbb{A}_n^m$ as

$$\mathbb{A}_n^m = \begin{bmatrix} \mathrm{A}_{m,n;1} & \cdots\cdots & \mathrm{A}_{m,n;p} \\ \mathrm{A}_{m,n;p+1} & \cdots\cdots & \mathrm{A}_{m,n;2p} \\ \vdots & \ddots & \vdots \\ \mathrm{A}_{m,n;p(p-1)+1} & \cdots\cdots & \mathrm{A}_{m,n;p^2} \end{bmatrix}, \tag{4.4}$$

$$A_{m,n;\alpha} = \sum_{k=1}^{p^{m-1}} A_{m,n;\alpha}^{(k)} \text{ where } A_{m,n;\alpha}^{(k)} = A_{n;j_1\cdot j_2}\cdot A_{n;j_2\cdot j_3}\cdot \ldots . A_{n;j_m\cdot j_{m+1}} \tag{4.5}$$

$$k = 1 + \sum_{i=2}^{m} p^{m-i}\cdot (j_i - 1) \text{ and } \alpha = p\cdot(j_1-1) + j_{m+1}. \tag{4.6}$$

Lemma 4.3.
Let $\mathbb{A}_n = [A_{n;ij}]_{p\times p} \in M_{p^n\times p^n}(\mathbb{Z})$, where $i,j \in \{1,2,...,p\}$, $A_{n;ij} \in M_{p^{n-1}\times p^{n-1}}(\mathrm{Z})$.
If $\mathbb{A}_n$ satisfies the following properties
(1) There exists an integer k and indices $1 \le i_0, i_1, \ldots\ldots, i_k \le p$ such that
 (a) $i_0 = i_k = l$, where $l \in \{1, 2, \ldots., p\}$;
 (b) $\prod_{i=1}^{k} A_{n;i_{j-1}i_j}$ is primitive.
(2) $A_{n;ll}$ is nonzero matrix, $A_{n;l\beta}$ has property C and $A_{n;\beta l}$ has property R for all $\beta \ne l$, $\beta \in \{1,2,...,p\}$.
Then $\mathbb{A}_n$ is primitive.

Proof. By the definition of primitive, it suffices to show that there exists $r \in \mathbb{N}$, such that $\mathbb{A}_n^r \ge E$. Firstly, for matrix multiplication, the indices of $A_{m,n;\alpha}$ are conveniently expressed as $(\mathbb{A}_n^m)_{ij}$, where $\alpha = \alpha(i,j) = p(i-1)+j$.

Let $m = k+1$, and consider $\mathbb{A}_n^m$. By(4.1) (4.2), we have

$$(\mathbb{A}_n^m)_{l\beta} = A_{m,n;\alpha} = \sum_{k=1}^{p^{m-1}} A_{m,n;\alpha}^{(k)} \ge \prod_{i=1}^{k} A_{n;i_{j-1}i_j}\cdot A_{n;l\beta}, \tag{4.7}$$

$$(\mathbb{A}_n^m)_{\beta l} = A_{m,n;\alpha} = \sum_{k=1}^{p^{m-1}} A_{m,n;\beta}^{(k)} \ge A_{n;\beta l}\cdot \prod_{i=1}^{k} A_{n;i_{j-1}i_j}, \tag{4.8}$$

$$(\mathbb{A}_n^m)_{ll} = A_{m,n;\alpha} = \sum_{k=1}^{p^{m-1}} A_{m,n;\alpha}^{(k)} \geq \prod_{i=1}^{k} A_{n;i_{j-1}i_j} \cdot A_{n;ll} \; + \; A_{n;ll} \cdot \prod_{i=1}^{k} A_{n;i_{j-1}i_j} . \quad (4.9)$$

From condition (2), we have $A_{n;ll}$ is nonzero matrix, so there exists

$$(A_{n;ll})_{st} \neq 0 \text{ for } s,t \in \{1,2,....,p^{n-1}\}. \quad (4.10)$$

Furthermore, since $(A_{n;ll})_{st} \neq 0$ and $\prod_{i=1}^{k} A_{n;i_{j-1}i_j} \geq E$, then for $\alpha = p \cdot (l-1) + l$, we have $(A_{m,n;\alpha})_{sj} \geq 1$ and $(A_{m,n;\alpha})_{it} \geq 1$ for all $i,j \in \{1,2,..,p^{n-1}\}$. Therefore, $(\mathbb{A}_n^m)_{ll}$ has property R and C. By condition (1) (2), (4.7), (4.8), and lemma 3.3, we have $(\mathbb{A}_n^m)_{l\beta} \geq E$ and $(A_n^m)_{\beta l} \geq E$ for all $\beta \neq l$, $\beta \in \{1,2,..,p\}$. Finally, choose r=2m, we have $\mathbb{A}_n^r \geq E$. This completes the proof of lemma 4.3. □

Lemma 4.4.
(a) If for any sequence $\{\alpha_k\}_{k=1}^{m(k)}$ belongs to Σ, $1 \leq k \leq m(k)$, $A_{2;\alpha_k}$ has property C then for any $n \geq 2$, $1 \leq k \leq m(k)$, $A_{n;\alpha_k}$ has property C.
(b) If for any sequence $\{\beta_k\}_{k=1}^{m(k)}$ belongs to Σ', $1 \leq k \leq m(k)$, $A_{2;\beta_k}$ has property R then for any $n \geq 2$, $1 \leq k \leq m(k)$, $A_{n;\beta_k}$ has property R.

Proof. Firstly, by recursive formulae (4.1), (4.2), (4.3), we have

$$A_{n+1;\alpha} = \begin{bmatrix} b_{\alpha;1}\mathrm{A}_{n;1} & \cdots & b_{\alpha;p}\mathrm{A}_{n;p} \\ b_{\alpha;p+1}\mathrm{A}_{n;p+1} & \cdots & b_{\alpha;2p}\mathrm{A}_{n;2p} \\ \vdots & \ddots & \vdots \\ b_{\alpha;p(p-1)+1}\mathrm{A}_{n;p(p-1)+1} & \cdots & b_{\alpha;p^2}\mathrm{A}_{n;p^2} \end{bmatrix} \quad (4.11)$$

for $\alpha \in \{1,..,p^2\}$, $n \geq 2$, where

$$\mathbb{A}_n = \begin{bmatrix} A_{n;1} & A_{n;2} & ... & A_{n;p} \\ A_{n;p+1} & A_{n;p+2} & .. & A_{n;2p} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n;p(p-1)+1} & \cdots & \cdots & A_{n;p^2} \end{bmatrix}. \quad (4.12)$$

Next, we prove (a) by induction on n.
When $n = 2$, by condition (a), it is trivial that the result holds for $n = 2$.

Now, assume that for any sequence $\{\alpha_k\}_{k=1}^{m(k)}$ belongs to Σ, $1 \leq k \leq m(k)$, $A_{n;\alpha_k}$ has property C; the goal is to show that it also holds for n+1. Firstly, we let $\{\alpha_j^{(1)}\}_{j=1}^{m}$ be a sequence in Σ and α_i be the $i-th$ term of $\{\alpha_j^{(1)}\}_{j=1}^{m}$, next we consider the following situations to show $A_{n+1;\alpha_i}$ has property C .
Case 1: $1 \leq i < m$
By condition (a), because $A_{2;\alpha_i}$ has property C, so $|C(\alpha_i)| = p$, and we denote it as $C(\alpha_i) = \{q_1, q_2, ..., q_p\}$ where $q_1, q_2, ..., q_p \in \{1, .., p\}$. i.e., $b_{\alpha_i;q_k} = 1$ for all $k \in \{1, .., p\}$. By condition (a), we have for any sequence $\{\alpha_k\}_{k=1}^{m(k)}$ belongs to Σ, $|C(\alpha_k)| = p$ and $\Sigma = \Sigma_e$. Therefore, it is trivial that there exists sequences $\{\alpha_j^{(2)}\}, \{\alpha_j^{(3)}\}, ..., \{\alpha_j^{(p)}\}$, which satisfy the following properties
(a)$\alpha_j^{(1)} = \alpha_j^{(2)} = ... = \alpha_j^{(p)}$, where $1 \leq j \leq i$,
(b)$\alpha_{i+1}^{(k)} = q_k$, for all $k \in \{1, 2, .., p\}$,
(c)$b_{\alpha_i;q_k} = 1$ for all $k \in \{1, .., p\}$,
(d)$A_{n;q_k}$ has property C for all $k \in \{1, .., p\}$.
Therefore by (4.11) $\mathrm{A}_{n+1;\alpha_i}$ has property C.
Case 2: $i = m$
Since $\{(\alpha_j)_{j=1}^{m}\}$ is an eventually periodic sequence, i.e., there exists $1 \leq M < m$ such that $\alpha_i = \alpha_m = \alpha_M$. By case 1, $A_{n+1;\alpha_M}$ has property C, i.e., $A_{n+1;\alpha_i}$ has property C.

Finally, using the same argument of case 1 and case 2, we obtain for any sequence $\{\alpha_k\}_{k=1}^{m(k)}$ belongs to Σ, $1 \leq k \leq m(k)$, $A_{n+1;\alpha_k}$ has property C for all $n \geq 2$.

In the same fashion of proof (a), we also have for any sequence $\{\beta_k\}_{k=1}^{m(k)}$ belongs to Σ', $1 \leq k \leq m(k)$, $A_{n;\beta_k}$ has property R, for all $n \geq 2$. This completes the proof of lemma 4.4. □

Next, we give $\mathbb{A}_2 \in M_{p^2 \times p^2}(\mathbb{Z})$ and write it as

$$\mathbb{A}_2 = \begin{bmatrix} A_{2;1} & \cdots & A_{2;p} \\ A_{2;p+1} & \cdots & A_{2;2p} \\ \vdots & \ddots & \vdots \\ A_{2;(p-1)p+1} & \cdots & A_{2;p^2} \end{bmatrix} = \begin{bmatrix} A_{2;11} & \cdots & A_{2;1p} \\ A_{2;21} & \cdots & A_{2;2p} \\ \vdots & \ddots & \vdots \\ A_{2;p1} & \cdots & A_{2;pp} \end{bmatrix} \text{ where } A_{2;\alpha} = \begin{bmatrix} b_{\alpha;1} & \cdots & b_{\alpha;p} \\ b_{\alpha;p+1} & \cdots & b_{\alpha;2p} \\ \vdots & \cdots & \vdots \\ b_{\alpha;p(p-1)+1} & \cdots & b_{\alpha;p^2} \end{bmatrix}, \ \alpha \in \{1, 2, ..., p^2\}.$$

And, we follow the recursive formulae for n-th order transition matrices $\mathbb{A}_n$ from (4.1). Then we prove the following Theorem.

Theorem 4.5. Given $\mathbb{A}_2 \in M_{p^2\times p^2}(\mathbb{Z})$, where $(A_{2;11})_{1j} = (A_{2;11})_{i1} = 1$, for all $i, j \in \{1, 2, .., p\}$. If $\mathbb{A}_2$ satisfies the following properties
(a) Every sequence $\{\alpha_k\}_{k=1}^{m(k)}$ in Σ, $A_{2;\alpha_k}$ has property C, $\forall\ 1 \leq k \leq m(k)$.
(b) Every sequence $\{\beta_k\}_{k=1}^{m(k)}$ in Σ', $A_{2;\beta_k}$ has property R, $\forall\ 1 \leq k \leq m(k)$.
Then $\mathbb{A}_n$ is primitive for all $n \geq 2$.

Proof. Firstly, for matrix multiplication, the indices of $A_{n;\alpha}$ are conveniently expressed as $A_{n;j_1j_2}$. Clearly, $A_{n;\alpha} = A_{n;j_1j_2}$, where

$$\alpha = \alpha(j_1, j_2) = 2(j_1 - 1) + j_2. \tag{4.13}$$

Next, we divide this proof into three steps.
Step 1: Because $(A_{2;1})_{1j} = 1$ for all $j \in \{1, 2, .., p\}$, so we have $C(1) = \{1, 2, .., p\}$, and it is trivial that there exists sequences $\{\alpha_j^{(k)}\}_{j=1}^{m(k)}$ which belong to Σ and satisfy $\alpha_1^{(k)} = 1$, $\alpha_2^{(k)} = k$, where $k = 1, 2, .., p$. Therefore, by condition (a) and lemma 4.4, $A_{n;\alpha}$ has property C for all $\alpha \in \{1, 2, .., p\}$, $n \geq 2$. By (4.13), we have $A_{n;1j}$ has property C, for all $j \in \{1, 2, ..., p\}$.
Step 2: Because $(A_{2;1})_{i1} = 1$ for all $i \in \{1, 2, .., p\}$, so we have $R(1) = \{1+(k-1)p \mid k \in \{1, 2, .., p\}\}$, then there exists sequences $\{\beta_j^{(k)}\}_{j=1}^{m(k)}$ which belong to Σ' and satisfy $\beta_1^{(k)} = 1$, $\beta_2^{(k)} = 1 + (k-1)p$, where $k = 1, 2, .., p$. Therefore, by condition (b) and lemma 4.4, $A_{n;\beta}$ has property R for all $\beta \in \{1 + (k-1)p \mid k \in \{1, 2, .., p\}\}$, $n \geq 2$. By (4.13), we have $A_{n;i1}$ has property R, for all $i \in \{1, 2, ..., p\}$.
Step 3: The goal is to show that there exists $k(n)$ such that $A_{n;11}^{k(n)} \geq E$ for all $n \geq 2$. This imply there exists an integer $k(n)$ and indices $i_0 = i_1 = ... = i_{k(n)} = 1$ such that

$$\prod_{i=1}^{k(n)} A_{i_{j-1},i_j} \geq E. \tag{4.14}$$

From (4.11) and (4.13), we have $A_{n+1;1} = A_{n+1;11}$, so to show that

$$A_{n+1;11}^{k(n+1)} \geq E \tag{4.15}$$

is equivalent to show that

$$\begin{bmatrix} \mathrm{A}_{n;11} & \mathrm{A}_{n;12} & \cdots & \mathrm{A}_{n;1p} \\ \mathrm{A}_{n;21} & b_{1;p+2}\mathrm{A}_{n;22} & \cdots & b_{1;2p}\mathrm{A}_{n;2p} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{A}_{n;p1} & b_{1;p(p-1)+2}\mathrm{A}_{n;p2} & \cdots & b_{1;p^2}\mathrm{A}_{n;pp} \end{bmatrix}^{k(n+1)} \geq E \tag{4.16}$$

We prove (4.15) by induction on n.
When n=1, we choose $k(2) = 2$, it is trivial that $A^2_{2;11} \geq E$.
When n=2, since $A^2_{2;11} \geq E$, $A_{2;1j}$ has property C and $A_{2;i1}$ has property R for all i , j $\in \{1, 2, 3, .., p\}$, by lemma 4.3 and (4.16) there exists $k(3)$ such that $A^{k(3)}_{3;11} \geq E$. Now, assume that holds for n, the goal is to show that it also holds for n+1. Since $A^{k(n)}_{n;11} \geq E$, $A_{n;1j}$ has property C and $A_{n;i1}$ has property R for all i j $\in \{1, 2, 3, .., p\}$, by lemma 4.3 and (4.16) there exists $k(n+1)$ such that $A^{k(n+1)}_{n+1;11} \geq E$.

Finally, by step 1, step 2, step 3 and lemma 4.3, we have $\mathbb{A}_n$ is primitive for all $n \geq 2$. This completes the proof of Theorem of 4.5 . □

Example 4.6. Consider $\mathbb{A}_2 = \begin{bmatrix} 1&1&1&1&1&1&1&1&0\\ 1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&1&1&1\\ 1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0 \end{bmatrix}$

By definition 4.1 we have
$\Sigma = \Sigma_e = \{\{1, 1\}, \{1, 2, 1\}, \{1, 2, 2\}, \{1, 2, 3, 1\}, \{1, 2, 3, 2\}, \{1, 2, 3, 9, 1\}, \{1, 2, 3, 9, 2\}, \{1, 2, 3, 9, 3\}, \{1, 3, 1\}, \{1, 3, 2, 1\}, \{1, 3, 2, 2\}, \{1, 3, 2, 3\}, \{1, 3, 9, 1\}, \{1, 3, 9, 2, 1\}, \{1, 3, 9, 2, 2\}\{1, 3, 9, 2, 3\}, \{1, 3, 9, 3\}\}$ and $\Sigma' = \Sigma'_e = \{\{1, 1\}, \{1, 4, 1\}, \{1, 4, 4\}, \{1, 4, 7, 1\}, \{1, 4, 7, 4\}, \{1, 4, 7, 7\}, \{1, 7, 1\}, \{1, 7, 4, 1\}, \{1, 7, 4, 4\}, \{1, 7, 4, 7\}, \{1, 7, 7\}\}$. From $\mathbb{A}_2$, we get $A_{2;1}$, $A_{2;2}$, $A_{2;3}$, $A_{2;9}$ have property C, and $A_{2;1}$, $A_{2;4}$, $A_{2;7}$ have property R. It is easily checked that (a) and (b) of Theorem 4.5 hold, then Theorem 4.5 is applied to show that $\mathbb{A}_n$ is primitive for all $n \geq 2$.

Corollary 4.7. Given $\mathbb{A}_2 \in M_{p^2 \times p^2}(\mathbb{Z})$, where $(A_{2;pp})_{pj} = (A_{2;pp})_{ip} = 1$, for all $i, j \in \{1, 2, .., p\}$. If $\mathbb{A}_2$ satisfies the following properties
(a) Every sequence $\{\alpha_k\}^{m(k)}_{k=1}$ in Γ, $A_{2;\alpha_k}$ has property C, $\forall\ 1 \leq k \leq m(k)$.
(b) Every sequence $\{\beta_k\}^{m(k)}_{k=1}$ in Γ', $A_{2;\beta_k}$ has property R, $\forall\ 1 \leq k \leq m(k)$.
Then $\mathbb{A}_n$ is primitive for all $n \geq 2$.

Proof. The proof is similar to Theorem 4.5, the details are omitted. □

Example 4.8. Consider $\mathbb{A}_2 = \begin{bmatrix} 0&0&1&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1\\ 1&1&1&1&1&1&1&1&1 \end{bmatrix}$

By definition 4.1 we have
$\Gamma = \{\{9,9\},\{9,7,7\},\{9,7,9\},\{9,7,8,7\},\{9,7,8,8\},\{9,7,8,9\},\{9,8,9\},\{9,8,8\},$ $\{9,8,7,7\},\{9,8,7,8\},\{9,8,7,9\}\}$ and $\Gamma' = \{\{9,9\},\{9,6,9\},\{9,6,6\},\{9,6,3,9\},$ $\{9,6,3,6\},\{9,6,3,1,3\},\{9,6,3,1,6\},\{9,6,3,1,9\},\{9,3,9\},\{9,3,6,3\},\{9,3,6,6\},$ $\{9,3,6,9\},\{9,3,1,9\},\{9,3,1,6,9\},\{9,3,1,6,6\},\{9,3,1,6,3\},\{9,3,1,3\}\}$. From $\mathbb{A}_2$, we get $A_{2;7}, A_{2;8}, A_{2;9}$ have property C, and $A_{2;1}, A_{2;3}, A_{2;6}, A_{2;9}$ have property R. It is easily checked that (a) and (b) of Corollary 4.7 hold, then Corollary 4.7 is applied to show that $\mathbb{A}_n$ is primitive for all $n \geq 2$.

Theorem 4.9. If $\mathbb{A}_2 \in M_{p^2\times p^2}(\mathbb{Z})$ satisfies the following properties
(a)There exists an integer $s \in \{1,2,..,p\}$ such that $(A_{2;ss})_{sj} = (A_{2;ss})_{is} = 1$ for all $i,j \in \{1,2,...,p\}$.
(b)$A_{2;ij}$ has property R and C for all $i,j \in \{1,2,...,p\}$.
Then $\mathbb{A}_n$ is primitive for all $n \geq 2$.

Proof. We divide this proof into two steps.

Step 1: By condition (b),(4.11) and (4.12), it is trivial that $A_{n;ij}$ has property R and C for all $i,j \in \{1,2,...,p\}$, $n \geq 2$. The details are omitted.

Step 2: The goal is to show that there exists $k(n)$, such that $A_{n;ss}^{k(n)} \geq E$. This imply there exists an integer $k(n)$, and indices $i_0 = i_1 = \cdots = i_{k(n)} = s$ such that

$$\prod_{i=1}^{k(n)} A_{i_{j-1},i_j} \geq E. \tag{4.17}$$

By condition (a), we have to show that

$$A_{n+1;ss}^{k(n+1)} \geq E \tag{4.18}$$

is equivalent to show that

$$\begin{bmatrix} a_{11}\mathrm{A}_{n;11} & \cdots & \mathrm{A}_{n;1s} & \cdots & a_{1p}\mathrm{A}_{n;1p} \\ a_{21}\mathrm{A}_{n;21} & \cdots & \mathrm{A}_{n;2s} & \cdots & a_{2p}\mathrm{A}_{n;2p} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathrm{A}_{n;s1} & \cdots & \mathrm{A}_{n;ss} & \cdots & \mathrm{A}_{n;sp} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{p1}\mathrm{A}_{n;p1} & \cdots & \mathrm{A}_{n;ps} & \cdots & a_{pp}\mathrm{A}_{n;pp} \end{bmatrix}^{k(n+1)} \geq E \tag{4.19}$$

We prove (4.18) by induction on n.
When n=1, we choose $k(2) = 2$, it is trivial that $A^2_{2;ss} \geq E$.
When n=2, since $A^2_{2;ss} \geq E$, $A_{2;sj}$ has property C and $A_{2;is}$ has property R for all $i\ ,j\ \in \{1, 2, 3, .., p\}$, by lemma 4.3 and (4.19) there exists $k(3)$ such that $A^{k(3)}_{3;ss} \geq E$. Now, assume that holds for n, the goal is to show that it also holds for n+1. Since $A^{k(n)}_{n;ss} \geq E$, $A_{n;sj}$ has property C and $A_{n;is}$ has property R for all $i\ j\ \in \{1, 2, 3, .., p\}$, by lemma 4.3 and (4.19) there exists $k(n+1)$ such that $A^{k(n+1)}_{n+1;ss} \geq E$.

Finally, by step 1, step 2 and lemma 4.3, we have $\mathbb{A}_n$ is primitive for all $n \geq 2$. This completes the proof of Theorem of 4.9 . □

Example 4.10. Consider $\mathbb{A}_2 == \begin{bmatrix} \mathrm{A}_{2;11} & \mathrm{A}_{2;12} & \mathrm{A}_{2;13} \\ \mathrm{A}_{2;21} & \mathrm{A}_{2;22} & \mathrm{A}_{2;23} \\ \mathrm{A}_{2;31} & \mathrm{A}_{2;32} & \mathrm{A}_{2;33} \end{bmatrix} = \begin{bmatrix} 1&0&0&0&1&0&1&0&0 \\ 0&1&0&1&0&0&0&1&0 \\ 0&0&1&0&0&1&0&0&1 \\ 0&1&0&0&1&0&0&1&0 \\ 1&0&0&1&1&1&1&0&0 \\ 0&0&1&0&1&0&0&0&1 \\ 0&0&1&0&1&0&0&0&1 \\ 0&1&0&1&0&0&0&1&0 \\ 1&0&0&0&0&1&1&0&0 \end{bmatrix}$

From $\mathbb{A}_2$, we have $(A_{2;22})_{2j} = (A_{2;22})_{i2} = 1$, $A_{2;ij}$ has property R and C for all $i, j \in \{1, 2, 3\}$; then Theorem 4.9 is applied to show that $\mathbb{A}_n$ is primitive for all $n \geq 2$.

References

[1] J.C. Ban, C.H. Hsu and S.S. Lin, Spatial disorder of Cellular Neural Network-with biased term, International J.of Bifurcation and Chaos, 12(2002), pp.525-53

[2] J.C. Ban, K.P. Chien, C.H. Hsu and S.S. Lin, Spatial disorder of CNN-with asymmetric output function, International J.of Bifurcation and Chaos, 11(2001), pp.2085-2095

[3] J.C. Ban and S.S. Lin Patterns Generation and Transition Matrices in Multi-Dimensional Lattices Models, submitted(2002)

[4] J.C. Ban and S.S. Lin and Y.H. Lin, Sufficient condition for the mixing property of 2-dimensional subshift of finite type, preprint(2005)

[5] J.C. Ban and S.S. Lin and C.W. Shih, Exact number of mosaic patterns in cellular neural networks, International J.of Bifurcation and Chaos, 11(2001), pp.1645-1653

[6] J.C. Ban and S.S. Lin and Y.H. Lin, Pattern generation and spatial entropy in two-dimensional lattices models, preprint(2005)

[7] P.W. Bates and A. Chmaj, A discrete convolution model for phase transitions, Arch.Rat.Mech.Anal, 150(1999), pp.281-305.

[8] J. Bell, Some threshold results for modes of myelinated nerves ,Math./biosci., 54(1981), pp.181-190.

[9] J. Bell and C. Cosner, Threshold behavior and propagation for nonlinear differential-difference systems motivated by modeling mydeling axons, Quart.Appl.Math., 42(1984), pp.1-14.

[10] R. Bellman, Introduction to matrix analysis, Mc Graw-Hill, N. Y.(1970)

[11] J.W. Cahn, Theory of crystal growth and interface motion in crystalline materials, Acta Metallurgica, 8(1960), pp.554-562

[12] L.O. Chua, K.R Crounse, M. Hasler and P. Thiran, Pattern formation properties of autonomous cellular neural networks, IEEE Trans.Circuits Systems, 42(1995), pp.757-774

[13] S.N. Chow, J.Mallet-Paret, Pattern formation and spatial chaos in lattice dynamical systems II, IEEETrans. Circuits Systems, 42(1995), pp.752-756.

[14] S.N Chow, J.Mallet-Paret and E.S. Van Vleck, Dynamics of lattice differential equations, International J.of Bifurcation and Chaos, 9(1996), pp.1605-1621.

[15] S.N. Chow, J.Mallet-Paret and E.S. Van Vleck, Pattern formation and spatial chaos in spatially discrete evolution equations, Random Comput.Dynam.,4(1996), pp.109-178.

[16] S.N. Chow and W. Shen, Dynamics in a discrete Nagugumo equation: Spatial topological chaos, SIAM J.Appl.Math, 55(1995), pp. 1764-1781.

[17] L.O.Chua, CNN: A paradigm for complexity .World Scientific Series on Nonlinear Science, Series A, 31.World Scietific , Singapore.(1998)

[18] L.O. Chua and L. Yang, Cellular neural networks: Theory, IEEE Trans. Circuits Systems, 35(1998), pp.1257-1272.

[19] L.O. Chua and L. Yang, Cellular neural networks : Applications, IEEE Trans. Circuits Systems, 35(1998), pp.1273-1290.

[20] L.O. Chua and T. Roska, the CNN paradigm, IEEE Trans. Circuits Systems, 40(1993), pp.147-156.

[21] H.E Cook, D.De Fontaine and J.E Hilliard, A model for diffusion on cubic lattices and its application to the early stages of ordering, Acta Metallurgica, 17(1969), pp. 765-773

[22] G.B Ermentrout, Stable periodic solutions to discrete and continuum arrays of weakly coupled nonlinear oscillators, SIAM J .Appl.Math., 52(1992), pp.1665-1687.

[23] G.B.Ermentrout and N.Kopell, Inhibition-produced patterning in chains of coupled nonlinear oscillators, SIAM J.APPL. Math.,54(1994), pp.478-507.

[24] G.B. Ermentrout and N. Kopell and T.L. Williams, On chains of oscillators forced at one end, SIAM J. Appl.Math.,51(1991), pp.1397-1417.

[25] T. Eveneux and J.P. Laplante, Propagation failure in arrays of coupled bistable chemical reactors, J.Phys.Chem., 96(1992), pp.4931-4934.

[26] W.J. Firth, Optical memory and spatial chaos, Phys. Rev. Lett., 61(1988), pp.329-332.

[27] M. Hillert, A solid-solution model for inhomogeneous systems, Acta Metallurgica, 9(1961), pp.525-535.

[28] J.P Keener, Propagation and its failure in coupled systems of discrete excitable cells, SIAM J. Appl. Math., 47(1987), pp.556-572.

[29] J.P Keener, The effects of discrete gap junction coupling on propagation in myocardium, J.Theor.Biol.,148(1991), pp.49-82.

[30] A.L. Kimball, A. Varghese and R.L. Winslow, Simulating cardiac sinus and atrial network dynamics on the connection machine, Phys.D, 64(1993), pp.281-298.

[31] S.S. Lin and P.J. Tsai, The mixing property of 2-dimensional of subshift of finite type, preprint(2005)

[32] D. Lind and B. Marcus, An introduction to symbolic dynamics and coding, Cambrige University Press, New York, 1995.

[33] R.S. Mackay and J.A. Sepulchre, Multistability in networks of weakly coupled bistable units, Phys.D, 82(1995), pp.243-254.

[34] J.Mallet-Paret and S.N. Chow, Pattern formation and spatial chaos in lattice dynamical systems I, IEEE Trans. Circuits Systems, 42(1995), pp.746-751.

[35] N.G. Markly and M.E. Paul, Matrix subshifts for Z^v symbolic dynamics, to be published in Proceedings of the London Mathematical Society.

[36] N.G. Markly and M.E. Paul, Maximal measures and entropy for Z^v subshifts of finite type.