國 立 交 通 大 學

資訊科學與工程研究所

博 士 論 文

連結網路上之路徑與迴圈嵌入

Path and Cycle Embedding on Interconnection Networks

研 究 生：龔自良

指導教授：梁 婷 博士

徐力行 博士

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研 究 生：龔自良　　　Student：Tzu-Liang Kung

指導教授：梁　婷 博士　　　Advisor：Dr. Tyne Liang

徐力行 博士　　　Dr. Lih-Hsing Hsu

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摘要

陣列與環是平行計算與分散式計算中兩種最基本的網路類型。一般而言，路徑與迴圈是普遍被使用來表示這兩種網路的拓樸結構。在本論文中，我們著眼於連結網路上的路徑與迴圈嵌入問題。由於網路上的任何元件隨時隨地都可能因損壞、損毀而使得整個網路無法正常運作，因此，網路的容錯能力是設計任何一個網路系統時所不能忽視的重要考量。所以，我們也深入討論與網路容錯相關的研究議題。藉由將網路抽象地以數學上的「圖」來表示，我們可以嚴謹地探討各種連結網路的特性。

首先，我們研究具有「超級容錯漢彌爾頓」性質的網路。若一個 k－正則漢彌爾頓連通網路在任意移除了 $k-2$ 個節點或連線之後仍然保有漢彌爾頓的特性，且在任意移除了 $k-3$ 個節點或連線之後仍然具有漢彌爾頓連通的特性，則我們稱此網路為「超級容錯漢彌爾頓」網路。給定 n 個具有相同節點數的 k－正則超級容錯漢彌爾頓網路，其中 $n\geq3$ 且 $k\geq4$，我們可以使用「迴圈複合」的架構來建構（$k+2$）－正則的超級容錯漢彌爾頓網路。

其次，我們研究漢彌爾頓迴圈問題的變形，文獻上稱之為「相互獨立漢彌爾頓迴圈」。給定一固定的節點作為起始點，若一個網路的任意 n 條漢彌爾頓迴圈從此固定的起始節點開始，在後續的每一個時間點上都剛好繞經不同的中間節

點，最後這 n 條漢彌爾頓迴圈又同時回到起始節點，則我們稱此 n 條漢彌爾頓迴圈具有相互獨立的特性。在本論文中，我們研究了如何在立方體、星狀圖及蝴蝶圖上嵌入相互獨立漢彌爾頓迴圈的方法。

最後，我們討論網路的條件式容錯能力。在本論文中，我們假定網路中的任一節點必須保有至少二個功能正常的相鄰節點或保有至少二個通訊功能正常的連線。在這個前提假設之下，我們針對立方體網路上的容錯式路徑嵌入問題進行深入的探討。相較於文獻中的既有的研究成果，我們證明了超立方體網路的容錯能力，在條件式損壞模型的假設之下其實是可以大幅增加的。

關鍵字：連結網路、立方體、星狀圖、蝴蝶圖、漢彌爾頓、漢彌爾頓連通、容錯、超級容錯漢彌爾頓、條件式損壞、陣列、環、迴圈嵌入、路徑嵌入

Path and Cycle Embedding on Interconnection Networks

Student: Tzu-Liang Kung Advisor: Dr. Tyne Liang
Dr. Lih-Hsing Hsu

Institute of Computer Science and Engineering
National Chiao Tung University

Abstract

In many parallel computer systems, processors are connected on the basis of interconnection networks, referred to as networks henceforth. Among various kinds of networks, linear arrays and rings are widely applied in parallel and distributed computation. In particular, paths and cycles are two topological structures commonly used to model linear arrays and rings, respectively. Therefore we investigate how to embed paths and cycles into some interconnection networks in this thesis. Because the components of a network may fail not only accidentally but frequently, it is of great importance for a network to be capable of tolerating as many faults as possible. In this thesis the fault-tolerance related issues are also concerned. With the graph representation of an interconnection network, we can discuss these issues in a formal way.

Firstly, we study a family of super fault-tolerant hamiltonian networks, namely cycle composition networks. A k-regular hamiltonian and hamiltonian connected network is super fault-tolerant hamiltonian if it remains hamiltonian after removing up to k-2 vertices and/or edges and remains hamiltonian connected after removing up to k-3 vertices and/or edges. Super fault-tolerant hamiltonian networks have an optimal flavor with regard to fault-tolerant hamiltonicity and fault-tolerant hamiltonian connectivity. For this motivation, we observe that the cycle composition is an effective framework to construct a (k+2)-regular super fault-tolerant hamiltonian network on the basis of n k-regular super fault-tolerant hamiltonian networks, containing the same number of vertices, provided that $n\geq 3$ and $k\geq 4$.

Secondly, we investigate a variant of hamiltonian cycles, namely mutually independent hamiltonian cycles, on some interconnection networks. A set of hamiltonian cycles, having the same start vertex, is said to be mutually independent if any two of these hamiltonian cycles traverse different vertices at every time step except the start-up and termination. In this thesis, we show that the maximum number of mutually independent hamiltonian cycles can be embedded onto the binary wrapped butterfly network. Moreover, embedding mutually independent hamiltonian cycles onto faulty hypercubes and onto faulty star networks are also addressed.

Next, we investigate the conditional-fault tolerance of hypercubes. There is one thing worth noting. That is, if components of a network fail independently, then it is unlikely that all failures would be close to each other. When faulty vertices are concerned, it is reasonable to require that every vertex should have at least g fault-free neighbors. Analogously, when faulty edges are concerned, it can be assumed that every vertex is still incident to at least g fault-free edges. In this thesis we first study the fault diameter of the n-cube only for g=1, and then we explore the feasibility of fault-tolerant path embedding on hypercubes when g=2.

Keywords: Interconnection network; Hypercube; Star graph; Butterfly graph; Hamiltonian; Hamiltonian connected; Fault tolerance; Super fault-tolerant hamiltonian; Conditional fault; Linear array; Ring; Cycle embedding; Path embedding.

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Chapter 1

Introduction

Many areas of human activity require enormous computational power; computer vision, robotics, air traffic control, weather prediction, stock market analysis, artificial intelligence, and numerous military applications are just few examples. The need to interconnect hundreds or more processing elements in computers solving such huge tasks in such a way that will ensure optimal network performance is paramount. Hence the interconnection network has been a critical factor affecting the system performance [24]. A multiprocessor/multicomputer/communication *interconnection network* is usually modeled as a graph, in which the vertices correspond to processors/computers, and the edges correspond to connections or communication links. Many issues, such as communication models, routing strategies, fault tolerance, reliability, fault diagnosis, etc. are intriguing around the theme of interconnection networks.

Network embedding is another interesting subject because the portability of the guest network into the host network would permit executing the guest specified algorithms on the host with as little modification as possible. By definition [43], embedding one guest network G into another host network H is a form of injective mapping, η, from the vertex set of G to the vertex set of H. An edge of G corresponds to a path of H under η. Often embedding takes cycles, paths, trees, or meshes as guest networks because these architectures are extensively applied in parallel systems. In this thesis, we mainly focus on path and cycle embedding. Before we proceed to go through the details of our research issues, we briefly introduce some graph-theoretic notions to be used later.

1.1 Graph-theoretic terminologies

Because the underlying topology of an interconnection network is modeled as a graph, we use the terms, graph and network, vertex and node, edge and link, interchangeably. Throughout this thesis, we concentrate on loopless undirected graphs. For the graph definitions and notations we follow the ones given by Hsu and Lin [30]. A graph G consists of a nonempty *vertex set* $V(G)$ and an *edge set* $E(G)$, which is a subset of $\{(u,v) \mid (u,v)$ is an unordered pair of $V(G)\}$. Two vertices, u and v, of G are adjacent if $(u,v) \in E(G)$. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let S be a nonempty subset of $V(G)$. The subgraph *induced* by S is the subgraph of G with its vertex set S and with its edge set that consists of those edges joining any two vertices in S. We use $G - S$ to denote the subgraph of G induced by $V(G) - S$. Analogously, the subgraph generated by a nonempty subset $F \subseteq E(G)$ is the subgraph of G with its edge set F and its vertex set consisting of those vertices of G incident with at least one edge of F. We use $G - F$ to denote the subgraph of G with vertex set $V(G)$ and edge set $E(G) - F$. The *degree* of a vertex u in G, denoted by $deg_G(u)$, is the number of edges incident to u. A graph G is *k-regular* if all its vertices have the same degree k. For any node u of G, its *neighborhood* $N_G(u)$ is defined by $N_G(u) = \{v \in V(G) \mid (u, v) \in E(G)\}$. A graph G is *bipartite* if its vertex set can be partitioned into two disjoint partite sets, $V_0(G)$ and $V_1(G)$, such that every edge joins a vertex of $V_0(G)$ and a vertex of $V_1(G)$.

A *matching* of size k in a graph G is a set of k edges with no shared endpoints. The vertices incident with the edges of a matching are called *saturated* by the matching; the others are unsaturated. A *perfect matching* is a matching that saturates every vertex of G. A *path* P of length k from vertex x to vertex y in a graph G is a sequence of distinct vertices $\langle v_1, v_2, \ldots, v_{k+1}\rangle$ such that $x = v_1$, $y = v_{k+1}$, and $(v_i, v_{i+1}) \in E(G)$ for every $1 \leq i \leq k$ if $k \geq 1$. More precisely, path P is represented as $\langle v_1, e_1, v_2, e_2, v_3, \ldots v_k, e_k, v_{k+1}\rangle$, where $e_i = (v_i, v_{i+1}) \in E(G)$ for every $1 \leq i \leq k$. A path of length 0, consisting of a single vertex x, is denoted by $\langle x\rangle$. For convenience, we write P as $\langle v_1, \ldots, v_i, Q, v_j, \ldots, v_{k+1}\rangle$, where $Q = \langle v_i, v_{i+1}, \ldots, v_j\rangle$. The i-th vertex of P is denoted by $P(i)$; i.e., $P(i) = v_i$. Moreover, we use P^{-1} to denote the path $\langle v_{k+1}, v_k, \ldots, v_1\rangle$. To emphasize the beginning and ending vertices of P, we also write P as $P[x, y]$. We use $\ell(P)$ to denote the length of P. For any two distinct vertices u and v of G, the distance between u and v, denoted by $d_G(u, v)$, is the length of the shortest path joining u and v in G. The diameter of G, denoted by $D(G)$, is defined to be $\max\{d_G(u, v) \mid u, v \in V(G)\}$. A *cycle* is a path with at least three vertices such that the last vertex is adjacent to the first one. For clarity, a cycle of length k is represented by $\langle v_1, v_2, \ldots, v_k, v_1\rangle$.

A path (or cycle) of a graph G is a *hamiltonian path* (or *hamiltonian cycle*) if it spans G. A graph is *hamiltonian* if it has a hamiltonian cycle. A graph is *hamiltonian connected* if there exists a hamiltonian path between every pair of distinct vertices. A bipartite graph is *hamiltonian laceable* [58] if there exists a hamiltonian path between any two vertices that are in different partite sets. Moreover, a hamiltonian laceable graph G is *hyper-hamiltonian laceable* [45] if, for any vertex $v \in V_i(G)$, there exists a hamiltonian path of $G - \{v\}$ between every pair of distinct vertices in $V_{1-i}(G)$. Later Hsieh et al. [27] introduced *strongly hamiltonian laceability*. A hamiltonian laceable graph G is strongly hamiltonian laceable if there exists a path of length $|V(G)| - 2$ between every pair of distinct vertices in the same partite set.

1.2 Some structured interconnection networks

Many interconnection networks have been proposed in research by [1, 13, 15, 18, 26, 43, 52, 55]. In this section, we introduce several of the most popular interconnection networks.

1.2.1 Hypercubes

Hypercube [55] is one of the most attractive interconnection networks already discovered for parallel computation. Not only is it ideally suited to both special-purpose and general-purpose tasks, but it can efficiently simulate many other networks [43]. The formal definition of hypercubes is given as follows.

For the sake of clarity, we use boldface letters to denote n-bit binary strings. Let $\mathbf{u} = b_{n-1} \dots b_i \dots b_0$ be an n-bit binary string. For $0 \le i \le n-1$, we use $(\mathbf{u})^i$ to denote the binary string $b_{n-1} \dots \bar{b}_i \dots b_0$. Moreover, we use $(\mathbf{u})_i$ to denote bit b_i of $\mathbf{u}$. The *Hamming weight* of $\mathbf{u}$, denoted by $w_H(\mathbf{u})$, is $|\{0 \le j \le n-1 \mid (\mathbf{u})_j = 1\}|$. The *n-dimensional hypercube* (or *n-cube* for short), Q_n, consists of 2^n nodes and $n2^{n-1}$ links. Each node corresponds to an n-bit binary string. Two nodes, $\mathbf{u}$ and $\mathbf{v}$, are adjacent if and only if $\mathbf{v} = (\mathbf{u})^i$ for some i, and we call the link $(\mathbf{u}, (\mathbf{u})^i)$ *i-dimensional*. We define $dim((\mathbf{u}, \mathbf{v})) = i$ if $\mathbf{v} = (\mathbf{u})^i$. The *Hamming distance* between two nodes $\mathbf{u}$ and $\mathbf{v}$, denoted by $h(\mathbf{u}, \mathbf{v})$, is defined to be $|\{0 \le j \le n-1 \mid (\mathbf{u})_j \ne (v)_j\}|$. Hence two nodes, $\mathbf{u}$ and $\mathbf{v}$, are adjacent if and only if $h(\mathbf{u}, \mathbf{v}) = 1$. It is well known that Q_n is a bipartite graph with partite sets $V_0(Q_n) = \{\mathbf{u} \in V(Q_n) \mid w_H(\mathbf{u}) \text{ is even}\}$ and $V_1(Q_n) = \{\mathbf{u} \in V(Q_n) \mid w_H(\mathbf{u}) \text{ is odd}\}$. Moreover, Q_n is both node-transitive and link-transitive [55].

A variety of issues on hypercubes have been addressed by many researchers [8, 23, 39, 46, 64, 66, 67]. For example, Latifi et al. [39] proved that an n-cube Q_n has a hamiltonian cycle even if it has $n-2$ faulty links. Moreover, Tseng [67] showed that a faulty n-cube, containing $f_e \le n-4$ faulty links and $f_v \le n-1$ faulty nodes with $f_e + f_v \le n-1$, has a fault-free cycle of length at least $2^n - 2f_v$. On the other hand, Tsai et al. [64] showed that Q_n $(n \ge 3)$ is both hamiltonian laceable and strongly hamiltonian laceable even if it has $n-2$ faulty links. In addition, Fu [23] investigated path embedding in an n-cube with up to $n-2$ faulty nodes.

1.2.2 Star networks

The star network was proposed by Akers and Krishnameurthy [1], as an attractive alternative to the n-cube topology for interconnecting processors in parallel computers. It can be defined as follows. Let n be a positive integer, and let $\langle n \rangle = \{1, \dots, n\}$. The *n-dimensional star network*, denoted by S_n, is a graph with vertex set $V(S_n) = \{u_1 \dots u_n \mid u_i \in \langle n \rangle \text{ and } u_i \ne u_j \text{ for } i \ne j\}$. Its adjacency is described as follows: a vertex $\mathbf{u} = u_1 \dots u_i \dots u_n$ is adjacent to another vertex $\mathbf{v} = v_1 \dots v_i \dots v_n$ through an edge of dimension i, $2 \le i \le n$, if $u_1 = v_i$, $v_1 = u_i$, and $u_j = v_j$ for $j \in \langle n \rangle - \{1, i\}$. By such definition, S_n is an $(n-1)$-regular graph with $n!$ vertices. Moreover, it is both vertex-transitive and edge-transitive [1]. Three star networks, S_2, S_3, and S_4, are illustrated in Figure 1.1.

The star network has also received many researchers' attention due to its nice topological properties. For example, the diameter and fault diameters were computed in [1, 40, 54]. The hamiltonian properties of star graphs were studied in [21, 22, 27, 36, 47, 68]. In particular, Fragopoulou and Akl [21, 22] studied the problem of embedding $n-1$ directed edge-disjoint

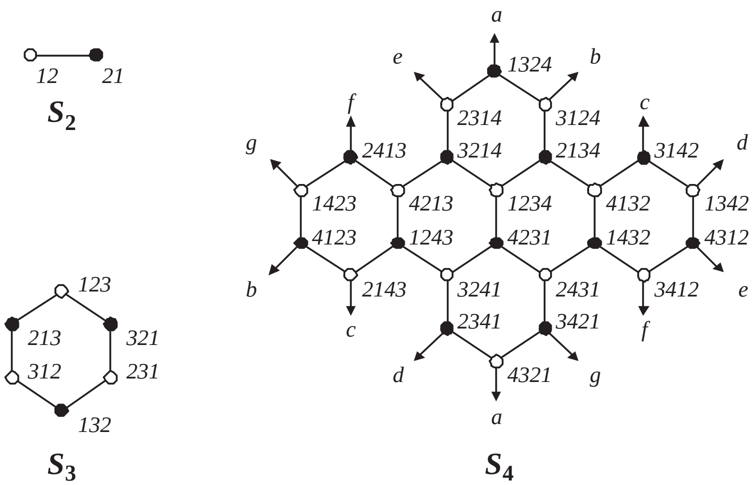

Figure 1.1: Illustrations for S_2, S_3, and S_4.

spanning trees onto an n-dimensional star network. These spanning trees could be used to design communication algorithms.

1.2.3 The binary wrapped butterfly networks

Among various kinds of popular network topologies, butterfly networks are very suitable for VLSI implementation and parallel computing. In particular, the binary wrapped butterfly graph has gained many researchers' efforts for its nice topological properties. For example, it belongs to the family of constant degree-four Cayley graphs [10, 69]. Therefore, it is vertex-transitive. In research by [25, 34, 61, 65, 70], embedding various topologies, such as rings, linear arrays, and binary trees, etc., into the butterfly networks were addressed. The definition of the binary wrapped butterfly graph is given as follows.

Let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ denote the set of integers modulo n. The n-dimensional *binary wrapped butterfly graph* (or *butterfly graph* for short) $BF(n)$ is a graph with vertex set $\mathbb{Z}_n \times \mathbb{Z}_2^n$. Each vertex is labeled by a two-tuple $\langle \ell, a_0 \ldots a_\ell \ldots a_{n-1} \rangle$ with a level $\ell \in \mathbb{Z}_n$ and an n-bit binary string $a_0 \ldots a_\ell \ldots a_{n-1} \in \mathbb{Z}_2^n$. A level-$\ell$ vertex $\langle \ell, a_0 \ldots a_\ell \ldots a_{n-1} \rangle$ is adjacent to two vertices, $\langle (\ell+1)_{\text{mod } n}, a_0 \ldots a_\ell \ldots a_{n-1} \rangle$ and $\langle (\ell-1)_{\text{mod } n}, a_0 \ldots a_{\ell-1} \ldots a_{n-1} \rangle$, by *straight edges*, and is adjacent to another two vertices, $\langle (\ell+1)_{\text{mod } n}, a_0 \ldots a_{\ell-1} \overline{a}_\ell a_{\ell+1} \ldots a_{n-1} \rangle$ and $\langle (\ell-1)_{\text{mod } n}, a_0 \ldots a_{\ell-2} \overline{a}_{\ell-1} a_\ell \ldots a_{n-1} \rangle$, by *cross edges*. More formally, the edges of $BF(n)$ can be defined in terms of four generators g, g^{-1}, f, and f^{-1} as follows [69]:

$$
\begin{aligned}
g(\langle \ell, a_0 \ldots a_\ell \ldots a_{n-1} \rangle) &= \langle (\ell+1)_{\text{mod } n}, a_0 \ldots a_\ell \ldots a_{n-1} \rangle, \\
f(\langle \ell, a_0 \ldots a_\ell \ldots a_{n-1} \rangle) &= \langle (\ell+1)_{\text{mod } n}, a_0 \ldots a_{\ell-1} \overline{a}_\ell a_{\ell+1} \ldots a_{n-1} \rangle, \\
g^{-1}(\langle \ell, a_0 \ldots a_\ell \ldots a_{n-1} \rangle) &= \langle (\ell-1)_{\text{mod } n}, a_0 \ldots a_\ell \ldots a_{n-1} \rangle, \\
f^{-1}(\langle \ell, a_0 \ldots a_{\ell-1} \ldots a_{n-1} \rangle) &= \langle (\ell-1)_{\text{mod } n}, a_0 a_1 \ldots a_{\ell-2} \overline{a}_{\ell-1} a_\ell \ldots a_{n-1} \rangle,
\end{aligned}
$$

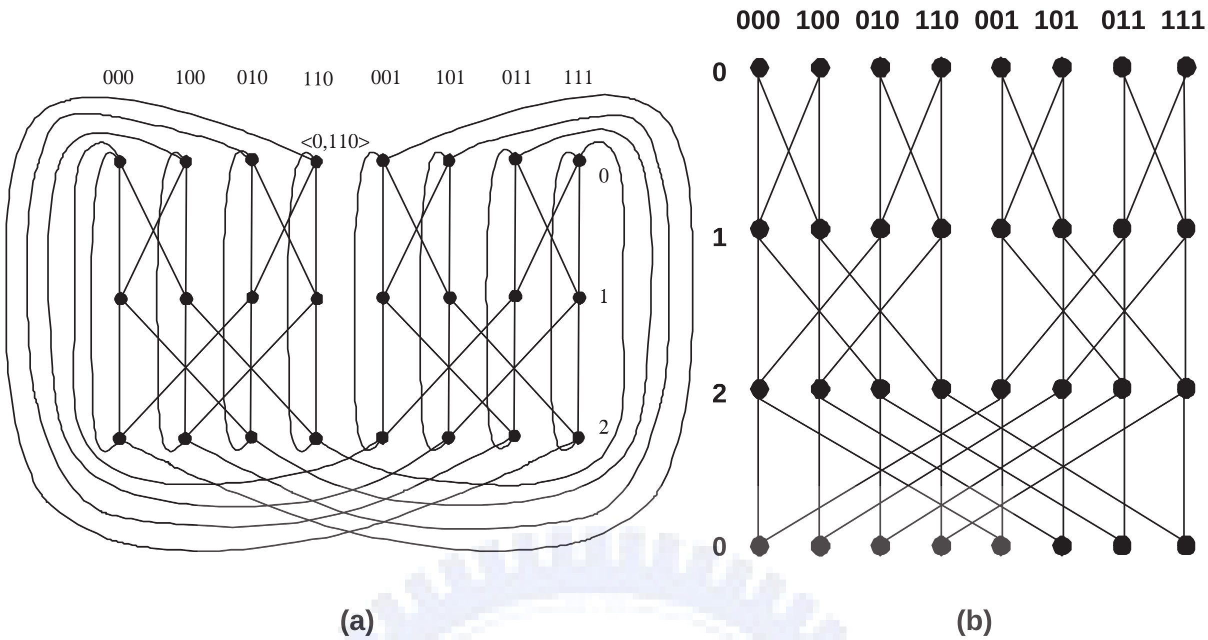

Figure 1.2: (a) $BF(3)$; (b) $BF(3)$ with level-0 vertices replicated to ease visualization.

where $\overline{a}_\ell \equiv a_\ell + 1 \pmod 2$. A level-$\ell$ edge of $BF(n)$ is an edge that joins a level-ℓ vertex and a level-$(\ell+1)_{\text{mod } n}$ vertex. To avoid the degenerate case, we concern only the case that $n \geq 3$. So, $BF(n)$ is 4-regular. Figure 1.2(a) depicts the structure of $BF(3)$, and Figure 1.2(b) is another layout of $BF(3)$ with the level-0 vertices replicated to ease visualization.

1.2.4 Cycle composition networks

The following framework, proposed by Chen et al. [12], recursively constructs a family of interconnection networks. Let $G_0, G_1, \ldots, G_{n-1}$ be n k-regular graphs with the same number of vertices. The *cycle composition network* $H = G(G_0, G_1, \ldots, G_{n-1}; M_{0,1}, M_{1,2}, \ldots, M_{n-2,n-1}, M_{n-1,0})$ is defined to be the graph with vertex set $V(H) = \bigcup_{i=0}^{n-1} V(G_i)$ and edge set $E(H) = \bigcup_{i=0}^{n-1}(E(G_i) \cup M_{i,i+1})$, where $M_{i,j}$ is an arbitrary perfect matching between the vertex set of G_i and the vertex set of G_j. It is noticed that both addition and subtraction will be taken modulo n. For convenience, we abbreviate $G(G_0, G_1, \ldots, G_{n-1}; M_{0,1}, M_{1,2}, \ldots, M_{n-2,n-1}, M_{n-1,0})$ as $G_{\langle 0,1,\ldots,n-1,0\rangle}$. See Figure 1.3 for illustration.

For instance, the k-ary n-cube, an extension of hypercubes, is constructed as a special case in this way. Many attractive topological properties of k-ary n-cubes were addressed in research [2, 5, 7, 73]. Similarly, the recursive circulant [52] are also constructed in the same fashion.

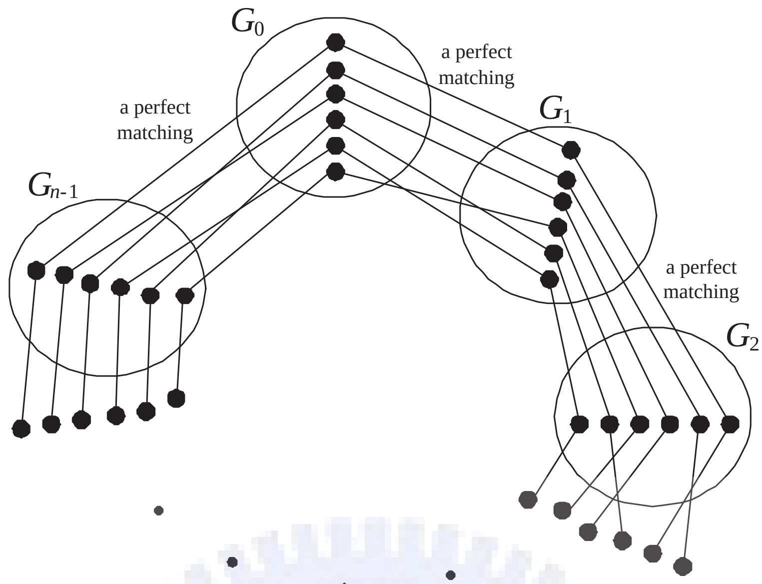

Figure 1.3: Illustration for $G_{\langle 0,1,\ldots,n-1,0\rangle}$.

1.3 Synopsis

Linear arrays and rings are two of the most fundamental networks for parallel and distributed computation. There is a wide range of efficient algorithms developed on the basis of these two topologies [43]. In particular, paths and cycles are two types of structures commonly used to model linear array and rings. Because the components of a network may fail not only accidentally but frequently, it is demanded to consider the fault-tolerance related issues on interconnection networks. For these two reasons, embedding paths and cycles into a faulty network is of crucial importance. Faults in a network may take various forms such as hardware/software errors, vertex/edge faults, etc. Throughout this thesis, vertex-faults and/or edge-faults are addressed.

First of all, we devote to fault-tolerant hamiltonian properties on cycle composition networks. A graph G is called l-fault-tolerant hamiltonian (respectively, l-fault-tolerant hamiltonian connected) if it remains hamiltonian (respectively, hamiltonian connected) after removing at most l vertices and/or edges. The fault-tolerant hamiltonicity of G, $\mathcal{H}_f(G)$, is defined to be the maximum integer l such that $G - F$ remains hamiltonian for every $F \subset V(G) \cup E(G)$ with $|F| \leq l$ if G is hamiltonian, and undefined otherwise. Obviously, $\mathcal{H}_f(G) \leq \delta(G) - 2$, where $\delta(G) = \min\{deg_G(v) \mid v \in V(G)\}$. A regular graph G is *optimal fault-tolerant hamiltonian* if $\mathcal{H}_f(G) = \delta(G) - 2$. The fault-tolerant hamiltonian connectivity of G, $\mathcal{H}_f^\kappa(G)$, is defined to be the maximum integer l such that $G - F$ remains hamiltonian connected for every $F \subset V(G) \cup E(G)$ with $|F| \leq l$ if G is hamiltonian connected, and undefined otherwise. Obviously, $\mathcal{H}_f^\kappa(G) \leq \delta(G) - 3$. A regular graph G is *optimal fault-tolerant hamiltonian connected* if $\mathcal{H}_f^\kappa(G) = \delta(G) - 3$. We say a regular graph G is *super* *fault-tolerant hamiltonian* if $\mathcal{H}_f(G) = \delta(G) - 2$ and $\mathcal{H}_f^\kappa(G) = \delta(G) - 3$. For instance, twisted-cubes, crossed-cubes, möbius cubes, and recursive circulant graphs are all super fault-tolerant hamiltonian [11, 31–33, 63]. Let $G_0, G_1, \ldots, G_{n-1}$ be n k-regular super fault-tolerant hamiltonian graphs with the same number of vertices. Then Chen et al. [12] proved that the cycle composition network $G_{\langle 0,1,\ldots,n-1,0\rangle}$ is super fault-tolerant hamiltonian, provided that $n \geq 3$ and $k \geq 5$. In this thesis, we will improve the previous result by showing that $G_{\langle 0,1,\ldots,n-1,0\rangle}$ is still super fault-tolerant hamiltonian even when $k = 4$.

Secondly, we investigate a variant of hamiltonian cycles, namely mutually independent hamiltonian cycles, on some interconnection networks. The mutually independent hamiltonian cycles are defined as follows [49, 59]. Let G be a graph with N vertices. A hamiltonian cycle C of G is described by $\langle u_1, u_2, \ldots, u_N, u_1\rangle$ to emphasize the order of vertices on C. Accordingly, u_1 is referred to as the beginning vertex. Two hamiltonian cycles of G beginning from a given vertex s, namely $C_1 = \langle u_1, u_2, \ldots, u_N, u_1\rangle$ and $C_2 = \langle v_1, v_2, \ldots, v_N, v_1\rangle$, are *independent* if $u_1 = v_1 = s$ and $u_i \neq v_i$ for $2 \leq i \leq N$. We say a set of m hamiltonian cycles $\{C_1, \ldots, C_m\}$ of G, beginning from the same vertex, is *m-mutually independent* if C_i and C_j are independent whenever $i \neq j$. In this thesis, we show that the maximum number of mutually independent hamiltonian cycles can be embedded onto the binary wrapped butterfly network. In particular, fault-tolerant embedding of mutually independent hamiltonian cycles onto faulty hypercubes and faulty star networks are also addressed.

Next, we turn our attention to fault distributions. It is worth noting that, if components of a network fail independently, then the likelihood that all failures would be close to each other becomes low. Motivated by this observation, Esfahanian [20] introduced the concept of *forbidden faulty sets*. The components of any forbidden faulty set cannot be faulty at the same time. In particular, for the n-cube, he has defined each forbidden faulty set to consist of all n neighbors of one processor; thus, there are 2^n forbidden faulty sets for an n-cube, each containing n processors. Later Latifi et al. [42] extended such a concept by defining the *conditional node-faults* which require every node to have at least g fault-free neighbors, $g \geq 1$. In this thesis, we concentrate mainly on $g = 2$.

The condition of having at least two fault-free neighbors for every node is statistically reasonable. We give the n-cube as an example under the consideration of at most $2n - 5$ faults. Suppose, with a random fault model, the probability of node failure is identical, and nodes fail independently. Let $P_N(n)$ denote the probability that every node of an n-cube, containing $2n - 5$ faulty nodes, is adjacent to at least two fault-free neighbors. Because Q_n has 2^n nodes, there are $\binom{2^n}{2n-5}$ ways to distribute $2n - 5$ faulty nodes. In the random fault model, all these fault distributions have equal probability of occurrence. Clearly, $P_N(3) = 1$ and $P_N(4) = 1 - \frac{2^4 \times \binom{4}{3}}{\binom{2^4}{3}} = \frac{31}{35}$, where $2^4 \times \binom{4}{3}$ is the number of faulty node distributions that there exists some node having three faulty neighbors. When $n \geq 5$, the number of faulty node distributions that there exists some node having n faulty neighbors is $2^n \times \binom{2^n - n}{n-5}$. Moreover, the number of faulty node distributions that there exists some node having exactly $n - 1$

faulty neighbors is $2^n \times \binom{n}{n-1}\binom{2^n-n}{n-4}$. Since $\binom{2^n-n}{n-4} \geq \binom{2^n-n}{n-5}$ for $n \geq 5$, we can derive that

$$\begin{aligned}
P_N(n) &= 1 - Pr(\text{some node has at least } n-1 \text{ faulty neighbors}) \\
&= 1 - \frac{2^n \times \binom{2^n-n}{n-5} + 2^n \times \binom{n}{n-1}\binom{2^n-n}{n-4}}{\binom{2^n}{2n-5}} \\
&\geq 1 - \frac{2^n \times (1+n) \times \binom{2^n-n}{n-4}}{\binom{2^n}{2n-5}} \\
&= 1 - \frac{2^n \times (1+n) \times (2^n-2n+5) \times \prod_{k=n-3}^{2n-5} k}{\prod_{k=2^n-n+1}^{2^n} k} \\
&= 1 - \frac{(n-3)(n-2)}{2^n-n+1} \times \frac{n-1}{2^n-n} \times \ldots \times \frac{2n-5}{2^n-3} \times \frac{n+1}{2^n-2} \times \frac{2^n-2n+5}{2^n-1} \triangleq L(n).
\end{aligned}$$

It is not difficult to compute $P_N(n)$ numerically, such as $P_N(5) = \frac{6157}{6293}$, $P_N(6) = \frac{9696527}{9706503}$, etc. Since $\lim_{n\to\infty} L(n) = 1$, $P_N(n)$ approaches to 1 as n increases. Under the condition of requiring every node to have at least two fault-free neighbors, we will explore the feasibility of embedding paths, as long as possible, into hypercubes if there are utmost $2n-5$ conditional node-faults.

On the other hand, *conditional link-faults*, which require that every node of a network will be incident to at least two fault-free links, can be addressed as well. This condition is also meaningful. Let $P_L(n)$ denote the probability that every node of an n-cube containing $2n-5$ faulty links is incident to at least two fault-free links. Suppose the probability of link failure is identical, and links fail independently. Then $P_L(n)$ can be computed as follows:

$$P_L(n) = \begin{cases} 1 & \text{if } n = 3, \\ 1 - \frac{2^n \times \binom{n}{2n-5}}{\binom{n\times 2^{n-1}}{2n-5}} & \text{if } n = 4, \\ 1 - \frac{2^n \times \binom{n\times 2^{n-1}-n}{n-5} + 2^n \times \binom{n}{n-1}\binom{n\times 2^{n-1}-n}{n-4}}{\binom{n\times 2^{n-1}}{2n-5}} & \text{if } n \geq 5. \end{cases}$$

Then $P_L(n)$ approaches to 1 as n increases. Accordingly, it is also intriguing to consider path embedding on hypercubes with conditional link-faults.

The rest of this thesis is organized as follows. In Chapter 2, we improve the result of Chen et al. [12] by showing that a cycle composition network is still super fault-tolerant hamiltonian even if it is constructed from a collection of 4-regular super fault-tolerant hamiltonian graphs. In Chapter 3 and Chapter 4, we study the problem of embedding mutually independent hamiltonian cycles onto butterfly graphs, faulty hypercubes, and faulty star networks, respectively. The fault diameter of a conditionally faulty n-cube, with hybrid node and link faults, is studied in Chapter 5. In Chapter 6 and Chapter 7, we investigate fault-tolerant path embedding in hypercubes with conditional link-faults and conditional node-faults, respectively. Finally, the concluding remarks are presented in Chapter 8.

Chapter 2

Fault-tolerant Hamiltonian Connectedness of Cycle Composition Networks

A suitable network is generally designed to satisfy some specified requirements. For example, the hamiltonian property is one of the major concerns for designing the network topology, and fault tolerance is desirable in massive parallel systems. So these two properties can be concerned simultaneously. A graph G is called l-fault-tolerant hamiltonian (respectively, l-fault-tolerant hamiltonian connected) if it remains hamiltonian (respectively, hamiltonian connected) after removing at most l vertices and/or edges. The fault-tolerant hamiltonicity of G, $\mathcal{H}_f(G)$, is defined to be the maximum integer l such that $G-F$ remains hamiltonian for every $F \subset V(G) \cup E(G)$ with $|F| \leq l$ if G is hamiltonian, and undefined otherwise. Obviously, $\mathcal{H}_f(G) \leq \delta(G) - 2$, where $\delta(G) = \min\{deg_G(v) \mid v \in V(G)\}$. A regular graph G is *optimal fault-tolerant hamiltonian* if $\mathcal{H}_f(G) = \delta(G) - 2$. The fault-tolerant hamiltonian connectivity of G, $\mathcal{H}_f^\kappa(G)$, is defined to be the maximum integer l such that $G - F$ remains hamiltonian connected for every $F \subset V(G) \cup E(G)$ with $|F| \leq l$ if G is hamiltonian connected, and undefined otherwise. Obviously, $\mathcal{H}_f^\kappa(G) \leq \delta(G) - 3$. A regular graph G is *optimal fault-tolerant hamiltonian connected* if $\mathcal{H}_f^\kappa(G) = \delta(G) - 3$. We say a regular graph G is said to be *super fault-tolerant hamiltonian* if $\mathcal{H}_f(G) = \delta(G) - 2$ and $\mathcal{H}_f^\kappa(G) = \delta(G) - 3$. For instance, twisted-cubes, crossed-cubes, möbius cubes, and recursive circulant graphs are all super fault-tolerant hamiltonian [11, 31–33, 63].

A network will have higher fault-tolerant capability if it is super fault-tolerant hamiltonian. With such motivation Chen et al. [12] proposed a systematic framework to recursively construct super fault-tolerant hamiltonian graphs. Let $G_0, G_1, \ldots, G_{n-1}$ be n k-regular super fault-tolerant hamiltonian graphs with the same number of vertices. The *cycle composition network* $H = G(G_0, G_1, \ldots, G_{n-1};\ M_{0,1}, M_{1,2}, \ldots, M_{n-2,n-1}, M_{n-1,0})$ is defined to be the graph with vertex set $V(H) = \bigcup_{i=0}^{n-1} V(G_i)$ and edge set $E(H) = \bigcup_{i=0}^{n-1}(E(G_i) \cup M_{i,i+1})$, where $M_{i,j}$ denotes an arbitrary perfect matching between $V(G_i)$ and $V(G_j)$. See Figure 1.3. It is noted that both addition and subtraction will be considered modulo n. Then Chen et al. [12] showed that $G(G_0, G_1, \ldots, G_{n-1}; M_{0,1}, M_{1,2}, \ldots, M_{n-2,n-1}, M_{n-1,0})$, abbreviated as

$G_{\langle 0,1,\ldots,n-1,0\rangle}$ henceforth, is super fault-tolerant hamiltonian for $n \geq 3$ and $k \geq 5$.

Theorem 2.1. *[12] Assume $n \geq 3$ and $k \geq 5$. Let $G_0, G_1, \ldots, G_{n-1}$ be n k-regular super fault-tolerant hamiltonian graphs with the same number of vertices. For any $0 \leq i \leq n-1$, let $M_{i,i+1}$ be a perfect matching between $V(G_i)$ and $V(G_{i+1})$. Then $G_{\langle 0,1,\ldots,n-1,0\rangle}$ is $(k+2)$-regular super fault-tolerant hamiltonian.*

For example, the recursive circulant graph, which was proposed by Park and Chwa [52], is essentially constructed as a special case in this way, and it is shown to be super fault-tolerant hamiltonian under a certain condition [63]. Similarly, k-ary n-cubes are also constructed using this framework [73]. In this chapter, we will improve Theorem 2.1 by showing that $G_{\langle 0,1,\ldots,n-1,0\rangle}$ is still super fault-tolerant hamiltonian even when $k = 4$. Such an extension is significant because only the remaining case of $k = 3$ needs to be concerned carefully or to be checked by computer, while the topological properties of cycle composition networks are investigated.

2.1 Fault-tolerant hamiltonicity

For the ease of exposition, the notations we use in this chapter are described in advance. We denote the graph $G(G_i, G_{i+1}, \ldots, G_j; M_{i,i+1}, M_{i+1,i+2}, \ldots, M_{j-1,j})$ by $G_{\langle i,i+1,\ldots,j\rangle}$. Let u be a vertex of G_i. We use $(u)^-$ to denote the vertex of G_{i-1} such that $((u)^-, u) \in M_{i-1,i}$, and use $(u)^+$ to denote the vertex of G_{i+1} such that $(u, (u)^+) \in M_{i,i+1}$. Hence we have $u = ((u)^-)^+ = ((u)^+)^-$. Moreover, all additions and subtractions are considered modulo n. In order to prove the main results, we need the following lemmas.

Lemma 2.1. *Assume that $n \geq 1$. Let $G_0, G_1, \ldots, G_{n-1}$ be n 4-regular super fault-tolerant hamiltonian graphs with the same number of vertices. For any $0 \leq i \leq n-2$, let $M_{i,i+1}$ be a perfect matching between $V(G_i)$ and $V(G_{i+1})$. Moreover, let $F_i \subseteq V(G_i) \cup E(G_i)$ with $|F_i| \leq 1$ for every $0 \leq i \leq n-1$ and let $X_{i,i+1} \subseteq M_{i,i+1}$ with $|X_{i,i+1}| \leq 1$ such that $|F_i| + |F_{i+1}| + |X_{i,i+1}| \leq 2$ is satisfied for all $0 \leq i \leq n-2$. Let u and v be two vertices of $G_0 - F_0$. Then there exists a hamiltonian path of $G_{\langle 0,\ldots,n-1\rangle} - ((\bigcup_{i=0}^{n-1} F_i) \cup (\bigcup_{i=0}^{n-2} X_{i,i+1}))$ joining u to v.*

Proof. For convenience, let $F = (\bigcup_{i=0}^{n-1} F_i) \cup (\bigcup_{i=0}^{n-2} X_{i,i+1})$. We prove this lemma by induction on n. Obviously, the result is trivial when $n = 1$. For any $n \geq 2$, suppose that the result holds for $n-1$. Depending on the value of $|V(G_0)|$, two cases are distinguished.

Case 1: Suppose that $|V(G_0)| = 5$. Thus G_0 is isomorphic to the complete graph of five vertices, K_5. Firstly, we assume $|F_0| = 0$. Since $|F_0| + |F_1| + |X_{0,1}| \leq 2$, we can choose two vertices x, y of G_0 such that $|\{x, y\} \cap \{u, v\}| \leq 1$ and $|F \cap \{(x)^+, (y)^+, (x, (x)^+), (y, (y)^+)\}| = 0$. Accordingly, we can construct a hamiltonian path $P = \langle u, P_1, x, y, P_2, v\rangle$ of G_0, where P_1 or P_2 may be a path of length 0. On the other hand, assume that $|F_0| = 1$. Since G_0 is 4-regular super fault-tolerant hamiltonian, there exists a hamiltonian path P of $G_0 - F_0$ joining u to v. Since $|F_0| + |F_1| + |X_{0,1}| \leq 2$ and $|F_0| = 1$, there exists an edge (x, y)

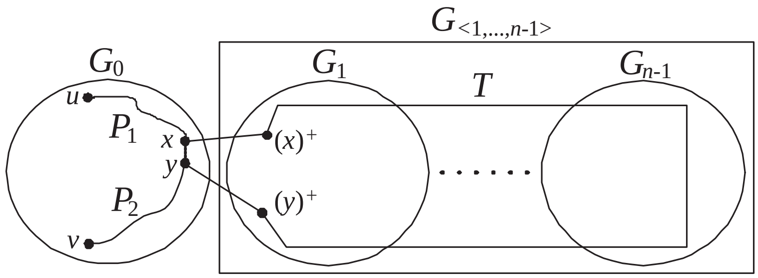

Figure 2.1: Illustration for Lemma 2.1.

on P such that $|F \cap \{(x)^+, (y)^+, (x,(x)^+), (y,(y)^+)\}| = 0$. Accordingly, we write $P = \langle u, P_1, x, y, P_2, v\rangle$, where P_1 or P_2 may be a path of length 0. By induction hypothesis, there exists a hamiltonian path T of $G_{\langle 1,\ldots,n-1\rangle} - ((\bigcup_{i=1}^{n-1} F_i) \cup (\bigcup_{i=1}^{n-2} X_{i,i+1}))$ joining $(x)^+$ to $(y)^+$. Then $\langle u, P_1, x, (x)^+, T, (y)^+, y, P_2, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1\rangle} - F$ joining u to v. See Figure 2.1 for illustration.

Case 2: Suppose that $|V(G_0)| \geq 6$. Since G_0 is super fault-tolerant hamiltonian, there exists a hamiltonian path P of $G_0 - F_0$ joining u to v. Since $|F_0|+|F_1|+|X_{0,1}| \leq 2$, there exists an edge (x,y) on P such that $|F \cap \{(x)^+, (y)^+, (x,(x)^+), (y,(y)^+)\}| = 0$. Accordingly, we write $P = \langle u, P_1, x, y, P_2, v\rangle$, where P_1 or P_2 may be a path of length 0. By induction hypothesis, there exists a hamiltonian path T of $G_{\langle 1,\ldots,n-1\rangle} - ((\bigcup_{i=1}^{n-1} F_i) \cup (\bigcup_{i=1}^{n-2} X_{i,i+1}))$ joining $(x)^+$ to $(y)^+$. Then $\langle u, P_1, x, (x)^+, T, (y)^+, y, P_2, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1\rangle} - F$ joining u to v. $\square$

Lemma 2.2. *Assume that $n \geq 1$. Let $G_0, G_1, \ldots, G_{n-1}$ be n 4-regular super fault-tolerant hamiltonian graphs with the same number of vertices. For any $0 \leq i \leq n-2$, let $M_{i,i+1}$ be a perfect matching between $V(G_i)$ and $V(G_{i+1})$. Moreover, let $F_i \subseteq V(G_i) \cup E(G_i)$ with $|F_i| \leq 1$ for every $0 \leq i \leq n-1$ and let $X_{i,i+1} \subseteq M_{i,i+1}$ with $|X_{i,i+1}| \leq 1$ for every $0 \leq i \leq n-2$ such that $|F_i| + |F_{i+1}| + |X_{i,i+1}| \leq 2$ is satisfied for all $0 \leq i \leq n-2$. Let u be a vertex of $G_0 - F_0$, and let v be a vertex of $G_t - F_t$ with $t \geq 0$. Then there exists a hamiltonian path of $G_{\langle 0,\ldots,n-1\rangle} - ((\bigcup_{i=0}^{n-1} F_i) \cup (\bigcup_{i=0}^{n-2} X_{i,i+1}))$ joining u to v.*

Proof. For convenience, let $F = (\bigcup_{i=0}^{n-1} F_i) \cup (\bigcup_{i=0}^{n-2} X_{i,i+1})$. When $t = 0$, the statement follows from Lemma 2.1. Hence we suppose $t > 0$ in the following. Since G_t is 4-regular, we have $|V(G_t)| \geq 5$. Moreover, since $|F_{t-1}| + |F_t| + |X_{t-1,t}| \leq 2$, we can choose a vertex w of $G_t - (F_t \cup \{v\})$ such that $|F \cap \{w, (w)^-, (w,(w)^-)\}| = 0$ and $(w)^- \neq u$.

Let $y_0 = u$ and $x_{t-1} = (w)^-$. Since every G_i, $0 \leq i \leq t-1$, is 4-regular and $|F_i| + |F_{i+1}| + |X_{i,i+1}| \leq 2$, we sequentially choose a vertex x_i of $G_i - F_i$ and denote $(x_i)^+$ by y_{i+1}, such that $x_i \neq y_i$ and $|F \cap \{x_i, y_{i+1}, (x_i, y_{i+1})\}| = 0$ from $i = 0$ to $i = t-3$ if $t \geq 3$. Next, we choose a vertex x_{t-2} of $G_{t-2} - (F_{t-2} \cup \{y_{t-2}\})$ and denote $(x_{t-2})^+$ by y_{t-1} such that $|F \cap \{x_{t-2}, y_{t-1}, (x_{t-2}, y_{t-1})\}| = 0$ and $y_{t-1} \neq x_{t-1}$ if $t \geq 2$. Since every G_i, $0 \leq i \leq t-1$, is super fault-tolerant hamiltonian, there exists a hamiltonian path P_i of $G_i - F_i$ joining y_i to x_i.

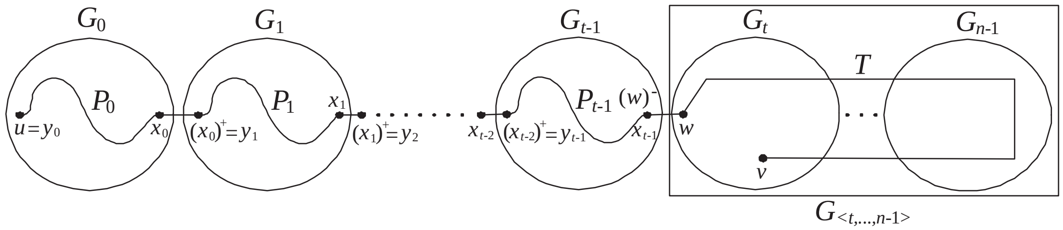

Figure 2.2: Illustration for Lemma 2.2.

By Lemma 2.1, there exists a hamiltonian path T of $G_{\langle t,\ldots,n-1\rangle} - ((\bigcup_{i=t}^{n-1} F_i) \cup (\bigcup_{i=t}^{n-2} X_{i,i+1}))$ joining w to v. Then $\langle u = y_0, P_0, x_0, (x_0)^+ = y_1, \ldots, x_{t-2}, (x_{t-2})^+ = y_{t-1}, P_{t-1}, x_{t-1} = (w)^-, w, T, v\rangle$ is a hamiltonian path of $G_{\langle 0,\ldots,n-1\rangle} - F$ joining u to v. See Figure 2.2 for illustration. □

Using Lemma 2.2, we prove the following theorem.

Theorem 2.2. *Assume that $n \geq 3$. Let $G_0, G_1, \ldots, G_{n-1}$ be n 4-regular super fault-tolerant hamiltonian graphs with the same number of vertices. For any $0 \leq i \leq n-1$, let $M_{i,i+1}$ be a perfect matching between $V(G_i)$ and $V(G_{i+1})$. Then $G_{\langle 0,1,\ldots,n-1,0\rangle}$ is optimal fault-tolerant hamiltonian.*

Proof. Obviously, $G_{\langle 0,1,\ldots,n-1,0\rangle}$ is 6-regular. Thus we are going to show that it is 4-fault-tolerant hamiltonian. Let F be a faulty set of $G_{\langle 0,1,\ldots,n-1,0\rangle}$ with $|F| \leq 4$. For convenience, let $F_i = F \cap (V(G_i) \cup E(G_i))$ for $0 \leq i \leq n-1$. Without loss of generality, we assume that $|F_0| \geq |F_i|$ for all $1 \leq i \leq n-1$. Depending on the value of $|F_0|$, five cases are distinguished.

Case 1: Suppose that $|F_0| = 4$. Let $F_0 = \{f_1, f_2, f_3, f_4\}$. Since G_0 is 2-fault-tolerant hamiltonian, there is a hamiltonian cycle C in $G_0 - \{f_3, f_4\}$.

Subcase 1.1: Suppose that both f_1 and f_2 are on C but they are not adjacent. Thus, we can write $C = \langle x_1, f_1, y_1, H_1, x_2, f_2, y_2, H_2, x_1\rangle$, where H_1 or H_2 may be a path of length 0. By Lemma 2.2, there exists a hamiltonian path $S_1[(x_1)^-, (y_1)^-]$ in G_{n-1}, and there exists a hamiltonian path $S_2[(x_2)^+, (y_2)^+]$ in $G_{\langle 1,\ldots,n-2\rangle}$. Then $\langle x_1, (x_1)^-, S_1, (y_1)^-, y_1, H_1, x_2,$ $(x_2)^+, S_2, (y_2)^+, y_2, H_2, x_1\rangle$ is a hamiltonian cycle of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$. See Figure 2.3(a) for illustration.

Subcase 1.2: Suppose that both f_1 and f_2 are on C, and they are adjacent. Thus we write $C = \langle x, R, y, f_1, f_2, x\rangle$. By Lemma 2.2, there exists a hamiltonian path H of $G_{\langle 1,\ldots,n-1\rangle}$ joining $(y)^+$ to $(x)^+$. Then $\langle x, R, y, (y)^+, H, (x)^+, x\rangle$ is a hamiltonian cycle of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$. See Figure 2.3(b) for illustration.

Subcase 1.3: Suppose that either f_1 or f_2 is on C. Without loss of generality, we assume that f_1 is on C. Thus we write C as $\langle x, R, y, f_1, x\rangle$. Then a hamiltonian cycle of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ can be formed in the same way as that used in Subcase 1.2.

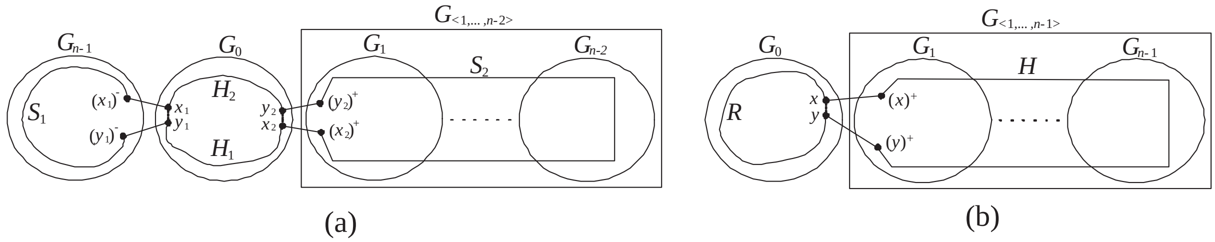

Figure 2.3: Illustration for Case 1 of Theorem 2.2.

Subcase 1.4: Suppose that neither f_1 nor f_2 is on C. Therefore we write C as $\langle x, R, y, x\rangle$ with any edge $(x, y) \in E(C)$. Then a hamiltonian cycle of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ can be formed in the same way as that used in Subcase 1.2.

Case 2: Suppose that $|F_0| = 3$. Let $F_0 = \{f_1, f_2, f_3\}$. Since G_0 is 2-fault-tolerant hamiltonian, there exists a hamiltonian cycle C in $G_0 - \{f_2, f_3\}$. Hence we have either $f_1 \notin V(C) \cup E(C)$ or $f_1 \in V(C) \cup E(C)$. Accordingly, we write $C = \langle x, R, y, x\rangle$ by picking any edge (x, y) on C if $f_1 \notin V(C) \cup E(C)$; we write $C = \langle x, R, y, f_1, x\rangle$ if f_1 is on C. Let $F' = F - F_0$. Since $|F| \leq 4$ and $|F_0| = 3$, $|F'| \leq 1$. Moreover, we have either $|\{(x)^+, (y)^+, (x, (x)^+), (y, (y)^+)\} \cap F| = 0$ or $|\{(x)^-, (y)^-, (x, (x)^-), (y, (y)^-)\} \cap F| = 0$. With symmetry, we assume that $|\{(x)^+, (y)^+, (x, (x)^+), (y, (y)^+)\} \cap F| = 0$. By Lemma 2.2, there exists a hamiltonian path H of $G_{\langle 1,\ldots,n-1\rangle} - F'$ joining $(y)^+$ to $(x)^+$. Then $\langle x, R, y, (y)^+, H, (x)^+, x\rangle$ is a hamiltonian cycle of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$.

Case 3: Suppose that $|F_0| = 2$ and $|F_i| = 2$ with any $1 \leq i \leq n-1$. Since both G_0 and G_i are 2-fault-tolerant hamiltonian, there exists a hamiltonian cycle C in $G_0 - F_0$, and there exists a hamiltonian cycle T in $G_i - F_i$. Since every G_j, $0 \leq j \leq n-1$, is 4-regular, we have $|V(G_j)| \geq 5$.

Subcase 3.1: Suppose that $i \in \{1, n-1\}$. With symmetry, we assume that $i = 1$. Apparently, there exists a vertex u in $G_0 - F_0$ such that $(u)^+$ is in $G_1 - F_1$. Without loss of generality, we write $C = \langle u, R_1, x, u\rangle$ and $T = \langle (u)^+, y, R_2, (u)^+\rangle$ so that $(y)^+$ is different from $(x)^-$. By Lemma 2.2, there exists a hamiltonian path H of $G_{\langle 2,\ldots,n-1\rangle} - F$ joining $(x)^-$ to $(y)^+$. Then $\langle u, R_1, x, (x)^-, H, (y)^+, y, R_2, (u)^+, u\rangle$ is a hamiltonian cycle of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$. See Figure 2.4(a).

Subcase 3.2: Suppose that $i \notin \{1, n-1\}$. Obviously, there exist a vertex u in $G_0 - F_0$ and a vertex v in $G_i - F_i$ such that $(u)^+ \neq (v)^-$. Without loss of generality, we write $C = \langle u, x, R_1, u\rangle$ and $T = \langle v, R_2, y, v\rangle$ so that $(y)^+$ is different from $(x)^-$. By Lemma 2.2, there exists a hamiltonian path P_1 of $G_{\langle 1,\ldots,i-1\rangle}$ joining $(u)^+$ to $(v)^-$. Similarly, there exists a hamiltonian path P_2 of $G_{\langle i+1,\ldots,n-1\rangle}$ joining $(y)^+$ to $(x)^-$. Then $\langle u, (u)^+, P_1, (v)^-, v, R_2, y, (y)^+, P_2, (x)^-, x, R_1, u\rangle$ is a hamiltonian cycle of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$. See Figure 2.4(b) for illustration.

Case 4: Suppose that $|F_0| = 2$ and $|F_i| \leq 1$ for every $1 \leq i \leq n-1$. Since G_0 is 2-fault-tolerant hamiltonian, there exists a hamiltonian cycle C in $G_0 - F_0$. Since G_0 is

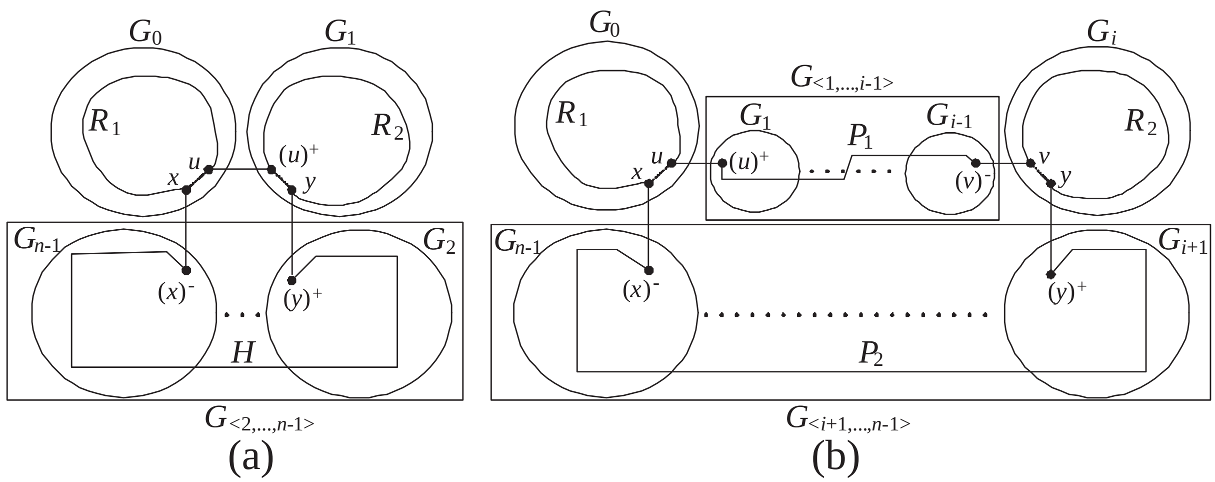

Figure 2.4: Illustration for Case 3 of Theorem 2.2.

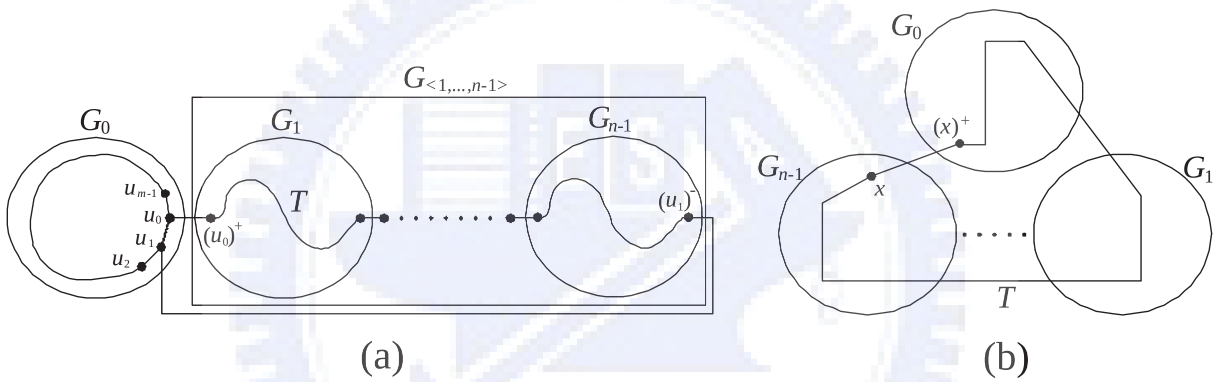

Figure 2.5: Illustration for Case 4 and Case 5 of Theorem 2.2.

4-regular, we have $|V(G_0 - F_0)| \geq 3$. For convenience, let $m = |V(G_0 - F_0)|$. Accordingly, we write $C = \langle u_0, u_1, u_2, \ldots, u_{m-1}, u_0\rangle$. Without loss of generality, we assume that $|F \cap \{(u_0)^+, (u_1)^-, (u_0, (u_0)^+), (u_1, (u_1)^-)\}| = 0$. Let $F' = F - F_0$. By Lemma 2.2, there exists a hamiltonian path T of $G_{\langle 1,\ldots,n-1\rangle} - F'$ joining $(u_0)^+$ to $(u_1)^-$. Then $\langle u_0,\, (u_0)^+,\, T,\, (u_1)^-,\, u_1, \ldots, u_{m-1},\, u_0\rangle$ is a hamiltonian cycle of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$. See Figure 2.5(a) for illustration.

Case 5: Suppose that $|F_0| \leq 1$. That is, $|F_i| \leq 1$ for all $0 \leq i \leq n-1$. For convenience, let $X_{i,i+1} = F \cap M_{i,i+1}$ for $0 \leq i \leq n-1$. Suppose that there exists an integer t of $\{0, 1, \ldots, n-1\}$ such that $|F_t| + |F_{t+1}| + |X_{t,t+1}| \geq 3$. Without loss of generality, t can be assumed to be $n-1$. Otherwise, t is fixed to be $n-1$. Accordingly, we have $|F_i| + |F_{i+1}| + |X_{i,i+1}| \leq 2$ for $0 \leq i \leq n-2$. Since $|F_{n-1}| + |F_0| + |X_{n-1,0}| \leq 4$, we can choose a vertex x of $G_{n-1} - F_{n-1}$ such that $|F \cap \{(x)^+, (x, (x)^+)\}| = 0$. Let $F' = F - X_{n-1,0}$. By Lemma 2.2, there exists a hamiltonian path T of $G_{\langle 0,1,\ldots,n-1\rangle} - F'$ joining x to $(x)^+$. Then $\langle x, T, (x)^+, x\rangle$ is a hamiltonian cycle of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$. See Figure 2.5(b) for illustration. □

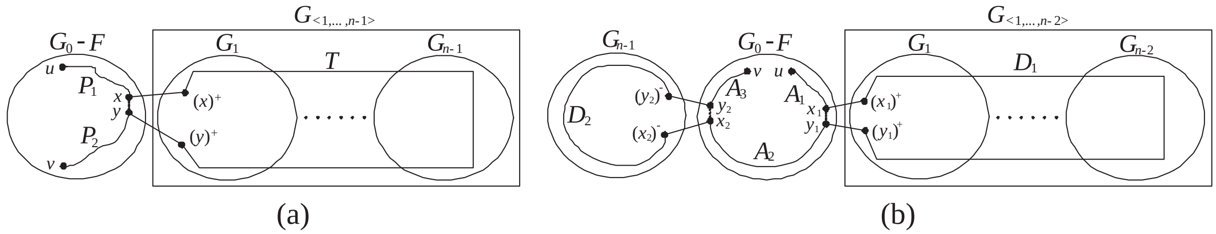

Figure 2.6: Illustration for Case 1 of Proposition 2.1.

2.2 Fault-tolerant hamiltonian connectedness

In this section, we are going to show that the cycle composition network is optimal fault-tolerant hamiltonian connected. This result is divided into three propositions.

Proposition 2.1. *Assume that $n \geq 1$. Let $G_0, G_1, \dots, G_{n-1}$ be n 4-regular super fault-tolerant hamiltonian graphs with the same number of vertices. For any $0 \leq i \leq n-1$, let $M_{i,i+1}$ be a perfect matching between $V(G_i)$ and $V(G_{i+1})$. Let F be a subset of $V(G_0) \cup E(G_0)$ with $|F| = 3$. Then $G_{\langle 0,1,\dots,n-1,0\rangle} - F$ is hamiltonian connected.*

Proof. Let $F = \{f_1, f_2, f_3\}$. Since G_0 is 2-fault-tolerant hamiltonian, there exists a hamiltonian cycle C in $G_0 - \{f_2, f_3\}$. Since G_0 is 4-regular, we have $|V(C)| \geq 3$. Let u and v be two vertices of $G_{\langle 0,1,\dots,n-1,0\rangle} - F$. Then we need to construct a hamiltonian path of $G_{\langle 0,1,\dots,n-1,0\rangle} - F$ joining u to v. The following cases are distinguished.

Case 1: Suppose that u and v are in $G_0 - F$. Since G_0 is 1-fault-tolerant hamiltonian connected, there exists a hamiltonian path H of $G_0 - \{f_3\}$ joining u to v. Suppose that f_1 and f_2 are exclusive from H. Thus we write $H = \langle u, P_1, x, y, P_2, v\rangle$ with any edge $(x, y) \in E(H)$. Suppose that either f_1 or f_2 is exclusive from H. Without loss of generality, we assume that f_2 is exclusive from H. Hence we may write $H = \langle u, P_1, x, f_1, y, P_2, v\rangle$. Suppose that both f_1 and f_2 are on H, and they are adjacent. Thus we write $H = \langle u, P_1, x, f_1, f_2, y, P_2, v\rangle$. By Lemma 2.2, there exists a hamiltonian path T of $G_{\langle 1,\dots,n-1\rangle}$ joining $(x)^+$ to $(y)^+$. Then $\langle u, P_1, x, (x)^+, T, (y)^+, y, P_2, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\dots,n-1,0\rangle} - F$ joining u to v. See Figure 2.6(a) for illustration.

Suppose that both f_1 and f_2 are on H, and they are not adjacent. Hence we may write $H = \langle u, A_1, x_1, f_1, y_1, A_2, x_2, f_2, y_2, A_3, v\rangle$. Using Lemma 2.2, we can find a hamiltonian path D_1 of $G_{\langle 1,\dots,n-2\rangle}$ joining $(x_1)^+$ to $(y_1)^+$. Similarly, there exists a hamiltonian path D_2 of G_{n-1} joining $(x_2)^-$ to $(y_2)^-$. Therefore, $\langle u, A_1, x_1, (x_1)^+, D_1, (y_1)^+, y_1, A_2, x_2, (x_2)^-, D_2, (y_2)^-, y_2, A_3, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\dots,n-1,0\rangle} - F$ joining u to v. See Figure 2.6(b) for illustration.

Case 2: Suppose that u and v are in G_i for some $1 \leq i \leq n-1$. With symmetry, we assume that $i \neq n-1$. Suppose that f_1 is on the hamiltonian cycle C of $G_0 - \{f_2, f_3\}$. Since $|V(C)| \geq 3$, we write $C = \langle x, P, y, f_1, x\rangle$. Otherwise, we write $C = \langle x, P, y, x\rangle$ with any edge

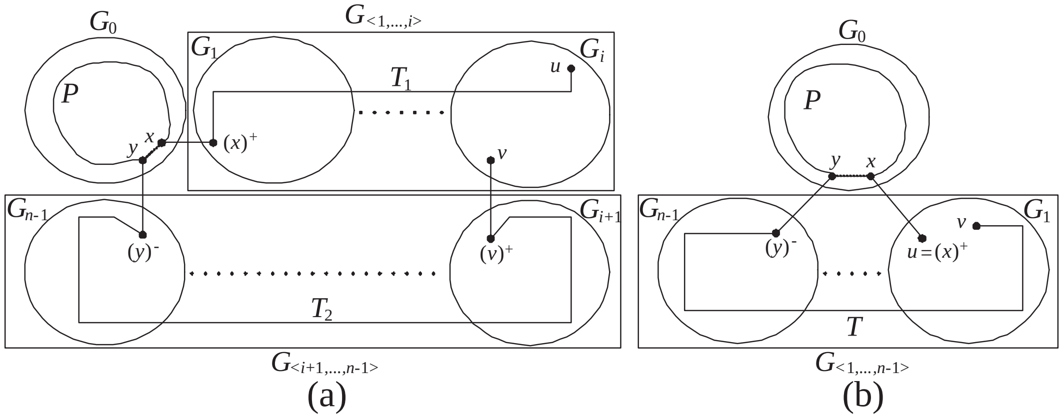

Figure 2.7: Illustration for Case 2 of Proposition 2.1.

$(x, y) \in E(C)$.

Subcase 2.1: Suppose that $(x)^+ \neq u$ and $(x)^+ \neq v$. Thus either $(y)^- \neq (u)^+$ or $(y)^- \neq (v)^+$. Without loss of generality, we assume that $(y)^- \neq (v)^+$. By Lemma 2.2, there exists a hamiltonian path T_1 of $G_{\langle 1,\ldots,i\rangle} - \{v\}$ joining u to $(x)^+$. Similarly, there exists a hamiltonian path T_2 of $G_{\langle i+1,\ldots,n-1\rangle}$ joining $(y)^-$ to $(v)^+$. Then $\langle u, T_1, (x)^+, x, P, y, (y)^-, T_2, (v)^+, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. See Figure 2.7(a) for illustration.

Subcase 2.2: Suppose that $(x)^+ = u$ or $(x)^+ = v$. Without loss of generality, we assume that $(x)^+ = u$. By Lemma 2.2, there exists a hamiltonian path T of $G_{\langle 1,\ldots,n-1\rangle} - \{u\}$ joining $(y)^-$ to v. Then $\langle u = (x)^+, x, P, y, (y)^-, T, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. See Figure 2.7(b) for illustration.

Case 3: Suppose that u is in $G_0 - F$, and v is in G_i with any $i > 0$. Since $i \neq 1$ or $i \neq n-1$, we may assume that $i \neq 1$. Since $|V(C)| \geq 3$, we write $C = \langle u, T, z, u\rangle$ with $z \neq u$. Moreover, T can be written as $\langle u, P_1, x, f_1, y, P_2, z\rangle$ if f_1 is on T, and can be written as $\langle u, P_1, x, y, P_2, z\rangle$ otherwise.

Subcase 3.1: Suppose that $(z)^- \neq v$. Since G_1 is 1-fault-tolerant hamiltonian connected, there exists a hamiltonian path H of G_1 joining $(x)^+$ to $(y)^+$. By Lemma 2.2, there exists a hamiltonian path R of $G_{\langle 2,\ldots,n-1\rangle}$ joining $(z)^-$ to v. Then $\langle u, P_1, x, (x)^+, H, (y)^+, y, P_2, z, (z)^-, R, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. See Figure 2.8(a).

Subcase 3.2: Suppose that $(z)^- = v$. By Lemma 2.2, there exists a hamiltonian path H of $G_{\langle 1,\ldots,n-1\rangle} - \{v\}$ joining $(x)^+$ to $(y)^+$. Then $\langle u, P_1, x, (x)^+, H, (y)^+, y, P_2, z, (z)^- = v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. See Figure 2.8(b) for illustration.

Case 4: Suppose that u is in G_i and v is in G_j for any $1 \leq i < j \leq n-1$. Suppose that f_1 is on C. Then we write $C = \langle x, P, y, f_1, x\rangle$. Otherwise, we write $C = \langle x, P, y, x\rangle$ with any

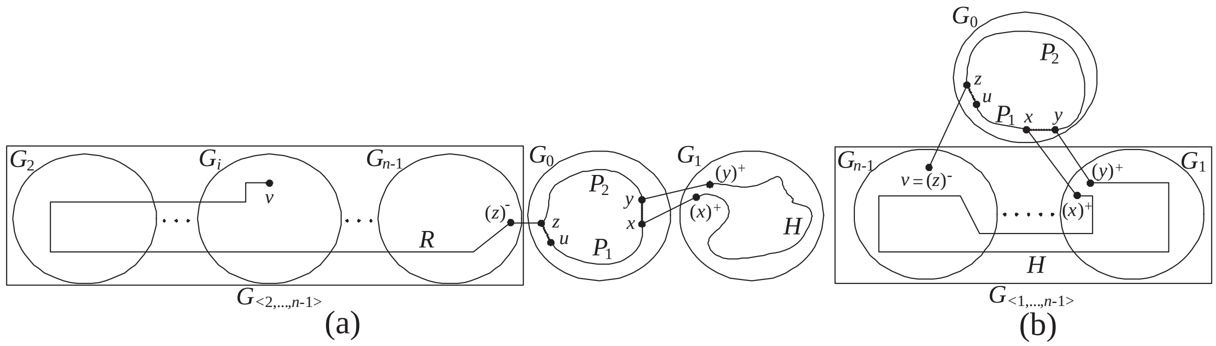

Figure 2.8: Illustration for Case 3 of Proposition 2.1.

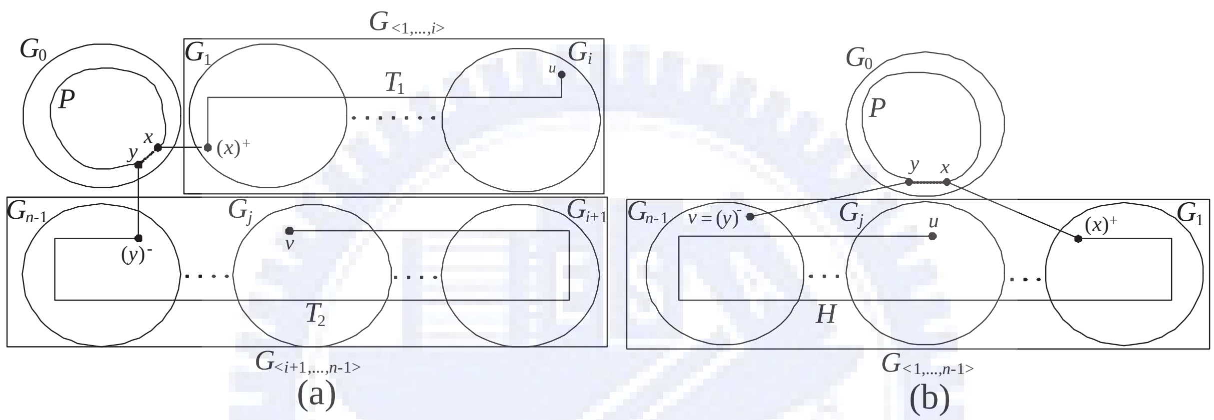

Figure 2.9: Illustration for Case 4 of Proposition 2.1.

$(x, y) \in E(C)$. Since $(x)^+ \neq u$ or $(y)^+ \neq u$, we may assume that $(x)^+ \neq u$.

Subcase 4.1: Suppose that $(y)^- \neq v$. By Lemma 2.2, there exists a hamiltonian path T_1 of $G_{\langle 1,\ldots,i\rangle}$ joining u to $(x)^+$. Similarly, there exists a hamiltonian path T_2 of $G_{\langle i+1,\ldots,n-1\rangle}$ joining $(y)^-$ to v. Then $\langle u, T_1, (x)^+, x, P, y, (y)^-, T_2, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. See Figure 2.9(a) for illustration.

Subcase 4.2: Suppose that $(y)^- = v$. By Lemma 2.2, there exists a hamiltonian path H of $G_{\langle 1,\ldots,n-1\rangle} - \{v\}$ joining u to $(x)^+$. Then $\langle u, H, (x)^+, x, P, y, (y)^- = v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. See Figure 2.9(b) for illustration. □

Proposition 2.2. *Assume that $n \geq 1$. Let $G_0, G_1, \ldots, G_{n-1}$ be n 4-regular super fault-tolerant hamiltonian graphs with the same number of vertices. For any $0 \leq i \leq n-1$, let $M_{i,i+1}$ be a perfect matching between $V(G_i)$ and $V(G_{i+1})$. Let F be a faulty set of $G_{\langle 0,1,\ldots,n-1,0\rangle}$ such that $|F| = 3$ and $|F \cap (V(G_0) \cup E(G_0))| = 2$. Then $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ is hamiltonian connected.*

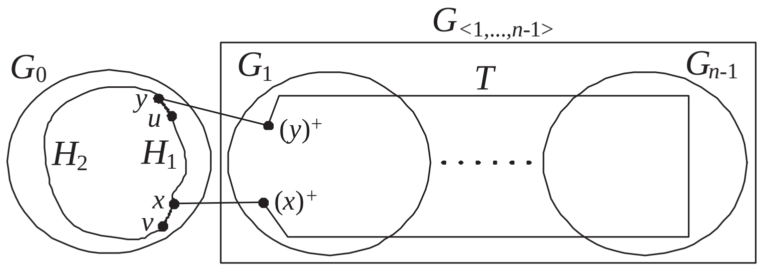

Figure 2.10: Illustration for Case 1 of Proposition 2.2.

Proof. For convenience, let $F_i = F \cap (V(G_i) \cup E(G_i))$ and $X_{i,i+1} = F \cap M_{i,i+1}$ for every $0 \le i \le n-1$. Moreover, let $F' = F - F_0$. Obviously, we have $|F_0| = 2$, $|F'| = 1$, and $|F_i| \le 1$ for all $1 \le i \le n-1$. Since G_0 is 4-regular, we have $|V(G_0)| \ge 5$ and $|V(G_0 - F_0)| \ge 3$. Moreover, since G_0 is 2-fault-tolerant hamiltonian, there exists a hamiltonian cycle C in $G_0 - F_0$. Let u and v be any two vertices of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$. Then we have to construct a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v.

Case 1: Suppose that u and v are in $G_0 - F_0$. Since $|V(G_0 - F_0)| \ge 3$, we may write $C = \langle u, P, y, u\rangle$, where $y \ne u$. Moreover, we may write $P = \langle u, H_1, x, v, H_2, y\rangle$. Note that the length of H_1 becomes zero if $u = x$. Since $|F'| = 1$, we have $|X_{0,1}| + |F_1| = 0$ or $|X_{n-1,0}| + |F_{n-1}| = 0$. With symmetry, we assume that $|X_{0,1}| + |F_1| = 0$. By Lemma 2.2, there exists a hamiltonian path T of $G_{\langle 1,\ldots,n-1\rangle} - F'$ joining $(x)^+$ to $(y)^+$. Then $\langle u, H_1, x, (x)^+, T, (y)^+, y, H_2^{-1}, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. See Figure 2.10 for illustration.

Case 2: Suppose that u and v are in either $G_1 - F_1$ or $G_{n-1} - F_{n-1}$. With symmetry, we assume that u and v are in $G_1 - F_1$.

Subcase 2.1: Suppose that $|X_{0,1}| + |F_1| = 1$. Since $|V(G_0 - F_0)| \ge 3$, we choose a vertex x of the hamiltonian cycle C such that $|F' \cap \{(x)^+, (x, (x)^+)\}| = 0$. Hence cycle C can be written as $\langle y, x, z, P, y\rangle$. Since $(x)^+ \ne u$ or $(x)^+ \ne v$, we assume that $(x)^+ \ne v$. Since G_1 is 1-fault-tolerant hamiltonian connected, there exists a hamiltonian path $Q[u, v]$ of $G_1 - F_1$. Since $(x)^+ \ne v$, we write $Q = \langle u, T_1, (x)^+, w, T_2, v\rangle$. Note that T_1 or T_2 may be a path of length 0. Moreover, we select a vertex from $\{y, z\}$, say y, such that $(y)^- \ne (w)^+$. By Lemma 2.2, there exists a hamiltonian path H of $G_{\langle 2,\ldots,n-1\rangle}$ joining $(y)^-$ to $(w)^+$. Then $\langle u, T_1, (x)^+, x, z, P, y, (y)^-, H, (w)^+, w, T_2, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. See Figure 2.11(a) for illustration.

Subcase 2.2: Suppose that $|X_{0,1}| + |F_1| = 0$. Thus we can choose a vertex x of C such that $|F' \cap \{(x)^+, (x, (x)^+)\}| = 0$ and $(x)^+ \notin \{u, v\}$. Hence the hamiltonian cycle C of $G_0 - F_0$ can be written as $C = \langle y, x, z, P, y\rangle$.

Subcase 2.2.1: Suppose that $|\{(y)^+, (z)^+\} \cap \{u, v\}| \ge 1$. Without loss of generality, we assume that $(z)^+ = u$. By Lemma 2.2, there exists a hamiltonian path T of $G_{\langle 1,\ldots,n-1\rangle} -$

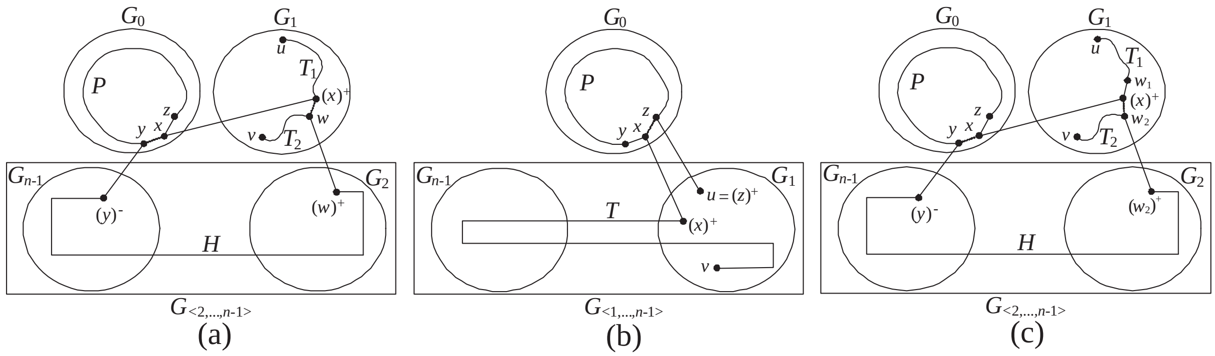

Figure 2.11: Illustration for Case 2 of Proposition 2.2.

$(F' \cup \{u\})$ joining $(x)^+$ to v. Then $\langle u = (z)^+, z, P, y, x, (x)^+, T, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. See Figure 2.11(b) for illustration.

Subcase 2.2.2: Suppose that $|\{(y)^+, (z)^+\} \cap \{u, v\}| = 0$. Since $|F' \cap \{(y)^-, (y, (y)^-)\}| = 0$ or $|F' \cap \{(z)^-, (z, (z)^-)\}| = 0$, we assume that $|F' \cap \{(y)^-, (y, (y)^-)\}| = 0$. Since G_1 is 1-fault-tolerant hamiltonian connected, there exists a hamiltonian path Q of $G_1 - \{((x)^+, ((y)^-)^-)\}$. Since $(x)^+ \notin \{u, v\}$, Q can be represented by $\langle u, T_1, w_1, (x)^+, w_2, T_2, v\rangle$. Note that T_1 or T_2 may be a path of length 0. Accordingly, we have that $|F' \cap \{(w_1)^+, (w_1, (w_1)^+)\}| = 0$ or $|F' \cap \{(w_2)^+, (w_2, (w_2)^+)\}| = 0$. Without loss of generality, we assume that $|F' \cap \{(w_2)^+, (w_2, (w_2)^+)\}| = 0$. By Lemma 2.2, there exists a hamiltonian path H of $G_{\langle 2,\ldots,n-1\rangle} - F'$ joining $(y)^-$ to $(w_2)^+$. Then $\langle u, T_1, w_1, (x)^+, x, z, P, y, (y)^-, H, (w_2)^+, w_2, T_2, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. See Figure 2.11(c).

Case 3: Suppose that u and v are in $G_i - F_i$ with $1 < i < n-1$. Without loss of generality, we assume that $\sum_{j=1}^{i-1} |F_j| + \sum_{j=0}^{i-1} |X_{j,j+1}| = 0$. Since $|V(G_0 - F_0)| \geq 3$, we first choose a vertex x of C such that $|F' \cap \{(x)^-, (x, (x)^-)\}| = 0$. Thus, we can write $C = \langle z, x, y, P, z\rangle$. Next, we choose a vertex t of $G_i - (F_i \cup \{u\})$ such that $|F' \cap \{(t)^+, (t, (t)^+)\}| = 0$ and $(t)^+ \neq (x)^-$. Since G_i is 1-fault-tolerant hamiltonian connected, there is a hamiltonian path H in $G_i - F_i$ joining u to t. Then H can be represented by $\langle u, R_1, w, v, R_2, t\rangle$, where R_1 or R_2 may be a path of length 0. Since $(y)^+ \neq (w)^-$ or $(z)^+ \neq (w)^-$, we assume that $(y)^+ \neq (w)^-$. By Lemma 2.2, there exists a hamiltonian path T_1 of $G_{\langle 0,\ldots,i-1\rangle} - F'$ joining $(w)^-$ to $(y)^+$. Similarly, there exists a hamiltonian path T_2 of $G_{\langle i+1,\ldots,n-1\rangle} - F'$ joining $(x)^-$ to $(t)^+$. As a result, $\langle u, R_1, w, (w)^-, T_1, (y)^+, y, P, z, x, (x)^-, T_2, (t)^+, t, R_2^{-1}, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. See Figure 2.12(a) for illustration.

Case 4: Suppose that u is in $G_0 - F_0$, and v is in $G_i - F_i$ with any $i > 0$. Since $|V(G_0 - F_0)| \geq 3$, we can write $C = \langle x, u, y, P, x\rangle$. Since $|F'| = 1$, we have $|X_{0,1}| + |F_1| = 0$ or $|X_{n-1,0}| + |F_{n-1}| = 0$. Without loss of generality, we assume $|X_{0,1}| + |F_1| = 0$. Hence, we have $(x)^+ \neq v$ or $(y)^+ \neq v$. Without loss of generality, we assume $(x)^+ \neq v$. By Lemma 2.2, there exists a hamiltonian path H of $G_{\langle 1,\ldots,n-1\rangle} - F'$ joining $(x)^+$ to v. Then $\langle u, y, P, x, (x)^+, H, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. See Figure 2.12(b) for illustration.

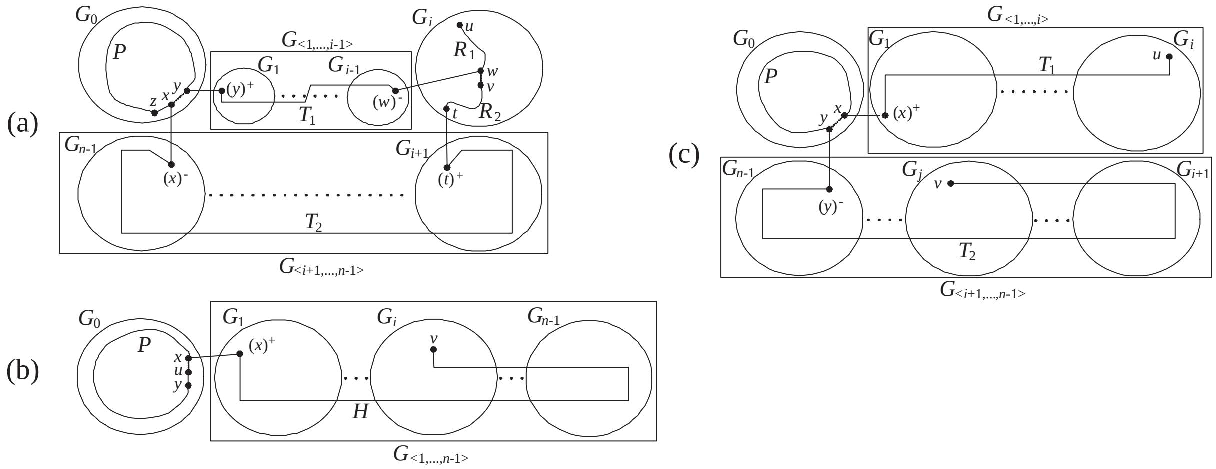

Figure 2.12: Illustration for Case 3, Case 4 and Case 5 of Proposition 2.2.

Case 5: Suppose that u is in $G_i - F_i$, and v is in $G_j - F_j$ for any $1 \leq i < j \leq n-1$. Since $|F'| = 1$, we have $|X_{0,1}| + |F_1| = 0$ or $|X_{n-1,0}| + |F_{n-1}| = 0$. Without loss of generality, we assume $|X_{n-1,0}| + |F_{n-1}| = 0$. Since $|V(G_0 - F_0)| \geq 3$, we can choose a vertex x of C such that $(x)^+ \neq u$ and $|F' \cap \{(x)^+, (x,(x)^+)\}| = 0$. Moreover, there exists at least one neighbor of x on C, namely y, satisfying $(y)^- \neq v$. Accordingly, we can write $C = \langle x, P, y, x\rangle$. By Lemma 2.2, there exists a hamiltonian path T_1 of $G_{\langle 1,\ldots,i\rangle} - F'$ joining u to $(x)^+$. Similarly, there exists a hamiltonian path T_2 of $G_{\langle i+1,\ldots,n-1\rangle} - F'$ joining $(y)^-$ to v. Then $\langle u, T_1, (x)^+, x, P, y, (y)^-, T_2, v\rangle$ is a hamiltonian path of $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ joining u to v. See Figure 2.12(c) for illustration. □

Lemma 2.3. *Assume that $n \geq 3$. Let $G_0, G_1, \ldots, G_{n-1}$ be n 4-regular super fault-tolerant hamiltonian graphs with the same number of vertices. For any $0 \leq i \leq n-2$, let $M_{i,i+1}$ be a perfect matching between $V(G_i)$ and $V(G_{i+1})$. Moreover, let $F_i \subseteq V(G_i) \cup E(G_i)$ with $|F_i| \leq 1$ for every $0 \leq i \leq n-1$ and let $X_{i,i+1} \subseteq M_{i,i+1}$ with $|X_{i,i+1}| \leq 1$ for every $0 \leq i \leq n-2$ such that $|F_i| + |F_{i+1}| + |F_{i+2}| + |X_{i,i+1}| + |X_{i+1,i+2}| \leq 2$ is satisfied for all $0 \leq i \leq n-3$. Let u and v be two vertices of $G_t - F_t$ with $0 < t < n-1$. Then there exists a hamiltonian path of $G_{\langle 0,\ldots,n-1\rangle} - ((\bigcup_{i=0}^{n-1} F_i) \cup (\bigcup_{i=0}^{n-2} X_{i,i+1}))$ joining u to v.*

Proof. For convenience, let $F = (\bigcup_{i=0}^{n-1} F_i) \cup (\bigcup_{i=0}^{n-2} X_{i,i+1})$. Since G_t is 4-regular super fault-tolerant hamiltonian, there exists a hamiltonian path P of $G_t - F_t$ joining u to v. Depending on the value of $|F_t|$, we distinguish the following two cases.

Case 1: Suppose that $|F_t| = 1$. We have $|V(G_t - F_t)| \geq 4$. Let $w_1 = u$. Thus we write P as $\langle u = w_1, w_2, w_3, w_4, R, v\rangle$. Since $|F_t| = 1$, we have $|F_{t-1}| + |F_{t+1}| + |X_{t-1,t}| + |X_{t,t+1}| \leq 1$. Hence, we select a vertex w_i from $\{w_2, w_3\}$ such that $|F \cap \{(w_i)^-, (w_i)^+, (w_i,(w_i)^-), (w_i,(w_i)^+)\}| = 0$. Accordingly, we can see that either $|F \cap \{ (w_{i-1})^+, (w_{i+1})^-, (w_{i-1}, (w_{i-1})^+), (w_{i+1}, (w_{i+1})^-) \}| = 0$ or $|F \cap \{(w_{i-1})^-, (w_{i+1})^+, (w_{i-1}, (w_{i-1})^-), (w_{i+1}, (w_{i+1})^+) \}| = 0$. Without loss of generality, we assume $|F \cap \{ (w_{i-1})^+, (w_{i+1})^-, (w_{i-1},$

$(w_{i-1})^+)$, $(w_{i+1}, (w_{i+1})^-)\}| = 0$. Hence we can further write P as $\langle u = w_1, P_1, w_{i-1}, w_i, w_{i+1}, P_2, v\rangle$. By Lemma 2.2, there exists a hamiltonian path T of $G_{\langle 0,\ldots,t-1\rangle} - ((\bigcup_{i=0}^{t-1} F_i) \cup (\bigcup_{i=0}^{t-2} X_{i,i+1}))$ joining $(w_i)^-$ to $(w_{i+1})^-$. Similarly, there exists a hamiltonian path Q of $G_{\langle t+1,\ldots,n-1\rangle} - ((\bigcup_{i=t+1}^{n-1} F_i) \cup (\bigcup_{i=t+1}^{n-2} X_{i,i+1}))$ joining $(w_{i-1})^+$ to $(w_i)^+$. Then $\langle u = w_1, P_1, w_{i-1}, (w_{i-1})^+, Q, (w_i)^+, w_i, (w_i)^-, T, (w_{i+1})^-, w_{i+1}, P_2, v\rangle$ is a hamiltonian path of $G_{\langle 0,\ldots,n-1\rangle} - F$ joining u to v.

Case 2: Suppose that $|F_t| = 0$. Firstly, assume that $|V(G_t)| \geq 6$. Hence we can select two adjacent edges $(x, y), (y, z) \in E(P)$ such that $|F \cap \{(x)^+, (y)^+, (y)^-, (z)^-, (x, (x)^+), (y, (y)^+), (y, (y)^-), (z, (z)^-)\}| = 0$ or $|F \cap \{(x)^-, (x, (x)^-), (y)^-, (y, (y)^-), (y, (y)^+), (z, (z)^+), (y)^+, (z)^+\}| = 0$. Without loss of generality, we assume that $|F \cap \{(x)^+, (y)^+, (y)^-, (z)^-, (x, (x)^+), (y, (y)^+), (y, (y)^-), (z, (z)^-)\}| = 0$. Accordingly, P can be written as $\langle u, P_1, x, y, z, P_2, v\rangle$, where P_1 or P_2 may be a path of length 0. By Lemma 2.2, there exists a hamiltonian path T of $G_{\langle 0,\ldots,t-1\rangle} - ((\bigcup_{i=0}^{t-1} F_i) \cup (\bigcup_{i=0}^{t-2} X_{i,i+1}))$ joining $(y)^-$ to $(z)^-$. Similarly, there exists a hamiltonian path Q of $G_{\langle t+1,\ldots,n-1\rangle} - ((\bigcup_{i=t+1}^{n-1} F_i) \cup (\bigcup_{i=t+1}^{n-2} X_{i,i+1}))$ joining $(x)^+$ to $(y)^+$. Then $\langle u, P_1, x, (x)^+, Q, (y)^+, y, (y)^-, T, (z)^-, z, P_2, v\rangle$ is a hamiltonian path of $G_{\langle 0,\ldots,n-1\rangle} - F$ joining u to v.

Secondly, assume that $|V(G_t)| = 5$. Thus G_t is isomorphic to the complete graph of five vertices, K_5. Let $V(G_t) = \{u = w_1, w_2, w_3, w_4, w_5 = v\}$. First of all, we choose a vertex from $\{w_2, w_3, w_4\}$, say w_2, such that $|F \cap \{(w_2)^-, (w_2)^+, (w_2, (w_2)^-), (w_2, (w_2)^+)\}| = 0$. Then we choose two vertices x, y from $\{w_3, w_4, w_5\}$ such that $|F \cap \{(x)^+, (x, (x)^+), (y)^-, (y, (y)^-)\}| = 0$. Accordingly, a hamiltonian path of G_t can be written as $\langle u = w_1, P_1, x, w_2, y, P_2, w_5 = v\rangle$. Then a hamiltonian path of $G_{\langle 0,\ldots,n-1\rangle} - F$ joining u to v can be formed in a way similar to that mentioned above. □

Lemma 2.4. *Assume that $n \geq 3$. Let $G_0, G_1, \ldots, G_{n-1}$ be n 4-regular super fault-tolerant hamiltonian graphs with the same number of vertices. For any $0 \leq i \leq n-2$, let $M_{i,i+1}$ be a perfect matching between $V(G_i)$ and $V(G_{i+1})$. Moreover, let $F_i \subseteq V(G_i) \cup E(G_i)$ with $|F_i| \leq 1$ for every $0 \leq i \leq n-1$, and let $X_{i,i+1} \subseteq M_{i,i+1}$ with $|X_{i,i+1}| \leq 1$ for every $0 \leq i \leq n-2$ such that $|F_i| + |F_{i+1}| + |F_{i+2}| + |X_{i,i+1}| + |X_{i+1,i+2}| \leq 2$ is satisfied for all $0 \leq i \leq n-3$. Let u be a vertex of $G_s - F_s$, and let v be a vertex of $G_t - F_t$ with $0 \leq s \leq t \leq n-1$. Then there exists a hamiltonian path of $G_{\langle 0,\ldots,n-1\rangle} - ((\bigcup_{i=0}^{n-1} F_i) \cup (\bigcup_{i=0}^{n-2} X_{i,i+1}))$ joining u to v.*

Proof. For convenience, let $F = (\bigcup_{i=0}^{n-1} F_i) \cup (\bigcup_{i=0}^{n-2} X_{i,i+1})$. When $s = 0$, the statement follows from Lemma 2.2. When $0 < s = t < n-1$, the statement follows from Lemma 2.3. So, we consider the case when $0 < s < t$ in the following. Since G_s is 4-regular, we have $|V(G_s)| \geq 5$. Moreover, since $|F_s| + |F_{s+1}| + |X_{s,s+1}| \leq 2$, we can choose a vertex x of $G_s - (F_s \cup \{u\})$ such that $|F \cap \{x, (x)^+, (x, (x)^+)\}| = 0$ and $(x)^+ \neq v$. By Lemma 2.2, there exists a hamiltonian path P of $G_{\langle 0,\ldots,s\rangle} - ((\bigcup_{i=0}^{s} F_i) \cup (\bigcup_{i=0}^{s-1} X_{i,i+1}))$ joining u to x. Similarly, there exists a hamiltonian path T of $G_{\langle s+1,\ldots,n-1\rangle} - ((\bigcup_{i=s+1}^{n-1} F_i) \cup (\bigcup_{i=s+1}^{n-2} X_{i,i+1}))$ joining $(x)^+$ to v. Then $\langle u, P, x, (x)^+, T, v\rangle$ is a hamiltonian path of $G_{\langle 0,\ldots,n-1\rangle} - F$ joining u to v. □

Proposition 2.3. *Assume that $n \geq 1$. Let $G_0, G_1, \ldots, G_{n-1}$ be n 4-regular super fault-tolerant hamiltonian graphs with the same number of vertices. For any $0 \leq i \leq n-1$, let $M_{i,i+1}$ be a perfect matching between $V(G_i)$ and $V(G_{i+1})$. Let F be a faulty set of $G_{\langle 0,1,\ldots,n-1,0\rangle}$ such that $|F| = 3$ and $|F \cap (V(G_i) \cup E(G_i))| \leq 1$ for $0 \leq i \leq n-1$. Then $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ is hamiltonian connected.*

Proof. Let u be a vertex of $G_a - F_a$, and let v be a vertex of $G_b - F_b$ for any $0 \leq a \leq b \leq n-1$. For convenience, let $F_i = F \cap (V(G_i) \cup E(G_i))$ and $X_{i,i+1} = F \cap M_{i,i+1}$ for every $0 \leq i \leq n-1$. Obviously, we have $|F_i| \leq 1$. Moreover, let t be an integer such that $|X_{t,t+1}| = \max\{|X_{i,i+1}| \mid 0 \leq i \leq n-1\}$. Depending on the value of $|X_{t,t+1}|$, two cases are distinguished.

Case 1: Suppose that $|X_{t,t+1}| \geq 1$. Without loss of generality, t can be assumed to be $n-1$. Accordingly, we have $|X_{i,i+1}| \leq 1$ for every $0 \leq i \leq n-2$. Let $F' = F - X_{n-1,0}$. Therefore we have $|F'| \leq 2$ and $|F_i| + |F_{i+1}| + |F_{i+2}| + |X_{i,i+1}| + |X_{i+1,i+2}| \leq 2$ for all $0 \leq i \leq n-3$. By Lemma 2.4, $G_{\langle 0,1,\ldots,n-1\rangle} - F'$ is hamiltonian connected.

Case 2: Suppose that $|X_{t,t+1}| = 0$. Then we set t to be $a-1$. Obviously, we have $|F_i| + |F_{i+1}| + |X_{i,i+1}| \leq 2$ for all $0 \leq i \leq n-2$. By Lemma 2.2, $G_{\langle a,a+1,\ldots,n-1,0,\ldots,a-1\rangle} - F$ is hamiltonian connected.

As a result, we conclude that $G_{\langle 0,1,\ldots,n-1,0\rangle} - F$ is hamiltonian connected. □

Theorem 2.3. *Suppose that $n \geq 3$. Let $G_0, G_1, \ldots, G_{n-1}$ be n 4-regular super fault-tolerant hamiltonian graphs with the same number of vertices. For any $0 \leq i \leq n-1$, let $M_{i,i+1}$ be a perfect matching between $V(G_i)$ and $V(G_{i+1})$. Then $G_{\langle 0,1,\ldots,n-1,0\rangle}$ is optimal fault-tolerant hamiltonian connected.*

Proof. Obviously, $G_{\langle 0,1,\ldots,n-1,0\rangle}$ is 6-regular. Thus, we need to show that $G_{\langle 0,1,\ldots,n-1,0\rangle}$ is 3-fault-tolerant hamiltonian connected. Let F be a faulty set of $G_{\langle 0,1,\ldots,n-1,0\rangle}$ with $|F| \leq 3$. For convenience, let $F_i = F \cap (V(G_i) \cup E(G_i))$ for $0 \leq i \leq n-1$. Without loss of generality, we assume that $|F_0| \geq |F_i|$ for all $1 \leq i \leq n-1$. Depending on the value of $|F_0|$, three cases are distinguished. The first case that $|F_0| = 3$ is proved by Proposition 2.1. The second case when $|F_0| = 2$ is proved by Proposition 2.2. Finally, the case for $|F_0| \leq 1$ follows from Proposition 2.3. □

According to Theorem 2.1, Theorem 2.2, and Theorem 2.3, we have the next corollary.

Corollary 2.1. *Suppose that $n \geq 3$ and $k \geq 4$. Let $G_0, G_1, \ldots, G_{n-1}$ be n k-regular super fault-tolerant hamiltonian graphs with the same number of vertices. For any $0 \leq i \leq n-1$, let $M_{i,i+1}$ be a perfect matching between $V(G_i)$ and $V(G_{i+1})$. Then $G_{\langle 0,1,\ldots,n-1,0\rangle}$ is $(k+2)$-regular super fault-tolerant hamiltonian.*

Chapter 3

Mutually Independent Hamiltonian Cycles in Butterfly Networks

It is well known that the problem of finding hamiltonian cycles in general graphs is NP-complete. Thus the *hamiltonicity* has gained many researchers' efforts, and has been discussed in many areas. For instance, hamiltonian cycles in Cayley graphs were widely addressed in computer science [38], in the study of word-hyperbolic groups and automatic groups [19], in creating Escher-like repeating patterns in hyperbolic plane [17], and in combinatorial designs [16]. Unlike the previous results, we would like to concern a variant of hamiltonian cycles, namely mutually independent hamiltonian cycles [59, 60], with regard to parallel and distributed computation.

The mutually independent hamiltonian cycles are defined as follows. Let G be a graph with N vertices. A hamiltonian cycle C of G is described by $\langle u_1, u_2, \ldots, u_N, u_1\rangle$ to emphasize its order of vertices. Accordingly, u_1 is referred to as the beginning vertex. Two hamiltonian cycles of G beginning from a given vertex s, namely $C_1 = \langle u_1, u_2, \ldots, u_N, u_1\rangle$ and $C_2 = \langle v_1, v_2, \ldots, v_N, v_1\rangle$, are *independent* if $u_1 = v_1 = s$ and $u_i \neq v_i$ for $2 \leq i \leq N$. Two hamiltonian paths of G, $P_1 = \langle u_1, u_2, \ldots, u_N\rangle$ and $P_2 = \langle v_1, v_2, \ldots, v_N\rangle$, are *independent* if $u_1 = v_1$, $u_N = v_N$, and $u_i \neq v_i$ for every $1 < i < N$; P_1 and P_2 are *fully independent* if $u_i \neq v_i$ for every $1 \leq i \leq N$. We say a set of m hamiltonian cycles $\{C_1, \ldots, C_m\}$ of G, beginning from the same vertex, is *m-mutually independent* if C_i and C_j are independent whenever $i \neq j$. A set of m hamiltonian paths $\{P_1, \ldots, P_m\}$ of G are *m-mutually independent* (respectively, *m-mutually fully independent*) if any two different hamiltonian paths are independent (respectively, fully independent). Moreover, the *mutually independent hamiltonicity* of G, denoted by $\mathcal{IHC}(G)$, is defined as the maximum integer m such that, for any vertex u, there exist m-mutually independent hamiltonian cycles of G beginning from u. Many popular interconnection networks, such as hypercubes [59], star graphs [49], pancake graphs [49], bubble-sort graphs [57], etc. have the maximum numbers of mutually independent hamiltonian cycles.

The concept of mutually independent hamiltonian cycles can be applied in many different areas. For example, communication applications on interconnection networks are often viewed as the interleaving of local computation stages and global communication stages. Such applications can be performed via a message routing protocol, by which information is transmitted along the communication links in packets of equal size. For the sake of simplification, the *store-and-forward all-port* communication model [35] has been widely adopted as one basic routing scheme, in which every processor is assumed to be capable of exchanging messages of fixed length with all its neighbors at each time step. Although routing messages over a spanning tree on the given network is intuitively the best strategy for message transmission, Baldi and Ofek [3] presented a systematic comparison between ring and tree embedding for group (many-to-many) multicast, and concluded that ring embedding remains a promising alternative. It is worth mentioning that there may be two potential shortcomings incurred by routing messages in a ring structured network [43]. Firstly, at least two message packets are likely to reside in the same processor, so as to provoke the contention for the local computation resources. Secondly, two or more message packets will contend for the use of some communication link (in the same direction). Clearly, the mutually independent hamiltonian cycles can ease the effects of such shortcomings.

As another example, a *Latin square* of order n is an $n \times n$ array containing the integers from 1 to n, arranged so that each integer appears exactly once in each row, and exactly once in each column. If we delete some rows from a Latin square, we will get a *Latin rectangle*. Obviously, a Latin square of order n can be thought of as the intermediate vertices of n mutually independent hamiltonian cycles on the complete graph with $n+1$ vertices. Thus the concept behind mutually independent hamiltonian cycles can be interpreted as a Latin square/rectangle for graphs. We can consider the following scenario. A tour agency will organize a 10-day tour to Japan in the Christmas vacation. Suppose that there will be many people joining this tour. However, the maximum number of people stay in each local area is limited, say 100 people, for the sake of a hotel contract. One trivial solution is based on the First-Come-First-Served intuition. So, only 100 people can join this tour. Note that we cannot schedule the tour in a pipelined manner because the holiday period is fixed. Fortunately, we observe that scheduling a tour is like a hamiltonian cycle of a graph, in which a vertex denotes a hotel and an edge denotes the connection between two hotels if they can be traveled in a reasonable time. Therefore, we can organize all the attendees into a number of subgroups; each subgroup has its own tour in such a way that no two subgroups will stay in the same area during the same time period. So any two different tours are indeed independent hamiltonian cycles. If there exist five mutually independent hamiltonian cycles, then we may allow up to 500 attendees to visit Japan on a Christmas vacation. Obviously, if we can find the maximum number of mutually independent hamiltonian cycles, the number of tour attendees would be maximized.

3.1 Topological structure of butterfly networks

For any $\ell \in \mathbb{Z}_n$ and $i \in \mathbb{Z}_2$, we use $BF_\ell^i(n)$ to denote the subgraph of $BF(n)$ induced by $\{\langle h, a_0 \dots a_\ell \dots a_{n-1} \rangle \in V(BF(n)) \mid a_\ell = i\}$. Obviously, $\{BF_\ell^0(n), BF_\ell^1(n)\}$ forms a partition of $BF(n)$. Moreover, $BF_{\ell_1}^i(n)$ is isomorphic to $BF_{\ell_2}^j(n)$ for any $i, j \in \mathbb{Z}_2$ and any $\ell_1, \ell_2 \in \mathbb{Z}_n$. With this observation, Wong [70] proposed a *stretching* operation to obtain $BF_\ell^i(n)$ from

$BF(n-1)$. More precisely, the stretching operation can be described as follows.

Let $i \in \mathbb{Z}_2$ and $\ell \in \mathbb{Z}_n$ for $n \geq 3$. Furthermore, let $\mathcal{G}_n$ denote the set of all subgraphs of $BF(n)$. Suppose that $G \in \mathcal{G}_n$. We define the following subsets of $V(BF(n+1))$ and $E(BF(n+1))$:

$$\begin{aligned}
V_1 &= \{\langle h, a_0 \ldots a_{\ell-1} i a_\ell \ldots a_{n-1}\rangle \mid 0 \leq h < \ell, \langle h, a_0 \ldots a_{\ell-1} a_\ell \ldots a_{n-1}\rangle \in V(G)\},\\
V_2 &= \{\langle h+1, a_0 \ldots a_{\ell-1} i a_\ell \ldots a_{n-1}\rangle \mid \ell < h \leq n-1, \langle h, a_0 \ldots a_{\ell-1} a_\ell \ldots a_{n-1}\rangle \in V(G)\},\\
V_3 &= \{\langle \ell, a_0 \ldots a_{\ell-1} i a_\ell \ldots a_{n-1}\rangle \mid \langle \ell, a_0 \ldots a_{\ell-1} a_\ell \ldots a_{n-1}\rangle \text{ is incident to}\\
&\qquad \text{a level-}(\ell-1)_{\text{mod } n} \text{ edge in } G\},\\
V_4 &= \{\langle \ell+1, a_0 \ldots a_{\ell-1} i a_\ell \ldots a_{n-1}\rangle \mid \langle \ell, a_0 \ldots a_{\ell-1} a_\ell \ldots a_{n-1}\rangle \text{ is incident to}\\
&\qquad \text{a level-}\ell \text{ edge in } G\},\\
E_1 &= \{(\langle h, a_0 \ldots a_{\ell-1} i a_\ell \ldots a_{n-1}\rangle, \langle h+1, b_0 \ldots b_{\ell-1} i b_\ell \ldots b_{n-1}\rangle) \mid 0 \leq h < \ell,\\
&\qquad (\langle h, a_0 \ldots a_{\ell-1} a_\ell \ldots a_{n-1}\rangle, \langle h+1, b_0 \ldots b_{\ell-1} b_\ell \ldots b_{n-1}\rangle) \in E(G)\},\\
E_2 &= \{(\langle h+1, a_0 \ldots a_{\ell-1} i a_\ell \ldots a_{n-1}\rangle, \langle (h+2)_{\text{mod } (n+1)}, b_0 \ldots b_{\ell-1} i b_\ell \ldots b_{n-1}\rangle) \mid \ell \leq h \leq n-1,\\
&\qquad (\langle h, a_0 \ldots a_{\ell-1} a_\ell \ldots a_{n-1}\rangle, \langle (h+1)_{\text{mod } n}, b_0 \ldots b_{\ell-1} b_\ell \ldots b_{n-1}\rangle) \in E(G)\},\\
E_3 &= \{(\langle \ell, a_0 \ldots a_{\ell-1} i a_\ell \ldots a_{n-1}\rangle, \langle \ell+1, a_0 \ldots a_{\ell-1} i a_\ell \ldots a_{n-1}\rangle) \mid\\
&\qquad \langle \ell, a_0 \ldots a_{\ell-1} a_\ell \ldots a_{n-1}\rangle \text{ is incident to at least one level-}(\ell-1)_{\text{mod } n} \text{ edge}\\
&\qquad \text{and at least one level-}\ell \text{ edge in } G\}.
\end{aligned}$$

The stretching function $\gamma_\ell^i : \bigcup_{n \geq 3} \mathcal{G}_n \to \bigcup_{n \geq 4} \mathcal{G}_n$ is defined by assigning $\gamma_\ell^i(G)$ as the graph with the vertex set $V_1 \cup V_2 \cup V_3 \cup V_4$ and the edge set $E_1 \cup E_2 \cup E_3$. Clearly γ_ℓ^i is well-defined and one-to-one. We have $\gamma_\ell^i(G) \in \mathcal{G}_{n+1}$ if $G \in \mathcal{G}_n$. In particular, $\gamma_\ell^i(BF(n)) = BF_\ell^i(n+1)$. In Figure 3.1, we illustrate a subgraph G of $BF(3)$, $\gamma_0^0(G)$ in $\gamma_0^0(BF(3))$, and $\gamma_1^0(G)$ in $\gamma_1^0(BF(3))$. Obviously, $\gamma_{\ell_1}^i(BF(n))$ is isomorphic to $\gamma_{\ell_2}^j(BF(n))$ for any $\ell_1, \ell_2 \in \mathbb{Z}_n$ and $i, j \in \mathbb{Z}_2$. Moreover, $\gamma_\ell^i(P)$ is a path in $BF(n+1)$ if P is a path in $BF(n)$.

In fact, $BF(n)$ can be further partitioned. Let m be an integer with $1 \leq m \leq n$. Assume that $\ell_1, \ldots, \ell_m \in \mathbb{Z}_n$ such that $\ell_1 < \ldots < \ell_m$. For any $i_1, \ldots, i_m \in \mathbb{Z}_2$, we use $BF_{\ell_1,\ldots,\ell_m}^{i_1,\ldots,i_m}(n)$ to denote the subgraph of $BF(n)$ induced by $\{\langle h, a_0 \ldots a_{n-1}\rangle \in V(BF(n)) \mid a_{\ell_j} = i_j \text{ for } 1 \leq j \leq m\}$. In Figure 3.2, we illustrate $BF_{0,1}^{0,0}(4)$, $BF_{0,2}^{0,0}(4)$, $BF_{0,3}^{0,0}(4)$, $BF_{0,2,3}^{0,0,0}(4)$, $BF_{0,1,3}^{0,0,0}(4)$, and $BF_{0,1,2}^{0,0,0}(4)$. Clearly $BF_{0,1}^{0,0}(4)$ is isomorphic with $BF_{0,3}^{0,0}(4)$; furthermore, $BF_{0,2,3}^{0,0,0}(4)$, $BF_{0,1,3}^{0,0,0}(4)$, and $BF_{0,1,2}^{0,0,0}(4)$ are also isomorphic. However, $BF_{0,1}^{0,0}(4)$ is not isomorphic to $BF_{0,2}^{0,0}(4)$.

Lemma 3.1. *Assume that $n \geq 3$ and $i, j, k \in \mathbb{Z}_2$. Then $BF_{0,1}^{i,j}(n)$ is isomorphic with $BF_{0,n-1}^{i,j}(n)$; $BF_{0,1,2}^{i,j,k}(n)$, $BF_{0,1,n-1}^{i,j,k}(n)$, and $BF_{0,n-2,n-1}^{i,j,k}(n)$ are isomorphic.*

Obviously, $\{BF_{\ell_1,\ldots,\ell_m}^{i_1,\ldots,i_m}(n) \mid i_1, \ldots, i_m \in \mathbb{Z}_2, \ell_1, \ldots, \ell_m \in \mathbb{Z}_n, \ell_1 < \ldots < \ell_m\}$ forms a partition of $BF(n)$ for any $1 \leq m \leq n$. To avoid the complication caused from modular

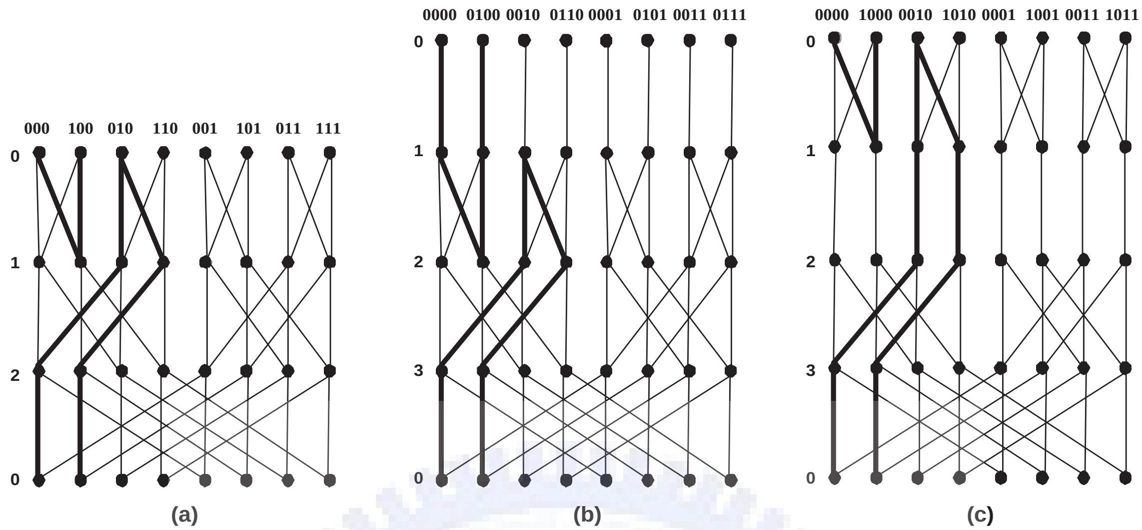

Figure 3.1: (a) A subgraph G of $BF(3)$; (b) $\gamma_0^0(G)$ in $\gamma_0^0(BF(3))$; (c) $\gamma_1^0(G)$ in $\gamma_1^0(BF(3))$.

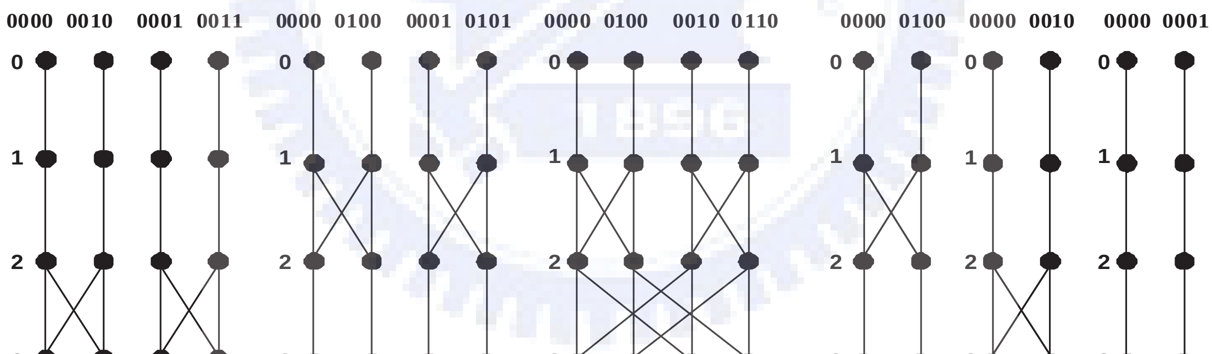

Figure 3.2: (a) $BF_{0,1}^{0,0}(4)$; (b) $BF_{0,2}^{0,0}(4)$; (c) $BF_{0,3}^{0,0}(4)$; (d) $BF_{0,2,3}^{0,0,0}(4)$; (e) $BF_{0,1,3}^{0,0,0}(4)$; (f) $BF_{0,1,2}^{0,0,0}(4)$.

arithmetic, we restrict our attention on the case that $1 \le m \le n-1$, $0 \le \ell_1 < \ldots < \ell_m$, and $\ell_j < n-m+j-1$ for each $1 \le j \le m$. The following two lemmas can be easily verified.

Lemma 3.2. *Let $1 \le m \le n-1$. Suppose that $i_1, \ldots, i_m \in \mathbb{Z}_2$ and $\ell_1, \ldots, \ell_m$ are integers such that $0 \le \ell_1 < \ldots < \ell_m$ and $\ell_j < n-m+j-1$ for each $1 \le j \le m$. Then*

$$BF^{i_1,\ldots,i_m}_{\ell_1,\ldots,\ell_m}(n) = \begin{cases} \gamma^{i_m}_{\ell_m} \circ \gamma^{i_{m-1}}_{\ell_{m-1}} \circ \ldots \circ \gamma^{i_3}_{\ell_3}(BF^{i_1,i_2}_{\ell_1,\ell_2}(3)) & \text{if } m = n-1, \\ \gamma^{i_m}_{\ell_m} \circ \gamma^{i_{m-1}}_{\ell_{m-1}} \circ \ldots \circ \gamma^{i_2}_{\ell_2}(BF^{i_1}_{\ell_1}(3)) & \text{if } m = n-2, \\ \gamma^{i_m}_{\ell_m} \circ \gamma^{i_{m-1}}_{\ell_{m-1}} \circ \ldots \circ \gamma^{i_1}_{\ell_1}(BF(n-m)) & \text{otherwise.} \end{cases}$$

Lemma 3.3. *Let G be a connected spanning subgraph of $BF^{i,j}_{0,1}(n)$, with $i, j \in \mathbb{Z}_2$ and $n \ge 3$. Assume that $2 \le \ell \le n-1$. Let*

$$\begin{aligned} F_0 &= \{\langle \ell, a_0 \ldots a_{n-1}\rangle \in V(G) \mid \langle \ell, a_0 \ldots a_{n-1}\rangle \text{ is not incident to any level-}(\ell-1) \text{ edge in } G\}, \\ F_1 &= \{\langle \ell, a_0 \ldots a_{n-1}\rangle \in V(G) \mid \langle \ell, a_0 \ldots a_{n-1}\rangle \text{ is not incident to any level-}\ell \text{ edge in } G\}. \end{aligned}$$

For any $p, q \in \mathbb{Z}_2$, let

$$\begin{aligned} \overline{F_0} &= \{\langle \ell, a_0 \ldots a_{\ell-1} p q a_\ell \ldots a_{n-1}\rangle \mid \langle \ell, a_0 \ldots a_{\ell-1} a_\ell \ldots a_{n-1}\rangle \in F_0\} \\ &\quad \cup \{\langle \ell+1, a_0 \ldots a_{\ell-1} p q a_\ell \ldots a_{n-1}\rangle \mid \langle \ell, a_0 \ldots a_{\ell-1} a_\ell \ldots a_{n-1}\rangle \in F_0\}, \\ \overline{F_1} &= \{\langle \ell+1, a_0 \ldots a_{\ell-1} p q a_\ell \ldots a_{n-1}\rangle \mid \langle \ell, a_0 \ldots a_{\ell-1} a_\ell \ldots a_{n-1}\rangle \in F_1\} \\ &\quad \cup \{\langle \ell+2, a_0 \ldots a_{\ell-1} p q a_\ell \ldots a_{n-1}\rangle \mid \langle \ell, a_0 \ldots a_{\ell-1} a_\ell \ldots a_{n-1}\rangle \in F_1\}, \\ M_0 &= \bigcup_{\langle \ell, a_0 \ldots a_{n-1}\rangle \notin F_0 \cup F_1} \{(\langle \ell, a_0 \ldots a_{\ell-1} p q a_\ell \ldots a_{n-1}\rangle, \langle \ell+1, a_0 \ldots a_{\ell-1} p q a_\ell \ldots a_{n-1}\rangle)\}, \text{ and} \\ M_1 &= \bigcup_{\langle \ell, a_0 \ldots a_{n-1}\rangle \notin F_0 \cup F_1} \{(\langle \ell+1, a_0 \ldots a_{\ell-1} p q a_\ell \ldots a_{n-1}\rangle, \langle \ell+2, a_0 \ldots a_{\ell-1} p q a_\ell \ldots a_{n-1}\rangle)\}. \end{aligned}$$

Then $F_0 \cap F_1 = \emptyset$, $\overline{F_0} \cap \overline{F_1} = \emptyset$, $\overline{F_0} \cup \overline{F_1} = V(BF^{i,j,p,q}_{0,1,\ell,\ell+1}(n+2)) - V(\gamma^q_{\ell+1} \circ \gamma^p_\ell(G))$, and $M_0 \cup M_1 \subseteq E(\gamma^q_{\ell+1} \circ \gamma^p_\ell(G))$.

3.2 Hamiltonian cycles and paths in butterfly networks

Let G be a subgraph of $BF(n)$. A cycle C in G is called an *ℓ-scheduled* cycle of G if every level-ℓ vertex of G is incident to a level-$(\ell-1)_{\bmod n}$ edge and a level-ℓ edge on C [70]. Furthermore, a cycle C in G is a *totally scheduled* cycle of G if it is an ℓ-scheduled cycle of G for all $\ell \in \mathbb{Z}_n$ [70]. Obviously, $\gamma^i_\ell(C)$ with $i \in \{0,1\}$ is a totally scheduled cycle of $\gamma^i_\ell(G)$ if C is a totally scheduled cycle of G.

Lemma 3.4. *[70] Let $n \ge 3$. Then $BF(n)$ has a totally scheduled hamiltonian cycle.*

By stretching operation, we have the following two corollaries.

Corollary 3.1. *Assume that $n \geq 3$ and $i, j, k \in \mathbb{Z}_2$. Then there exists a totally scheduled hamiltonian cycle of $BF^{i,j,k}_{0,1,2}(n)$ including all straight edges of level 0, level 1, and level 2.*

Corollary 3.2. *Assume that $n \geq 4$ and $i, j, p, q \in \mathbb{Z}_2$. Then there exists a totally scheduled hamiltonian cycle of $BF^{i,j,p,q}_{0,1,2,3}(n)$ including all straight edges of level 0, level 1, level 2, and level 3 in $BF^{i,j,p,q}_{0,1,2,3}(n)$.*

Suppose that $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ are either any two cross edges of $BF(n)$ or any two straight edges of $BF(n)$. Since $BF(n)$ is vertex-transitive, there exists an isomorphism μ over $V(BF(n))$ such that $u_2 = \mu(u_1)$ and $v_2 = \mu(v_1)$. Clearly, every hamiltonian cycle of $BF(n)$ includes at least one cross edge and at least one straight edge.

Lemma 3.5. *For any edge e of $BF(n)$ with $n \geq 3$, there exists a totally scheduled hamiltonian cycle of $BF(n)$ including e.*

Lemma 3.6. *Assume that $i, j, k \in \mathbb{Z}_2$. Let e be any edge of $BF^{i,j,k}_{0,1,2}(4)$ such that $e \notin \{ (\langle 3, ijk0\rangle, \langle 0, ijk0\rangle), (\langle 3, ijk1\rangle, \langle 0, ijk1\rangle)\}$. Then there exists a totally scheduled hamiltonian cycle C of $BF^{i,j,k}_{0,1,2}(4)$ such that $e \in E(C)$.*

Proof. Obviously, $\langle\ \langle 0, ijk0\rangle, \langle 1, ijk0\rangle, \langle 2, ijk0\rangle, \langle 3, ijk0\rangle, \langle 0, ijk1\rangle, \langle 1, ijk1\rangle, \langle 2, ijk1\rangle, \langle 3, ijk1\rangle, \langle 0, ijk0\rangle\ \rangle$ is the unique hamiltonian cycle of $BF^{i,j,k}_{0,1,2}(4)$. Thus, this lemma is proved. □

By stretching operation and Corollary 3.1, we have the following corollary.

Corollary 3.3. *Suppose that $n \geq 5$. Let e be any edge of $BF^{i,j,k}_{0,1,2}(n)$ with $i, j, k \in \mathbb{Z}_2$. Then there exists a totally scheduled hamiltonian cycle of $BF^{i,j,k}_{0,1,2}(n)$ including e.*

A path P of $BF(n)$ is *weakly ℓ-scheduled* if there is at least one non-terminal level-ℓ vertex v of P such that v is incident to a level-$(\ell - 1)_{\bmod n}$ edge and a level-ℓ edge on P. Figure 3.3 illustrates two weakly 2-scheduled hamiltonian paths P_1 and P_2 of $BF^{i,j}_{0,1}(4)$ and their images $\gamma^0_3 \circ \gamma^0_2(P_1)$ and $\gamma^0_3 \circ \gamma^0_2(P_2)$ on $\gamma^0_3 \circ \gamma^0_2(BF^{i,j}_{0,1}(4)) = BF^{i,j,0,0}_{0,1,2,3}(6)$, respectively.

Lemma 3.7. *Let $n \geq 4$ and $i, j \in \mathbb{Z}_2$. Suppose that s is any level-1 vertex of $BF^{i,j}_{0,1}(n)$ and d is any level-2 vertex of $BF^{i,j}_{0,1}(n)$. Then there exists a weakly 2-scheduled hamiltonian path of $BF^{i,j}_{0,1}(n)$, joining s to d.*

Proof. Without loss of generality, we assume that $s = \langle 1, ij0^{n-2}\rangle$ and $d = \langle 2, ijpqx\rangle$ with $p, q \in \mathbb{Z}_2$ and $x \in \mathbb{Z}^{n-4}_2$. We prove this lemma by induction on n. The induction bases are listed in Table 3.1 and Table 3.2.

As the inductive hypothesis, we assume that the statement holds for $BF^{i,j}_{0,1}(n-2)$ with $n \geq 6$. Now we partition $BF^{i,j}_{0,1}(n)$ into $\{BF^{i,j,h,k}_{0,1,2,3}(n) \mid h, k \in \mathbb{Z}_2\}$. By the inductive hypothesis,

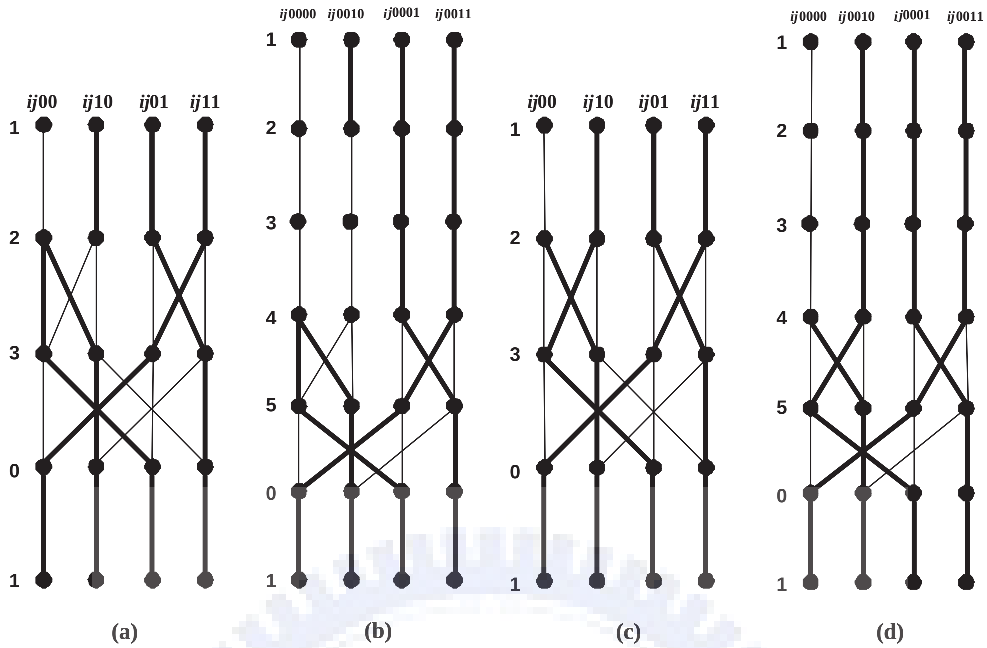

Figure 3.3: (a) A weakly 2-scheduled hamiltonian path P_1 of $BF^{i,j}_{0,1}(4)$ joins $\langle 1, ij00\rangle$ to $\langle 2, ij10\rangle$; (b) $\gamma^0_3 \circ \gamma^0_2(P_1)$ in $BF^{i,j,0,0}_{0,1,2,3}(6) = \gamma^0_3 \circ \gamma^0_2(BF^{i,j}_{0,1}(4))$; (c) a weakly 2-scheduled hamiltonian path P_2 of $BF^{i,j}_{0,1}(4)$ joins $\langle 1, ij00\rangle$ to $\langle 2, ij00\rangle$; (d) $\gamma^0_3 \circ \gamma^0_2(P_2)$ in $BF^{i,j,0,0}_{0,1,2,3}(6)$.

there exists a weakly 2-scheduled hamiltonian path P^{00} of $BF^{i,j}_{0,1}(n-2)$ joining $\langle 1, ij0^{n-4}\rangle$ to $\langle 2, ijx\rangle$. Hence, there is at least one non-terminal level-2 vertex of P^{00}, say $v = \langle 2, ijy\rangle$ with $y \neq x$, such that v is incident to a level-1 edge and a level-2 edge on P^{00}. By Lemma 3.2, we have $BF^{i,j,0,0}_{0,1,2,3}(n) = \gamma^0_3 \circ \gamma^0_2 \circ \gamma^j_1(BF^i_0(n-3)) = \gamma^0_3 \circ \gamma^0_2(BF^{i,j}_{0,1}(n-2))$. Thus, $\gamma^0_3 \circ \gamma^0_2(P^{00})$ is a path on $BF^{i,j,0,0}_{0,1,2,3}(n)$ joining s to $\langle 2, ij00x\rangle$ or joining s to $\langle 4, ij00x\rangle$. By Corollary 3.2, there is a totally scheduled hamiltonian cycle C^{hk} of $BF^{i,j,h,k}_{0,1,2,3}(n)$ including all straight edges of level 2 and level 3 for any $h, k \in \mathbb{Z}_2$.

Let $F_k = \{\langle 2, ijw\rangle \in V(P^{00}) \mid \langle 2, ijw\rangle$ is not incident to any level-$(k+1)$ edge on $P^{00}\}$ with $k \in \{0, 1\}$. Obviously, P^{00} is a connected spanning subgraph of $BF^{i,j}_{0,1}(n-2)$. By Lemma 3.3, we have $V(\gamma^0_3 \circ \gamma^0_2(P^{00})) = V(BF^{i,j,0,0}_{0,1,2,3}(n)) - (\overline{F_0} \cup \overline{F_1})$, where $\overline{F_0} = \{\langle 2, ij00w\rangle \mid \langle 2, ijw\rangle \in F_0\} \cup \{\langle 3, ij00w\rangle \mid \langle 2, ijw\rangle \in F_0\}$ and $\overline{F_1} = \{\langle 3, ij00w\rangle \mid \langle 2, ijw\rangle \in F_1\} \cup \{\langle 4, ij00w\rangle \mid \langle 2, ijw\rangle \in F_1\}$. In addition, we have $\overline{F_0} \cap \overline{F_1} = \emptyset$. If $\gamma^0_3 \circ \gamma^0_2(P^{00})$ joins s to $\langle 2, ij00x\rangle$, let $\overline{P^{00}} = \gamma^0_3 \circ \gamma^0_2(P^{00})$ and $\widetilde{F_0} = \overline{F_0}$. Otherwise, let $\overline{P^{00}} = \langle s, \gamma^0_3 \circ \gamma^0_2(P^{00}), \langle 4, ij00x\rangle, \langle 3, ij00x\rangle, \langle 2, ij00x\rangle\rangle$ and $\widetilde{F_0} = \overline{F_0} - \{\langle 2, ij00x\rangle, \langle 3, ij00x\rangle\}$. For any

$h, k \in \mathbb{Z}_2$, let

$$\begin{aligned}
X_0^{hk} &= \{(\langle 2, ijhkw\rangle, \langle 3, ijhkw\rangle) \mid \langle 2, ij00w\rangle \text{ and } \langle 3, ij00w\rangle \text{ are in } \widetilde{F_0}\}, \\
Y_0^{hk} &= \{(\langle 2, ijhkw\rangle, \langle 3, ij\bar{h}kw\rangle) \mid \langle 2, ij00w\rangle \text{ and } \langle 3, ij00w\rangle \text{ are in } \widetilde{F_0}\}, \\
X_1^{hk} &= \{(\langle 3, ijhkw\rangle, \langle 4, ijhkw\rangle) \mid \langle 3, ij00w\rangle \text{ and } \langle 4, ij00w\rangle \text{ are in } \overline{F_1}\}, \text{ and} \\
Y_1^{hk} &= \{(\langle 3, ijhkw\rangle, \langle 4, ijh\bar{k}w\rangle) \mid \langle 3, ij00w\rangle \text{ and } \langle 4, ij00w\rangle \text{ are in } \overline{F_1}\}.
\end{aligned}$$

Then we consider the following four cases.

Case 1: If $pq = 00$, then $d = \langle 2, ij00x\rangle$. It is noticed that $v \notin F_0 \cup F_1$. Let

$$\begin{aligned}
A &= \{(\langle 2, ij10y\rangle, \langle 3, ij00y\rangle), (\langle 2, ij00y\rangle, \langle 3, ij10y\rangle), (\langle 2, ij11y\rangle, \langle 3, ij01y\rangle), \\
&\quad (\langle 2, ij01y\rangle, \langle 3, ij11y\rangle), (\langle 3, ij11y\rangle, \langle 4, ij10y\rangle), (\langle 3, ij10y\rangle, \langle 4, ij11y\rangle)\} \text{ and} \\
B &= \{(\langle 2, ij00y\rangle, \langle 3, ij00y\rangle), (\langle 2, ij10y\rangle, \langle 3, ij10y\rangle), (\langle 2, ij01y\rangle, \langle 3, ij01y\rangle), \\
&\quad (\langle 2, ij11y\rangle, \langle 3, ij11y\rangle), (\langle 3, ij10y\rangle, \langle 4, ij10y\rangle), (\langle 3, ij11y\rangle, \langle 4, ij11y\rangle)\}.
\end{aligned}$$

It follows from Lemma 3.3 that $(\langle 2, ij00y\rangle, \langle 3, ij00y\rangle) \in E(\overline{P^{00}})$. By Corollary 3.2, we have $(\langle 2, ij10y\rangle, \langle 3, ij10y\rangle) \in E(C^{10})$, $(\langle 2, ij01y\rangle, \langle 3, ij01y\rangle) \in E(C^{01})$, $(\langle 2, ij11y\rangle, \langle 3, ij11y\rangle) \in E(C^{11})$, $(\langle 3, ij10y\rangle, \langle 4, ij10y\rangle) \in E(C^{10})$, and $(\langle 3, ij11y\rangle, \langle 4, ij11y\rangle) \in E(C^{11})$. Then the subgraph P of $BF_{0,1}^{i,j}(n)$, generated by $(E(\overline{P^{00}}) \cup E(C^{10}) \cup E(C^{01}) \cup E(C^{11}) \cup A) - B$, forms a weakly 2-scheduled path of $BF_{0,1}^{i,j}(n)$ between s and d. Clearly, we have $V(P) = V(BF_{0,1}^{i,j}(n)) - (\overline{F_0} \cup \widetilde{F_1})$. Since C^{hk} includes all straight edges of level 2 and level 3 in $BF_{0,1,2,3}^{i,j,h,k}(n)$, we have $X_0^{10} \subset E(C^{10})$ and $X_1^{01} \subset E(C^{01})$. Moreover, we have $(X_0^{10} \cup X_1^{01}) \cap B = \emptyset$. Therefore, it follows that $(X_0^{10} \cup X_1^{01}) \subset E(P)$. Let P' be the subgraph generated by $(E(P) \cup (X_0^{00} \cup Y_0^{00} \cup Y_0^{10}) \cup (X_1^{00} \cup Y_1^{00} \cup Y_1^{01})) - (X_0^{10} \cup X_1^{01})$. Then P' is a weakly 2-scheduled hamiltonian path of $BF_{0,1}^{i,j}(n)$ joining s to d. See Figure 3.4 for illustration, in which $\gamma_3^0 \circ \gamma_2^0(P^{00})$ is supposed to join s and $\langle 2, ij00x\rangle$.

Case 2: If $pq = 10$, then $d = \langle 2, ij10x\rangle$. Let

$$\begin{aligned}
A &= \{(\langle 2, ij00x\rangle, \langle 3, ij10x\rangle), (\langle 2, ij11y\rangle, \langle 3, ij01y\rangle), (\langle 2, ij01y\rangle, \langle 3, ij11y\rangle), \\
&\quad (\langle 3, ij11y\rangle, \langle 4, ij10y\rangle), (\langle 3, ij10y\rangle, \langle 4, ij11y\rangle)\} \text{ and} \\
B &= \{(\langle 2, ij10x\rangle, \langle 3, ij10x\rangle), (\langle 2, ij01y\rangle, \langle 3, ij01y\rangle), (\langle 2, ij11y\rangle, \langle 3, ij11y\rangle), \\
&\quad (\langle 3, ij10y\rangle, \langle 4, ij10y\rangle), (\langle 3, ij11y\rangle, \langle 4, ij11y\rangle)\}.
\end{aligned}$$

Obviously, the subgraph P, generated by $(E(\overline{P^{00}}) \cup E(C^{10}) \cup E(C^{01}) \cup E(C^{11}) \cup A) - B$, forms a weakly 2-scheduled path of $BF_{0,1}^{i,j}(n)$ between s and d. Moreover, the subgraph P', generated by $(E(P) \cup (X_0^{00} \cup Y_0^{00} \cup Y_0^{10}) \cup (X_1^{00} \cup Y_1^{00} \cup Y_1^{01})) - (X_0^{10} \cup X_1^{01})$, is a weakly 2-scheduled hamiltonian path of $BF_{0,1}^{i,j}(n)$ joining s to d.

Case 3: If $pq = 01$, then $d = \langle 2, ij01x \rangle$. Let

$$\begin{aligned} A &= \{(\langle 2, ij00x \rangle, \langle 3, ij10x \rangle), (\langle 2, ij11x \rangle, \langle 3, ij01x \rangle), (\langle 3, ij11x \rangle, \langle 4, ij10x \rangle)\} \text{ and} \\ B &= \{(\langle 2, ij01x \rangle, \langle 3, ij01x \rangle), (\langle 2, ij11x \rangle, \langle 3, ij11x \rangle), (\langle 3, ij10x \rangle, \langle 4, ij10x \rangle)\}. \end{aligned}$$

Obviously, the subgraph P, generated by $(E(\overline{P^{00}}) \cup E(C^{10}) \cup E(C^{01}) \cup E(C^{11}) \cup A) - B$, forms a weakly 2-scheduled path of $BF_{0,1}^{i,j}(n)$, between s and d. Moreover, the subgraph P', generated by $(E(P) \cup (X_0^{00} \cup Y_0^{00} \cup Y_0^{10}) \cup (X_1^{00} \cup Y_1^{00} \cup Y_1^{01})) - (X_0^{10} \cup X_1^{01})$, is a weakly 2-scheduled hamiltonian path of $BF_{0,1}^{i,j}(n)$ joining s to d.

Case 4: If $pq = 11$, then $d = \langle 2, ij11x \rangle$. Let

$$\begin{aligned} A = \{&(\langle 2, ij00x \rangle, \langle 3, ij10x \rangle), (\langle 3, ij11x \rangle, \langle 4, ij10x \rangle), (\langle 3, ij01y \rangle, \langle 4, ij00y \rangle), \\ &(\langle 3, ij00y \rangle, \langle 4, ij01y \rangle)\} \text{ and} \\ B = \{&(\langle 3, ij10x \rangle, \langle 4, ij10x \rangle), (\langle 3, ij00y \rangle, \langle 4, ij00y \rangle), (\langle 3, ij01y \rangle, \langle 4, ij01y \rangle), \\ &(\langle 2, ij11x \rangle, \langle 3, ij11x \rangle)\}. \end{aligned}$$

The subgraph P, generated by $(E(\overline{P^{00}}) \cup E(C^{10}) \cup E(C^{01}) \cup E(C^{11}) \cup A) - B$, forms a weakly 2-scheduled path of $BF_{0,1}^{i,j}(n)$ between s and d. Moreover, the subgraph P', generated by $(E(P) \cup (X_0^{00} \cup Y_0^{00} \cup Y_0^{10}) \cup (X_1^{00} \cup Y_1^{00} \cup Y_1^{01})) - (X_0^{10} \cup X_1^{01})$, is a weakly 2-scheduled hamiltonian path of $BF_{0,1}^{i,j}(n)$ joining s to d. □

Table 3.1: Hamiltonian paths of $BF_{0,1}^{i,j}(4)$ between $\langle 1, ij00 \rangle$ and $\langle 2, ijpq \rangle$ for any $p, q \in \mathbb{Z}_2$.

$\langle\langle 1, ij00\rangle, \langle 0, ij00\rangle, \langle 3, ij01\rangle, \langle 2, ij11\rangle, \langle 1, ij11\rangle, \langle 0, ij11\rangle, \langle 3, ij11\rangle, \langle 2, ij01\rangle, \langle 1, ij01\rangle, \langle 0, ij01\rangle, \langle 3, ij00\rangle, \langle 2, ij10\rangle, \langle 1, ij10\rangle, \langle 0, ij10\rangle, \langle 3, ij10\rangle, \langle 2, ij00\rangle\rangle$
$\langle\langle 1, ij00\rangle, \langle 0, ij00\rangle, \langle 3, ij00\rangle, \langle 2, ij00\rangle, \langle 3, ij10\rangle, \langle 0, ij11\rangle, \langle 1, ij11\rangle, \langle 2, ij11\rangle, \langle 3, ij01\rangle, \langle 0, ij01\rangle, \langle 1, ij01\rangle, \langle 2, ij01\rangle, \langle 3, ij11\rangle, \langle 0, ij10\rangle, \langle 1, ij10\rangle, \langle 2, ij10\rangle\rangle$
$\langle\langle 1, ij00\rangle, \langle 0, ij00\rangle, \langle 3, ij00\rangle, \langle 2, ij00\rangle, \langle 3, ij10\rangle, \langle 2, ij10\rangle, \langle 1, ij10\rangle, \langle 0, ij10\rangle, \langle 3, ij11\rangle, \langle 0, ij11\rangle, \langle 1, ij11\rangle, \langle 2, ij11\rangle, \langle 3, ij01\rangle, \langle 0, ij01\rangle, \langle 1, ij01\rangle, \langle 2, ij01\rangle\rangle$
$\langle\langle 1, ij00\rangle, \langle 0, ij00\rangle, \langle 3, ij01\rangle, \langle 2, ij01\rangle, \langle 1, ij01\rangle, \langle 0, ij01\rangle, \langle 3, ij00\rangle, \langle 2, ij00\rangle, \langle 3, ij10\rangle, \langle 2, ij10\rangle, \langle 1, ij10\rangle, \langle 0, ij10\rangle, \langle 3, ij11\rangle, \langle 0, ij11\rangle, \langle 1, ij11\rangle, \langle 2, ij11\rangle\rangle$

By symmetry, the next corollary can be proved in the way similar to Lemma 3.7.

Corollary 3.4. *Assume that $n \geq 4$ and $i, j \in \mathbb{Z}_2$. Let s be any level-1 vertex of $BF_{0,1}^{i,j}(n)$ and d be any level-0 vertex of $BF_{0,1}^{i,j}(n)$. Then there exists a weakly 0-scheduled hamiltonian path of $BF_{0,1}^{i,j}(n)$ joining s to d.*

Lemma 3.8. *Assume that $n \geq 4$. Let $s = \langle 1, 0^n \rangle$, $d_1 = \langle 2, 0^2 1 0^{n-3} \rangle$, and $d_2 = \langle 0, 0^n \rangle$. Then there exist two hamiltonian paths H_1 and H_2 of $BF_{0,1}^{0,0}(n)$ such that the following conditions are all satisfied: (i) H_1 joins s to d_1, (ii) H_2 joins s to d_2, and (iii) $H_1(1) = H_2(1) = s$ and $H_1(t) \neq H_2(t)$ for each $2 \leq t \leq \left|V\left(BF_{0,1}^{0,0}(n)\right)\right| = n2^{n-2}$.*

Proof. Let $u_1 = g(s) = \langle 2, 0^n \rangle$, $u_2 = f(u_1) = g(d_1) = \langle 3, 0^2 1 0^{n-3} \rangle$, $u_3 = g^{-1}(d_1) = \langle 1, 0^2 1 0^{n-3} \rangle$, $u_4 = f(u_2)$, and $u_5 = g(u_1) = f(d_1) = \langle 3, 0^n \rangle$. Note that $u_4 = \langle 0, 0011 \rangle$ if

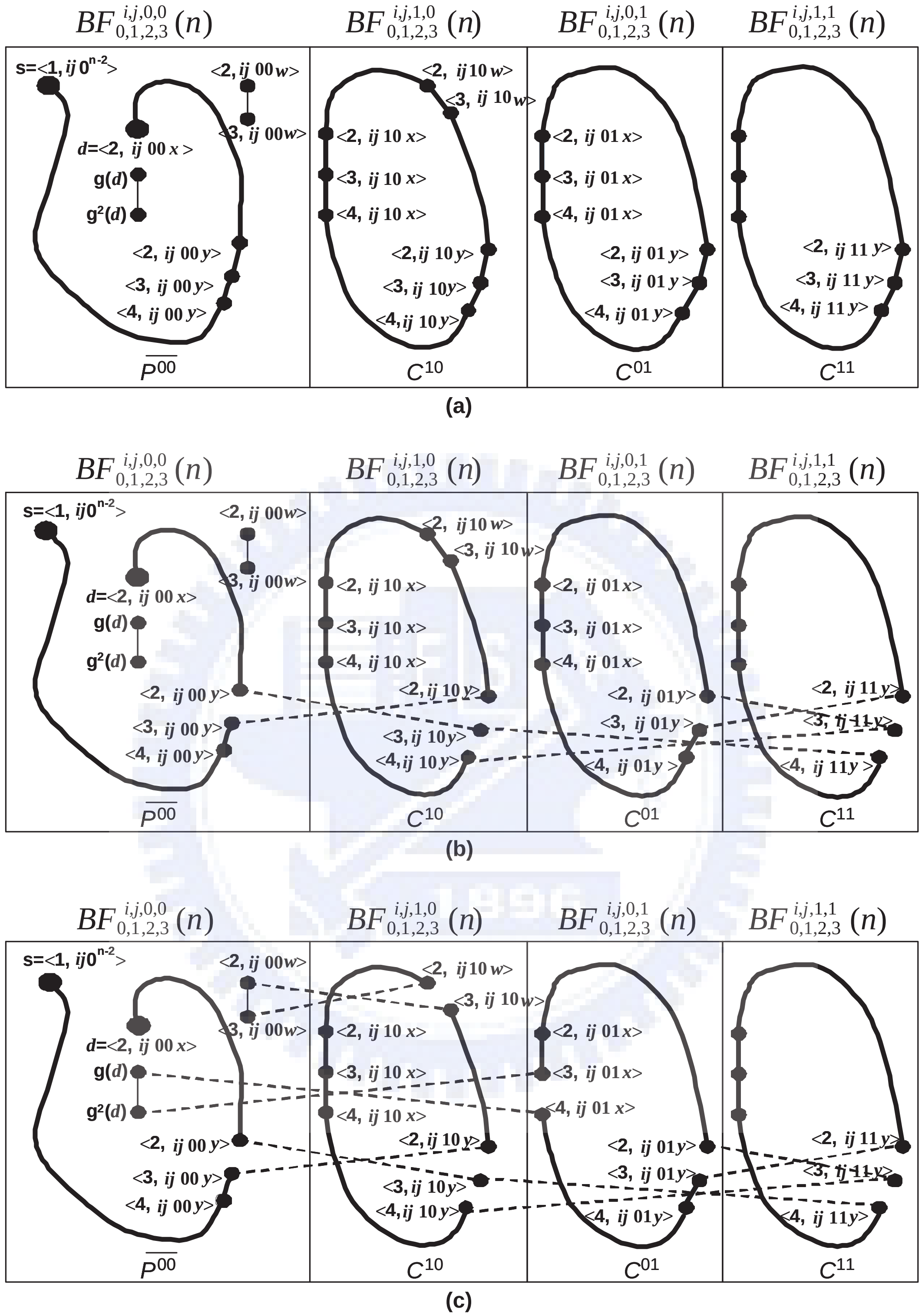

Figure 3.4: (a) $\overline{P^{00}} = \gamma_3^0 \circ \gamma_2^0(P^{00})$, C^{10}, C^{01}, and C^{11}; (b) the path P generated by $(E(\overline{P^{00}}) \cup E(C^{10}) \cup E(C^{01}) \cup E(C^{11}) \cup A) - B$; (c) the path P' generated by $(E(P) \cup (X_0^{00} \cup Y_0^{00} \cup Y_0^{10}) \cup (X_1^{00} \cup Y_1^{00} \cup Y_1^{01})) - (X_0^{10} \cup X_1^{01})$ to cover all vertices of $\widetilde{F_0} \cup \overline{F_1}$.

Table 3.2: Hamiltonian paths of $BF_{0,1}^{i,j}(5)$ between $\langle 1, ij000\rangle$ and $\langle 2, ijpqx\rangle$ for any $p, q, x \in \mathbb{Z}_2$.

$\langle\langle 1, ij000\rangle, \langle 0, ij000\rangle, \langle 4, ij001\rangle, \langle 3, ij011\rangle, \langle 2, ij111\rangle, \langle 1, ij111\rangle, \langle 0, ij111\rangle, \langle 4, ij111\rangle, \langle 3, ij111\rangle, \langle 2, ij011\rangle,$ $\langle 1, ij011\rangle, \langle 0, ij011\rangle, \langle 4, ij011\rangle, \langle 3, ij001\rangle, \langle 2, ij101\rangle, \langle 1, ij101\rangle, \langle 0, ij101\rangle, \langle 4, ij101\rangle, \langle 3, ij101\rangle, \langle 2, ij001\rangle,$ $\langle 1, ij001\rangle, \langle 0, ij001\rangle, \langle 4, ij000\rangle, \langle 3, ij010\rangle, \langle 2, ij110\rangle, \langle 1, ij110\rangle, \langle 0, ij110\rangle, \langle 4, ij110\rangle, \langle 3, ij110\rangle, \langle 2, ij010\rangle,$ $\langle 1, ij010\rangle, \langle 0, ij010\rangle, \langle 4, ij010\rangle, \langle 3, ij000\rangle, \langle 2, ij100\rangle, \langle 1, ij100\rangle, \langle 0, ij100\rangle, \langle 4, ij100\rangle, \langle 3, ij100\rangle, \langle 2, ij000\rangle\rangle$
$\langle\langle 1, ij000\rangle, \langle 0, ij000\rangle, \langle 4, ij001\rangle, \langle 3, ij011\rangle, \langle 2, ij111\rangle, \langle 1, ij111\rangle, \langle 0, ij111\rangle, \langle 4, ij111\rangle, \langle 3, ij111\rangle, \langle 2, ij011\rangle,$ $\langle 1, ij011\rangle, \langle 0, ij011\rangle, \langle 4, ij011\rangle, \langle 3, ij001\rangle, \langle 2, ij101\rangle, \langle 1, ij101\rangle, \langle 0, ij101\rangle, \langle 4, ij101\rangle, \langle 3, ij101\rangle, \langle 2, ij001\rangle,$ $\langle 1, ij001\rangle, \langle 0, ij001\rangle, \langle 4, ij000\rangle, \langle 3, ij010\rangle, \langle 2, ij110\rangle, \langle 1, ij110\rangle, \langle 0, ij110\rangle, \langle 4, ij110\rangle, \langle 3, ij110\rangle, \langle 2, ij010\rangle,$ $\langle 1, ij010\rangle, \langle 0, ij010\rangle, \langle 4, ij010\rangle, \langle 3, ij000\rangle, \langle 2, ij000\rangle, \langle 3, ij100\rangle, \langle 4, ij100\rangle, \langle 0, ij100\rangle, \langle 1, ij100\rangle, \langle 2, ij100\rangle\rangle$
$\langle\langle 1, ij000\rangle, \langle 0, ij000\rangle, \langle 4, ij001\rangle, \langle 3, ij011\rangle, \langle 2, ij111\rangle, \langle 1, ij111\rangle, \langle 0, ij111\rangle, \langle 4, ij111\rangle, \langle 3, ij111\rangle, \langle 2, ij011\rangle,$ $\langle 1, ij011\rangle, \langle 0, ij011\rangle, \langle 4, ij011\rangle, \langle 3, ij001\rangle, \langle 2, ij101\rangle, \langle 1, ij101\rangle, \langle 0, ij101\rangle, \langle 4, ij101\rangle, \langle 3, ij101\rangle, \langle 2, ij001\rangle,$ $\langle 1, ij001\rangle, \langle 0, ij001\rangle, \langle 4, ij000\rangle, \langle 3, ij000\rangle, \langle 2, ij000\rangle, \langle 3, ij100\rangle, \langle 2, ij100\rangle, \langle 1, ij100\rangle, \langle 0, ij100\rangle, \langle 4, ij100\rangle,$ $\langle 3, ij110\rangle, \langle 4, ij110\rangle, \langle 0, ij110\rangle, \langle 1, ij110\rangle, \langle 2, ij110\rangle, \langle 3, ij010\rangle, \langle 4, ij010\rangle, \langle 0, ij010\rangle, \langle 1, ij010\rangle, \langle 2, ij010\rangle\rangle$
$\langle\langle 1, ij000\rangle, \langle 0, ij000\rangle, \langle 4, ij001\rangle, \langle 3, ij011\rangle, \langle 2, ij111\rangle, \langle 1, ij111\rangle, \langle 0, ij111\rangle, \langle 4, ij111\rangle, \langle 3, ij111\rangle, \langle 2, ij011\rangle,$ $\langle 1, ij011\rangle, \langle 0, ij011\rangle, \langle 4, ij011\rangle, \langle 3, ij001\rangle, \langle 2, ij101\rangle, \langle 1, ij101\rangle, \langle 0, ij101\rangle, \langle 4, ij101\rangle, \langle 3, ij101\rangle, \langle 2, ij001\rangle,$ $\langle 1, ij001\rangle, \langle 0, ij001\rangle, \langle 4, ij000\rangle, \langle 3, ij010\rangle, \langle 2, ij010\rangle, \langle 1, ij010\rangle, \langle 0, ij010\rangle, \langle 4, ij010\rangle, \langle 3, ij000\rangle, \langle 2, ij000\rangle,$ $\langle 3, ij100\rangle, \langle 2, ij100\rangle, \langle 1, ij100\rangle, \langle 0, ij100\rangle, \langle 4, ij100\rangle, \langle 3, ij110\rangle, \langle 4, ij110\rangle, \langle 0, ij110\rangle, \langle 1, ij110\rangle, \langle 2, ij110\rangle\rangle$
$\langle\langle 1, ij000\rangle, \langle 0, ij000\rangle, \langle 4, ij000\rangle, \langle 3, ij000\rangle, \langle 2, ij000\rangle, \langle 3, ij100\rangle, \langle 2, ij100\rangle, \langle 1, ij100\rangle, \langle 0, ij100\rangle, \langle 4, ij100\rangle,$ $\langle 3, ij110\rangle, \langle 2, ij010\rangle, \langle 1, ij010\rangle, \langle 0, ij010\rangle, \langle 4, ij010\rangle, \langle 3, ij010\rangle, \langle 2, ij110\rangle, \langle 1, ij110\rangle, \langle 0, ij110\rangle, \langle 4, ij110\rangle,$ $\langle 0, ij111\rangle, \langle 1, ij111\rangle, \langle 2, ij111\rangle, \langle 3, ij011\rangle, \langle 4, ij011\rangle, \langle 0, ij011\rangle, \langle 1, ij011\rangle, \langle 2, ij011\rangle, \langle 3, ij111\rangle, \langle 4, ij111\rangle,$ $\langle 3, ij101\rangle, \langle 4, ij101\rangle, \langle 0, ij101\rangle, \langle 1, ij101\rangle, \langle 2, ij101\rangle, \langle 3, ij001\rangle, \langle 4, ij001\rangle, \langle 0, ij001\rangle, \langle 1, ij001\rangle, \langle 2, ij001\rangle\rangle$
$\langle\langle 1, ij000\rangle, \langle 0, ij000\rangle, \langle 4, ij000\rangle, \langle 3, ij000\rangle, \langle 2, ij000\rangle, \langle 3, ij100\rangle, \langle 2, ij100\rangle, \langle 1, ij100\rangle, \langle 0, ij100\rangle, \langle 4, ij100\rangle,$ $\langle 3, ij110\rangle, \langle 2, ij010\rangle, \langle 1, ij010\rangle, \langle 0, ij010\rangle, \langle 4, ij010\rangle, \langle 3, ij010\rangle, \langle 2, ij110\rangle, \langle 1, ij110\rangle, \langle 0, ij110\rangle, \langle 4, ij110\rangle,$ $\langle 0, ij111\rangle, \langle 1, ij111\rangle, \langle 2, ij111\rangle, \langle 3, ij011\rangle, \langle 4, ij001\rangle, \langle 0, ij001\rangle, \langle 1, ij001\rangle, \langle 2, ij001\rangle, \langle 3, ij001\rangle, \langle 4, ij011\rangle,$ $\langle 0, ij011\rangle, \langle 1, ij011\rangle, \langle 2, ij011\rangle, \langle 3, ij111\rangle, \langle 4, ij111\rangle, \langle 3, ij101\rangle, \langle 4, ij101\rangle, \langle 0, ij101\rangle, \langle 1, ij101\rangle, \langle 2, ij101\rangle\rangle$
$\langle\langle 1, ij000\rangle, \langle 0, ij000\rangle, \langle 4, ij000\rangle, \langle 3, ij000\rangle, \langle 2, ij000\rangle, \langle 3, ij100\rangle, \langle 2, ij100\rangle, \langle 1, ij100\rangle, \langle 0, ij100\rangle, \langle 4, ij100\rangle,$ $\langle 3, ij110\rangle, \langle 2, ij010\rangle, \langle 1, ij010\rangle, \langle 0, ij010\rangle, \langle 4, ij010\rangle, \langle 3, ij010\rangle, \langle 2, ij110\rangle, \langle 1, ij110\rangle, \langle 0, ij110\rangle, \langle 4, ij110\rangle,$ $\langle 0, ij111\rangle, \langle 1, ij111\rangle, \langle 2, ij111\rangle, \langle 3, ij111\rangle, \langle 4, ij111\rangle, \langle 3, ij101\rangle, \langle 4, ij101\rangle, \langle 0, ij101\rangle, \langle 1, ij101\rangle, \langle 2, ij101\rangle,$ $\langle 3, ij001\rangle, \langle 2, ij001\rangle, \langle 1, ij001\rangle, \langle 0, ij001\rangle, \langle 4, ij001\rangle, \langle 3, ij011\rangle, \langle 4, ij011\rangle, \langle 0, ij011\rangle, \langle 1, ij011\rangle, \langle 2, ij011\rangle\rangle$
$\langle\langle 1, ij000\rangle, \langle 0, ij000\rangle, \langle 4, ij000\rangle, \langle 3, ij010\rangle, \langle 2, ij110\rangle, \langle 1, ij110\rangle, \langle 0, ij110\rangle, \langle 4, ij110\rangle, \langle 3, ij110\rangle, \langle 2, ij010\rangle,$ $\langle 1, ij010\rangle, \langle 0, ij010\rangle, \langle 4, ij010\rangle, \langle 3, ij000\rangle, \langle 2, ij000\rangle, \langle 3, ij100\rangle, \langle 2, ij100\rangle, \langle 1, ij100\rangle, \langle 0, ij100\rangle, \langle 4, ij100\rangle,$ $\langle 0, ij101\rangle, \langle 1, ij101\rangle, \langle 2, ij101\rangle, \langle 3, ij001\rangle, \langle 4, ij011\rangle, \langle 0, ij011\rangle, \langle 1, ij011\rangle, \langle 2, ij011\rangle, \langle 3, ij011\rangle, \langle 4, ij001\rangle,$ $\langle 0, ij001\rangle, \langle 1, ij001\rangle, \langle 2, ij001\rangle, \langle 3, ij101\rangle, \langle 4, ij101\rangle, \langle 3, ij111\rangle, \langle 4, ij111\rangle, \langle 0, ij111\rangle, \langle 1, ij111\rangle, \langle 2, ij111\rangle\rangle$

$n = 4$ and $u_4 = \langle 4, 0^2 1^2 0^{n-4}\rangle$ if $n \geq 5$. We partition $BF_{0,1}^{0,0}(n)$ into $\{BF_{0,1,2}^{0,0,0}(n), BF_{0,1,2}^{0,0,1}(n)\}$. By Corollary 3.1, there is a hamiltonian cycle C_0 of $BF_{0,1,2}^{0,0,0}(n)$ including all straight edges of level 2. Thus, we have $(u_1, u_5) \in E(C_0)$. By Lemma 3.6 and Corollary 3.3, there is a hamiltonian cycle C_1 of $BF_{0,1,2}^{0,0,1}(n)$ such that $(u_2, u_4) \in E(C_1)$. It is noticed that s and d_1 are vertices of degree two in $BF_{0,1,2}^{0,0,0}(n)$ and $BF_{0,1,2}^{0,0,1}(n)$, respectively. Therefore, we can write $C_0 = \langle s, u_1, u_5, P_0, d_2, s\rangle$ and $C_1 = \langle d_1, u_2, u_4, P_1, u_3, d_1\rangle$. As an illustrative example, Figure 3.5(a) depicts C_0 and C_1 on $BF_{0,1}^{0,0}(4)$. Figure 3.5(b) illustrates the abstraction of C_0 and C_1 for general n. Since $\{(u_1, u_2), (d_1, u_5)\} \subset E(BF_{0,1}^{0,0}(n))$, we set

$$\begin{aligned} H_1 &= \langle s, d_2, P_0^{-1}, u_5, u_1, u_2, u_4, P_1, u_3, d_1\rangle \text{ and} \\ H_2 &= \langle s, u_1, u_2, u_4, P_1, u_3, d_1, u_5, P_0, d_2\rangle. \end{aligned}$$

Then it can be verified, as shown on Figure 3.5(c), that H_1 and H_2 satisfy the conditions. □

Lemma 3.9. *Given any $k \in \{0, 1\}$ and $n \geq 4$, let (b_1, w_1) be a level-1 straight edge of $BF_{0,1,n-1}^{1,1,k}(n)$ and (b_2, w_2) be a level-0 straight edge of $BF_{0,1,n-1}^{1,1,k}(n)$ such that w_1 and w_2 are two distinct level-1 vertices. Then there exist two hamiltonian paths H_1 and H_2 of $BF_{0,1}^{1,1}(n)$ such that the following conditions are all satisfied:*

(i) $H_1(1) = b_1$ and $H_1(n2^{n-2}) = w_1$,

(ii) $H_2(1) = b_2$ and $H_2(n2^{n-2}) = w_2$, and

(iii) $H_1(t) \neq H_2(t)$ for each $1 \leq t \leq n2^{n-2}$.

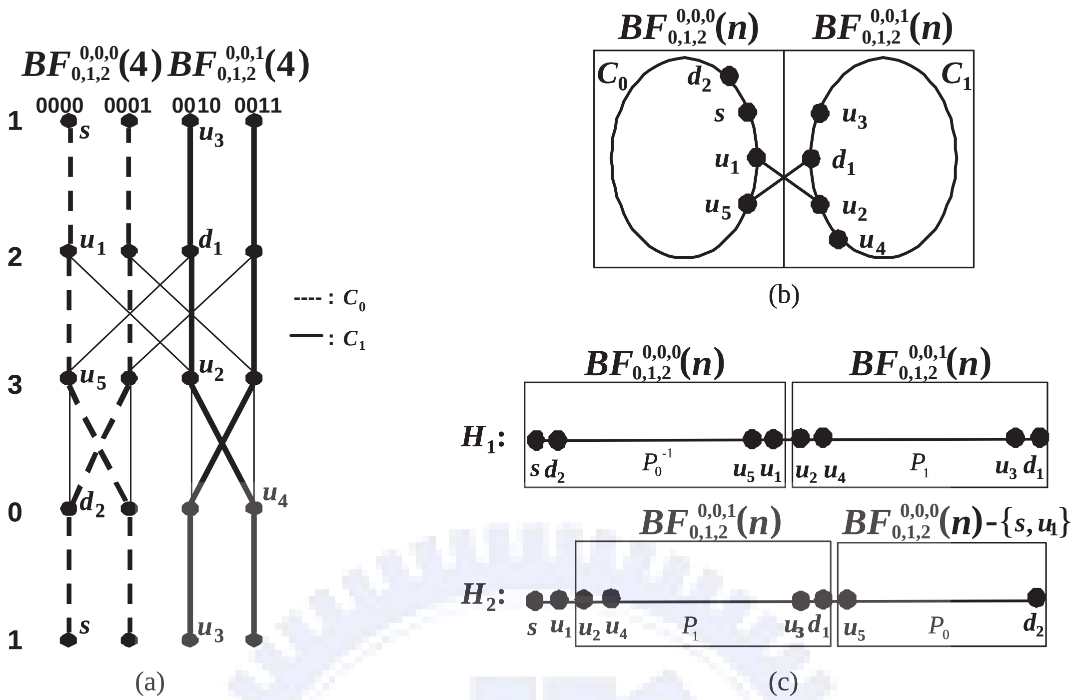

Figure 3.5: Illustration for Lemma 3.8.

Proof. Without loss of generality, we assume that $k = 0$. Let $u_1 = g^{n-3}(b_1)$, $u_2 = f(u_1)$, $u_3 = g(u_2)$, $u_4 = g(u_3)$, $u_5 = g^{n-3}(u_4) = g^{-1}(u_2)$, $u_6 = f(u_5) = g^{-1}(w_1)$, $v_1 = f^{-1}(b_2)$, $v_2 = g^{-n+3}(v_1)$, $v_3 = g^{-1}(v_2)$, $v_4 = g^{-1}(v_3) = g(v_1)$, $v_5 = f^{-1}(v_4) = g^{-1}(b_2)$, and $v_6 = g^{-n+3}(v_5) = g(w_2)$. By Corollary 3.1, $BF^{1,1,0}_{0,1,2}(n)$ has a totally scheduled hamiltonian cycle. By Lemma 3.1, $BF^{1,1,0}_{0,1,n-1}(n)$ is isomorphic with $BF^{1,1,0}_{0,1,2}(n)$. Hence, there also exists a totally scheduled hamiltonian cycle C_0 of $BF^{1,1,0}_{0,1,n-1}(n)$. It is noticed that w_1 is adjacent to u_6. Moreover, w_1, u_6, b_2, and w_2 are all vertices of degree two in $BF^{1,1,0}_{0,1,n-1}(n)$. Accordingly, C_0 can be written as $C_0 = \langle w_1, b_1, P_0, u_1, u_6, w_1\rangle$, where $P_0 = \langle b_1, P_{01}, v_5, b_2, w_2, v_6, P_{02}, u_1\rangle$.

By Lemma 3.6, $BF^{1,1,1}_{0,1,2}(4)$ has a totally scheduled hamiltonian cycle C such that $e \in E(C)$ if $e \in E(BF^{1,1,1}_{0,1,2}(4)) - \{(\langle 3, 1110\rangle, \langle 0, 1110\rangle), (\langle 3, 1111\rangle, \langle 0, 1111\rangle)\}$. By Lemma 3.1, $BF^{1,1,1}_{0,1,3}(4)$ is isomorphic with $BF^{1,1,1}_{0,1,2}(4)$. Hence, $BF^{1,1,1}_{0,1,3}(4)$ has a totally scheduled hamiltonian cycle C such that $e \in E(C)$ if $e \in E(BF^{1,1,1}_{0,1,3}(4)) - \{(\langle 2, 1101\rangle, \langle 3, 1101\rangle), (\langle 2, 1111\rangle, \langle 3, 1111\rangle)\}$. Obviously, (u_5, u_2) is a level-$(n-3)$ for any $n \geq 4$. Therefore, we have $(u_5, u_2) \in E(BF^{1,1,1}_{0,1,3}(4)) - \{ (\langle 2, 1101\rangle, \langle 3, 1101\rangle), (\langle 2, 1111\rangle, \langle 3, 1111\rangle)\}$. It follows that $BF^{1,1,1}_{0,1,3}(4)$ has a totally scheduled hamiltonian cycle C_1 such that $(u_5, u_2) \in E(C_1)$. By Corollary 3.3, $BF^{1,1,1}_{0,1,2}(n)$, $n \geq 5$, has a totally scheduled hamiltonian cycle including any required edge. Since $BF^{1,1,1}_{0,1,n-1}(n)$ is isomorphic with $BF^{1,1,1}_{0,1,2}(n)$, it has a totally scheduled hamiltonian cycle C_1 such that $(u_5, u_2) \in E(C_1)$ if $n \geq 5$. In short, by Lemma 3.6 and Corollary 3.3, there is a totally scheduled hamiltonian cycle C_1 of $BF^{1,1,1}_{0,1,n-1}(n)$ such that $(u_5, u_2) \in E(C_1)$. Since u_2, u_3, v_3, and v_4 are vertices of degree two in $BF^{1,1,1}_{0,1,n-1}(n)$, we write $C_1 = \langle u_3, u_4, P_1, u_5, u_2, u_3\rangle$,

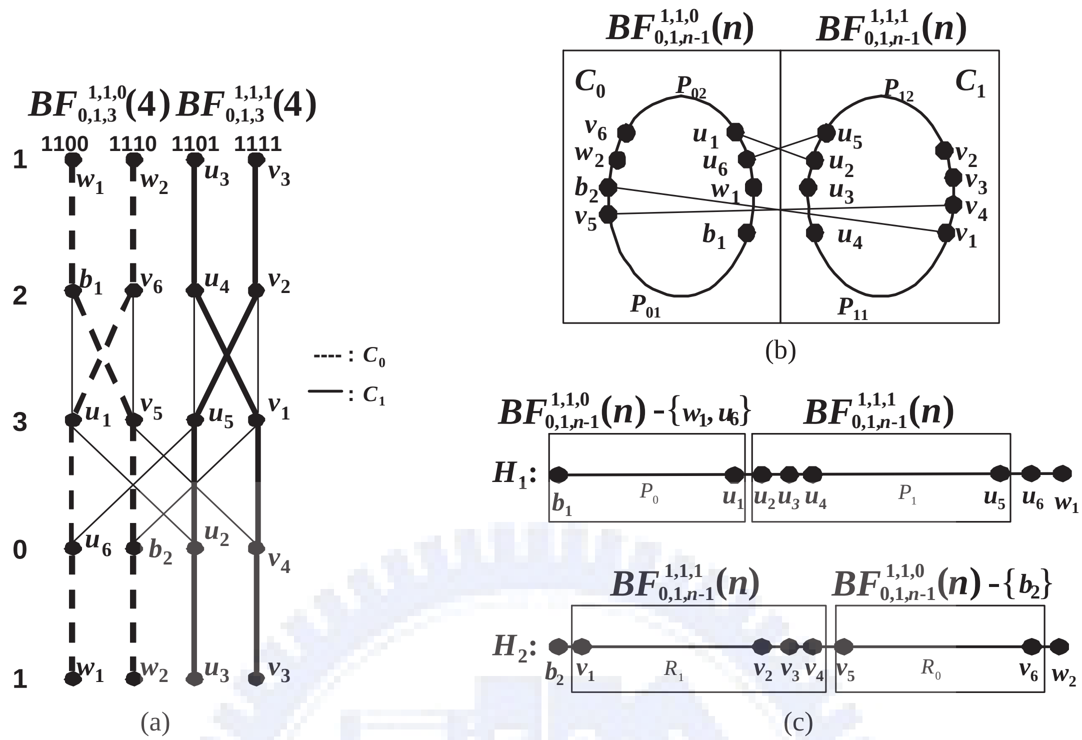

Figure 3.6: Illustration for Lemma 3.9. In (a), $(b_1, w_1) = (\langle 2, 1100\rangle, \langle 1, 1100\rangle)$ and $(b_2, w_2) = (\langle 0, 1110\rangle, \langle 1, 1110\rangle)$ are assumed. In (c), we let $R_1 = \langle v_1,\ P_{11}^{-1},\ u_4,\ u_3,\ u_2,\ u_5,\ P_{12}^{-1},\ v_2\rangle$ and $R_0 = \langle\ v_5,\ P_{01}^{-1},\ b_1,\ w_1,\ u_6,\ u_1,\ P_{02}^{-1},\ v_6\rangle$.

where $P_1 = \langle u_4, P_{11}, v_1, v_4, v_3, v_2, P_{12}, u_5\rangle$. Figure 3.6(a) depicts C_0 and C_1 on $BF^{1,1}_{0,1}(4)$. Figure 3.6(b) illustrates the abstraction of C_0 and C_1 for general n. Then we set

$$\begin{aligned} H_1 &= \langle b_1, P_{01}, v_5, b_2, w_2, v_6, P_{02}, u_1, u_2, u_3, u_4, P_{11}, v_1, v_4, v_3, v_2, P_{12}, u_5, u_6, w_1\rangle \text{ and} \\ H_2 &= \langle b_2, v_1, P_{11}^{-1}, u_4, u_3, u_2, u_5, P_{12}^{-1}, v_2, v_3, v_4, v_5, P_{01}^{-1}, b_1, w_1, u_6, u_1, P_{02}^{-1}, v_6, w_2\rangle. \end{aligned}$$

Since $w_1 \neq w_2$, $u_2 \neq v_2$, $u_3 \neq v_3$, $u_4 \neq v_4$, and $u_6 \neq v_6$, it can be checked that H_1 and H_2 satisfy the conditions. See Figure 3.6(c) for illustration. □

3.3 Cycle embedding

Theorem 3.1. *For all $n \geq 3$, $\mathcal{IHC}(BF(n)) = 4$.*

Proof. It is trivial that $\mathcal{IHC}(BF(n)) \leq \delta(BF(n)) = 4$. Suppose that $n = 3$. Since $BF(3)$ is vertex-transitive, we only find 4-mutually independent hamiltonian cycles starting from vertex $\langle 0, 000\rangle$. A set $\{C_1, C_2, C_3, C_4\}$ of four hamiltonian cycles is listed in Table 3.3. It is easy to check that they are mutually independent.

For $n \geq 4$, we partition $BF(n)$ into $\{BF_{0,1}^{i,j}(n) \mid i,j \in \mathbb{Z}_2\}$. Since $BF(n)$ is vertex-transitive, we assume that the beginning vertex is $s = \langle 1, 0^n \rangle$. Let $u_1 = \langle 2, 0^2 10^{n-3} \rangle$, $u_2 = f^{-1}(u_1) = \langle 1, 01^2 0^{n-3} \rangle$, $u_3 = g^{-1}(u_2) = \langle 0, 01^2 0^{n-3} \rangle$, $u_4 = f(u_3) = \langle 1, 1^3 0^{n-3} \rangle$, $u_5 = g(u_4) = \langle 2, 1^3 0^{n-3} \rangle$, $u_6 = f^{-1}(u_5) = \langle 1, 1010^{n-3} \rangle$, $u_7 = f^{-1}(s) = \langle 0, 10^{n-1} \rangle$, $v_1 = g^{-1}(s) = \langle 0, 0^n \rangle$, $v_2 = f(v_1) = \langle 1, 10^{n-1} \rangle$, $v_3 = g(v_2) = \langle 2, 10^{n-1} \rangle$, $v_4 = f^{-1}(v_3) = \langle 1, 1^2 0^{n-2} \rangle$, $v_5 = g^{-1}(v_4) = \langle 0, 1^2 0^{n-2} \rangle$, $v_6 = f(v_5) = \langle 1, 010^{n-2} \rangle$, and $v_7 = g(v_6) = f(s) = \langle 2, 010^{n-2} \rangle$. Obviously, $\{u_1, u_2, u_3, u_4, u_5, u_6, u_7, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ consists of 14 different vertices of $BF(n)$ such that all (u_1, u_2), (u_3, u_4), (u_5, u_6), (u_7, s), (v_1, v_2), (v_3, v_4), (v_5, v_6), and (v_7, s) are in $E(BF(n))$. By Lemma 3.8, there exist two hamiltonian paths P_1 and P_2 of $BF_{0,1}^{0,0}(n)$ such that (1) P_1 joins s to u_1, (2) P_2 joins s to v_1, and (3) $P_1(1) = P_2(1) = s$ and $P_1(t) \neq P_2(t)$ for each $2 \leq t \leq n2^{n-2}$. By Corollary 3.4, there is a hamiltonian path Q_1 of $BF_{0,1}^{0,1}(n)$ joining u_2 to u_3. Similarly, there is a hamiltonian path R_1 of $BF_{0,1}^{1,0}(n)$ joining u_6 to u_7. By Lemma 3.7, there is a hamiltonian path Q_2 of $BF_{0,1}^{1,0}(n)$ joining v_2 to v_3. Again, there is a hamiltonian path R_2 of $BF_{0,1}^{0,1}(n)$ joining v_6 to v_7. Applying Lemma 3.9, we can find two hamiltonian paths S_1 and S_2 of $BF_{0,1}^{1,1}(n)$ such that (1) S_1 joins u_4 to u_5, (2) S_2 joins v_4 to v_5, and (3) $S_1(t) \neq S_2(t)$ for each $1 \leq t \leq n2^{n-2}$. We set $C_1 = \langle s, P_1, u_1, u_2, Q_1, u_3, u_4, S_1, u_5, u_6, R_1, u_7, s \rangle$ and $C_2 = \langle s, P_2, v_1, v_2, Q_2, v_3, v_4, S_2, v_5, v_6, R_2, v_7, s \rangle$. Figure 3.7(a) and Figure 3.7(b) illustrate C_1 and C_2, respectively. Obviously, C_1 and C_2 are both hamiltonian cycles of $BF(n)$.

In what follows, we claim that C_1 and C_2 are independent: firstly, Lemma 3.8 guarantees that $C_1(t) \neq C_2(t)$ for all $2 \leq t \leq n2^{n-2}$. Next, we have $C_1(t) \neq C_2(t)$ for $n2^{n-2} + 1 \leq t \leq n2^{n-1}$ because C_1 and C_2 pass through the vertices of $BF_{0,1}^{0,1}(n)$ and $BF_{0,1}^{1,0}(n)$, respectively. Moreover, Lemma 3.9 guarantees that $C_1(t) \neq C_2(t)$ for all $n2^{n-1}+1 \leq t \leq 3 \times n2^{n-2}$. Finally, we have $C_1(t) \neq C_2(t)$ for $3 \times n2^{n-2} + 1 \leq t \leq n2^n$ since C_1 and C_2 pass through the vertices of $BF_{0,1}^{1,0}(n)$ and $BF_{0,1}^{0,1}(n)$, respectively. As a consequence, C_1 and C_2 are independent.

Let $u_3' = \langle 0, 01^2 0^{n-4} 1 \rangle$, $u_4' = f(u_3') = \langle 1, 1^3 0^{n-4} 1 \rangle$, $u_5' = g(u_4') = \langle 2, 1^3 0^{n-4} 1 \rangle$, $u_6' = f^{-1}(u_5') = \langle 1, 1010^{n-4} 1 \rangle$, $v_3' = \langle 2, 10^{n-2} 1 \rangle$, $v_4' = f^{-1}(v_3') = \langle 1, 1^2 0^{n-3} 1 \rangle$, $v_5' = g^{-1}(v_4') = \langle 0, 1^2 0^{n-3} 1 \rangle$, and $v_6' = f(v_5') = \langle 1, 010^{n-3} 1 \rangle$. Obviously, $u_i' \neq u_i$ and $v_i' \neq v_i$ for $3 \leq i \leq 6$. By Corollary 3.4, there is a hamiltonian path Q_3 of $BF_{0,1}^{0,1}(n)$ joining u_2 to u_3'. Similarly, there is a hamiltonian path R_3 of $BF_{0,1}^{1,0}(n)$ joining u_6' to u_7. By Lemma 3.7, there is a hamiltonian path Q_4 of $BF_{0,1}^{1,0}(n)$ joining v_2 to v_3'. Similarly, there is a hamiltonian path R_4 of $BF_{0,1}^{0,1}(n)$ joining v_6' to v_7. We apply Lemma 3.9 to construct two hamiltonian paths S_3 and S_4 of $BF_{0,1}^{1,1}(n)$ such that (1) S_3 joins u_4' to u_5', (2) S_4 joins v_4' to v_5', and (3) $S_3(t) \neq S_4(t)$ for all $1 \leq t \leq n2^{n-2}$. Then we set $O_1 = \langle s, P_1, u_1, u_2, Q_3, u_3', u_4', S_3, u_5', u_6', R_3, u_7, s \rangle$ and $O_2 = \langle s, P_2, v_1, v_2, Q_4, v_3', v_4', S_4, v_5', v_6', R_4, v_7, s \rangle$. Similar to C_1 and C_2, O_1 and O_2 are independent.

Let $C_3 = O_1^{-1}$ and $C_4 = O_2^{-1}$. For clarity, we list C_1, C_2, C_3, and C_4 as follows.

$$\begin{aligned}
C_1 &= \langle s, P_1, u_1, u_2, Q_1, u_3, u_4, S_1, u_5, u_6, R_1, u_7, s \rangle, \\
C_2 &= \langle s, P_2, v_1, v_2, Q_2, v_3, v_4, S_2, v_5, v_6, R_2, v_7, s \rangle, \\
C_3 &= \langle s, u_7, R_3^{-1}, u_6', u_5', S_3^{-1}, u_4', u_3', Q_3^{-1}, u_2, u_1, P_1^{-1}, s \rangle, \text{ and} \\
C_4 &= \langle s, v_7, R_4^{-1}, v_6', v_5', S_4^{-1}, v_4', v_3', Q_4^{-1}, v_2, v_1, P_2^{-1}, s \rangle.
\end{aligned}$$

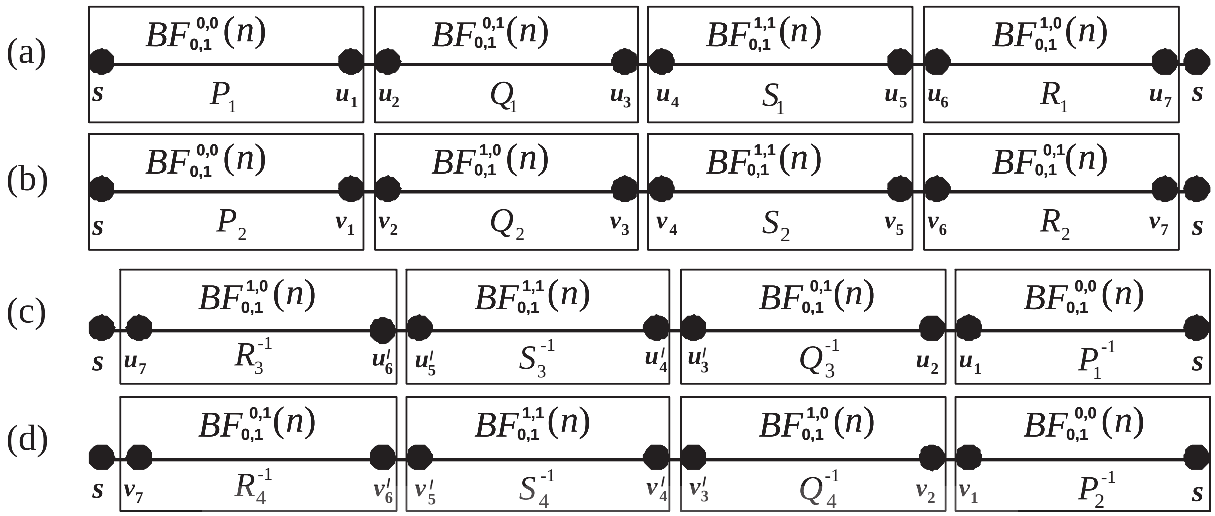

Figure 3.7: Illustration for Theorem 3.1. (a) C_1; (b) C_2; (c) C_3; (d) C_4.

Then it is easy to check that C_1, C_2, C_3, and C_4 are 4-mutually independent hamiltonian cycles of $BF(n)$ starting from vertex s. See Figures 3.7 for illustration. □

Table 3.3: 4-mutually independent hamiltonian cycles C_1, C_2, C_3, C_4 of $BF(3)$ starting from vertex $\langle 0, 000\rangle$.

C_1	$\langle\langle 0,000\rangle, \langle 2,001\rangle, \langle 0,001\rangle, \langle 1,001\rangle, \langle 2,011\rangle, \langle 0,011\rangle, \langle 1,011\rangle, \langle 0,111\rangle, \langle 2,111\rangle, \langle 1,111\rangle, \langle 2,101\rangle, \langle 1,101\rangle,$ $\langle 0,101\rangle, \langle 2,100\rangle, \langle 0,100\rangle, \langle 1,100\rangle, \langle 2,110\rangle, \langle 0,110\rangle, \langle 1,110\rangle, \langle 0,010\rangle, \langle 2,010\rangle, \langle 1,010\rangle, \langle 2,000\rangle, \langle 1,000\rangle, \langle 0,000\rangle\rangle$
C_2	$\langle\langle 0,000\rangle, \langle 1,000\rangle, \langle 2,000\rangle, \langle 0,001\rangle, \langle 1,001\rangle, \langle 2,011\rangle, \langle 0,011\rangle, \langle 1,111\rangle, \langle 2,101\rangle, \langle 0,101\rangle, \langle 1,101\rangle, \langle 2,111\rangle,$ $\langle 0,110\rangle, \langle 1,010\rangle, \langle 2,010\rangle, \langle 0,010\rangle, \langle 1,110\rangle, \langle 2,100\rangle, \langle 0,100\rangle, \langle 1,100\rangle, \langle 2,110\rangle, \langle 0,111\rangle, \langle 1,011\rangle, \langle 2,001\rangle, \langle 0,000\rangle\rangle$
C_3	$\langle\langle 0,000\rangle, \langle 1,100\rangle, \langle 2,100\rangle, \langle 0,100\rangle, \langle 1,000\rangle, \langle 2,010\rangle, \langle 0,010\rangle, \langle 1,110\rangle, \langle 2,110\rangle, \langle 0,111\rangle, \langle 1,111\rangle, \langle 0,011\rangle,$ $\langle 2,011\rangle, \langle 1,011\rangle, \langle 2,001\rangle, \langle 0,001\rangle, \langle 1,001\rangle, \langle 0,101\rangle, \langle 2,101\rangle, \langle 1,101\rangle, \langle 2,111\rangle, \langle 0,110\rangle, \langle 1,010\rangle, \langle 2,000\rangle, \langle 0,000\rangle\rangle$
C_4	$\langle\langle 0,000\rangle, \langle 2,000\rangle, \langle 1,000\rangle, \langle 2,010\rangle, \langle 0,010\rangle, \langle 1,010\rangle, \langle 0,110\rangle, \langle 2,110\rangle, \langle 1,110\rangle, \langle 2,100\rangle, \langle 0,101\rangle, \langle 1,001\rangle,$ $\langle 2,001\rangle, \langle 0,001\rangle, \langle 1,101\rangle, \langle 2,111\rangle, \langle 0,111\rangle, \langle 1,011\rangle, \langle 2,011\rangle, \langle 0,011\rangle, \langle 1,111\rangle, \langle 2,101\rangle, \langle 0,100\rangle, \langle 1,100\rangle, \langle 0,000\rangle\rangle$

Chapter 4

Mutually Independent Hamiltonian Cycles in Faulty Networks

As we have introduced in the preceding chapter, many popular interconnection networks have the maximum numbers of mutually independent hamiltonian cycles. In this chapter, we will show that such a promising property can be preserved even if there are some faulty edges in networks. In particular, we concern both faulty hypercubes and faulty star networks. To simplify our discussion, we permit faulty edges to take place everywhere.

4.1 Faulty hypercubes

As Latifi et al. [39] showed, an n-cube has a hamiltonian cycle even if it has $n-2$ faulty edges. As usual let Q_n denote an n-cube. By definition, we know that Q_n is n-regular. It is also known that Q_n has a recursive construction; that is, it can be decomposed into two $(n-1)$-dimensional subcubes [55]. Let Q_n^j be the subgraph of Q_n induced by $\{\mathbf{u} \in V(Q_n) \mid (\mathbf{u})_{n-1} = j\}$ for $j \in \{0,1\}$. Obviously, Q_n^j is isomorphic to Q_{n-1}. Then an $(n-1)$-*partition* of Q_n divides the Q_n along dimension n into Q_n^0 and Q_n^1. The set of crossing edges between Q_n^0 and Q_n^1, denoted by $E_c = \{(\mathbf{u},\mathbf{v}) \in E(Q_n) \mid \mathbf{u} \in V(Q_n^0), \mathbf{v} \in V(Q_n^1)\}$, consists of all $(n-1)$-dimensional edges of Q_n. Besides the recursive structure, Q_n is both vertex-transitive and edge-transitive [55]. For convenience, we use $\mathbf{e}$ to denote the identity vertex 0^n of Q_n.

Sun et al. [59] proved that $\mathcal{IHC}(Q_n) = n-1$ if $n \in \{1,2,3\}$, and $\mathcal{IHC}(Q_n) = n$ if $n \geq 4$. In this section, we would like to show that Q_n contains $(n-1-f)$-mutually independent hamiltonian cycles even if $f \leq n-2$ faulty edges occur accidentally.

Theorem 4.1. *[59] $\mathcal{IHC}(Q_n) = n-1$ if $n \in \{1,2,3\}$ and $\mathcal{IHC}(Q_n) = n$ if $n \geq 4$.*

The following results are fault-tolerant properties of hypercubes.

Theorem 4.2. *[64] Let $n \geq 3$. Suppose that $F \subseteq E(Q_n)$ is a set of at most $n-2$ faulty links. Then $Q_n - F$ is both hamiltonian laceable and strongly hamiltonian laceable.*

Theorem 4.3. *[64] Let $n \geq 3$. Suppose that $F \subseteq E(Q_n)$ is a set of at most $n-3$ faulty edges. Then $Q_n - F$ is hyper-hamiltonian laceable.*

Lemma 4.1. *[59] Let $n \geq 4$. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are any two vertices from different partite sets of Q_n. Then $Q_n - \{\mathbf{x}, \mathbf{v}\}$ is hamiltonian laceable.*

4.1.1 Mutually fully-independent hamiltonian paths in faulty hypercubes

To embed mutually independent hamiltonian cycles into faulty hypercubes, we need the following lemmas.

Lemma 4.2. *[59] Let Q_n be an n-cube for $n \geq 2$. Suppose that $\{(\mathbf{w}_i, \mathbf{b}_i) \in E(Q_n) \mid \mathbf{w}_i \in V_0(Q_n), \mathbf{b}_i \in V_1(Q_n), 1 \leq i \leq n-1\}$ consists of $n-1$ distinct edges with no shared endpoints. Then Q_n contains $(n-1)$-mutually fully independent hamiltonian paths $P_1[\mathbf{w}_1, \mathbf{b}_1], \ldots, P_{n-1}[\mathbf{w}_{n-1}, \mathbf{b}_{n-1}]$.*

Let F be a set of faulty edges of Q_n. Suppose that Q_n is partitioned along dimension n into Q_n^0 and Q_n^1, and E_c is the set of crossing edges between Q_n^0 and Q_n^1. Then we define $F_0 = F \cap E(Q_n^0)$, $F_1 = F \cap E(Q_n^1)$, and $F_c = F \cap E_c$. Moreover, we set $\delta = n - 1 - |F|$ in the remainder of this chapter. To tolerate faulty edges in hypercubes, we have the next lemma.

Lemma 4.3. *Let $F \subseteq E(Q_n)$ be a set of at most $n-2$ faulty edges for $n \geq 3$. Suppose that $A = \{(\mathbf{w}_i, \mathbf{b}_i) \in E(Q_n) \mid \mathbf{w}_i \in V_0(Q_n), \mathbf{b}_i \in V_1(Q_n), 1 \leq i \leq \delta\}$ consists of δ distinct edges with no shared endpoints. Then $Q_n - F$ contains δ-mutually fully independent hamiltonian paths $P_1[\mathbf{w}_1, \mathbf{b}_1], \ldots, P_\delta[\mathbf{w}_\delta, \mathbf{b}_\delta]$.*

Proof. This proof proceeds by induction on n. First suppose $|F| = 0$. Then this case follows from Lemma 4.2. Suppose $|F| = n-2$. Then we have $\delta = n - 1 - (n-2) = 1$. By Theorem 4.2, $Q_n - F$ has a hamiltonian path between any two vertices from different partite sets. Obviously, the statement holds for Q_3, as the induction basis. In what follows we only consider $1 \leq |F| \leq n-3$ and $n \geq 4$. As the inductive hypothesis, suppose that the result is true for Q_{n-1}.

Since $\delta + |F| = n - 1 < n$, there must exist a dimension d of $\{0, 1, \ldots, n-1\}$ such that $A \cup F$ contains no d-dimensional edges. Since Q_n is edge-transitive, we can assume $d = n-1$. Then we partition Q_n into Q_n^0 and Q_n^1 along dimension $n-1$. Thus each edge of $A \cup F$ is in either Q_n^0 or Q_n^1. Let $r_0 = |\{(\mathbf{w}_i, \mathbf{b}_i) \in E(Q_n^0) \mid 1 \leq i \leq \delta\}|$ and $r_1 = |\{(\mathbf{w}_i, \mathbf{b}_i) \in E(Q_n^1) \mid 1 \leq i \leq \delta\}|$. Clearly, $r_0 + r_1 = \delta$. Without loss of generality, we assume $\{(\mathbf{w}_1, \mathbf{b}_1), \ldots, (\mathbf{w}_{r_0}, \mathbf{b}_{r_0})\} \subset E(Q_n^0)$ and $\{(\mathbf{w}_{r_0+1}, \mathbf{b}_{r_0+1}), \ldots, (\mathbf{w}_\delta, \mathbf{b}_\delta)\} \subset E(Q_n^1)$. Since $n - 1 = \delta + |F| = r_0 + r_1 + |F_0| + |F_1|$, we have $r_i + |F_j| \leq n-1$ for any $i, j \in \{0, 1\}$. Then we have to take the following cases into account.

Case 1: Suppose $r_i + |F_j| \leq n-2$ for any $i, j \in \{0, 1\}$. Since $r_0 + |F_0| \leq n-2$, $r_0 \leq n - 2 - |F_0| = (n-1) - 1 - |F_0|$. By the inductive hypothesis, $Q_n^0 - F_0$ has r_0-mutually fully independent hamiltonian paths $H_i[\mathbf{w}_i, \mathbf{b}_i]$, $1 \le i \le r_0$. Obviously, $H_i[\mathbf{w}_i, \mathbf{b}_i]$ can be represented as $\langle \mathbf{w}_i, H'_i, \mathbf{u}_i, \mathbf{b}_i \rangle$, where $\mathbf{u}_i$ is some vertex adjacent to $\mathbf{b}_i$. Similarly, $Q^1_n - F_1$ has r_1-mutually fully independent hamiltonian paths $H_i[\mathbf{w}_i, \mathbf{b}_i] = \langle \mathbf{w}_i, H'_i, \mathbf{u}_i, \mathbf{b}_i \rangle$, $r_0+1 \le i \le \delta$.

Next, we construct r_0 paths in $Q^1_n - F_1$ to incorporate the previously established r_0 paths of $Q^0_n - F_0$. Since $r_0 + |F_1| \le n-2$, we have $r_0 \le n-2-|F_1|$. By the inductive hypothesis, $Q^1_n - F_1$ also contains r_0-mutually fully independent hamiltonian paths $R_1[(\mathbf{u}_1)^{n-1}, (\mathbf{b}_1)^{n-1}], \ldots, R_{r_0}[(\mathbf{u}_{r_0})^{n-1}, (\mathbf{b}_{r_0})^{n-1}]$. Similarly, $Q^0_n - F_0$ also contains r_1-mutually fully independent hamiltonian paths $R_{r_0+1}[(\mathbf{u}_{r_0+1})^{n-1}, (\mathbf{b}_{r_0+1})^{n-1}]$, $\ldots$, $R_\delta[(\mathbf{u}_\delta)^{n-1}, (\mathbf{b}_\delta)^{n-1}]$. Accordingly, we set $P_i[\mathbf{w}_i, \mathbf{b}_i] = \langle \mathbf{w}_i, H'_i, \mathbf{u}_i, (\mathbf{u}_i)^{n-1}, R_i, (\mathbf{b}_i)^{n-1}, \mathbf{b}_i \rangle$ for every $1 \le i \le \delta$. Thus, $\{P_1, \ldots, P_\delta\}$ forms a set of δ-mutually fully independent hamiltonian paths in $Q_n - F$. See Figure 4.1(a) for illustration.

Case 2: Suppose $r_i + |F_i| = n-1$ for some $i \in \{0,1\}$. Without loss of generality, we assume $r_0 + |F_0| = n-1$. Since $r_0 = n-1-|F_0| \ge n-1-|F| = \delta$, we have $r_0 = \delta$ and $|F_0| = |F| \le n-3$. Note that $r_0 - 1 = \delta - 1 = n-2-|F| = (n-1)-1-|F_0|$. By the inductive hypothesis, $Q^0_n - F_0$ has (r_0-1)-mutually fully independent hamiltonian paths $H_i[\mathbf{w}_i, \mathbf{b}_i]$, $2 \le i \le r_0$. Again, $H_i[\mathbf{w}_i, \mathbf{b}_i]$ can be represented as $\langle \mathbf{w}_i, H'_i, \mathbf{u}_i, \mathbf{b}_i \rangle$, where $\mathbf{u}_i$ is some vertex adjacent to $\mathbf{b}_i$.

Subcase 2.1: Suppose $n = 4$. Thus we have $r_0 = 2$. By Theorem 4.3, $Q^0_4 - F_0$ has a hamiltonian path $H_1[\mathbf{w}_1, \mathbf{b}_1] = \langle \mathbf{w}_1, \mathbf{u}_1, H'_1, (\mathbf{b}_1)^j, \mathbf{b}_1 \rangle$, where $\mathbf{u}_1$ is a vertex adjacent to $\mathbf{w}_1$, and j is some integer of $\{0,1,2,3\}$. Let $X = \{((\mathbf{u}_1)^3, (\mathbf{u}_2)^3)\}$. Similarly, there are two hamiltonian paths $R_1[(\mathbf{w}_1)^3, (\mathbf{u}_1)^3]$ and $R_2[(\mathbf{u}_2)^3, (\mathbf{b}_2)^3]$ in $Q^1_4 - X$. Obviously, we see that $R_1(7) \ne R_2(1)$ and $R_1(8) \ne R_2(2)$. Then we set $P_1[\mathbf{w}_1, \mathbf{b}_1] = \langle \mathbf{w}_1, (\mathbf{w}_1)^3, R_1, (\mathbf{u}_1)^3, \mathbf{u}_1, H'_1, (\mathbf{b}_1)^j, \mathbf{b}_1 \rangle$ and $P_2[\mathbf{w}_2, \mathbf{b}_2] = \langle \mathbf{w}_2, H'_2, \mathbf{u}_2, (\mathbf{u}_2)^3, R_2, (\mathbf{b}_2)^3, \mathbf{b}_2 \rangle$. Consequently, $\{P_1, P_2\}$ forms a set of 2-mutually fully independent hamiltonian paths in $Q_4 - F$. See Figure 4.1(b) for illustration.

Subcase 2.2: Suppose $n \ge 5$. We first consider $|F_0| \le n-4$. By the inductive hypothesis, Q^1_n has (r_0-1)-mutually fully independent hamiltonian paths $R_i[(\mathbf{u}_i)^{n-1}, (\mathbf{b}_i)^{n-1}]$, $2 \le i \le r_0$. Then we can choose an integer j of $\{0,1,\ldots,n-2\}$ such that both $(\mathbf{b}_1)^j \ne \mathbf{w}_1$ and $((\mathbf{b}_1)^j)^{n-1} \notin \{R_i(2^{n-1}-1) \mid 2 \le i \le r_0\}$ are satisfied. Since $r_0 = n-1-|F| \le n-2$, such an integer exists. By Theorem 4.3, $Q^0_n - (F_0 \cup \{\mathbf{b}_1\})$ has a hamiltonian path $H_1[\mathbf{w}_1, (\mathbf{b}_1)^j] = \langle \mathbf{w}_1, \mathbf{u}_1, H'_1, (\mathbf{b}_1)^j \rangle$, where $\mathbf{u}_1$ is some vertex adjacent to $\mathbf{w}_1$. By Lemma 4.1, there exists a hamiltonian path $R_1[(\mathbf{w}_1)^{n-1}, (\mathbf{u}_1)^{n-1}]$ in $Q^1_n - \{(\mathbf{b}_1)^{n-1}, ((\mathbf{b}_1)^j)^{n-1}\}$. Then we set $P_1[\mathbf{w}_1, \mathbf{b}_1] = \langle \mathbf{w}_1, (\mathbf{w}_1)^{n-1}, R_1, (\mathbf{u}_1)^{n-1}, \mathbf{u}_1, H'_1, (\mathbf{b}_1)^j, ((\mathbf{b}_1)^j)^{n-1}, (\mathbf{b}_1)^{n-1}, \mathbf{b}_1 \rangle$ and $P_i[\mathbf{w}_i, \mathbf{b}_i] = \langle \mathbf{w}_i, H'_i, \mathbf{u}_i, (\mathbf{u}_i)^{n-1}, R_i, (\mathbf{b}_i)^{n-1}, \mathbf{b}_i \rangle$ for $2 \le i \le r_0$. As a result, $\{P_1, \ldots, P_{r_0}\}$ forms a set of r_0-mutually fully independent hamiltonian paths in $Q_n - F$. See Figure 4.1(c) for illustration.

Next, we consider $|F_0| = n-3$. Thus, we have $r_0 = 2$. By Theorem 4.2, $Q^0_n - F_0$ has a hamiltonian path $H_1[\mathbf{w}_1, \mathbf{b}_1] = \langle \mathbf{w}_1, \mathbf{u}_1, H'_1, (\mathbf{b}_1)^j, \mathbf{b}_1 \rangle$, where $\mathbf{u}_1$ is a vertex adjacent to $\mathbf{w}_1$, and j is some integer of $\{0,1,\ldots,n-2\}$. By Lemma 4.1, there exists a hamiltonian path $R_1[(\mathbf{w}_1)^{n-1}, (\mathbf{u}_1)^{n-1}]$ in $Q^1_n - \{(\mathbf{b}_1)^{n-1}, ((\mathbf{b}_1)^j)^{n-1}\}$. By the inductive hypothesis, $Q^1_n - \{((\mathbf{b}_2)^{n-1}, ((\mathbf{b}_1)^j)^{n-1})\}$ has a hamiltonian path $R_2[(\mathbf{u}_2)^{n-1}, (\mathbf{b}_2)^{n-1}]$. Obviously, we have $R_2(2^{n-1}-1) \ne ((\mathbf{b}_1)^j)^{n-1}$. Again, we set $P_1[\mathbf{w}_1, \mathbf{b}_1] = \langle \mathbf{w}_1, (\mathbf{w}_1)^{n-1}, R_1, (\mathbf{u}_1)^{n-1}, \mathbf{u}_1, H'_1,$

$(\mathbf{b}_1)^j, ((\mathbf{b}_1)^j)^{n-1}, (\mathbf{b}_1)^{n-1}, \mathbf{b}_1\rangle$ and $P_2[\mathbf{w}_2, \mathbf{b}_2] = \langle \mathbf{w}_2, H_2', \mathbf{u}_2, (\mathbf{u}_2)^{n-1}, R_2, (\mathbf{b}_2)^{n-1}, \mathbf{b}_2\rangle$. Hence $\{P_1, P_2\}$ forms a set of 2-mutually fully independent hamiltonian paths in $Q_n - F$. See Figure 4.1(c).

Case 3: Suppose that $r_i + |F_{1-i}| = n-1$ for some $i \in \{0,1\}$. Without loss of generality, we assume $r_1 + |F_0| = n-1$. Since $r_1 = n-1-|F_0| \geq n-1-|F| = \delta$, we have $r_1 = \delta$ and $F_0 = F$. By the inductive hypothesis, Q_n^1 has (r_1-1)-mutually fully independent hamiltonian paths $H_i[\mathbf{w}_i, \mathbf{b}_i] = \langle \mathbf{w}_i, H_i', \mathbf{u}_i, \mathbf{b}_i\rangle$, where $\mathbf{u}_i$ is some vertex adjacent to $\mathbf{b}_i$ with $1 \leq i \leq r_1 - 1$. Since $r_1 - 1 = \delta - 1 = n-2-|F| = (n-1)-1-|F_0|$, $Q_n^0 - F_0$ has (r_1-1)-mutually fully independent hamiltonian paths $R_i[(\mathbf{u}_i)^{n-1}, (\mathbf{b}_i)^{n-1}]$, $1 \leq i \leq r_1 - 1$. Then we set $P_i[\mathbf{w}_i, \mathbf{b}_i] = \langle \mathbf{w}_i, H_i', \mathbf{u}_i, (\mathbf{u}_i)^{n-1}, R_i, (\mathbf{b}_i)^{n-1}, \mathbf{b}_i\rangle$ with $1 \leq i \leq r_1 - 1$. Next, we have to choose a vertex $\mathbf{v}$ of $V_0(Q_n^0)$, and construct a hamiltonian path $R_{r_1}[(\mathbf{w}_{r_1})^{n-1}, \mathbf{v}]$ in $Q_n^0 - F_0$ such that $v \neq R_i(2)$ and $R_{r_1}(2^{n-1}-1) \neq (\mathbf{u}_i)^{n-1}$ for every $1 \leq i \leq r_1 - 1$. We distinguish the following subcases.

Subcase 3.1: Suppose $n \neq 5$ or $|F| > 1$. One can see that $(\mathbf{u}_1)^{n-1}, \ldots, (\mathbf{u}_{r_1-1})^{n-1}$ have at most $(r_1-1)(n-1)$ neighbors in Q_n^0. Since $|V_0(Q_n^0)| = 2^{n-2} > (r_1-1)(n-1) = (n-2-|F|)(n-1)$ in this subcase, we can choose $\mathbf{v}$ other than all neighbors of $(\mathbf{u}_1)^{n-1}, \ldots, (\mathbf{u}_{r_1-1})^{n-1}$. Obviously, we have $\mathbf{v} \neq R_i(2)$ for $1 \leq i \leq r_1 - 1$. By Theorem 4.2, there exists a hamiltonian path $R_{r_1}[(\mathbf{w}_{r_1})^{n-1}, \mathbf{v}]$ in $Q_n^0 - F_0$. Since $\mathbf{v}$ is not adjacent to any node of $\{(\mathbf{u}_1)^{n-1}, \ldots, (\mathbf{u}_{r_1-1})^{n-1}\}$, we have $R_{r_1}(2^{n-1}-1) \neq (\mathbf{u}_i)^{n-1}$ for every $1 \leq i \leq r_1 - 1$. By Theorem 4.3, there exists a hamiltonian path $H_{r_1}[(\mathbf{v})^{n-1}, \mathbf{b}_{r_1}]$ in $Q_n^1 - \{\mathbf{w}_{r_1}\}$. Then we set $P_{r_1} = \langle \mathbf{w}_{r_1}, (\mathbf{w}_{r_1})^{n-1}, R_{r_1}, \mathbf{v}, (\mathbf{v})^{n-1}, H_{r_1}, \mathbf{b}_{r_1}\rangle$. Consequently, $\{P_1, \ldots, P_{r_1}\}$ forms a set of r_1-mutually fully independent hamiltonian paths in $Q_n - F$. See Figure 4.1(d) for illustration.

In the following, we consider $n = 5$ and $|F| = 1$; that is, $r_1 = 3$.

Subcase 3.2: Suppose that $n = 5$, $|F| = 1$, and $(\mathbf{u}_1)^{n-1}$ and $(\mathbf{u}_2)^{n-1}$ have at least one common neighbor. Since $|V_0(Q_n^0)| = 2^{n-2} = 8 > 7 = (r_1-1)(n-1)-1$, we still can choose a vertex $\mathbf{v}$ from $V_0(Q_n^0)$ other than all neighbors of $(\mathbf{u}_1)^{n-1}$ and $(\mathbf{u}_2)^{n-1}$. Obviously, we have $\mathbf{v} \neq R_i(2)$ for $1 \leq i \leq r_1 - 1$. By Theorem 4.2, there exists a hamiltonian path $R_{r_1}[(\mathbf{w}_{r_1})^{n-1}, \mathbf{v}]$ in $Q_n^0 - F_0$ such that $R_{r_1}(2^{n-1}-1) \neq (\mathbf{u}_i)^{n-1}$ for every $1 \leq i \leq r_1 - 1$. By Theorem 4.3, there exists a hamiltonian path $H_{r_1}[(\mathbf{v})^{n-1}, \mathbf{b}_{r_1}]$ in $Q_n^1 - \{\mathbf{w}_{r_1}\}$. Similarly, we set $P_{r_1} = \langle \mathbf{w}_{r_1}, (\mathbf{w}_{r_1})^{n-1}, R_{r_1}, \mathbf{v}, (\mathbf{v})^{n-1}, H_{r_1}, \mathbf{b}_{r_1}\rangle$. Then $\{P_1, \ldots, P_{r_1}\}$ forms a set of r_1-mutually fully independent hamiltonian paths in $Q_n - F$. See Figure 4.1(d).

Subcase 3.3: Suppose that $n = 5$, $|F| = 1$, and $(\mathbf{u}_1)^{n-1}$ and $(\mathbf{u}_2)^{n-1}$ have no common neighbors. Then we assign the vertex $\mathbf{v}$ as the one that is adjacent to $(\mathbf{u}_1)^{n-1}$ but not identical to $R_1(2)$. Obviously, we have $\mathbf{v} \neq R_i(2)$ for $1 \leq i \leq r_1 - 1$. By Theorem 4.2, $Q_n^0 - (F_0 \cup \{(\mathbf{v}, (\mathbf{u}_1)^{n-1})\})$ remains hamiltonian laceable. Thus there exists a hamiltonian path $R_{r_1}[(\mathbf{w}_{r_1})^{n-1}, \mathbf{v}]$ of $Q_n^0 - (F_0 \cup \{(\mathbf{v}, (\mathbf{u}_1)^{n-1})\})$ such that $R_{r_1}(2^{n-1}-1) \neq (\mathbf{u}_i)^{n-1}$ for every $1 \leq i \leq r_1 - 1$. By Theorem 4.3, there exists a hamiltonian path $H_{r_1}[(\mathbf{v})^{n-1}, \mathbf{b}_{r_1}]$ in $Q_n^1 - \{\mathbf{w}_{r_1}\}$. Similarly, we set $P_{r_1} = \langle \mathbf{w}_{r_1}, (\mathbf{w}_{r_1})^{n-1}, R_{r_1}, \mathbf{v}, (\mathbf{v})^{n-1}, H_{r_1}, \mathbf{b}_{r_1}\rangle$. Then $\{P_1, \ldots, P_{r_1}\}$ forms a set of r_1-mutually fully independent hamiltonian paths in $Q_n - F$. See Figure 4.1(d). □

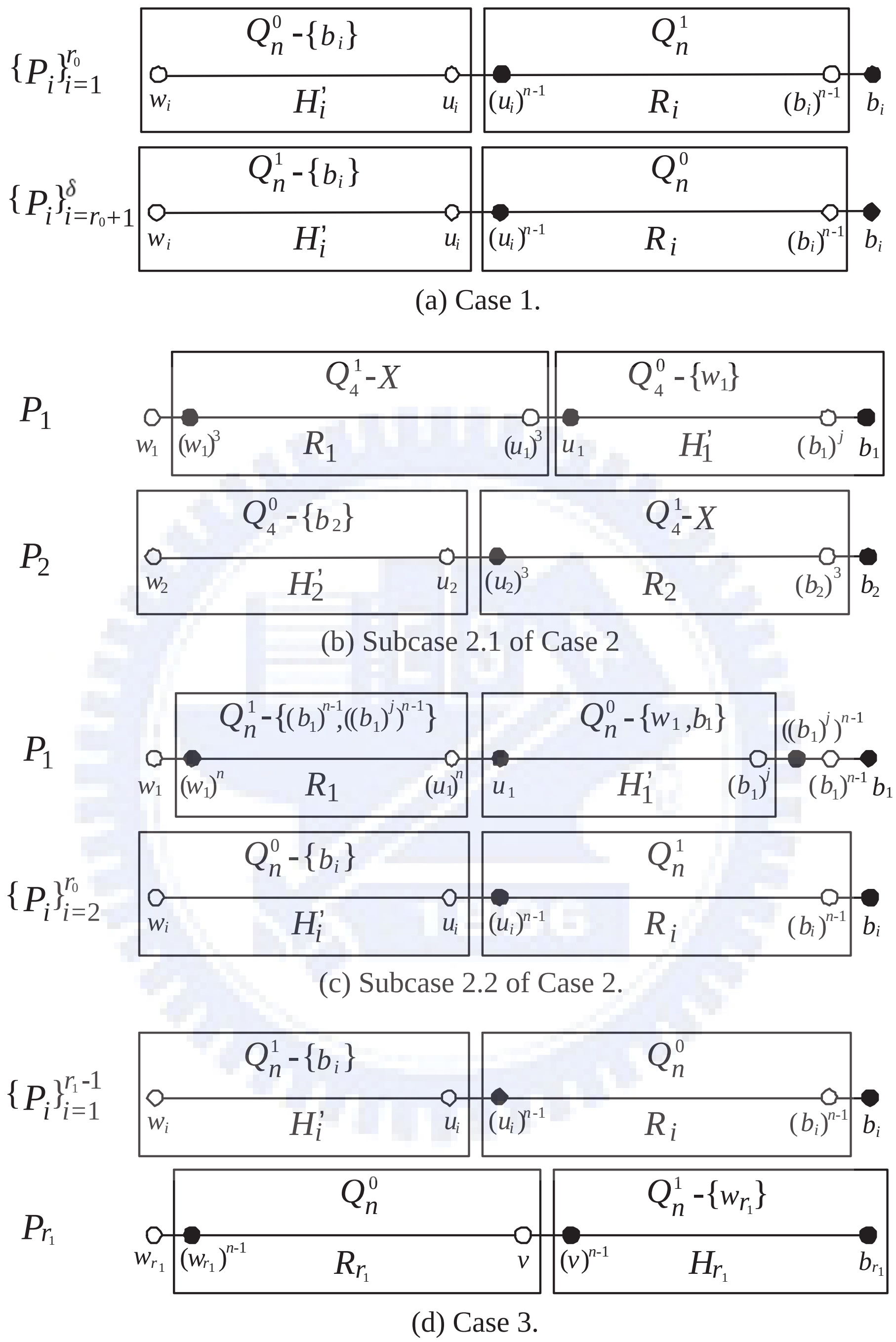

Figure 4.1: Illustration for the proof of Lemma 4.3.

4.1.2 The main theorem

With Lemma 4.3, we can construct the maximum number of mutually independent hamiltonian cycles on faulty hypercubes.

Theorem 4.4. *Let $n \geq 3$. Suppose that $F \subseteq E(Q_n)$ consists of at most $n-2$ faulty edges. Then $Q_n - F$ contains $(n-1-|F|)$-mutually independent hamiltonian cycles beginning from any vertex.*

Proof. Since Q_n is vertex-transitive, we only need to construct δ-mutually independent hamiltonian cycles beginning from $\mathbf{e} = 0^n$. Suppose $|F| = 0$. Then the statement follows from Theorem 4.1. Thus, we only consider the situation that F is nonempty. Furthermore, since Q_n is edge-transitive, we assume that at least one faulty edge is an n-dimensional edge.

The proof idea is based on the partition of Q_n. As discussed previously, Q_n can be partitioned into $\{Q_n^0, Q_n^1\}$. Obviously, $\mathbf{e}$ is located in Q_n^0. Recall that F_0 and F_1 denote the sets of faulty edges in Q_n^0 and Q_n^1, respectively. Then the proof idea is outlined as follows:

(1) We first build δ-mutually independent hamiltonian cycles $C_1, C_2, \ldots, C_\delta$ beginning from $\mathbf{e}$ in $Q_n^0 - F_0$.

(2) Next, we have to claim that there must exist an integer t, $1 \leq t \leq 2^{n-2}$, so that the crossing edges $(C_i(2t-1), (C_i(2t-1))^{n-1})$ and $(C_i(2t), (C_i(2t))^{n-1})$ are fault-free for all $1 \leq i \leq \delta$. For convenience, let $\mathbf{x}_i = C_i(2t-1)$ and $\mathbf{y}_i = C_i(2t)$.

(3) By Lemma 4.3, $Q_n^1 - F_1$ contains δ-mutually fully independent hamiltonian paths $R_1[(\mathbf{x}_1)^{n-1}, (\mathbf{y}_1)^{n-1}], \ldots, R_\delta[(\mathbf{x}_\delta)^{n-1}, (\mathbf{y}_\delta)^{n-1}]$.

(4) Finally, we obtain the desired hamiltonian cycles from combining C_i and R_i, $1 \leq i \leq \delta$. See Figure 4.2 for illustration.

More precisely, the proof is by induction on n. It is trivial that the statement holds for Q_3, as the induction basis. When $n \geq 4$, we assume that the statement holds for Q_{n-1}. Now we consider how to build δ-mutually independent hamiltonian cycles in $Q_n - F$. Since we assume there is at least one n-dimensional faulty edge, we partition Q_n into $\{Q_n^0, Q_n^1\}$ along dimension n. Accordingly, we have $|F_0| \leq |F| - 1 \leq n-3$, $|F_1| \leq |F| - 1 \leq n-3$, and $(n-1) - 1 - |F_0| \geq (n-1) - 1 - (|F|-1) = n - 1 - |F| = \delta$. Thus, by the inductive hypothesis, $Q_n^0 - F_0$ contains δ-mutually independent hamiltonian cycles $C_1, C_2, \ldots, C_\delta$ beginning from $\mathbf{e}$. For convenience, we assume that the vertices on each cycle are indexed sequentially from 1 to 2^{n-1}; that is, the beginning vertex $\mathbf{e}$ has index 1. Next, we claim that there must exist an integer t, $1 \leq t \leq 2^{n-2}$, so that the crossing edges $(C_i(2t-1), (C_i(2t-1))^{n-1})$ and $(C_i(2t), (C_i(2t))^{n-1})$ are fault-free for all $1 \leq i \leq \delta$. If such edges do not exist, then we have $|F| \geq |F_c| \geq 2^{n-2}/\delta > |F|$ for $n \geq 3$, leading to an immediate contradiction. Let $\mathbf{x}_i = C_i(2t-1)$ and $\mathbf{y}_i = C_i(2t)$. Accordingly, C_i can be represented as $\langle \mathbf{e}, P_i, \mathbf{x}_i, \mathbf{y}_i, H_i, \mathbf{e} \rangle$, $1 \leq i \leq \delta$. By the definition of hypercubes, $(\mathbf{x}_i)^{n-1}$ and $(\mathbf{y}_i)^{n-1}$ are adjacent in Q_n^1. By Lemma 4.3, $Q_n^1 - F_1$ contains δ-mutually

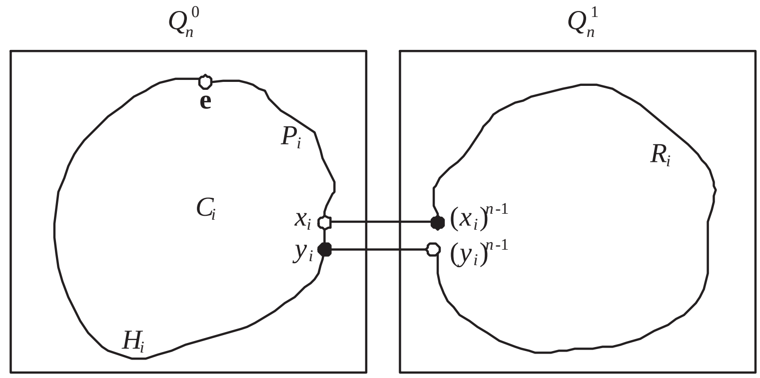

Figure 4.2: Illustration for the proof of Theorem 4.4. Without loss of generality, we assume $\mathbf{x}_i \in V_0(Q_n)$ for $1 \leq i \leq \delta$.

fully independent hamiltonian paths $R_1[(\mathbf{x}_1)^{n-1}, (\mathbf{y}_1)^{n-1}], \ldots, R_\delta[(\mathbf{x}_\delta)^{n-1}, (\mathbf{y}_\delta)^{n-1}]$. Therefore, $\{\langle \mathbf{e}, P_i, \mathbf{x}_i, (\mathbf{x}_i)^{n-1}, R_i, (\mathbf{y}_i)^{n-1}, \mathbf{y}_i, H_i, \mathbf{e}\rangle \mid 1 \leq i \leq \delta\}$ forms a set of δ-mutually independent hamiltonian cycles beginning from $\mathbf{e}$. □

4.2 Faulty star networks

Tseng et al. [68] addressed fault-tolerant ring embedding in an injured star network, and showed that an injured n-dimensional star network is still hamiltonian when no more than $n-3$ edge faults occur. As Lin et al. [49] showed, $\mathcal{IHC}(S_n) = n-2$ if $n \in \{3,4\}$, and $\mathcal{IHC}(S_n) = n-1$ if $n \geq 5$. Let $F \subseteq E(S_n)$ with $|F| \leq n-3$. In this section, we aim to prove that there exist $(n-2-|F|)$-mutually independent hamiltonian cycles of $S_n - F$ beginning from any vertex of S_n if $n \in \{3,4\}$, and there exist $(n-1-|F|)$-mutually independent hamiltonian cycles of $S_n - F$ beginning from any vertex of S_n if $n \geq 5$. Before proceeding, we recite the definition of an n-dimensional star network and introduce its basic properties.

4.2.1 Definition and basic properties of star networks

For the sake of clarity, we recall the definition of star networks in advance. Let n be a positive integer. We use $\langle n\rangle$ to denote the set of integers from 1 to n. A *permutation* on $\langle n\rangle$, namely $u_1u_2\ldots u_n$, is a sequence of all elements of $\langle n\rangle$. An *inversion* of $u_1u_2\ldots u_n$ is a pair of integers (i_1, i_2) such that $u_{i_1} < u_{i_2}$ and $i_1 > i_2$. An *even permutation* is a permutation with an even number of inversions, and an *odd permutation* is a permutation with an odd number of inversions. The n-dimensional star network, denoted by S_n, is a graph with vertex set $V(S_n) = \{u_1\ldots u_n \mid \ u_i \in \langle n\rangle$ and $u_i \neq u_j$ for $i \neq j\}$. Its adjacency is defined as follows: $u_1\ldots u_i\ldots u_n$ is adjacent to $v_1\ldots v_i\ldots v_n$ through an edge of dimension i with $2 \leq i \leq n$ if $u_1 = v_i$, $v_1 = u_i$, and $u_j = v_j$ for $j \in \langle n\rangle - \{1, i\}$. Obviously, S_n is both vertex-transitive and edge-transitive [1]. Three star networks S_2, S_3, and S_4 are illustrated in Figure 1.1.

We use a boldface letter to denote any vertex of S_n. Moreover, we use $\mathbf{e}$ to denote the vertex $12 \ldots n$. It is known that S_n is a bipartite graph with one partite set $V_0(S_n)$ consisting of those vertices corresponding to even permutations and the other partite set $V_1(S_n)$ consisting of those vertices corresponding to odd permutations. Let $\mathbf{u} = u_1 u_2 \ldots u_n$ be a vertex of S_n. Then u_i is the i-th coordinate of $\mathbf{u}$, denoted by $(\mathbf{u})_i$, for $1 \leq i \leq n$. According to the definition of S_n, there is exactly one neighbor $\mathbf{v}$ of $\mathbf{u}$ such that $\mathbf{u}$ and $\mathbf{v}$ are adjacent through an edge in the i-th dimension for $2 \leq i \leq n$. Therefore, we use $(\mathbf{u})^i$ to denote the unique i-neighbor of $\mathbf{u}$. Obviously, $((\mathbf{u})^i)^i = \mathbf{u}$. For every $1 \leq i \leq n$, let $S_n^{\{i\}}$ be a subgraph of S_n induced by those vertices $\mathbf{u}$ with $(\mathbf{u})_n = i$. Then S_n can be partitioned into n vertex-disjoint subgraphs $S_n^{\{1\}}, \ldots, S_n^{\{n\}}$, and each of these subgraphs is isomorphic to S_{n-1}. For this reason, star networks can be constructed recursively. Let $I \subseteq \langle n \rangle$. We use S_n^I or $\cup_{i \in I} S_n^{\{i\}}$ to denote a subgraph of S_n induced by $\cup_{i \in I} V(S_n^{\{i\}})$. For $1 \leq i \neq j \leq n$, we use $E^{i,j}$ to denote the set of edges between $S_n^{\{i\}}$ and $S_n^{\{j\}}$.

Theorem 4.5. *[68] Let $F \subset E(S_n)$ with $|F| \leq n-3$ for $n \geq 3$. Then $S_n - F$ is hamiltonian.*

Li et al. [47] introduced the *edge-fault-tolerant hamiltonian laceability* of a graph G, which is the integer f such that, for any $F \subseteq E(G)$ with $|F| \leq f$, $G - F$ is still hamiltonian laceable and there exists a subset F' of $E(G)$ with $|F'| = f + 1$ such that $G - F'$ is not hamiltonian laceable. Moreover, the *edge-fault-tolerant hyper-hamiltonian laceability* of G is defined as the integer f such that, for any $F \subseteq E(G)$ with $|F| \leq f$, $G - F$ is hyper-hamiltonian laceable and there exists a subset F' of $E(G)$ with $|F'| = f + 1$ such that $G - F'$ is no longer-hyper hamiltonian laceable.

Theorem 4.6. *[47] The S_n is $(n-3)$-edge fault tolerant hamiltonian laceable and $(n-4)$-edge fault tolerant hyper-hamiltonian laceable for $n \geq 4$.*

Lemma 4.4. *[53] Assume that $n \geq 3$. Then $|E^{i,j}| = (n-2)!$ for any $1 \leq i \neq j \leq n$. Moreover, there are $(n-2)!/2$ edges joining vertices of $V_0(S_n^{\{i\}})$ to vertices of $V_1(S_n^{\{j\}})$.*

Lemma 4.5. *For $n \geq 3$, let $\mathbf{u}$ and $\mathbf{v}$ be two distinct vertices of S_n with $d(\mathbf{u}, \mathbf{v}) \leq 2$. Then $(\mathbf{u})_1 \neq (\mathbf{v})_1$.*

Lemma 4.6. *Let $n \geq 5$. Assume that $F \subset E(S_n)$ with $|F| \leq n-4$, and assume that $I = \{a_1, \ldots, a_r\}$ is a subset of r elements of $\langle n \rangle$ for some $r \in \langle n \rangle$. Suppose that $\mathbf{u} \in V_0(S_n^{\{a_1\}})$ and $\mathbf{v} \in V_1(S_n^{\{a_r\}})$. Then there exists a hamiltonian path $H = \langle \mathbf{u} = \mathbf{x}_1, P_1, \mathbf{y}_1, \mathbf{x}_2, P_2, \mathbf{y}_2, \ldots, \mathbf{x}_r, P_r, \mathbf{y}_r = \mathbf{v} \rangle$ of $S_n^I - F$ joining $\mathbf{u}$ to $\mathbf{v}$ such that $\mathbf{x}_1 = \mathbf{u}$, $\mathbf{y}_r = \mathbf{v}$, and P_i is a hamiltonian path of $S_n^{\{a_i\}} - F$ joining $\mathbf{x}_i$ to $\mathbf{y}_i$ for every $1 \leq i \leq r$.*

Proof. By Theorem 4.6, this statement holds on $r = 1$. Suppose that $r \geq 2$ and we set $\mathbf{x}_1 = \mathbf{u}$ and $\mathbf{y}_r = \mathbf{v}$. By Lemma 4.4, there are $(n-2)!/2 > n-4$ edges joining vertices of $V_1(S_n^{\{a_i\}})$ to vertices of $V_0(S_n^{\{a_{i+1}\}})$ for every $i \in \langle r-1 \rangle$. Therefore, we choose

$(\mathbf{y}_i, \mathbf{x}_{i+1}) \in E^{a_i,a_{i+1}} - F$ with $\mathbf{y}_i \in V_1(S_n^{\{a_i\}})$ and $\mathbf{x}_{i+1} \in V_0(S_n^{\{a_{i+1}\}})$ for $i \in \langle r-1 \rangle$. By Theorem 4.6, there exists a hamiltonian path P_i of $S_n^{\{a_i\}} - F$ joining $\mathbf{x}_i$ to $\mathbf{y}_i$ for every $i \in \langle r \rangle$. As a result, $\langle \mathbf{u} = \mathbf{x}_1, P_1, \mathbf{y}_1, \mathbf{x}_2, P_2, \mathbf{y}_2, \ldots, \mathbf{x}_r, P_r, \mathbf{y}_r = \mathbf{v} \rangle$ forms a desired hamiltonian path of $S_n^I - F$ joining $\mathbf{u}$ to $\mathbf{v}$. □

Lemma 4.7. *Let $n \geq 5$. Assume that $F \subset E(S_n)$ with $|F| \leq n-4$ and $|F \cap S_n^{\{i\}}| \leq n-5$ for every $i \in \langle n \rangle$. Moreover, assume that $I = \{a_1, \ldots, a_r\}$ is a subset of r elements of $\langle n \rangle$ for some $2 \leq r \leq n$. Suppose that $\mathbf{u} \in V_0(S_n^{\{a_1\}})$, $\mathbf{w} \in V_1(S_n^{\{a_1\}})$, and $\mathbf{v} \in V_0(S_n^{\{a_r\}})$. Then there exists a hamiltonian path H of $S_n^I - (F \cup \{\mathbf{w}\})$ joining $\mathbf{u}$ to $\mathbf{v}$.*

Proof. By Lemma 4.4, there are $(n-2)!/2 > n-3$ edges joining vertices of $V_0(S_n^{\{a_1\}})$ to vertices of $V_1(S_n^{\{a_2\}})$. Thus, we choose a vertex $\mathbf{x}$ of $V_0(S_n^{\{a_1\}}) - \{\mathbf{u}\}$ with $(\mathbf{x})_1 = a_2$ and $(\mathbf{x}, (\mathbf{x})^n) \notin F$. By Theorem 4.6, there exists a hamiltonian path P of $S_n^{\{a_1\}} - (F \cup \{\mathbf{w}\})$ joining $\mathbf{u}$ to $\mathbf{x}$. By Lemma 4.6, there exists a hamiltonian path Q of $S_n^{I-\{a_1\}} - F$ joining $(\mathbf{x})^n$ to $\mathbf{v}$. As a result, $\langle \mathbf{u}, P, \mathbf{x}, (\mathbf{x})^n, Q, \mathbf{v} \rangle$ forms a desired hamiltonian path. □

Lemma 4.8. *[48] Assume that $\mathbf{w}$ and $\mathbf{b}$ are two adjacent vertices of S_n with $n \geq 4$. For any vertex $\mathbf{u}$ in $V_0(S_n) - \{\mathbf{w}, \mathbf{b}\}$ and for any $i \in \langle n \rangle$, there exists a hamiltonian path P of $S_n - \{\mathbf{w}, \mathbf{b}\}$ joining $\mathbf{u}$ to some vertex $\mathbf{v}$ of $V_1(S_n) - \{\mathbf{w}, \mathbf{b}\}$ with $(\mathbf{v})_1 = i$.*

Lemma 4.9. *Let $i \in \langle n \rangle$ and $F \subset E(S_n)$ with $|F| \leq n-4$ for $n \geq 4$. Suppose that $\mathbf{w}$ and $\mathbf{b}$ are two adjacent vertices of S_n and $\mathbf{u} \in V_0(S_n) - \{\mathbf{w}, \mathbf{b}\}$. Then there exists a hamiltonian path of $S_n - (F \cup \{\mathbf{w}, \mathbf{b}\})$ joining $\mathbf{u}$ to some vertex $\mathbf{v}$ of $V_1(S_n) - \{\mathbf{w}, \mathbf{b}\}$ with $(\mathbf{v})_1 = i$.*

Proof. Since S_n is vertex-transitive and edge-transitive, we assume that $\mathbf{w} = \mathbf{e}$ and $\mathbf{b} = (\mathbf{e})^j$ with some $j \in \langle n \rangle - \{1\}$. We set $F_k = F \cap E(S_n^{\{k\}})$ for every $k \in \langle n \rangle$. Then we prove this lemma by induction on n. The induction bases depend upon Lemma 4.8. Suppose that this statement holds on S_{n-1} with $n \geq 5$. We consider the dimensions of all edges of $F \cup \{(\mathbf{e}, (\mathbf{e})^j)\}$. If there exists an edge of F whose dimension, say j', is different from j, we can partition S_n over dimension j'. Otherwise, every edge of F has the same dimension as j.

Case 1: There exists an edge of F whose dimension, say j', is different from j. Since S_n is edge-transitive, we assume $j' = n$. Thus, $(\mathbf{e}, (\mathbf{e})^j) \in E(S_n^{\{n\}})$ and $|F_k| \leq n-5$ for every $k \in \langle n \rangle$.

Subcase 1.1: Suppose that $\mathbf{u} \in V_0(S_n^{\{n\}})$. Since $|F| \leq n-4$, we can choose an integer $r \in \langle n-1 \rangle$ such that $|F \cap E^{r,n}| = 0$. By induction hypothesis, there exists a hamiltonian path P of $S_n^{\{n\}} - (F_n \cup \{\mathbf{e}, (\mathbf{e})^j\})$ joining $\mathbf{u}$ to a vertex $\mathbf{x} \in V_1(S_n^{\{n\}})$ with $(\mathbf{x})_1 = r$. We choose a vertex $\mathbf{v}$ in $V_1(S_n^{\langle n-1 \rangle - \{r\}})$ with $(\mathbf{v})_1 = i$. By Lemma 4.6, there exists a hamiltonian path Q of $S_n^{\langle n-1 \rangle} - F$ joining $(\mathbf{x})^n$ to $\mathbf{v}$. Then $\langle \mathbf{u}, P, \mathbf{x}, (\mathbf{x})^n, Q, \mathbf{v} \rangle$ is a desired path.

Subcase 1.2: Suppose that $\mathbf{u} \in V_0(S_n^{\{k\}})$ for some $k \in \langle n-1 \rangle$. By Lemma 4.4, there are $(n-2)!/2 > n-3$ edges joining vertices of $V_1(S_n^{\{k\}})$ to vertices of $V_0(S_n^{\{n\}})$. We choose a vertex $\mathbf{y}$ of $V_1(S_n^{\{k\}})$ such that $(\mathbf{y})^n \in V_0(S_n^{\{n\}}) - \{\mathbf{e}\}$ and $(\mathbf{y}, (\mathbf{y})^n) \notin F$. By Theorem 4.6, there exists a hamiltonian path H of $S_n^{\{k\}} - F_k$ joining $\mathbf{u}$ to $\mathbf{y}$. We choose an integer r of $\langle n-1 \rangle - \{k\}$ such that $|F \cap E^{r,n}| = 0$. By induction hypothesis, there exists a hamiltonian path P of $S_n^{\{n\}} - (F_n \cup \{\mathbf{e}, (\mathbf{e})^j\})$ joining $(\mathbf{y})^n$ to a vertex $\mathbf{x}$ of $V_1(S_n^{\{n\}}) - \{(\mathbf{e})^j\}$ with $(\mathbf{x})_1 = r$. Besides, we choose a vertex $\mathbf{v}$ of $V_1(S_n^{\langle n-1 \rangle - \{k,r\}})$ with $(\mathbf{v})_1 = i$. By Lemma 4.6, there exists a hamiltonian path Q of $S_n^{\langle n-1 \rangle - \{k\}} - F$ joining $(\mathbf{x})^n$ to $\mathbf{v}$. Then $\langle \mathbf{u}, H, \mathbf{y}, (\mathbf{y})^n, P, \mathbf{x}, (\mathbf{x})^n, Q, \mathbf{v} \rangle$ forms a desired path.

Case 2: Every edge of F has the same dimension j. Without loss of generality, we may assume $j = n$. Thus, $|F_t| = 0$ for every $t \in \langle n \rangle$.

Subcase 2.1: Suppose that $\mathbf{u} \in V_0(S_n^{\{k\}})$ for some $k \in \langle n-1 \rangle - \{1\}$. By Lemma 4.4, there are $(n-2)!/2 > n-4$ edges joining vertices of $V_1(S_n^{\{k\}})$ to vertices of $V_0(S_n^{\{1\}})$. Thus, we can choose a vertex $\mathbf{x}$ of $V_1(S_n^{\{k\}})$ with $(\mathbf{x})_1 = 1$ and $(\mathbf{x}, (\mathbf{x})^n) \notin F$. By Theorem 4.6, there exists a hamiltonian path H of $S_n^{\{k\}}$ joining $\mathbf{u}$ to $\mathbf{x}$. Similarly, we can choose a vertex $\mathbf{y}$ of $V_0(S_n^{\{1\}})$ with $(\mathbf{y})_1 = n$ and $(\mathbf{y}, (\mathbf{y})^n) \notin F$. By Theorem 4.6, there exists a hamiltonian path P of $S_n^{\{1\}} - \{(\mathbf{e})^n\}$ joining $(\mathbf{x})^n$ to $\mathbf{y}$. Let $\mathbf{v}$ be a vertex in $V_1(S_n^{\langle n-1 \rangle - \{1,k\}})$ with $(\mathbf{v})_1 = i$. By Lemma 4.7, there exists a hamiltonian path Q of $S_n^{\langle n \rangle - \{1,k\}} - (F \cup \{\mathbf{e}\})$ joining $(\mathbf{y})^n$ to $\mathbf{v}$. Then $\langle \mathbf{u}, H, \mathbf{x}, (\mathbf{x})^n, P, \mathbf{y}, (\mathbf{y})^n, Q, \mathbf{v} \rangle$ forms a desired path.

Subcase 2.2: Suppose that $\mathbf{u} \in V_0(S_n^{\{1\}})$. By Lemma 4.4, there are $(n-2)!/2 > n-4$ edges joining vertices of $V_0(S_n^{\{1\}})$ to vertices of $V_1(S_n^{\{n\}})$. Thus, we can choose a vertex $\mathbf{x}$ of $V_0(S_n^{\{1\}}) - \{\mathbf{u}\}$ with $(\mathbf{x})_1 = n$ and $(\mathbf{x}, (\mathbf{x})^n) \notin F$. By Theorem 4.6, there exists a hamiltonian path H of $S_n^{\{1\}} - \{(\mathbf{e})^n\}$ joining $\mathbf{u}$ to $\mathbf{x}$. We choose a vertex $\mathbf{v}$ of $V_1(S_n^{\langle n-1 \rangle - \{1\}})$ with $(\mathbf{v})_1 = i$. By Lemma 4.7, there exists a hamiltonian path Q of $S_n^{\langle n \rangle - \{1\}} - (F \cup \{\mathbf{e}\})$ joining $(\mathbf{x})^n$ to $\mathbf{v}$. Then $\langle \mathbf{u}, H, \mathbf{x}, (\mathbf{x})^n, Q, \mathbf{v} \rangle$ forms a desired path.

Subcase 2.3: Suppose that $\mathbf{u} \in V_0(S_n^{\{n\}})$. Since $|F| \leq n-4$, we can choose two integers k_1 and k_2 in $\langle n-1 \rangle - \{1\}$ such that $((\mathbf{e})^{k_1}, ((\mathbf{e})^{k_1})^n) \notin F$ and $((\mathbf{e})^{k_2}, ((\mathbf{e})^{k_2})^n) \notin F$. Let $X = \{(\mathbf{e}, (\mathbf{e})^t) \mid t \in \langle n-1 \rangle - \{1, k_1, k_2\}\}$. Obviously, $|X| = n-4$. Moreover, we can choose a vertex $\mathbf{x} \in V_1(S_n^{\{n\}})$ such that $(\mathbf{x})_1 \in \langle n-1 \rangle - \{1, k_1, k_2\}$ and $(\mathbf{x}, (\mathbf{x})^n) \notin F$. Since $(\mathbf{x})_1 \neq k_1$ and $(\mathbf{x})_1 \neq k_2$, we have $\mathbf{x} \neq (\mathbf{e})^{k_1}$ and $\mathbf{x} \neq (\mathbf{e})^{k_2}$. By Theorem 4.6, there exists a hamiltonian path $H = \langle \mathbf{u}, H_1, (\mathbf{e})^{k_1}, \mathbf{e}, (\mathbf{e})^{k_2}, H_2, \mathbf{x} \rangle$ of $S_n^{\{n\}} - X$ joining $\mathbf{u}$ to $\mathbf{x}$. Let $\mathbf{y} = (\mathbf{e})^{k_2}$. Since $(\mathbf{y})_1 \neq (\mathbf{x})_1$, we have $i \neq (\mathbf{x})_1$ or $i \neq (\mathbf{y})_1$.

Subcase 2.3.1: Suppose that $i \neq (\mathbf{x})_1$. Let $k_3 = (\mathbf{x})_1$. We choose a vertex $\mathbf{v}$ of $V_1(S_n^{\{k_3\}})$ with $(\mathbf{v})_1 = i$. By Lemma 4.4, there are $(n-2)!/2 > n-4$ edges joining vertices of $V_1(S_n^{\{k_1\}})$ to vertices of $V_0(S_n^{\{1\}})$. Thus, we can choose a vertex $\mathbf{z}$ of $V_1(S_n^{\{k_1\}})$ with $(\mathbf{z})_1 = 1$ and $(\mathbf{z}, (\mathbf{z})^n) \notin F$. By Theorem 4.6, there exists a hamiltonian path T of $S_n^{\{k_3\}}$ joining $(\mathbf{x})^n$ to $\mathbf{v}$. Similarly, there is a hamiltonian path P of $S_n^{\{k_1\}}$ joining $((\mathbf{e})^{k_1})^n$ to $\mathbf{z}$. By Lemma 4.7, there exists a hamiltonian path Q of $S_n^{\langle n-1\rangle-\{k_1,k_3\}} - (F \cup \{(\mathbf{e})^n\})$ joining $(\mathbf{z})^n$ to $(\mathbf{y})^n$. Then $\langle \mathbf{u}, H_1, (\mathbf{e})^{k_1}, ((\mathbf{e})^{k_1})^n, P, \mathbf{z}, (\mathbf{z})^n, Q, (\mathbf{y})^n, \mathbf{y}, H_2, \mathbf{x}, (\mathbf{x})^n, T, \mathbf{v}\rangle$ a the desired path.

Subcase 2.3.2: Suppose that $i \neq (\mathbf{y})_1$. Let $k_3 = (\mathbf{y})_1$. Then the proof of this case is similar to that of **Subcase 2.3.1**. □

Lemma 4.10. *Let $\{a, b\} \subset \langle n\rangle$ with $a < b$ and let $F \subset E(S_n)$ with $|F| \leq n-4$ for $n \geq 4$. Suppose that $\mathbf{x} \in V_0(S_n)$, and assume that $\mathbf{x}_1$ and $\mathbf{x}_2$ are two distinct neighbors of $\mathbf{x}$. Then $S_n - (F \cup \{\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2\})$ has a hamiltonian path between two vertices $\mathbf{u}$ and $\mathbf{v}$ in $V_0(S_n) - \{\mathbf{x}\}$ such that $(\mathbf{u})_1 = a$ and $(\mathbf{v})_1 = b$.*

Proof. Since S_n is vertex-transitive and edge-transitive, we assume that $\mathbf{x} = \mathbf{e}$, $\mathbf{x}_1 = (\mathbf{e})^{i_1}$, and $\mathbf{x}_2 = (\mathbf{e})^{i_2}$ with some $\{i_1, i_2\} \subset \{2, 3, \ldots, n\}$. We prove this lemma by induction on n.

Suppose that $n = 4$. Thus, we have $|F| = 0$. Since S_4 is edge-transitive, we assume that $\mathbf{x}_1 = (\mathbf{e})^2 = 2134$ and $\mathbf{x}_2 = (\mathbf{e})^3 = 3214$. The required paths of $S_4 - \{1234, 2134, 3214\}$ are listed in Table 4.1.

$a = 1$ and $b = 2$	$\langle 1324, 3142, 4132, 1432, 3412, 4312, 2314, 1324, 3124, 4123, 2143, 1243, 4213, 2413, 1423, 3421, 4321, 2341, 3241, 4231, 2431\rangle$
$a = 1$ and $b = 3$	$\langle 1423, 2413, 4213, 1243, 2143, 4123, 3124, 1324, 2314, 4312, 3412, 1432, 4132, 3142, 1342, 2341, 4321, 3421, 2431, 4231, 3241\rangle$
$a = 1$ and $b = 4$	$\langle 1324, 3142, 4132, 1432, 3412, 4312, 2314, 1324, 3124, 4123, 2143, 1243, 4213, 2413, 1423, 3421, 2431, 4231, 3241, 2341, 4321\rangle$
$a = 2$ and $b = 3$	$\langle 2314, 1324, 3124, 4123, 2143, 1243, 4213, 2413, 1423, 3421, 4321, 2341, 3241, 4231, 2431, 1432, 4132, 3142, 1342, 4312, 3412\rangle$
$a = 2$ and $b = 4$	$\langle 2314, 1324, 3124, 4123, 2143, 1243, 4213, 2413, 1423, 3421, 4321, 2341, 3241, 4231, 2431, 1432, 3412, 4312, 1342, 3142, 4132\rangle$
$a = 3$ and $b = 4$	$\langle 3124, 1324, 2314, 4312, 3412, 1432, 4132, 3142, 1342, 2341, 4321, 3421, 2431, 4231, 3241, 1243, 2143, 4123, 1423, 2413, 4213\rangle$

Table 4.1: The required hamiltonian path of $S_4 - \{1234, 2134, 3214\}$.

Suppose that the statement holds on S_{n-1} with $n \geq 5$. Let $F_k = F \cap E(S_n^{\{k\}})$ for every $k \in \langle n\rangle$. Without loss of generality, suppose that there is at least one edge of F in dimension n. Thus, $|F_k| \leq n-5$ for every $k \in \langle n\rangle$. Because $a < b$, we have $a \neq n$ and $b \neq 1$. Since $|F| \leq n-4$, we can choose an integer c in $\langle n-1\rangle - \{1, a\}$ such that $|F \cap E^{c,n}| = 0$. Moreover, we choose a vertex $\mathbf{v}$ of $V_0(S_n^{\{1\}})$ with $(\mathbf{v})_1 = b$.

Case 1: Suppose that $i_1 \neq n$ and $i_2 \neq n$. By induction hypothesis, there is a hamiltonian path H of $S_n^{\{n\}} - (F_n \cup \{\mathbf{e}, (\mathbf{e})^{i_1}, (\mathbf{e})^{i_2}\})$ joining a vertex $\mathbf{u}$ of $V_0(S_n^{\{n\}})$ with $(\mathbf{u})_1 = a$ to a vertex $\mathbf{y}$ of $V_0(S_n^{\{n\}})$ with $(\mathbf{y})_1 = c$. By Lemma 4.6, there exists a hamiltonian path R of $S_n^{\langle n-1\rangle} - F$ joining $(\mathbf{y})^n$ to $\mathbf{v}$. As a result, $\langle \mathbf{u}, H, \mathbf{y}, (\mathbf{y})^n, R, \mathbf{v}\rangle$ forms the desired path in $S_n - (F \cup \{\mathbf{e}, (\mathbf{e})^{i_1}, (\mathbf{e})^{i_2}\})$.

Case 2: Either $i_1 = n$ or $i_2 = n$. Without loss of generality, we assume $i_2 = n$. We choose a vertex $\mathbf{u} \in V_0(S_n^{\{n\}})$ with $(\mathbf{u})_1 = a$. By Lemma 4.9, there exists a hamiltonian path H of $S_n^{\{n\}} - (F_n \cup \{\mathbf{e}, (\mathbf{e})^{i_1}\})$ joining a vertex $\mathbf{u}$ to some vertex $\mathbf{y}$ of $V_1(S_n^{\{n\}})$ with $(\mathbf{y})_1 = c$. By Lemma 4.7, there exists a hamiltonian path Q of $S_n^{\langle n-1\rangle} - (F \cup \{(\mathbf{e})^n\})$ joining $(\mathbf{y})^n$ to $\mathbf{v}$. As a result, $\langle \mathbf{u}, H, \mathbf{y}, (\mathbf{y})^n, Q, \mathbf{v}\rangle$ forms a desired path. □

4.2.2 The main results

Theorem 4.7. *[49] $\mathcal{IHC}(S_n) = n-2$ if $n \in \{3,4\}$; $\mathcal{IHC}(S_n) = n-1$ if $n \geq 5$.*

Lemma 4.11. *Let $f \in E(S_4)$. Then $\mathcal{IHC}(S_4 - \{f\}) = 1$.*

Proof. Since S_4 is vertex-transitive, we only consider the mutually independent hamiltonian cycles of $S_4 - \{f\}$ beginning from 1234. Suppose that $f = (1234, 4231)$. We list all hamiltonian cycles of $S_4 - \{(1234, 4231)\}$, beginning from 1234, in Table 4.2. By brute force, there do not exist 2-mutually independent hamiltonian cycles of $S_4 - \{(1234, 4231)\}$ beginning from 1234. Thus, $\mathcal{IHC}(S_4 - \{(1234, 4231)\}) \leq 1$. By Theorem 4.5, there exists a hamiltonian cycle in $S_4 - \{(1234, 4231)\}$. Hence, $\mathcal{IHC}(S_4 - \{f\}) = 1$. □

⟨1234, 2134, 3124, 1324, 2314, 4312, 3412, 1432, 4132, 3142, 1342, 2341, 4321, 3421, 2431, 4231, 3241, 1243, 2143, 4123, 1423, 2413, 4213, 3214, 1234⟩
⟨1234, 2134, 3124, 1324, 4321, 2341, 3241, 4231, 2431, 3421, 1423, 4123, 2143, 1243, 4213, 2413, 3412, 1432, 4132, 3142, 1342, 4312, 2314, 3214, 1234⟩
⟨1234, 2134, 3124, 4123, 1423, 2413, 4213, 1243, 2143, 3142, 4132, 1432, 3412, 4312, 1342, 2341, 3241, 4231, 2431, 3421, 4321, 1324, 2314, 3214, 1234⟩
⟨1234, 2134, 4132, 1432, 2431, 4231, 3241, 1243, 2143, 3142, 1342, 2341, 4321, 3421, 1423, 4123, 3124, 1324, 2314, 4312, 3412, 2413, 4213, 3214, 1234⟩
⟨1234, 2134, 4132, 3142, 1342, 4312, 3412, 1432, 2431, 4231, 3241, 2341, 4321, 3421, 1423, 2413, 4213, 1243, 2143, 4123, 3124, 1324, 2314, 3214, 1234⟩
⟨1234, 2134, 4132, 3142, 2143, 4123, 3124, 1324, 2314, 4312, 1342, 2341, 4321, 3421, 1423, 2413, 3412, 1432, 2431, 4231, 3241, 1243, 4213, 3214, 1234⟩
⟨1234, 3214, 2314, 1324, 3124, 4123, 2143, 1243, 4213, 2413, 1423, 3421, 4321, 2341, 3241, 4231, 2431, 1432, 3412, 4312, 1342, 3142, 4132, 2134, 1234⟩
⟨1234, 3214, 2314, 1324, 4321, 3421, 2431, 4231, 3241, 2341, 1342, 4312, 3412, 1432, 4132, 3142, 2143, 1243, 4213, 2413, 1423, 4123, 3124, 2134, 1234⟩
⟨1234, 3214, 2314, 4312, 1342, 3142, 4132, 1432, 3412, 2413, 4213, 1243, 2143, 4123, 1423, 3421, 2431, 4231, 3241, 2341, 4321, 1324, 3124, 2134, 1234⟩
⟨1234, 3214, 4213, 2413, 1423, 4123, 2143, 1243, 3241, 4231, 2431, 3421, 4321, 2341, 1342, 3142, 4132, 1432, 3412, 4312, 2314, 1324, 3124, 2134, 1234⟩
⟨1234, 3214, 4213, 2413, 3412, 4312, 2314, 1324, 3124, 4123, 1423, 3421, 4321, 2341, 1342, 3142, 2143, 1243, 3241, 4231, 2431, 1432, 4132, 2134, 1234⟩
⟨1234, 3214, 4213, 1243, 3241, 4231, 2431, 1432, 3412, 2413, 1423, 3421, 4321, 2341, 1342, 4312, 2314, 1324, 3124, 4123, 2143, 3142, 4132, 2134, 1234⟩

Table 4.2: All hamiltonian cycles of $S_4 - \{(1234, 4231)\}$, beginning from 1234.

Lemma 4.12. *Suppose that $n \geq 5$ and $F \subset E(S_n)$ with $|F| = n-3$. Let $\mathbf{u} \in V(S_n)$. Then there exist 2-mutually independent hamiltonian cycles of $S_n - F$ beginning from $\mathbf{u}$.*

Proof. Since S_n is vertex-transitive and edge-transitive, we assume that $\mathbf{u} = \mathbf{e}$ and also that F contains at least one edge in dimension n. Let $F_k = F \cap E(S_n^{\{k\}})$ for every $k \in \langle n \rangle$. As a result, $|F_k| \leq n-4$ for every $k \in \langle n \rangle$.

Case 1: Suppose that $(\mathbf{e}, (\mathbf{e})^n) \notin F$. Let $B = (b_{i,j})$ be the $2 \times n$ matrix with

$$b_{i,j} = \begin{cases} j & \text{if } i = 1, \\ n & \text{if } i = 2 \text{ and } j = 1, \\ j+1 & \text{if } i = 2 \text{ and } 2 \leq j \leq n-2, \\ 2 & \text{if } i = 2 \text{ and } j = n-1, \\ 1 & \text{if } i = 2 \text{ and } j = n. \end{cases}$$

By Lemma 4.6, there exists a hamiltonian path P of $(\cup_{j=1}^{n} S_n^{\{b_{1,j}\}}) - F$ joining $(\mathbf{e})^n$ to $\mathbf{e}$. Similarly, there exists a hamiltonian path H of $(\cup_{j=1}^{n} S_n^{\{b_{2,j}\}}) - F$ joining $\mathbf{e}$ to $(\mathbf{e})^n$. Then we set $C_1 = \langle \mathbf{e}, (\mathbf{e})^n, P, \mathbf{e} \rangle$ and $C_2 = \langle \mathbf{e}, H, (\mathbf{e})^n, \mathbf{e} \rangle$. Obviously, $\{C_1, C_2\}$ forms a set of 2-mutually independent hamiltonian cycles of $S_n - F$ beginning from $\mathbf{e}$. See Figure 4.3(a) for illustration.

Case 2: Suppose that $(\mathbf{e}, (\mathbf{e})^n) \in F$ and $|F_n| = n-4$. Obviously, $|F_k| = 0$ for every $k \in \langle n-1 \rangle$. By Theorem 4.5, there exists a hamiltonian cycle $H = \langle \mathbf{e}, R, \mathbf{q}, \mathbf{p}, \mathbf{e} \rangle$ of

$S_n^{\{n\}} - F_n$. Accordingly, we have that $(\mathbf{p}, (\mathbf{p})^n) \notin F$ and $(\mathbf{q}, (\mathbf{q})^n) \notin F$. By Lemma 4.5, $(\mathbf{p})_1 \neq (\mathbf{q})_1$. We set $(\mathbf{p})_1 = i_{n-1}$ and $(\mathbf{q})_1 = i_1$. Let $i_2 i_3 \ldots i_{n-2}$ be an arbitrary permutation of $\langle n-1 \rangle - \{i_1, i_{n-1}\}$.

For $1 \leq k \leq n-2$, let $\mathbf{x}_k$ be a vertex of $V_0(S_n^{\{i_k\}})$ such that $(\mathbf{x}_k)_1 = i_{k+1}$ and $(\mathbf{x}_k, (\mathbf{x}_k)^n) \notin F$. By Theorem 4.6, there exists a hamiltonian path P_1 of $S_n^{\{i_1\}}$ joining $(\mathbf{q})^n$ to $\mathbf{x}_1$. Similarly, there exists a hamiltonian path P_k of $S_n^{\{i_k\}}$ joining $(\mathbf{x}_{k-1})^n$ to $\mathbf{x}_k$ for $2 \leq k \leq n-2$ and there exists a hamiltonian path P_{n-1} of $S_n^{\{i_{n-1}\}}$ joining $(\mathbf{x}_{n-2})^n$ to $(\mathbf{p})^n$. Then we set $C_1 = \langle \mathbf{e}, R, \mathbf{q}, (\mathbf{q})^n, P_1, \mathbf{x}_1, (\mathbf{x}_1)^n, P_2, \mathbf{x}_2, (\mathbf{x}_2)^n, \ldots, \mathbf{x}_{n-2}, (\mathbf{x}_{n-2})^n, P_{n-1}, (\mathbf{p})^n, \mathbf{p}, \mathbf{e} \rangle$.

Obviously, we can choose a vertex $\mathbf{y}_{n-1}$ of $V_1(S_n^{\{i_{n-1}\}})$ such that $(\mathbf{y}_{n-1})_1 = i_2$ and $(\mathbf{y}_{n-1}, (\mathbf{y}_{n-1})^n) \notin F$. For $2 \leq k \leq n-3$, $|\{\mathbf{u} \in V_1(S_n^{\{i_k\}}) \mid (\mathbf{u})_1 = i_{k+1} \text{ and } d(\mathbf{u}, (\mathbf{x}_{k-1})^n) = 2\}| = n-3 < (n-2)!/2$ if $n \geq 5$. Thus, we choose a vertex $\mathbf{y}_k$ of $V_1(S_n^{\{i_k\}})$ such that $d(\mathbf{y}_k, (\mathbf{x}_{k-1})^n) > 2$, $(\mathbf{y}_k)_1 = i_{k+1}$, and $(\mathbf{y}_k, (\mathbf{y}_k)^n) \notin F$ for $2 \leq k \leq n-3$. Since $|\{\mathbf{u} \in V_1(S_n^{\{i_{n-2}\}}) \mid (\mathbf{u})_1 = i_1 \text{ and } d(\mathbf{u}, (\mathbf{x}_{n-3})^n) = 2\}| = n-3 < (n-2)!/2$ if $n \geq 5$, we choose a vertex $\mathbf{y}_{n-2}$ of $V_1(S_n^{\{i_{n-2}\}})$ such that $d(\mathbf{y}_{n-2}, (\mathbf{x}_{n-3})^n) > 2$, $(\mathbf{y}_{n-2})_1 = i_1$, and $(\mathbf{y}_{n-2}, (\mathbf{y}_{n-2})^n) \notin F$. By Theorem 4.6, there exists a hamiltonian path Q_1 of $S_n^{\{i_1\}}$ joining $(\mathbf{y}_{n-2})^n$ to $(\mathbf{q})^n$. Again, there is a hamiltonian path Q_2 of $S_n^{\{i_2\}}$ joining $(\mathbf{y}_{n-1})^n$ to $\mathbf{y}_2$, there is a hamiltonian path Q_{n-1} of $S_n^{\{i_{n-1}\}}$ joining $(\mathbf{p})^n$ to $\mathbf{y}_{n-1}$, and there exists a hamiltonian path Q_k of $S_n^{\{i_k\}}$ joining $(\mathbf{y}_{k-1})^n$ to $\mathbf{y}_k$ for $3 \leq k \leq n-2$. Then we set $C_2 = \langle \mathbf{e}, \mathbf{p}, (\mathbf{p})^n, Q_{n-1}, \mathbf{y}_{n-1}, (\mathbf{y}_{n-1})^n, Q_2, \mathbf{y}_2, (\mathbf{y}_2)^n, Q_3, \mathbf{y}_3, (\mathbf{y}_3)^n, \ldots, (\mathbf{y}_{n-2})^n, Q_1, (\mathbf{q})^n, \mathbf{q}, R^{-1}, \mathbf{e} \rangle$.

In summary, $\{C_1, C_2\}$ forms a set of 2-mutually independent hamiltonian cycles of $S_n - F$ beginning from $\mathbf{e}$. Figure 4.3(b) illustrates C_1 and C_2 in S_5.

Case 3: Suppose that $(\mathbf{e}, (\mathbf{e})^n) \in F$ and $|F_n| \leq n-5$. Since $|F| = n-3$, there must exist an integer i_{n-1} of $\langle n-1 \rangle - \{1\}$ such that $|F \cap E^{i_{n-1}, n}| = 0$. Assume that i_1 and i_2 are two integers of $\langle n-1 \rangle - \{i_{n-1}\}$ such that $|F \cap E^{i_1, i_2}| = \max\{|F \cap E^{s,t}| \mid s, t \in \langle n-1 \rangle - \{i_{n-1}\}\}$. Moreover, let $i_3 i_4 \ldots i_{n-2}$ be an arbitrary permutation of $\langle n-1 \rangle - \{i_1, i_2, i_{n-1}\}$. Since $(\mathbf{e}, (\mathbf{e})^n) \in F$, we have $|F \cap E^{i_1, i_2}| \leq n-4$. Thus, $|F \cap E^{i_{n-2}, i_1}| \leq n-5$ and $|F \cap E^{i_k, i_{k+1}}| \leq n-5$ for $2 \leq k \leq n-3$.

By Lemma 4.4, there are $(n-2)!/2 > n-3$ edges joining vertices of $V_0(S_n^{\{n\}})$ to vertices of $V_1(S_n^{\{i_1\}})$. Thus, we can choose a vertex $\mathbf{w} \in V_0(S_n^{\{n\}}) - \{\mathbf{e}\}$ such that $(\mathbf{w})_1 = i_1$ and $(\mathbf{w}, (\mathbf{w})^n) \notin F$. By Theorem 4.6, there exists a hamiltonian path R of $S_n^{\{n\}} - (F_n \cup \{(\mathbf{e})^{i_{n-1}}\})$ joining $\mathbf{e}$ to $\mathbf{w}$. For $1 \leq k \leq n-2$, let $\mathbf{x}_k$ be a vertex of $V_0(S_n^{\{i_k\}})$ such that $(\mathbf{x}_k)_1 = i_{k+1}$ and $(\mathbf{x}_k, (\mathbf{x}_k)^n) \notin F$. By Theorem 4.6, there exists a hamiltonian path P_1 of $S_n^{\{i_1\}} - F_{i_1}$ joining $(\mathbf{w})^n$ to $\mathbf{x}_1$. Similarly, there exists a hamiltonian path P_k of $S_n^{\{i_k\}} - F_{i_k}$ joining $(\mathbf{x}_{k-1})^n$ to $\mathbf{x}_k$ for $2 \leq k \leq n-2$, and there exists a hamiltonian path P_{n-1} of $S_n^{\{i_{n-1}\}} - F_{i_{n-1}}$ joining $(\mathbf{x}_{n-2})^n$ to $((\mathbf{e})^{i_{n-1}})^n$. Then we set $C_1 = \langle \mathbf{e}, R, \mathbf{w}, (\mathbf{w})^n, P_1, \mathbf{x}_1, (\mathbf{x}_1)^n, P_2, \mathbf{x}_2, (\mathbf{x}_2)^n, \ldots, (\mathbf{x}_{n-2})^n, P_{n-1}, ((\mathbf{e})^{i_{n-1}})^n, (\mathbf{e})^{i_{n-1}}, \mathbf{e} \rangle$.

Obviously, we can choose a vertex $\mathbf{y}_{n-1}$ of $V_1(S_n^{\{i_{n-1}\}})$ such that $(\mathbf{y}_{n-1})_1 = i_2$ and $(\mathbf{y}_{n-1}, (\mathbf{y}_{n-1})^n) \notin F$. For $2 \leq k \leq n-3$, $|\{\mathbf{u} \in V_1(S_n^{\{i_k\}}) \mid (\mathbf{u})_1 = i_{k+1}$ and $d(\mathbf{u}, (\mathbf{x}_{k-1})^n) = 2\}| = n-3$. By Lemma 4.4, there are $(n-2)!/2$ edges joining vertices of $V_1(S_n^{\{i_k\}})$ to vertices of $V_0(S_n^{\{i_{k+1}\}})$. We emphasize that $(n-2)!/2 > (n-3)+(n-5) = 2n-8$ if $n \geq 5$. Thus, we choose a vertex $\mathbf{y}_k$ of $V_1(S_n^{\{i_k\}})$ such that $d(\mathbf{y}_k, (\mathbf{x}_{k-1})^n) > 2$, $(\mathbf{y}_k)_1 = i_{k+1}$, and $(\mathbf{y}_k, (\mathbf{y}_k)^n) \notin F$ for $2 \leq k \leq n-3$. Since $(n-2)!/2 > |\{\mathbf{u} \in V_1(S_n^{\{i_{n-2}\}}) \mid (\mathbf{u})_1 = i_1$ and $d(\mathbf{u}, (\mathbf{x}_{n-3})^n) = 2\}| + (n-5) = (n-3)+(n-5) = 2n-8$ if $n \geq 5$, we choose a vertex $\mathbf{y}_{n-2}$ of $V_1(S_n^{\{i_{n-2}\}})$ such that $d(\mathbf{y}_{n-2}, (\mathbf{x}_{n-3})^n) > 2$, $(\mathbf{y}_{n-2})_1 = i_1$, and $(\mathbf{y}_{n-2}, (\mathbf{y}_{n-2})^n) \notin F$. By Theorem 4.6, there exists a hamiltonian path Q_1 of $S_n^{\{i_1\}} - F_{i_1}$ joining $(\mathbf{y}_{n-2})^n$ to $(\mathbf{w})^n$. Again, there exists a hamiltonian path Q_2 of $S_n^{\{i_2\}} - F_{i_2}$ joining $(\mathbf{y}_{n-1})^n$ to $\mathbf{y}_2$, there exists a hamiltonian path Q_{n-1} of $S_n^{\{i_{n-1}\}} - F_{i_{n-1}}$ joining $((\mathbf{e})^{i_{n-1}})^n$ to $\mathbf{y}_{n-1}$, and there exists a hamiltonian path Q_k of $S_n^{\{i_k\}} - F_{i_k}$ joining $(\mathbf{y}_{k-1})^n$ to $\mathbf{y}_k$ for $3 \leq k \leq n-2$. We set $C_2 = \langle \mathbf{e}$, $(\mathbf{e})^{i_{n-1}}$, $((\mathbf{e})^{i_{n-1}})^n$, Q_{n-1}, $\mathbf{y}_{n-1}$, $(\mathbf{y}_{n-1})^n$, Q_2, $\mathbf{y}_2$, $(\mathbf{y}_2)^n$, Q_3, $\mathbf{y}_3$, $(\mathbf{y}_3)^n, \ldots, (\mathbf{y}_{n-2})^n$, Q_1, $(\mathbf{w})^n$, $\mathbf{w}$, R^{-1}, $\mathbf{e}\rangle$.

As a result, $\{C_1, C_2\}$ forms a set of 2-mutually independent hamiltonian cycles of $S_n - F$ beginning from $\mathbf{e}$. Figure 4.3(c) illustrates C_1 and C_2 in S_5. □

Lemma 4.13. *Let f be any integer of $\langle n-4\rangle$ with $n \geq 5$. Suppose that $F \subset E(S_n)$ with $|F| = f$. Let $\mathbf{u} \in V(S_n)$. Then there exist $(n-1-f)$-mutually independent hamiltonian cycles of $S_n - F$ beginning from $\mathbf{u}$.*

Proof. Since S_n is vertex-transitive and edge-transitive, we assume that $\mathbf{u} = \mathbf{e}$ and F contains at least one edge in dimension n. Let $F_k = F \cap E(S_n^{\{k\}})$ for every $k \in \langle n\rangle$. Thus, $|F_k| \leq n-5$ for every $k \in \langle n\rangle$. Moreover, let $A_1 = E^{1,n} - \{(\mathbf{e}, (\mathbf{e})^n)\}$ and let $A_i = E^{i,n} \cup \{(\mathbf{e}, (\mathbf{e})^i)\}$ for $2 \leq i \leq n-1$.

Case 1: Suppose that $(\mathbf{e}, (\mathbf{e})^n) \in F$. We emphasize that there are at least $n-1-f$ elements of $\{|F \cap A_2|, |F \cap A_3|, \ldots, |F \cap A_{n-1}|\}$ equal to 0. Without loss of generality, we assume that $|F \cap (\cup_{i=f+1}^{n-1} A_i)| = 0$. Thus, at least one of $\{|F \cap A_1|, \ldots, |F \cap A_f|\}$ equals to 0.

Subcase 1.1: Suppose that $|F \cap A_1| = 0$. Let $B = (b_{i,j})$ be the $(n-1-f) \times n$ matrix with

$$b_{i,j} = \begin{cases} f+i+j & \text{if } f+i+j \leq n, \\ f+i+j-n & \text{otherwise.} \end{cases}$$

Note that $b_{i,n-f-i} = n$ for every $1 \leq i \leq n-1-f$. Then we construct $(n-1-f)$-mutually independent hamiltonian cycles $\{C_1, C_2, \ldots, C_{n-1-f}\}$ of $S_n - F$ beginning from $\mathbf{e}$ as follows.

Let $i \in \langle n-2-f\rangle$. We set $t_i = n-f-i$. By Lemma 4.10, there exists a hamiltonian path Q_i of $S_n^{\{b_{i,t_i}\}} - (F_{b_{i,t_i}} \cup \{\mathbf{e}, (\mathbf{e})^{b_{i,1}}, (\mathbf{e})^{b_{i,n}}\})$ joining two vertices $\mathbf{x}_i$ and $\mathbf{y}_i$ in $V_0(S_n^{\{b_{i,t_i}\}}) - \{\mathbf{e}\}$ such that $(\mathbf{x}_i)_1 = b_{i,t_i-1}$ and $(\mathbf{y}_i)_1 = b_{i,t_i+1}$. By Lemma 4.6, there exists a hamiltonian path P_i of $(\cup_{j=1}^{t_i-1} S_n^{\{b_{i,j}\}}) - F$ joining $((\mathbf{e})^{b_{i,1}})^n$ to $(\mathbf{x}_i)^n$. Similarly, there exists a hamiltonian path R_i of $(\cup_{j=t_i+1}^{n} S_n^{\{b_{i,j}\}}) - F$ joining $(\mathbf{y}_i)^n$ to $((\mathbf{e})^{b_{i,n}})^n$. Then we set $C_i = \langle \mathbf{e}, (\mathbf{e})^{b_{i,1}}, ((\mathbf{e})^{b_{i,1}})^n, P_i, (\mathbf{x}_i)^n, \mathbf{x}_i, Q_i, \mathbf{y}_i, (\mathbf{y}_i)^n, R_i, ((\mathbf{e})^{b_{i,n}})^n, (\mathbf{e})^{b_{i,n}}, \mathbf{e}\rangle$.

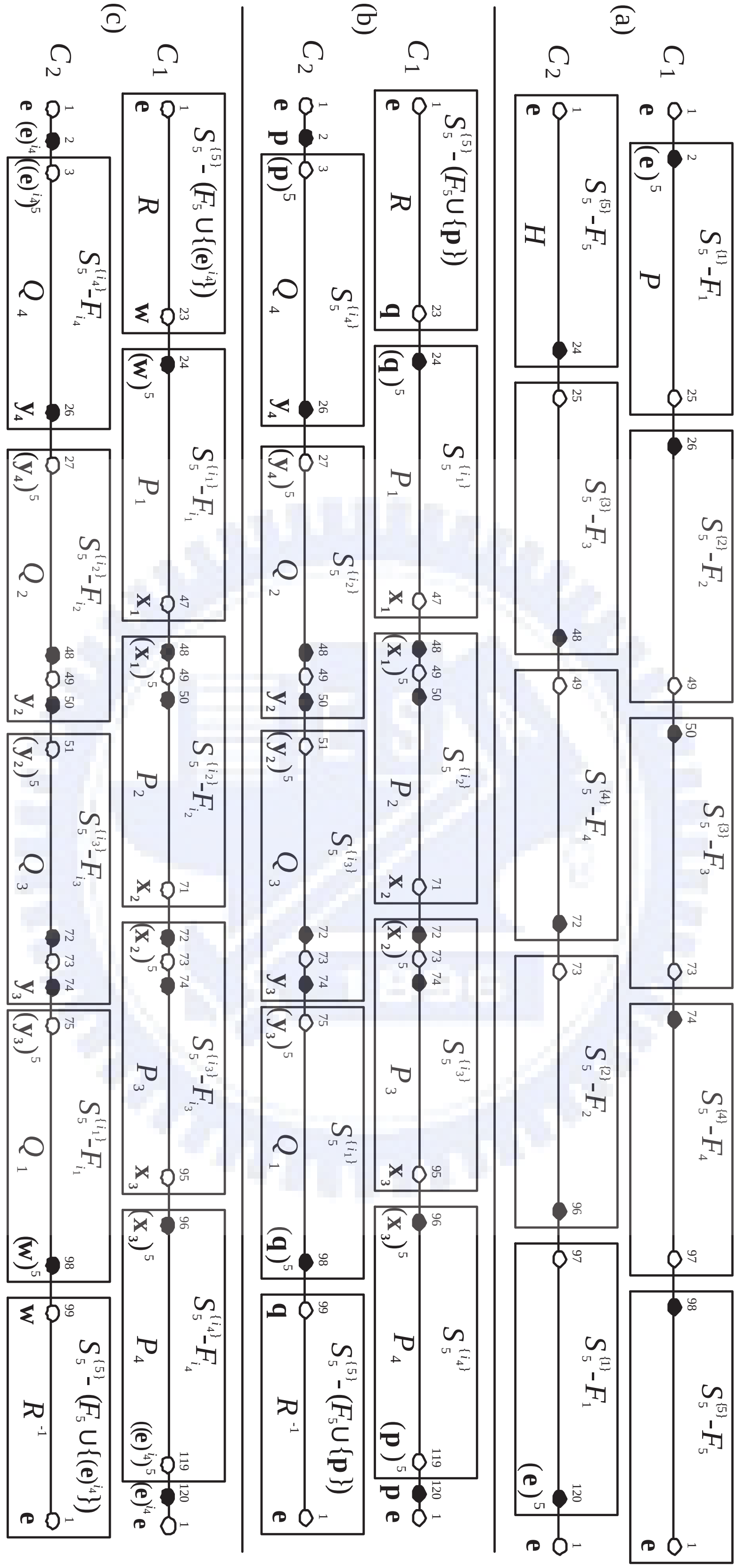

Figure 4.3: The 2-mutually independent hamiltonian cycles in $S_5 - F$ for Lemma 4.12.

By Lemma 4.9, there exists a hamiltonian path T of $S_n^{\{b_{n-1-f,1}\}} - (F_{b_{n-1-f,n}} \cup \{\mathbf{e}, (\mathbf{e})^{b_{n-1-f,n}}\})$ joining $(\mathbf{e})^{b_{1,n}}$ to a vertex $\mathbf{z}$ of $V_0(S_n^{\{b_{n-1-f,1}\}}) - \{\mathbf{e}\}$ with $(\mathbf{z})_1 = b_{n-1-f,2}$. By Lemma 4.6, there exists a hamiltonian path W of $(\cup_{j=2}^{n} S_n^{\{b_{n-1-f,j}\}}) - F$ joining $(\mathbf{z})^n$ to $((\mathbf{e})^{b_{n-1-f,n}})^n$. Then we set $C_{n-1-f} = \langle \mathbf{e}, (\mathbf{e})^{b_{1,n}}, T, \mathbf{z}, (\mathbf{z})^n, W, ((\mathbf{e})^{b_{n-1-f,n}})^n, (\mathbf{e})^{b_{n-1-f,n}}, \mathbf{e} \rangle$.

As a result, $\{C_1, \ldots, C_{n-2-f}, C_{n-1-f}\}$ forms a set of $(n-1-f)$-mutually independent hamiltonian cycles of $S_n - F$ beginning from $\mathbf{e}$. Figure 4.4 illustrates $\{C_1, C_2, C_3, C_4\}$ in $S_6 - F$ with $|F| = f = 1$.

Subcase 1.2: Suppose that $|F \cap A_1| > 0$. We emphasize that $f \geq 2$ in this subcase. Thus, at least one of $\{|F \cap A_2|, \ldots, |F \cap A_f|\}$ equals to 0. Without loss of generality, we assume that $|F \cap A_2| = 0$. Let $B = (b_{i,j})$ be the $(n-1-f) \times n$ matrix with

$$b_{i,j} = \begin{cases} f+i+j & \text{if } f+i+j \leq n, \\ 2 & \text{if } f+i+j = n+1, \\ 1 & \text{if } f+i+j = n+2, \\ f+i+j-n & \text{otherwise.} \end{cases}$$

Then we build $(n-1-f)$-mutually independent hamiltonian cycles $\{C_1, C_2, \ldots, C_{n-1-f}\}$ of $S_n - F$ beginning from $\mathbf{e}$ in the same manner as that of **Subcase 1.1**.

Case 2: Suppose that $(\mathbf{e}, (\mathbf{e})^n) \notin F$. We emphasize that there are at least $n-2-f$ elements of $\{|F \cap A_2|, |F \cap A_3| \ldots, |F \cap A_{n-1}|\}$ equaling to 0. Without loss of generality, we assume that $|F \cap (\cup_{i=f+2}^{n-1} A_i)| = 0$. Thus, at least one of $\{|F \cap A_1|, \ldots, |F \cap A_{f+1}|\}$ is 0.

Subcase 2.1: Suppose that $|F \cap A_1| = 0$. Let $B_n = (b_{i,j})$ be the $(n-1-f) \times n$ matrix with

$$B_5 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 4 & 2 & 3 & 1 \end{bmatrix},$$

and for $n \geq 6$,

$$b_{i,j} = \begin{cases} j & \text{if } i = 1, \\ f+i+j & \text{if } 2 \leq i \leq n-2-f \text{ and } f+i+j \leq n, \\ f+i+j-n & \text{if } 2 \leq i \leq n-2-f \text{ and } f+i+j > n, \\ n & \text{if } i = n-1-f \text{ and } j = 1, \\ 3 & \text{if } i = n-1-f \text{ and } j = 2, \\ 2 & \text{if } i = n-1-f \text{ and } j = 3, \\ n-1 & \text{if } i = n-1-f \text{ and } j = 4, \\ j-1 & \text{if } i = n-1-f \text{ and } 5 \leq j \leq n-1, \\ 1 & \text{if } i = n-1-f \text{ and } j = n. \end{cases}$$

Then we build $(n-1-f)$-mutually independent hamiltonian cycles $\{C_1, C_2, \ldots, C_{n-1-f}\}$ of $S_n - F$ beginning from $\mathbf{e}$ as follows.

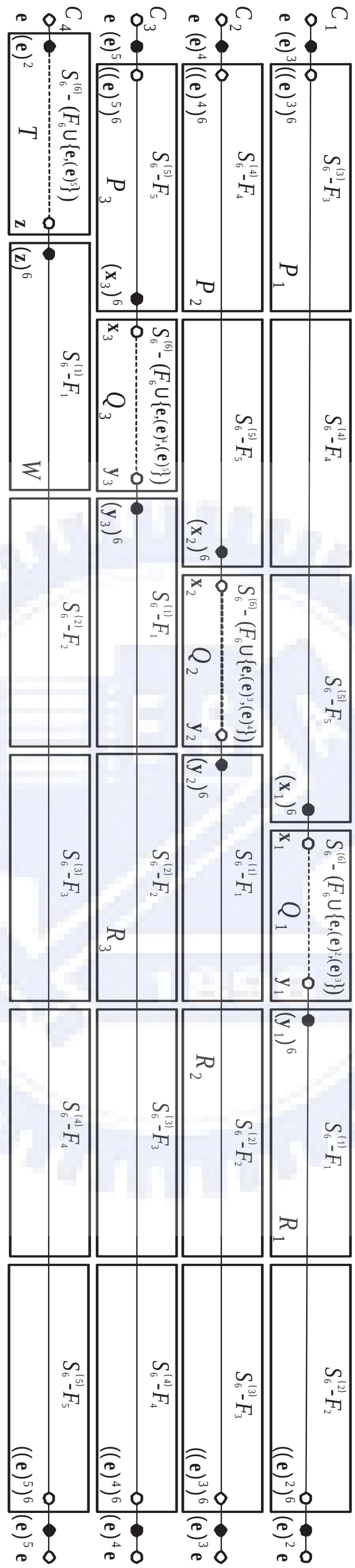

Figure 4.4: Mutually independent hamiltonian cycles in $S_6 - F$ with $|F| = 1$ for Subcase 1.1 of Lemma 4.13.

We choose a vertex $\mathbf{v}$ of $V_1(S_n^{\{b_{1,n}\}}) - \{(\mathbf{e})^{b_{n-2-f,1}}\}$ with $(\mathbf{v})_1 = b_{1,n-1}$. By Theorem 4.6, there exists a hamiltonian path W of $S_n^{\{b_{1,n}\}} - (F_{b_{1,n}} \cup \{\mathbf{e}\})$ joining $\mathbf{v}$ to $(\mathbf{e})^{b_{n-2-f,1}}$. By Lemma 4.6, there exists a hamiltonian path D of $(\cup_{j=1}^{n-1} S_n^{\{b_{1,j}\}}) - F$ joining $(\mathbf{e})^n$ to $(\mathbf{v})^n$. We set $C_1 = \langle \mathbf{e}, (\mathbf{e})^n, D, (\mathbf{v})^n, \mathbf{v}, W, (\mathbf{e})^{b_{n-2-f,1}}, \mathbf{e} \rangle$.

Let $i \in \langle n-2-f \rangle - \{1\}$. We set $t_i = n - f - i$. By Lemma 4.10, there exists a hamiltonian path Q_i of $S_n^{\{b_{i,t_i}\}} - (F_{b_{i,t_i}} \cup \{\mathbf{e}, (\mathbf{e})^{b_{i,1}}, (\mathbf{e})^{b_{i,n}}\})$ joining two vertices $\mathbf{x}_i$ and $\mathbf{y}_i$ in $V_0(S_n^{\{b_{i,t_i}\}}) - \{\mathbf{e}\}$ such that $(\mathbf{x}_i)_1 = b_{i,t_i-1}$ and $(\mathbf{y}_i)_1 = b_{i,t_i+1}$. By Lemma 4.6, there exists a hamiltonian path P_i of $(\cup_{j=1}^{t_i-1} S_n^{\{b_{i,j}\}}) - F$ joining $((\mathbf{e})^{b_{i,1}})^n$ to $(\mathbf{x}_i)^n$. Similarly, there exists a hamiltonian path R_i of $(\cup_{j=t_i+1}^{n} S_n^{\{b_{i,j}\}}) - F$ joining $(\mathbf{y}_i)^n$ to $((\mathbf{e})^{b_{i,n}})^n$. Then we set $C_i = \langle \mathbf{e}, (\mathbf{e})^{b_{i,1}}, ((\mathbf{e})^{b_{i,1}})^n, P_i, (\mathbf{x}_i)^n, \mathbf{x}_i, Q_i, \mathbf{y}_i, (\mathbf{y}_i)^n, R_i, ((\mathbf{e})^{b_{i,n}})^n, (\mathbf{e})^{b_{i,n}}, \mathbf{e} \rangle$.

By Lemma 4.4, there are $(n-2)!/2 > n-3$ edges joining vertices of $V_0(S_n^{\{b_{n-1-f,k}\}})$ to vertices of $V_1(S_n^{\{b_{n-1-f,k-1}\}})$ for $3 \le k \le n-1$. Thus, we choose a vertex $\mathbf{z}_k$ of $V_0(S_n^{\{b_{n-1-f,k}\}})$ such that $(\mathbf{z}_k)_1 = b_{n-1-f,k-1}$, $(\mathbf{z}_k, (\mathbf{z}_k)^n) \notin F$, and $\mathbf{z}_k \neq C_1((k-1)(n-1)!+1)$. By Lemma 4.7, there exists a hamiltonian path T of $(\cup_{j=1}^{2} S_n^{\{b_{n-1-f,j}\}}) - (F \cup \{\mathbf{e}\})$ joining $(\mathbf{e})^{b_{2,n}}$ to $(\mathbf{z}_3)^n$. By Theorem 4.6, there is a hamiltonian path H_k of $S_n^{\{b_{n-1-f,k}\}} - F_{b_{n-1-f,k}}$ joining $\mathbf{z}_k$ to $(\mathbf{z}_{k+1})^n$ for $3 \le k \le n-2$. By Lemma 4.6, there exists a hamiltonian path H_{n-1} of $(\cup_{j=n-1}^{n} S_n^{\{b_{n-1-f,j}\}}) - F$ joining $\mathbf{z}_{n-1}$ to $(\mathbf{e})^n$. Then we set $C_{n-1-f} = \langle \mathbf{e}, (\mathbf{e})^{b_{2,n}}, T, (\mathbf{z}_3)^n, \mathbf{z}_3, H_3, (\mathbf{z}_4)^n, \ldots, \mathbf{z}_{n-2}, H_{n-2}, (\mathbf{z}_{n-1})^n, \mathbf{z}_{n-1}, H_{n-1}, (\mathbf{e})^n, \mathbf{e} \rangle$.

Consequently, $\{C_1, C_2, \ldots, C_{n-2-f}, C_{n-1-f}\}$ forms a set of $(n-1-f)$-mutually independent hamiltonian cycles of $S_n - F$ beginning from $\mathbf{e}$. Figure 4.5(a) illustrates $\{C_1, C_2, C_3, C_4\}$ in $S_6 - F$ with $|F| = f = 1$.

Subcase 2.2: Suppose that $|F \cap A_1| > 0$. Thus, at least one of $\{|F \cap A_2|, \ldots, |F \cap A_{f+1}|\}$ equals to 0. Without loss of generality, we assume that $|F \cap A_2| = 0$. Let $B_n = (b_{i,j})$ be the $(n-1-f) \times n$ matrix with

$$b_{i,j} = \begin{cases} n & \text{if } i = 1 \text{ and } j = 1, \\ j+1 & \text{if } i = 1 \text{ and } 2 \le j \le n-2, \\ 2 & \text{if } i = 1 \text{ and } j = n-1, \\ 1 & \text{if } i = 1 \text{ and } j = n, \\ f+i+j & \text{if } 2 \le i \le n-2-f \text{ and } f+i+j \le n, \\ 2 & \text{if } 2 \le i \le n-2-f \text{ and } f+i+j = n+1, \\ 1 & \text{if } 2 \le i \le n-2-f \text{ and } f+i+j = n+2, \\ f+i+j-n & \text{if } 2 \le i \le n-2-f \text{ and } f+i+j \ge n+3, \\ j & \text{if } i = n-1-f. \end{cases}$$

By Lemma 4.4, there are $(n-2)!/2 > n-3$ edges joining vertices of $V_0(S_n^{\{b_{1,2}\}})$ to vertices of $V_1(S_n^{\{b_{1,1}\}})$. Thus, we choose a vertex $\mathbf{z}$ of $V_0(S_n^{\{b_{1,2}\}})$ such that $(\mathbf{z})_1 = b_{1,1}$, $(\mathbf{z}, (\mathbf{z})^n) \notin F$, and $(\mathbf{z})^n \neq (\mathbf{e})^{b_{2,n}}$. By Theorem 4.6, there exists a hamiltonian path T of $S_n^{\{b_{1,1}\}} - (F_{b_{1,1}} \cup \{\mathbf{e}\})$ joining $(\mathbf{e})^{b_{2,n}}$ to $(\mathbf{z})^n$. By Lemma 4.6, there exists a hamiltonian path H of $(\cup_{j=2}^{n} S_n^{\{b_{1,j}\}}) - F$ joining $\mathbf{z}$ to $(\mathbf{e})^n$. Then we set $C_1 = \langle \mathbf{e}, (\mathbf{e})^{b_{2,n}}, T, (\mathbf{z})^n, \mathbf{z}, H, (\mathbf{e})^n, \mathbf{e} \rangle$.

Let $i \in \langle n-2-f \rangle - \{1\}$. We set $t_i = n - f - i$. By Lemma 4.10, there exists a hamiltonian path Q_i of $S_n^{\{b_{i,t_i}\}} - (F_{b_{i,t_i}} \cup \{\mathbf{e}, (\mathbf{e})^{b_{i,1}}, (\mathbf{e})^{b_{i,n}}\})$ joining two vertices $\mathbf{x}_i$ and $\mathbf{y}_i$ in $V_0(S_n^{\{b_{i,t_i}\}}) - \{\mathbf{e}\}$ such that $(\mathbf{x}_i)_1 = b_{i,t_i-1}$ and $(\mathbf{y}_i)_1 = b_{i,t_i+1}$. By Lemma 4.6, there exists a hamiltonian path P_i of $(\cup_{j=1}^{t_i-1} S_n^{\{b_{i,j}\}}) - F$ joining $((\mathbf{e})^{b_{i,1}})^n$ to $(\mathbf{x}_i)^n$. Similarly, there exists a hamiltonian path R_i of $(\cup_{j=t_i+1}^{n} S_n^{\{b_{i,j}\}}) - F$ joining $(\mathbf{y}_i)^n$ to $((\mathbf{e})^{b_{i,n}})^n$. Then we set $C_i = \langle \mathbf{e}, (\mathbf{e})^{b_{i,1}}, ((\mathbf{e})^{b_{i,1}})^n, P_i, (\mathbf{x}_i)^n, \mathbf{x}_i, Q_i, \mathbf{y}_i, (\mathbf{y}_i)^n, R_i, ((\mathbf{e})^{b_{i,n}})^n, (\mathbf{e})^{b_{i,n}}, \mathbf{e} \rangle$.

By Lemma 4.4, there are $(n-2)!/2 > n-3$ edges joining vertices of $V_0(S_n^{\{b_{n-1-f,2}\}})$ to vertices of $V_1(S_n^{\{b_{n-1-f,3}\}})$. Thus, we choose a vertex $\mathbf{w}$ of $V_0(S_n^{\{b_{n-1-f,2}\}})$ such that $(\mathbf{w})_1 = b_{n-1-f,3}$, $(\mathbf{w}, (\mathbf{w})^n) \notin F$, and $d(\mathbf{w}, (\mathbf{y}_{n-2-f})^n) > 1$. Moreover, we choose a vertex $\mathbf{v}$ of $V_1(S_n^{\{b_{n-1-f,n}\}})$ such that $(\mathbf{v})_1 = b_{n-1-f,n-1}$ and $(\mathbf{v}, (\mathbf{v})^n) \notin F$. By Lemma 4.6, there exists a hamiltonian path D_1 of $(\cup_{j=1}^{2} S_n^{\{b_{n-1-f,j}\}}) - F$ joining $(\mathbf{e})^n$ to $\mathbf{w}$. Similarly, there exists a hamiltonian path D_2 of $(\cup_{j=3}^{n-1} S_n^{\{b_{n-1-f,j}\}}) - F$ joining $(\mathbf{w})^n$ to $(\mathbf{v})^n$. By Theorem 4.6, there exists a hamiltonian path W of $S_n^{\{b_{n-1-f,n}\}} - (F_{b_{n-1-f,n}} \cup \{\mathbf{e}\})$ joining $\mathbf{v}$ to $(\mathbf{e})^{b_{n-2-f,1}}$. Then we set $C_{n-1-f} = \langle \mathbf{e}, (\mathbf{e})^n, D_1, \mathbf{w}, (\mathbf{w})^n, D_2, (\mathbf{v})^n, \mathbf{v}, W, (\mathbf{e})^{b_{n-2-f,1}}, \mathbf{e} \rangle$.

Hence, $\{C_1, C_2, \ldots, C_{n-2-f}, C_{n-1-f}\}$ forms a set of $(n-1-f)$-mutually independent hamiltonian cycles of $S_n - F$ beginning from $\mathbf{e}$. Figure 4.5(b) illustrates $\{C_1, C_2, C_3, C_4\}$ in $S_6 - F$ with $|F| = f = 1$. □

According to Lemma 4.11, Lemma 4.12, and Lemma 4.13, coupled with the result of Lin et al. [49], we summarize the embedding of mutually independent hamiltonian cycles in star networks as follows:

Theorem 4.8. *Let $F \subset E(S_n)$ with $|F| \leq n-3$ for $n \geq 3$, and let $\mathbf{u} \in V(S_n)$. Then there exist $(n-2-|F|)$-mutually independent hamiltonian cycles of $S_n - F$ beginning from $\mathbf{u}$ if $n \in \{3,4\}$, and there exist $(n-1-|F|)$-mutually independent hamiltonian cycles of $S_n - F$ beginning from $\mathbf{u}$ if $n \geq 5$.*

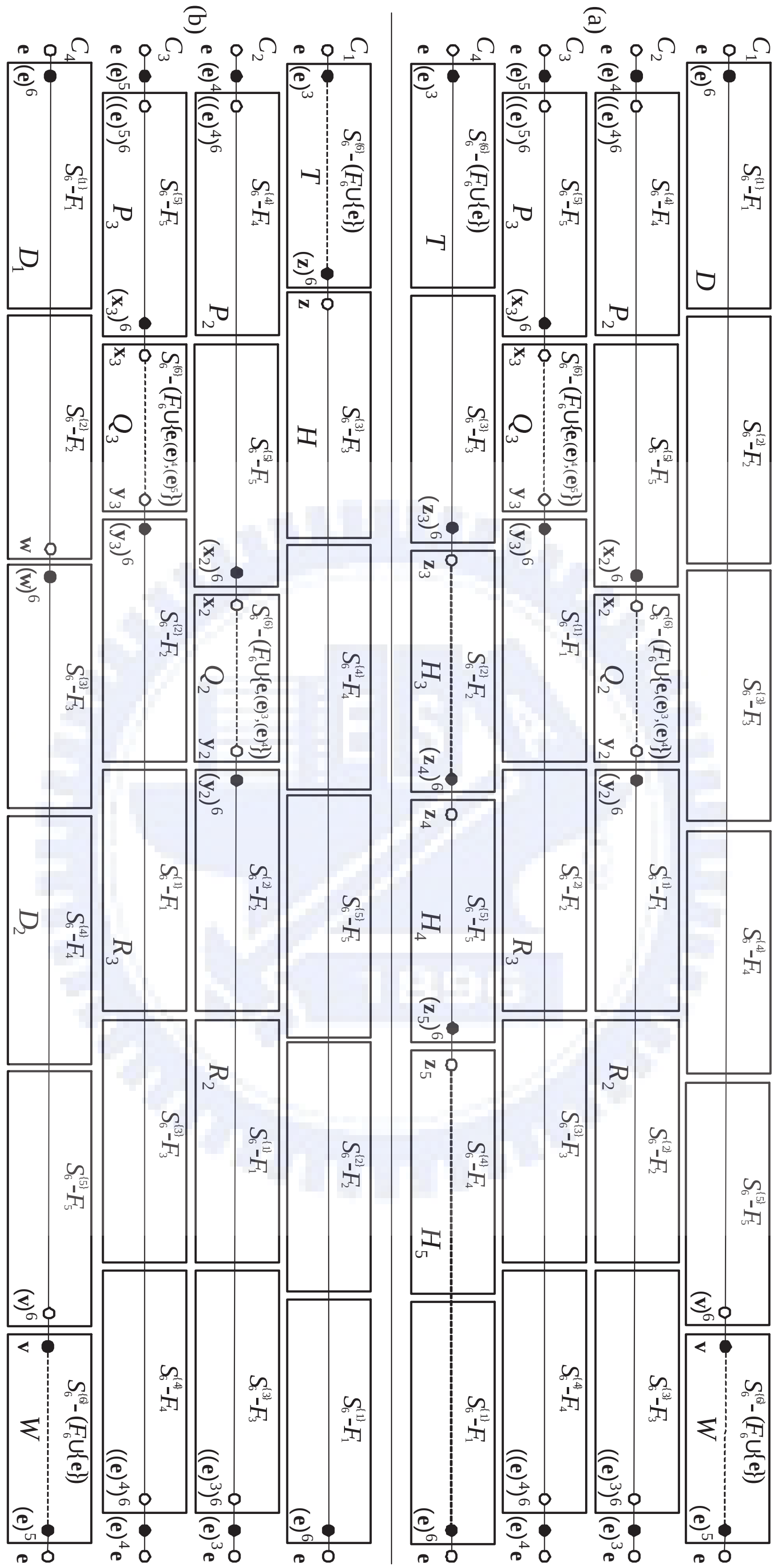

Figure 4.5: Mutually independent hamiltonian cycles in $S_6 - F$ with $|F| = 1$ for Case 2 of Lemma 4.13.

Chapter 5

Fault Diameter of Hypercubes

A variety of graph parameters, such as connectivity [50], wide diameter [29], fault diameter [37], etc. can be used to measure fault tolerance of networks: The *connectivity* of a graph G, written by $\kappa(G)$, is the minimum size of a vertex set S such that $G - S$ is disconnected or has only one vertex; a graph G is said to have *fault diameter* of $D_f(G)$ if its diameter increases from $D(G)$ to $D_f(G)$ as a consequence of f hybrid node and/or link faults.

Esfahanian [20] observed that the likelihood of having a disconnected n-cube due to n faulty processors is negligible and asymptotically zero. Motivated by this observation, he introduced the concept of *forbidden faulty sets*. The components of any forbidden faulty set cannot be faulty at the same time. For the n-cube, each forbidden faulty set is defined to consist of all n neighbors of one processor; thus, there are 2^n forbidden faulty sets in an n-cube, each containing n processors. Later the *conditional node-faults* [42] were defined in such a way that every node is required to have at least g fault-free neighbors. It is also intuitive to extend this concept by defining *conditional link-faults*, which require that every node will be incident to at least g fault-free links. In this chapter, we allow node-faults and link-faults can take place simultaneously. Moreover, we concentrate only on $g = 1$. Suppose that u is an arbitrary node of a graph G, and v is a neighbor of u in G. We say v is a *reachable* neighbor of u if both v and (u, v) are fault-free; otherwise, v is an *unreachable* neighbor of u. We will compute the fault diameter of an n-cube, in which every node is required to have at least one reachable neighbor.

5.1 Basic properties of hypercubes

Before we proceed to obtain the main results, we introduce some basic properties of hypercubes. Again, we use a boldface letter to denote any node of hypercube. Let $Q_n^{i,j}$ be a subgraph of Q_n induced by $\{\mathbf{u} \in V(Q_n) \mid (\mathbf{u})_i = j\}$ for $0 \le i \le n-1$ and $j \in \{0, 1\}$. Obviously, $Q_n^{i,j}$ is isomorphic to Q_{n-1}. Then the node partition of Q_n into subgraphs $Q_n^{j,0}$ and $Q_n^{j,1}$ is called *j-partition*. The set of crossing links between $Q_n^{i,0}$ and $Q_n^{i,1}$, denoted by $E_c^i = \{(\mathbf{u}, \mathbf{v}) \in E(Q_n) \mid \mathbf{u} \in V(Q_n^{i,0}), \mathbf{v} \in V(Q_n^{i,1})\}$, consists of all i-dimensional links of Q_n. In order to clearly indicate the faulty elements in network G, we use $F(G)$ to denote the set of all faulty elements in G.

The following lemma characterizes a collection of n disjoint paths in Q_n.

Lemma 5.1. *[55] For any two nodes,* $\mathbf{u}$ *and* $\mathbf{v}$*, of* Q_n*, there exist* n *internally node-disjoint paths joining* $\mathbf{u}$ *and* $\mathbf{v}$*,* $h(\mathbf{u}, \mathbf{v})$ *of which are of length* $h(\mathbf{u}, \mathbf{v})$*, and the other* $n - h(\mathbf{u}, \mathbf{v})$ *of which are of length* $h(\mathbf{u}, \mathbf{v}) + 2$*.*

The next corollary follows directly from Lemma 5.1.

Corollary 5.1. *Let* F *be a set of* $n-1$ *node-faults and/or link-faults in* Q_n*. For any pair* $\mathbf{u}, \mathbf{v}$ *of distinct nodes in* $Q_n - F$*, then* $d_{Q_n - F}(\mathbf{u}, \mathbf{v}) \leq h(\mathbf{u}, \mathbf{v}) + 2$*.*

By computing the upper bound of distance between any pair of distinct nodes, Latifi [41] investigated the fault diameter of Q_n under the assumption that every node has at least one fault-free neighbor. It is noticed that only node-faults were addressed in [41].

Theorem 5.1. *[41] Let* F *be a set of* $2n - 3$ *faulty nodes in* Q_n *such that every node of* Q_n *has at least one fault-free neighbor. For any pair* $\mathbf{u}, \mathbf{v}$ *of distinct nodes in* $Q_n - F$*, then* $d_{Q_n - F}(\mathbf{u}, \mathbf{v}) \leq h(\mathbf{u}, \mathbf{v}) + 4$*.*

Theorem 5.2. *[41] Let* F *be a set of faulty nodes in* Q_n *such that every node of* Q_n *has at least one fault-free neighbor. Then the diameter of* $Q_n - F$ *is computed as follows:*

$$D(Q_n - F) = \begin{cases} n & \text{if } |F| \leq n-2, \\ n+1 & \text{if } |F| = n-1, \\ n+2 & \text{if } |F| = 2n-3. \end{cases}$$

5.2 Shortest paths in faulty hypercubes

We can improve Theorem 5.1, mentioned earlier, by proving the next three propositions.

Proposition 5.1. *Suppose that* $\mathbf{u}$ *and* $\mathbf{v}$ *are any two distinct nodes of* Q_n *with* $h(\mathbf{u}, \mathbf{v}) = n$*. Let* F *be a set of* $2n - 3$ *hybrid node-faults and/or link-faults in* Q_n *such that both* $\mathbf{u}$ *and* $\mathbf{v}$ *are fault-free with at least one reachable neighbor. Then* $d_{Q_n - F}(\mathbf{u}, \mathbf{v}) = n$*.*

Proof. It is not difficult to verify that this proposition holds for $n = 2$. Hence we concern only the case that $n \geq 3$. Let $I_u = \{i_1, \ldots, i_p\}$ be a set of p distinct integers of $\{0, 1, \ldots, n-1\}$ such that $(\mathbf{u})^{i_1}, \ldots, (\mathbf{u})^{i_p}$ are reachable neighbors of $\mathbf{u}$. Similarly, let $I_v = \{i'_1, \ldots, i'_q\} \subseteq \{0, 1, \ldots, n-1\}$ be a set of q distinct integers such that $(\mathbf{v})^{i'_1}, \ldots, (\mathbf{v})^{i'_q}$ are reachable neighbors of $\mathbf{v}$. We distinguish the following two cases.

Case 1: Suppose that $I_u \cap I_v \neq \emptyset$. Let $j \in I_u \cap I_v$. Then we partition Q_n into $Q_n^{j,0}$ and $Q_n^{j,1}$. For convenience, let $F_0 = F(Q_n^{j,0})$ and $F_1 = F(Q_n^{j,1})$. Since $h(\mathbf{u}, \mathbf{v}) = n$, nodes $\mathbf{u}$ and $\mathbf{v}$ are located in different subcubes. Moreover, we have $h(\mathbf{u}, (\mathbf{v})^j) = n - 1$. By the pigeonhole principle, we have $|F_0| \leq n-2$ or $|F_1| \leq n-2$. Without loss of generality, we assume that $|F_0| \leq n-2$. Moreover, we assume $\mathbf{u} \in V(Q_n^{j,0})$. By Lemma 5.1, $Q_n^{j,0}$ has at least one fault-free path L of length $n-1$ between $\mathbf{u}$ and $(\mathbf{v})^j$. Hence $\langle \mathbf{u}, L, (\mathbf{v})^j, \mathbf{v} \rangle$ forms a fault-free path of length n between $\mathbf{u}$ and $\mathbf{v}$.

Case 2: Suppose that $I_u \cap I_v = \emptyset$. Since $|F| = 2n-3$, we can conclude that $3 \leq p+q \leq n$. Without loss of generality, we assume that $p \geq q$. Thus we have $p \geq 2$.

Suppose first that $n = 3$. We have $p = 2$ and $q = 1$. Let $j \in I_v$. Without loss of generality, we assume that $\mathbf{u} \in V(Q_n^{j,0})$. Obviously, $Q_n^{j,0}$ is fault-free, and it has a fault-free path L of length two between $\mathbf{u}$ and $(\mathbf{v})^j$. Then $\langle \mathbf{u}, L, (\mathbf{v})^j, \mathbf{v} \rangle$ is a fault-free path of length 3. See Figure 5.1(a).

Suppose that $n \geq 4$. Let $j \in I_u$. Since $I_u \cap I_v = \emptyset$, $(\mathbf{u})^j$ is a reachable neighbor of $\mathbf{u}$, whereas $(\mathbf{v})^j$ is an unreachable neighbor of $\mathbf{v}$. Again, we assume $\mathbf{u} \in V(Q_n^{j,0})$. Let $F_0 = F(Q_n^{j,0})$ and $F_1 = F(Q_n^{j,1})$. If $|F_1| \leq n-2$, Lemma 5.1 ensures that $Q_n^{j,1}$ has a fault-free path R of length $n-1$ between $(\mathbf{u})^j$ and $\mathbf{v}$. Hence $\langle \mathbf{u}, (\mathbf{u})^j, R, \mathbf{v} \rangle$ is a fault-free path of length n between $\mathbf{u}$ and $\mathbf{v}$. See Figure 5.1(b).

Suppose that $|F_1| \geq n-1$. Thus we have $|F_0| + |E_c^j| \leq n-2$. Let $\tilde{I}_v = \{k \in I_v \mid ((\mathbf{v})^k)^j \in N_{Q_n - F}((\mathbf{v})^k)\}$, where $N_{Q_n-F}((\mathbf{v})^k)$ is the set of all reachable neighbors of $(\mathbf{v})^k$.

Subcase 2.1: Suppose that $\hat{I}_v \neq \emptyset$. Let $k \in \tilde{I}_v$ and Θ be a subgraph of Q_n induced by $\{\mathbf{x} \in V(Q_n) \mid (\mathbf{x})_j = (\mathbf{u})_j, (\mathbf{x})_k = (\mathbf{u})_k\}$. Then Θ is an $(n-2)$-cube inside $Q_n^{j,0}$. Because $(\mathbf{v})^j$ is an unreachable neighbor of $\mathbf{v}$ and it is outside Θ, there are utmost $n-3$ faulty elements in Θ. By Lemma 5.1, Θ has a fault-free path L of length $n-2$ between $\mathbf{u}$ and $((\mathbf{v})^k)^j$. So $\langle \mathbf{u}, L, ((\mathbf{v})^k)^j, (\mathbf{v})^k, \mathbf{v} \rangle$ is a fault-free path of length n. See Figure 5.1(c).

Subcase 2.2: Suppose that $\hat{I}_v = \emptyset$. Let $k_1 \in I_v$. Since $|F| \leq 2n-3$ and $p+q \leq n$, there exists an integer $k_2 \in \{0, 1, \ldots, n-1\} - \{j, k_1\}$ such that $((\mathbf{v})^{k_1})^{k_2}$ is a reachable neighbor of $(\mathbf{v})^{k_1}$ and $(((\mathbf{v})^{k_1})^{k_2})^j$ is a reachable neighbor of $((\mathbf{v})^{k_1})^{k_2}$. Let $\mathbf{w} = ((\mathbf{v})^{k_1})^{k_2}$ and Ω be a subgraph of Q_n induced by $\{\mathbf{x} \in V(Q_n) \mid (\mathbf{x})_j = (\mathbf{u})_j, (\mathbf{x})_{k_1} = (\mathbf{u})_{k_1}, (\mathbf{x})_{k_2} = (\mathbf{u})_{k_2}\}$. Then Ω is an $(n-3)$-cube inside $Q_n^{j,0}$. Obviously, $(\mathbf{u})^{k_1}$, $(\mathbf{v})^j$, and $((\mathbf{v})^{k_1})^j$ are unreachable neighbors of $\mathbf{u}$, $\mathbf{v}$, and $(\mathbf{v})^{k_1}$, respectively. Since $(\mathbf{u})^{k_1}$, $(\mathbf{v})^j$, and $((\mathbf{v})^{k_1})^j$ are outside Ω, there are utmost $n-4$ faulty elements in Ω. It follows from Lemma 5.1 that Ω has a fault-free path L of length $n-3$ between $\mathbf{u}$ and $(\mathbf{w})^j$. So $\langle \mathbf{u}, L, (\mathbf{w})^j, \mathbf{w}, (\mathbf{w})^{k_2} = (\mathbf{v})^{k_1}, \mathbf{v} \rangle$ is a fault-free path of length n between $\mathbf{u}$ and $\mathbf{v}$. See Figure 5.1(d).

In summary, we conclude that $d_{Q_n-F}(\mathbf{u}, \mathbf{v}) = n$, and the proof is completed. □

Proposition 5.2. *Suppose that* $\mathbf{u}$ *and* $\mathbf{v}$ *are any two distinct nodes of* Q_n, $n \geq 3$. *Let* F *be a set of utmost* $2n-4$ *hybrid node-faults and/or link-faults in* Q_n *such that both* $\mathbf{u}$ *and* $\mathbf{v}$

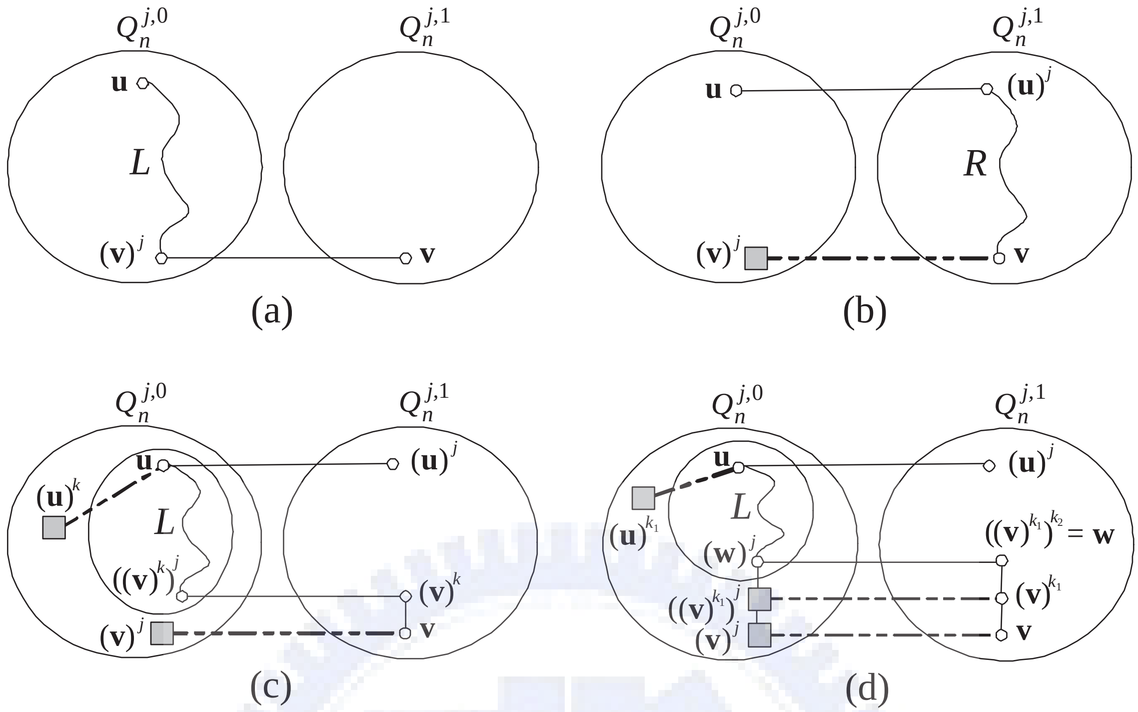

Figure 5.1: Illustration for Proposition 5.1.

are fault-free with at least one reachable neighbor. Then

$$d_{Q_n-F}(\mathbf{u},\mathbf{v}) \leq \begin{cases} n+1 & \text{if } h(\mathbf{u},\mathbf{v}) = n-1 \text{ and } n \geq 3, \\ n & \text{if } h(\mathbf{u},\mathbf{v}) = n-2 \text{ and } n \neq 4, \\ n+2 & \text{if } h(\mathbf{u},\mathbf{v}) = n-2 \text{ and } n = 4, \\ h(\mathbf{u},\mathbf{v})+4 & \text{if } h(\mathbf{u},\mathbf{v}) \leq n-3 \text{ and } n \geq 4. \end{cases}$$

Proposition 5.3. *Suppose that* **u** *and* **v** *are any two distinct nodes of* Q_n, $n \geq 2$. *Let* F *be a set of utmost* $2n-3$ *hybrid node-faults and/or link-faults in* Q_n *such that both* **u** *and* **v** *are fault-free with at least one reachable neighbor. Then*

$$d_{Q_n-F}(\mathbf{u},\mathbf{v}) \leq \begin{cases} n+1 & \text{if } h(\mathbf{u},\mathbf{v}) = n-1 \text{ and } n \geq 2, \\ h(\mathbf{u},\mathbf{v})+4 & \text{if } h(\mathbf{u},\mathbf{v}) \leq n-2 \text{ and } n \geq 3. \end{cases}$$

Proof. For the sake of clarity, we prove Proposition 5.2 and Proposition 5.3 simultaneously. The proof is by induction on n. Obviously, the result is true for $n = 2$. As our inductive hypothesis, we assume that the result holds for Q_{n-1} with $n \geq 3$. Since $h(\mathbf{u},\mathbf{v}) \leq n-1$, we can partition Q_n along some dimension j such that both **u** and **v** are in the same subcube. By transitivity, we assume that $j = 0$. Without loss of generality, we assume that $\mathbf{u},\mathbf{v} \in V(Q_n^{0,0})$. For convenience, let $F_0 = F(Q_n^{0,0})$ and $F_1 = F(Q_n^{0,1})$. Then we distinguish two cases.

Case 1: Suppose that $|F_1| \leq 2n-5 = 2(n-1)-3$. First, we consider the case that both $\mathbf{u}$ and $\mathbf{v}$ have at least one reachable neighbor in $Q_n^{0,1}$. Then it follows from the inductive hypothesis that $d_{Q_n-F}(\mathbf{u},\mathbf{v}) = d_{Q_n^{0,1}-F_1}(\mathbf{u},\mathbf{v}) = n-1$ if $h(\mathbf{u},\mathbf{v}) = n-1$ for $n \geq 3$, $d_{Q_n-F}(\mathbf{u},\mathbf{v}) \leq d_{Q_n^{0,1}-F_1}(\mathbf{u},\mathbf{v}) \leq n$ if $h(\mathbf{u},\mathbf{v}) = n-2$ for $n \geq 3$, and $d_{Q_n-F}(\mathbf{u},\mathbf{v}) \leq d_{Q_n^{0,1}-F_1}(\mathbf{u},\mathbf{v}) \leq h(\mathbf{u},\mathbf{v})+4$ if $h(\mathbf{u},\mathbf{v}) \leq n-3$ for $n \geq 4$. See Figure 5.2(a).

Now we consider the case that either $\mathbf{u}$ or $\mathbf{v}$ has no reachable neighbors in $Q_n^{0,1}$. Thus, we have $|F_1| \geq n-1$ and $|F_0| + |E_c^0| \leq n-2$. Since $n-1 \leq |F_1| \leq 2n-5$, we have $n \geq 4$. Without loss of generality, we assume that $\mathbf{u}$ has no reachable neighbors in $Q_n^{0,1}$. Accordingly, $(\mathbf{u})^0$ is the unique reachable neighbor of $\mathbf{u}$.

Suppose first that $h(\mathbf{u},\mathbf{v}) = n-1$. Since $h((\mathbf{u})^0,\mathbf{v}) = n$, it follows from Proposition 5.1 that $d_{Q_n-F}((\mathbf{u})^0,\mathbf{v}) = n$. Let P be a fault-free path of length n between $(\mathbf{u})^0$ and $\mathbf{v}$. Obviously, we have $\mathbf{u} \notin V(P)$. Hence $\langle \mathbf{u},(\mathbf{u})^0,P,\mathbf{v}\rangle$ turns out to be a fault-free path of length $n+1$. See Figure 5.2(b).

Suppose that $h(\mathbf{u},\mathbf{v}) \leq n-2$. If $(\mathbf{v})^0$ is a reachable neighbor of $\mathbf{v}$, then it follows from Corollary 5.1 that $d_{Q_n^{0,0}-F_0}((\mathbf{u})^0,(\mathbf{v})^0) \leq h((\mathbf{u})^0,(\mathbf{v})^0)+2 = h(\mathbf{u},\mathbf{v})+2$ since $|F_0| \leq n-2$. Let R be a shortest path between $(\mathbf{u})^0$ and $(\mathbf{v})^0$ in $Q_n^{0,0}-F_0$. Then $\langle \mathbf{u},(\mathbf{u})^0,R,(\mathbf{v})^0,\mathbf{v}\rangle$ forms a fault-free path of length at most $h(\mathbf{u},\mathbf{v})+4$. See Figure 5.2(b). In particular, we have $|F_0| \leq n-3$ if $|F| = 2n-4$. Therefore, $Q_n^{0,0}-F_0$ has a path R of length $n-2$ between $(\mathbf{u})^0$ and $(\mathbf{v})^0$ if $h(\mathbf{u},\mathbf{v}) = n-2$ and $|F| = 2n-4$. As a result, $\langle \mathbf{u},(\mathbf{u})^0,R,(\mathbf{v})^0,\mathbf{v}\rangle$ is a fault-free path of length n. On the other hand, if $(\mathbf{v})^0$ is an unreachable neighbor of $\mathbf{v}$, then we have $(\mathbf{v})^0 \in F$ or $(\mathbf{v},(\mathbf{v})^0) \in F$. By Lemma 5.1, $Q_n^{0,0}$ has $n-1$ internally node-disjoint paths $L_1,\ldots,L_{n-1}$ between $(\mathbf{u})^0$ and $(\mathbf{v})^0$. For clarity, L_i can be written as $\langle (\mathbf{u})^0,L_i',((\mathbf{v})^0)^i,(\mathbf{v})^0\rangle$ for $1 \leq i \leq n-1$. Let $T_i = \langle (\mathbf{u})^0,L_i',((\mathbf{v})^0)^i,(\mathbf{v})^i,\mathbf{v}\rangle$ with $1 \leq i \leq n-1$. Then $\{T_1,\ldots,T_{n-1}\}$ is a set of $n-1$ internally node-disjoint paths between $(\mathbf{u})^0$ and $\mathbf{v}$. We distinguish two subcases.

Subcase 1.1: One of $\{T_1,\ldots,T_{n-1}\}$, say T_i, is fault-free. Hence $\langle \mathbf{u},(\mathbf{u})^0,T_i,\mathbf{v}\rangle$ is a path of length at most $h(\mathbf{u},\mathbf{v})+4$ between $\mathbf{u}$ and $\mathbf{v}$. See Figure 5.2(d). In particular, we consider the case that $h(\mathbf{u},\mathbf{v}) = n-2$. Clearly, $n-2$ paths of $\{T_1,\ldots,T_{n-1}\}$ are of length $n-1$. When $n \geq 5$, $\mathbf{u}$ and $\mathbf{v}$ have no common neighbors. Since

$$\left(\{(\mathbf{v})^0,(\mathbf{v},(\mathbf{v})^0)\} \cup \bigcup_{i=1}^{n-1}\{(\mathbf{u})^i,(\mathbf{u},(\mathbf{u})^i)\}\right) \cap \left(\bigcup_{i=1}^{n-1} V(T_i) \cup E(T_i)\right) = \emptyset,$$

at most $n-3$ faults may appear on $T_1,\ldots,T_{n-1}$. Hence there still exists a fault-free path T_k of $\{T_1,\ldots,T_{n-1}\}$ such that $\ell(T_k) = n-1$ if $n \geq 5$. Then $\langle \mathbf{u},(\mathbf{u})^0,T_k,\mathbf{v}\rangle$ is a fault-free path of length n.

Subcase 1.2: None of $\{T_1,\ldots,T_{n-1}\}$ is fault-free. We claim first that $h(\mathbf{u},\mathbf{v}) = 2$. Moreover, it is noticed that $|F| = 2n-3$ in this subcase. Because $T_1,\ldots,T_{n-1}$ are internally node-disjoint and $\mathbf{u}$ has $n-1$ unreachable neighbors in $Q_n^{0,1}$, we conclude that T_i, $1 \leq i \leq n-1$, contains exactly one faulty element. Since $V(T_i) \cap V(Q_n^{0,1}) = \{\mathbf{v},(\mathbf{v})^i\}$ for

$1 \le i \le n-1$, there exist two distinct integers t_1 and t_2, $1 \le t_1, t_2 \le n-1$, such that $F(T_{t_1}) = \{(\mathbf{v})^{t_1}\} = \{(\mathbf{u})^{t_2}\}$ and $F(T_{t_2}) = \{(\mathbf{v})^{t_2}\} = \{(\mathbf{u})^{t_1}\}$. By transitivity, we assume that $t_1 = n-1$ and $t_2 = n-2$. Again, Lemma 5.1 ensures that $Q_n^{0,1}$ has $n-1$ internally node-disjoint paths $R_1, \ldots, R_{n-1}$ of length at most 4 between $\mathbf{u}$ and $\mathbf{v}$. For clarity, we can write R_i as $\langle \mathbf{u}, R'_i, (\mathbf{v})^i, \mathbf{v} \rangle$ for $1 \le i \le n-1$. Thus we have $\ell(R_{n-2}) = \ell(R_{n-1}) = 2$ and $\ell(R_i) = 4$ for $1 \le i \le n-3$. Because $(\mathbf{v})^0$ is an unreachable neighbor of $\mathbf{v}$, thus $\mathbf{v}$ has a reachable neighbor in $Q_n^{0,1}$, say $(\mathbf{v})^k$ with some $k \in \{1, \ldots, n-3\}$. To be precise, we write $R_k = \langle \mathbf{u}, \mathbf{x}_k, \mathbf{y}_k, (\mathbf{v})^k, \mathbf{v} \rangle$ and $L_k = \langle (\mathbf{u})^0, (\mathbf{x}_k)^0, (\mathbf{y}_k)^0, ((\mathbf{v})^k)^0, (\mathbf{v})^0 \rangle$, where $\mathbf{x}_k$ is some neighbor of $\mathbf{u}$, and $\mathbf{y}_k$ is a common neighbor of $\mathbf{x}_k$ and $(\mathbf{v})^k$.

Subcase 1.2.1: Suppose that $((\mathbf{v})^k)^0$ is an unreachable neighbor of $(\mathbf{v})^k$. Let $S_k^{(1)} = \langle (\mathbf{u})^0, (\mathbf{x}_k)^0, (\mathbf{y}_k)^0 \rangle$ and $S_k^{(2)} = \langle (\mathbf{y}_k)^0, \mathbf{y}_k, (\mathbf{v})^k \rangle$ be two paths. Because T_k has only one faulty element, path $S_k^{(1)}$ is fault-free. Since $\Big(V(S_k^{(2)}) \cup E(S_k^{(2)})\Big) \cap \Big(\bigcup_{i \ne k} V(T_i) \cup E(T_i)\Big) = \emptyset$, path $S_k^{(2)}$ is also fault-free. Then $\langle \mathbf{u}, (\mathbf{u})^0, S_k^{(1)}, (\mathbf{y}_k)^0, S_k^{(2)}, (\mathbf{v})^k, \mathbf{v} \rangle$ turns out to be a fault-free path of length 6. See Figure 5.2(e).

Subcase 1.2.2: Suppose that $((\mathbf{v})^k)^0$ is a reachable neighbor of $(\mathbf{v})^k$. Let Θ be a subgraph of $Q_n^{0,0}$ induced by $\{\mathbf{x} \in V(Q_n^{0,0}) \mid (\mathbf{x})_p = (\mathbf{u})_p, p \in \{1, \ldots, n-3\} - \{k\}\}$. Obviously, Θ is isomorphic to Q_3. Then we claim that $|F(\Theta)| \le 2$. Since $|F_0| \le n-2$, this claim holds for $n = 4$ trivially. In what follows, we concern that $n \ge 5$. It is easy to see that L_k, L_{n-2}, and L_{n-1} are inside Θ. Moreover, we have $(V(T_i) \cup E(T_i)) \cap (V(\Theta) \cup E(\Theta)) = \{(\mathbf{u})^0\}$ for $i \in \{1, \ldots, n-3\} - \{k\}$. Since T_i contains one faulty element for each $1 \le i \le n-1$, at least $n-4$ faulty elements are outside Θ; i.e., $|F(\Theta)| \le 2$. Since $h((\mathbf{u})^0, ((\mathbf{v})^k)^0) = 3$, it follows from Lemma 5.1 that Θ has a fault-free path S of length 3 between $(\mathbf{u})^0$ and $((\mathbf{v})^k)^0$. As a result, $\langle \mathbf{u}, (\mathbf{u})^0, S, ((\mathbf{v})^k)^0, (\mathbf{v})^k, \mathbf{v} \rangle$ is a fault-free path of length 6. See Figure 5.2(f).

Case 2: Suppose that $|F_0| \ge 2n-4$. Thus, we have $|F_1| + |E_c^0| \le 1$.

Subcase 2.1: Suppose that $(\mathbf{u})^0$ and $(\mathbf{v})^0$ are reachable neighbors of $\mathbf{u}$ and $\mathbf{v}$, respectively. Since $|F_1| \le 1$, it follows from Lemma 5.1 that $Q_n^{0,1}$ has a fault-free path R of length at most $h(\mathbf{u}, \mathbf{v}) + 2$ between $(\mathbf{u})^0$ and $(\mathbf{v})^0$. Then $\langle \mathbf{u}, (\mathbf{u})^0, R, (\mathbf{v})^0, \mathbf{v} \rangle$ is a fault-free path of length at most $h(\mathbf{u}, \mathbf{v}) + 4$ between $\mathbf{u}$ and $\mathbf{v}$. See Figure 5.2(g). Obviously, we have $\ell(R) = h(\mathbf{u}, \mathbf{v})$ if $|F| \le 2n-4$. Therefore, $\langle \mathbf{u}, (\mathbf{u})^0, R, (\mathbf{v})^0, \mathbf{v} \rangle$ turns out to be a fault-free path of length $h(\mathbf{u}, \mathbf{v}) + 2$.

Subcase 2.2: Suppose that $(\mathbf{u})^0$ or $(\mathbf{v})^0$ is an unreachable neighbor of $\mathbf{u}$ or $\mathbf{v}$, respectively. If $|F| \le 2n-4$, then $|F_1| + |E_c^0| = 0$. Thus we have $|F| = 2n-3$ in this case. Since $|F_1| + |E_c^0| \le 1$, we assume that $(\mathbf{u})^0$ is an unreachable neighbor of $\mathbf{u}$ and $(\mathbf{v})^0$ is a reachable neighbor of $\mathbf{v}$. Let $(\mathbf{u})^k$ be a reachable neighbor of $\mathbf{u}$ with some $k \in \{1, \ldots, n-1\}$. If $(\mathbf{u})^k = \mathbf{v}$, then we have $d_{Q_n - F}(\mathbf{u}, \mathbf{v}) = h(\mathbf{u}, \mathbf{v}) = 1$. In what follows we assume $(\mathbf{u})^k \ne \mathbf{v}$. Since $|F_1| + |E_c^0| \le 1$, node $((\mathbf{u})^k)^j$ is a reachable neighbor of $(\mathbf{u})^k$. If $(\mathbf{u})_k \ne (\mathbf{v})_k$, then $h((\mathbf{u})^k, \mathbf{v}) = h(\mathbf{u}, \mathbf{v}) - 1$. By Lemma 5.1, $Q_n^{0,1}$ has a fault-free path R of length at most $h((\mathbf{u})^k, \mathbf{v}) + 2 = h(\mathbf{u}, \mathbf{v}) + 1$ between $((\mathbf{u})^k)^0$ and $(\mathbf{v})^0$. Then $\langle \mathbf{u}, (\mathbf{u})^k, ((\mathbf{u})^k)^0, R, (\mathbf{v})^0, \mathbf{v} \rangle$ is a fault-free path of length at most $h(\mathbf{u}, \mathbf{v}) + 4$.

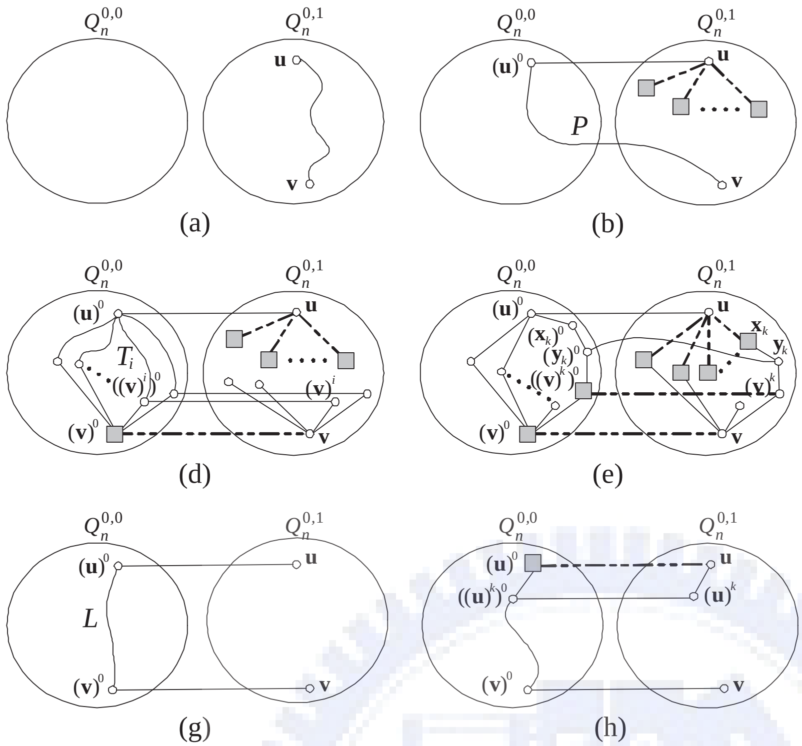

Figure 5.2: Illustration for Proposition 5.3.

Otherwise, if $(\mathbf{u})_k = (\mathbf{v})_k$, then $h((\mathbf{u})^k, \mathbf{v}) = h(\mathbf{u}, \mathbf{v}) + 1$. Since $|F_1| \leq 1$, Lemma 5.1 ensures that $Q_n^{0,1}$ has a fault-free path R of length $h(\mathbf{u}, \mathbf{v}) + 1$ between $((\mathbf{u})^k)^0$ and $(\mathbf{v})^0$. Then $\langle \mathbf{u}, (\mathbf{u})^k, ((\mathbf{u})^k)^0, R, (\mathbf{v})^0, \mathbf{v} \rangle$ is a fault-free path of length at most $h(\mathbf{u}, \mathbf{v}) + 4$. See Figure 5.2(h).

The proof is completed. □

According to Lemma 5.1 and Propositions 5.1−5.3, we can compute the fault diameter of hypercubes as follows.

Theorem 5.3. *Let F be a set of hybrid node-faults and/or link-faults in Q_n, $n \geq 3$, such that every node of Q_n has at least one reachable neighbor. Then the diameter of $Q_n - F$ is computed as follows:*

$$D(Q_n - F) = \begin{cases} n & \text{if } |F| \leq n-2 \text{ and } n \geq 3, \\ n+1 & \text{if } n-1 \leq |F| \leq 2n-4 \text{ and } n \neq 4, \\ n+1 & \text{if } |F| = 3 \text{ and } n = 4, \\ n+2 & \text{if } |F| = 4 \text{ and } n = 4, \\ n+2 & \text{if } |F| = 2n-3 \text{ and } n \geq 3. \end{cases}$$

Proof. Suppose first that $n \neq 4$. The results are direct consequences from Lemma 5.1 and

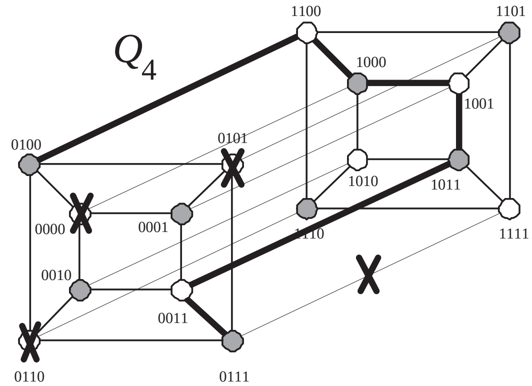

Figure 5.3: The distance between 0100 and 0111 in the faulty 4-cube is 6.

Propositions 5.1−5.3.

Suppose that $n = 4$. Applying Lemma 5.1 and Propositions 5.1−5.3, we have $D(Q_4 - F) = 4$ if $|F| \leq 2$, $D(Q_4 - F) = 5$ if $|F| = 3$, $D(Q_4 - F) \leq 6$ if $|F| = 4$, and $D(Q_4 - F) = 6$ if $|F| = 5$. Let $F = \{0000, 0101, 0110, (0111, 1111)\}$. Then $d_{Q_4 - F}(0100, 0111) = 6$. See Figure 5.3. Therefore, $D(Q_4 - F) = 6$ if $|F| = 4$.

The proof is completed. □

Chapter 6

Paths of Variable Lengths in Hypercubes with Conditional Link-faults

The minimum degree of a graph G, denoted by $\delta(G)$, is defined to be $\min\{deg_G(v) \mid v \in V(G)\}$. Again we use $F(G)$ to denote the set of all faulty elements in graph G. Clearly, a graph G will have no hamiltonian cycles if $\delta(G-F(G)) < 2$. Moreover, a graph G will have no hamiltonian paths between some pair of distinct nodes if $\delta(G-F(G)) = 2$. To understand more about networks' fault-tolerant capabilities in the perspective on path embedding, we concern the model of conditional faults as introduced in the preceding chapter. Throughout out this chapter, a graph G is said to be *conditionally faulty* if and only if $\delta(G-F(G)) \geq 2$. In order to simplify our discussion, only link-faults are addressed in this chapter. Thus a network is conditionally faulty if its every node is incident to at least two fault-free links.

We focus only on hypercubes. As before we denote by $Q_n^{i,j}$ a subgraph of Q_n induced by $\{\mathbf{u} \in V(Q_n) \mid (\mathbf{u})_i = j\}$, for $0 \leq i \leq n-1$ and $j \in \{0,1\}$. Let $\mathbf{u}$ be any node of Q_n, and let $\mathbf{v} = ((\mathbf{u})^0)^1$. Suppose that $F = \{(\mathbf{u}, (\mathbf{u})^i) \mid 2 \leq i \leq n-1\} \cup \{(\mathbf{v}, (\mathbf{v})^i) \mid 2 \leq i \leq n-1\}$ is a set of $2n-4$ faulty links in Q_n. Obviously, $Q_n - F$ has no hamiltonian paths between $\mathbf{u}$ and $(\mathbf{u})^1$. For this reason, we concern $2n-5$ faulty links only.

The condition of requiring every node to be incident to at least two fault-free links is meaningful. Suppose that the probabilities of link-failures are independent and identical. Let $P_L(n)$ denote the probability that every node of an n-cube, containing $2n-5$ faulty links, is incident to at least two fault-free links. As discussed in Section 1.3 of Chapter 1, we have $P_L(3) = 1$, $P_L(4) = 1 - \frac{2^4 \times \binom{4}{3}}{\binom{4\times 2^3}{3}}$, and

$$P_L(n) = 1 - \frac{2^n \times \binom{n\times 2^{n-1}-n}{n-5} + 2^n \times \binom{n}{n-1}\binom{n\times 2^{n-1}-n}{n-4}}{\binom{n\times 2^{n-1}}{2n-5}}$$

for $n \geq 5$. Apparently $P_L(n)$ approaches to 1 as n increases. See Table 6.1.

Table 6.1: Values of $P_L(n)$.

n	$P_L(n)$
5	> 0.999
6	> 0.99999
7	$> 1 - 8 \times 10^{-9}$
8	$> 1 - 4 \times 10^{-12}$
9	$> 1 - 4 \times 10^{-16}$
10	$> 1 - 9 \times 10^{-21}$
11	$> 1 - 6 \times 10^{-26}$
12	$> 1 - 9 \times 10^{-32}$
13	$> 1 - 4 \times 10^{-38}$
14	$> 1 - 3 \times 10^{-45}$
15	$> 1 - 7 \times 10^{-53}$

Under the consideration of $2n - 5$ conditional link-faults, Chan and Lee [8] discussed the existence of hamiltonian cycles in an n-cube with $2n - 5$ conditional link-faults. In addition, Tsai [66] showed that an injured n-cube contains a fault-free cycle of every even length from 4 to 2^n inclusive even if it has up to $2n - 5$ conditional link-faults. It was also proved in [66] that an n-cube with $2n-5$ conditional link-faults is hamiltonian laceable and strongly hamiltonian laceable. As Shih [56] showed, any fault-free link of a faulty n-cube lies on a cycle of even length in the range from 6 to 2^n when up to $2n-5$ conditional link-faults may occur. In other words, there exists a path of odd length from 1 to $2^n - 1$, excluding 3, between any two adjacent nodes in a faulty n-cube with $2n - 5$ conditional link-faults. We are curious whether paths of variable lengths still can be constructed to join two arbitrary nodes of distance greater than 1. Later we will show that a conditionally faulty n-cube, with $2n - 5$ faulty links, actually contains a path of length l between any pair $\mathbf{u}, \mathbf{v}$ of distinct nodes, with distance $d^* > 1$, for each integer l satisfying both $d^* \leq l \leq 2^n - 1$ and $2|(l - d^*)$.

6.1 Partition of a faulty n-cube

Suppose that Q_n, $n \geq 4$, is conditionally faulty with $2n - 5$ faulty links. For convenience, let $F = F(Q_n)$ and F_i denote the set of faulty i-dimensional links. Since $|F| = 2n-5$, there are utmost two nodes of Q_n incident to $n-2$ faulty links. For any two distinct nodes, $\mathbf{u}$ and $\mathbf{v}$, of Q_n, the procedure Partition(Q_n, F, $\mathbf{u}$, $\mathbf{v}$) determines a dimension j according to the following rules:

(1) Suppose that there are exactly two nodes incident to $n-2$ faulty links. Then the two nodes must be connected by a faulty link $(\mathbf{w}, (\mathbf{w})^j)$ with some $j \in \{0, 1, \ldots, n-1\}$. Obviously, both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty with $n-3$ faulty links.

(2) Suppose that there is only one node, namely $\mathbf{z}$, incident to $n-2$ faulty links. Let $S = \{0 \leq i \leq n-1 \mid (\mathbf{z}, (\mathbf{z})^i) \in F\} = \{k_3, \ldots, k_n\}$ and $\{0, 1, \ldots, n-1\} - S = \{k_1, k_2\}$. Then both $Q_n^{i,0}$ and $Q_n^{i,1}$ are conditionally faulty for each $i \in S$.

(2.1) If there exists a dimension j of S such that $|F_j| > 1$, then we partition Q_n along dimension j. Otherwise, if there exists a dimension j of S such that $|F(Q_n^{j,0})| \cdot |F(Q_n^{j,1})| > 0$, then we partition Q_n along dimension j. Obviously, both $Q_n^{j,0}$ and $Q_n^{j,1}$ contain $2n-7$ or less faulty links.

(2.2) Suppose that $|F_i| = 1$ and $|F(Q_n^{i,0})| \cdot |F(Q_n^{i,1})| = 0$ for every $i \in S$. That is, for any $i \in S$, either $|F(Q_n^{i,0})|$ or $|F(Q_n^{i,1})|$ remains $2n-6$. Hence, for any $(\mathbf{x}, \mathbf{y}) \in F - \{(\mathbf{z}, (\mathbf{z})^i) \mid i \in S\}$, we have $(\mathbf{x})_i = (\mathbf{y})_i = (\mathbf{z})_i$ for every $i \in S$. That is, for $(\mathbf{x}, \mathbf{y}) \in F - \{(\mathbf{z}, (\mathbf{z})^i) \mid i \in S\}$, we have $\mathbf{x}, \mathbf{y} \in \{\mathbf{z}, (\mathbf{z})^{k_1}, (\mathbf{z})^{k_2}, ((\mathbf{z})^{k_1})^{k_2}\}$. Because both $(\mathbf{z}, (\mathbf{z})^{k_1})$ and $(\mathbf{z}, (\mathbf{z})^{k_2})$ are fault-free, it follows that $F - \{(\mathbf{z}, (\mathbf{z})^i) \mid i \in S\} \subseteq \{((\mathbf{z})^{k_1}, ((\mathbf{z})^{k_1})^{k_2}), ((\mathbf{z})^{k_2}, ((\mathbf{z})^{k_1})^{k_2})\}$. Since $|F - \{(\mathbf{z}, (\mathbf{z})^i) \mid i \in S\}| = n - 3 \le 2$, we obtain $n \in \{4, 5\}$. The faulty links are distributed as illustrated in Figure 6.1.

(2.2.1) If there exists a dimension j of S such that $(\mathbf{z})^j$ is neither $\mathbf{u}$ nor $\mathbf{v}$, then we partition Q_n along dimension j.

(2.2.2) Otherwise, we have $\{\mathbf{u}, \mathbf{v}\} = \{(\mathbf{z})^i \mid i \in S\}$; thus, we have $n = 4$. In this case, we partition Q_4 along any dimension $j \in S$. Clearly, $\mathbf{u}$ and $\mathbf{v}$ belong to the same partite set of Q_4.

(3) Suppose that every node is incident to utmost $n-3$ faulty links. Obviously, every $(n-1)$-cube in Q_n is conditionally faulty. Let $S = \{0 \le i \le n-1 \mid F_i \ne \emptyset\}$.

(3.1) Suppose that $|F_j| \ge 2$ with some $j \in S$. Then both $Q_n^{j,0}$ and $Q_n^{j,1}$ contain $2n-7$ or less faulty links.

(3.2) Suppose that $|F_i| \le 1$ for each $i \in S$. Clearly we have $2n-5 = |F| = \left|\bigcup_{i \in S} F_i\right| = \sum_{i \in S} |F_i| \le n$; i.e., $n \le 5$. Then a dimension j of S can be chosen so that both $Q_n^{j,0}$ and $Q_n^{j,1}$ contain $2n-7$ or less faulty links.

(3.2.1) When $n = 5$, we claim that $|F(Q_n^{j,0})| \cdot |F(Q_n^{j,1})| > 0$ for some $j \in S$. Let $e_i = (b_{i4} \ldots b_{ii} \ldots b_{i0},\ b_{i4} \ldots \bar{b}_{ii} \ldots b_{i0})$ be an i-dimensional link of Q_5 for $i \in \{0, 1, 2, 3, 4\}$. Suppose that $F = \{e_0, e_1, e_2, e_3, e_4\}$ is a faulty set of Q_5 such that $|F(Q_5^{i,0})| \cdot |F(Q_5^{i,1})| = 0$ for each $i \in \{0, 1, 2, 3, 4\}$. Then we have $b_{0i} = b_{1i} = b_{2i} = b_{3i} = b_{4i}$ for each $i \in \{0, 1, 2, 3, 4\}$; i.e., all faulty links are incident with an identical node. This contradicts the assumption that every node is incident to utmost $n-3$ faulty links.

(3.2.2) Similarly, there exists an integer $j \in S$ such that $|F(Q_4^{j,0})| \cdot |F(Q_4^{j,1})| > 0$.

In summary, the proposed procedure determines a j-partition of Q_n such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty with $|F(Q_n^{j,0})| + |F(Q_n^{j,1})| \le 2n-6$.

6.2 Path embedding in faulty hypercubes

The following theorem characterizes a property of some shortest paths in a faulty n-cube.

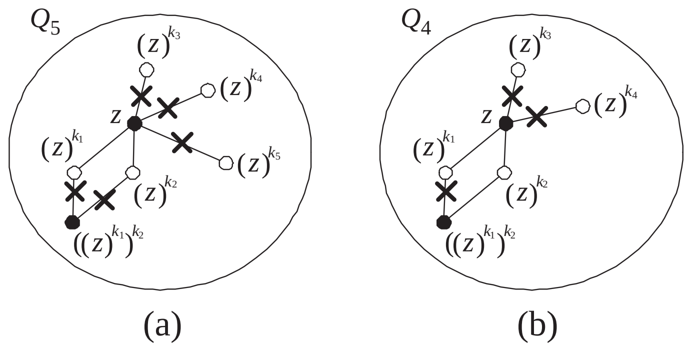

Figure 6.1: The distributions of faulty links indicated in (2.2).

Theorem 6.1. *Let F be a set of $2n-5$ faulty links in Q_n such that every node of $Q_n - F$ has at least two neighbors. Moreover, let j be an integer of $\{0, 1, \ldots, n-1\}$ such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty with $2n-7$ or less faulty links. Suppose that $\mathbf{u}$ is a node of $Q_n^{j,0}$, and $\mathbf{v}$ is a node of $Q_n^{j,1}$. Then there exists a shortest path P^* between $\mathbf{u}$ and $\mathbf{v}$ in $Q_n - F$ such that P^* crosses the dimension j exactly once.*

Proof. Since $|F(Q_n^{j,0})| + |F(Q_n^{j,1})| \leq |F| = 2n-5$, we assume that $|F(Q_n^{j,1})| \leq n-3$. Since $(\mathbf{u})_j \neq (\mathbf{v})_j$, every shortest path between $\mathbf{u}$ and $\mathbf{v}$ crosses the dimension j an odd number of times. If there exists a shortest path between $\mathbf{u}$ and $\mathbf{v}$ crossing the dimension j exactly once, the proof is done. Thus, we assume that one shortest path between $\mathbf{u}$ and $\mathbf{v}$, namely P, crosses the dimension j more than once. Accordingly, the shortest path P can be represented as $\langle \mathbf{u}, P_0, \mathbf{x}_1, (\mathbf{x}_1)^j, P_1, (\mathbf{x}_2)^j, \mathbf{x}_2, P_2, \mathbf{x}_3, (\mathbf{x}_3)^j, \ldots, \mathbf{x}_r, (\mathbf{x}_r)^j, P_r, \mathbf{v} \rangle$ with odd integer $r \geq 3$. For convenience, let $H = \langle (\mathbf{x}_1)^j, P_1, (\mathbf{x}_2)^j, \mathbf{x}_2, P_2, \mathbf{x}_3, (\mathbf{x}_3)^j, \ldots, \mathbf{x}_r, (\mathbf{x}_r)^j, P_r, \mathbf{v} \rangle$. By Corollary 5.1, we have $d_{Q_n^{j,1} - F(Q_n^{j,1})}((\mathbf{x}_1)^j, \mathbf{v}) \leq h((\mathbf{x}_1)^j, \mathbf{v}) + 2$. Suppose that R is a shortest path between $(\mathbf{x}_1)^j$ and $\mathbf{v}$ in $Q_n^{j,1} - F(Q_n^{j,1})$. Then we have $\ell(H) \leq \ell(R)$. Since $r \geq 3$, we have $\ell(H) \geq h((\mathbf{x}_1)^j, \mathbf{v}) + 2 \geq \ell(R)$. As a result, $P^* = \langle \mathbf{u}, P_0, \mathbf{x}_1, (\mathbf{x}_1)^j, R, \mathbf{v} \rangle$ happens to be a shortest path between $\mathbf{u}$ and $\mathbf{v}$ and it crosses the dimension j exactly once. □

Theorem 6.2 is proved in [71].

Theorem 6.2. *[71] Let F be a set of $n-2$ faulty links in Q_n ($n \geq 2$). Suppose that $\mathbf{u}$ and $\mathbf{v}$ are any two different nodes of $Q_n - F$. Then $Q_n - F$ contains a path of length l between $\mathbf{u}$ and $\mathbf{v}$ for every l satisfying $d_{Q_n - F}(\mathbf{u}, \mathbf{v}) \leq l \leq 2^n - 1$ and $2 | (l - d_{Q_n-F}(\mathbf{u}, \mathbf{v}))$.*

As Tsai [66] showed, an n-cube with $2n-5$ conditional link-faults is both hamiltonian laceable and strongly hamiltonian laceable.

Theorem 6.3. *[66] Let F be a set of faulty links in Q_n ($n \geq 3$) such that every node of $Q_n - F$ has at least two neighbors. Then $Q_n - F$ is both hamiltonian laceable and strongly hamiltonian laceable if $|F| \leq 2n-5$.*

The next two lemmas will be applied to prove Theorem 7.3.

Lemma 6.1. *[66] Assume that $n \geq 2$. Let $\mathbf{x}$ and $\mathbf{u}$ be two distinct nodes of $V_0(Q_n)$; let $\mathbf{y}$ and $\mathbf{v}$ be two distinct nodes of $V_1(Q_n)$. Then there exist two node-disjoint paths P_1 and P_2 such that the following conditions are satisfied: (1) P_1 joins $\mathbf{x}$ to $\mathbf{y}$, (2) P_2 joins $\mathbf{u}$ to $\mathbf{v}$, and (3) $V(P_1) \cup V(P_2) = V(Q_n)$.*

Lemma 6.2. *Let $\mathbf{v}$ be any node of Q_n ($n \geq 3$), and let $(\mathbf{w}, \mathbf{b})$ be any link of $Q_n - \{\mathbf{v}\}$. For every odd integer l in the range from 1 to $2^n - 3$, $Q_n - \{\mathbf{v}\}$ has a path of length l between $\mathbf{w}$ and $\mathbf{b}$.*

Proof. Since Q_n is node-transitive, we assume that $\mathbf{v} = 0^n$. We prove this lemma by the induction on n. The induction base depends on Q_3. With the link-transitivity, the required paths are listed in Table 6.2.

Table 6.2: The paths of variable lengths between $\mathbf{w}$ and $\mathbf{b}$ in $Q_3 - \{000\}$.

$(w, b) = (011, 001)$	$\langle 011, 111, 101, 001\rangle, \langle 011, 111, 110, 100, 101, 001\rangle$
$(w, b) = (011, 111)$	$\langle 011, 001, 101, 111\rangle, \langle 011, 001, 101, 100, 110, 111\rangle$
$(w, b) = (101, 001)$	$\langle 101, 111, 011, 001\rangle, \langle 101, 100, 110, 111, 011, 001\rangle$
$(w, b) = (101, 100)$	$\langle 101, 111, 110, 100\rangle, \langle 101, 111, 011, 010, 110, 100\rangle$
$(w, b) = (101, 111)$	$\langle 101, 100, 110, 111\rangle, \langle 101, 100, 110, 010, 011, 111\rangle$

When $n \geq 4$, we assume that the result is true for Q_{n-1}. Then we partition Q_n along some dimension p other than $dim((\mathbf{w}, \mathbf{b}))$. Obviously, $\mathbf{v}$ is located in $Q_n^{p,0}$.

Case 1: Suppose that $(\mathbf{w}, \mathbf{b})$ is in $Q_n^{p,0}$. By the inductive hypothesis, $Q_n^{p,0} - \{\mathbf{v}\}$ has a path of odd length l_0 between $\mathbf{w}$ and $\mathbf{b}$ for any odd integer l_0 from 1 to $2^{n-1} - 3$. Let H be a path of length $2^{n-1} - 3$ between $\mathbf{w}$ and $\mathbf{b}$ in $Q_n^{p,0} - \{\mathbf{v}\}$. Since $2^{n-1} - 3 > 1$, we can represent H as $\langle \mathbf{w}, \mathbf{u}, H_0, \mathbf{b}\rangle$. By Theorem 6.2, $Q_n^{p,1}$ has a path H_1 of odd length l_1 between $(\mathbf{w})^p$ and $(\mathbf{b})^p$ for any odd integer l_1 from 1 to $2^{n-1} - 1$. As a result, $\langle \mathbf{w}, (\mathbf{w})^p, H_1, (\mathbf{u})^p, \mathbf{u}, H_0, \mathbf{b}\rangle$ is a path of odd length $2^{n-1} - 2 + l_1$, in the range from $2^{n-1} - 1$ to $2^n - 3$.

Case 2: Suppose that $(\mathbf{w}, \mathbf{b})$ is in $Q_n^{p,1}$. By Theorem 6.2, $Q_n^{p,1}$ has a path of odd length l_1 between $\mathbf{w}$ and $\mathbf{b}$ for any odd integer l_1 from 1 to $2^{n-1} - 1$. Let H be a path of length $2^{n-1} - 1$ between $\mathbf{w}$ and $\mathbf{b}$ in $Q_n^{p,1}$. Then we can choose a link $(\mathbf{x}, \mathbf{y})$ on H such that $\mathbf{v} \notin \{(\mathbf{x})^p, (\mathbf{y})^p\}$. Hence, we can represent H as $\langle \mathbf{w}, H_1', \mathbf{x}, \mathbf{y}, H_1'', \mathbf{b}\rangle$. By the inductive hypothesis, $Q_n^{p,0} - \{\mathbf{v}\}$ has a path H_0 of odd length l_0 between $(\mathbf{x})^p$ and $(\mathbf{y})^p$ for any odd integer l_0 from 1 to $2^{n-1} - 3$. As a result, $\langle \mathbf{w}, H_1', \mathbf{x}, (\mathbf{x})^p, H_0, (\mathbf{y})^p, \mathbf{y}, H_1'', \mathbf{b}\rangle$ is a path of odd length $2^{n-1} + l_0$, in the range from $2^{n-1} + 1$ to $2^n - 3$. □

As Shih [56] showed, any fault-free link of Q_n lies on a cycle of even length from 6 to 2^n when up to $2n - 5$ conditional link-faults may occur.

Theorem 6.4. *[56] Let F be a set of $2n-5$ faulty links in Q_n such that every node of $Q_n - F$ has at least two neighbors. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are any two adjacent nodes of $Q_n - F$. Then $Q_n - F$ contains a path of odd length l between $\mathbf{u}$ and $\mathbf{v}$ if l is in the range from 1 to $2^n - 1$ excluding 3.*

In the following discussion, we devote to constructing paths between any two nodes with distance greater than 1.

Theorem 6.5. *Let F be a set of $2n-5$ faulty links in Q_n ($n \geq 3$) such that every node of $Q_n - F$ has at least two neighbors. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are two arbitrary nodes of $Q_n - F$ with distance $d^* = d_{Q_n - F}(\mathbf{u}, \mathbf{v}) \geq 2$. Then $Q_n - F$ contains a path of length l between $\mathbf{u}$ and $\mathbf{v}$ for every integer l satisfying both $d^* \leq l \leq 2^n - 1$ and $2|(l - d^*)$.*

Proof. Applying procedure Partition(Q_n, F, $\mathbf{u}$, $\mathbf{v}$), we can determine a j-partition of Q_n such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty with $|F(Q_n^{j,0})| + |F(Q_n^{j,1})| \leq 2n-6$. As a result, the proof can proceed by the induction on n. The induction base, depending upon Q_3, follows from Theorem 6.2. As our inductive hypothesis, we assume that the result holds for Q_{n-1} when $n \geq 4$.

Case I: Suppose that $\mathbf{u}$ and $\mathbf{v}$ are in the different partite sets of Q_n. Without loss of generality, we assume that $\mathbf{u} \in V_0(Q_n)$ and $\mathbf{v} \in V_1(Q_n)$. By Theorem 6.3, $Q_n - F$ is hamiltonian laceable. Moreover, a shortest path between $\mathbf{u}$ and $\mathbf{v}$ can be easily obtained by a simple breadth-first search. Therefore, we mainly concentrate on the paths of odd lengths in the range from $d^* + 2$ to $2^n - 3$.

Subcase I.1: Suppose that $|F(Q_n^{j,0})| \leq 2n-7$ and $|F(Q_n^{j,1})| \leq 2n-7$. Without loss of generality, we assume that $|F(Q_n^{j,0})| \geq |F(Q_n^{j,1})|$; thus, $|F(Q_n^{j,1})| \leq n-3$.

Subcase I.1.1: Suppose that both $\mathbf{u}$ and $\mathbf{v}$ are in $Q_n^{j,0}$. By the inductive hypothesis, $Q_n^{j,0} - F(Q_n^{j,0})$ contains a path H_0 of length $2^{n-1} - 1$ between $\mathbf{u}$ and $\mathbf{v}$. Let $A = \{(H_0(i), H_0(i+1)) \mid 1 \leq i \leq 2^{n-1}, i \equiv 1 \pmod 2\}$ be a set of disjoint links on H_0. Since $|A| = \lceil \frac{2^{n-1}-1}{2} \rceil > 2n-5$ for any $n \geq 4$, there exists a link $(\mathbf{w}, \mathbf{b})$ of A such that $(\mathbf{w}, (\mathbf{w})^j)$, $(\mathbf{b}, (\mathbf{b})^j)$, and $((\mathbf{w})^j, (\mathbf{b})^j)$ are all fault-free. Hence, H_0 can be written as $\langle \mathbf{u}, H_0', \mathbf{w}, \mathbf{b}, H_0'', \mathbf{v} \rangle$. Since $|F(Q_n^{j,1})| \leq n-3$, it follows from Theorem 6.2 that $Q_n^{j,1} - F(Q_n^{j,1})$ contains a path H_1 of odd length l_1 between $(\mathbf{w})^j$ and $(\mathbf{b})^j$ for any odd integer l_1 from 1 to $2^{n-1} - 1$. As a result, $\langle \mathbf{u}, H_0', \mathbf{w}, (\mathbf{w})^j, H_1, (\mathbf{b})^j, \mathbf{b}, H_0'', \mathbf{v} \rangle$ is a path of odd length $2^{n-1} + l_1$, in the range from $2^{n-1} + 1$ to $2^n - 1$. See Figure 6.2(a) for illustration.

The paths of lengths less than $2^{n-1}+1$ can be obtained as follows. By Proposition 5.3, we have $d^* = d_{Q_n - F}(\mathbf{u}, \mathbf{v}) \leq h(\mathbf{u}, \mathbf{v}) + 4$ and $d_{Q_n^{j,0} - F(Q_n^{j,0})}(\mathbf{u}, \mathbf{v}) \leq h(\mathbf{u}, \mathbf{v}) + 4$. By the inductive hypothesis, $Q_n^{j,0} - F(Q_n^{j,0})$ has a path T_0 of length l_0 between $\mathbf{u}$ and $\mathbf{v}$ for any odd integer l_0 in the range from $d_{Q_n^{j,0} - F(Q_n^{j,0})}(\mathbf{u}, \mathbf{v})$ to $2^{n-1} - 1$. If $d^* = h(\mathbf{u}, \mathbf{v})$ or $d^* = h(\mathbf{u}, \mathbf{v}) + 4$, then $d_{Q_n^{j,0} - F(Q_n^{j,0})}(\mathbf{u}, \mathbf{v}) = d^*$. Otherwise, if $d^* = h(\mathbf{u}, \mathbf{v}) + 2$, then $d_{Q_n^{j,0} - F(Q_n^{j,0})}(\mathbf{u}, \mathbf{v}) \leq d^* + 2$.

Subcase I.1.2: Suppose that both $\mathbf{u}$ and $\mathbf{v}$ are in $Q_n^{j,1}$. Since $|F(Q_n^{j,1})| \leq n-3$, it

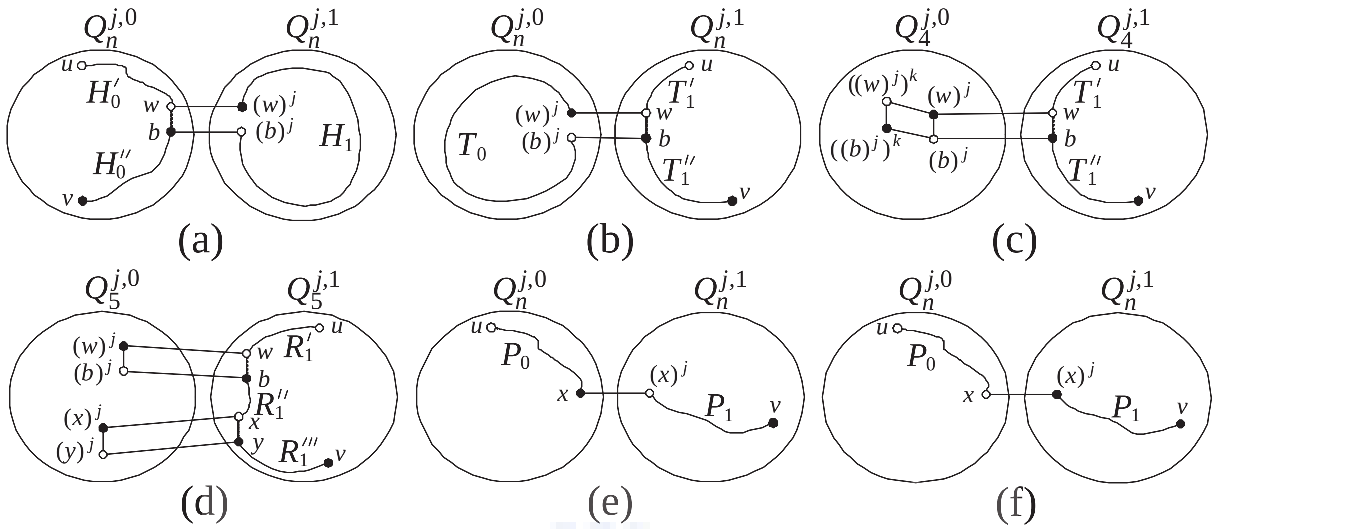

Figure 6.2: Illustration for Subcase I.1.

follows from Corollary 5.1 that $d^* \leq d_{Q_n^{j,1}-F(Q_n^{j,1})}(\mathbf{u},\mathbf{v}) \leq h(\mathbf{u},\mathbf{v})+2$. Thus, there exists a shortest path between $\mathbf{u}$ and $\mathbf{v}$ in $Q_n - F$ that does not cross the dimension j. By inductive hypothesis, $Q_n^{j,1} - F(Q_n^{j,1})$ contains a path T_1 of odd length l_1 between $\mathbf{u}$ and $\mathbf{v}$ for each odd integer l_1 from d^* to $2^{n-1}-1$. Let $\overline{T}_1$ be a path of length $2^{n-1}-1$ between $\mathbf{u}$ and $\mathbf{v}$ in $Q_n^{j,1} - F(Q_n^{j,1})$. Moreover, let $A = \{(\overline{T}_1(i), \overline{T}_1(i+1)) \mid 1 \leq i \leq 2^{n-1}, i \equiv 1 \pmod 2\}$ be a set of disjoint links on $\overline{T}_1$. Since $|A| = \lceil \frac{2^{n-1}-1}{2} \rceil > 2n-5$ for $n \geq 4$, there exists a link $(\mathbf{w},\mathbf{b})$ of A such that $(\mathbf{w},(\mathbf{w})^j)$, $(\mathbf{b},(\mathbf{b})^j)$, and $((\mathbf{w})^j,(\mathbf{b})^j)$ are all fault-free. Hence, $\overline{T}_1$ can be written as $\langle \mathbf{u}, T_1', \mathbf{w}, \mathbf{b}, T_1'', \mathbf{v}\rangle$. Since $|F(Q_n^{j,0})| \leq 2n-7$, it follows from Theorem 6.4 that $Q_n^{j,0} - F(Q_n^{j,0})$ contains a path T_0 of odd length l_0 between $(\mathbf{w})^j$ and $(\mathbf{b})^j$ for any odd integer l_0 in the range from 1 to $2^{n-1}-1$ excluding 3. As a result, $\langle \mathbf{u}, T_1', \mathbf{w}, (\mathbf{w})^j, T_0, (\mathbf{b})^j, \mathbf{b}, T_1'', \mathbf{v}\rangle$ is a path of odd length $2^{n-1}+l_0$, in the range from $2^{n-1}+1$ to 2^n-1 excluding $2^{n-1}+3$. See Figure 6.2(b) for illustration.

The path of length $2^{n-1}+3$ is discussed as follows. When $n=4$, we have $|F(Q_n^{j,0})| \leq 1$. Thus, there exists an integer k of $\{0,1,2,3\} - \{j, dim((\mathbf{w},\mathbf{b}))\}$ such that $((\mathbf{w})^j, ((\mathbf{w})^j)^k)$, $((\mathbf{b})^j, ((\mathbf{b})^j)^k)$, and $(((\mathbf{w})^j)^k, ((\mathbf{b})^j)^k)$ are all fault-free. Hence, $\langle \mathbf{u}, T_1', \mathbf{w}, (\mathbf{w})^j, ((\mathbf{w})^j)^k, ((\mathbf{b})^j)^k, (\mathbf{b})^j, \mathbf{b}, T_1'', \mathbf{v}\rangle$ is a path of length 11. See Figure 6.2(c) for illustration. When $n \geq 5$, we have $|A| - |F| = |A| - (2n-5) = \lceil \frac{2^{n-1}-1}{2} \rceil - (2n-5) \geq 2$. Thus, there is a link $(\mathbf{x},\mathbf{y})$ of A, other than $(\mathbf{w},\mathbf{b})$, such that $(\mathbf{x},\mathbf{y})$ and $(\mathbf{w},\mathbf{b})$ have no shared endpoints and $(\mathbf{x},(\mathbf{x})^j)$, $(\mathbf{y},(\mathbf{y})^j)$, and $((\mathbf{x})^j,(\mathbf{y})^j)$ are all fault-free. Without loss of generality, $\overline{T}_1$ can be written as $\langle \mathbf{u}, R_1', \mathbf{w}, \mathbf{b}, R_1'', \mathbf{x}, \mathbf{y}, R_1''', \mathbf{v}\rangle$. Hence, $\langle \mathbf{u}, R_1', \mathbf{w}, (\mathbf{w})^j, (\mathbf{b})^j, \mathbf{b}, R_1'', \mathbf{x}, (\mathbf{x})^j, (\mathbf{y})^j, \mathbf{y}, R_1''', \mathbf{v}\rangle$ is a path of length $2^{n-1}+3$. See Figure 6.2(d).

Subcase I.1.3: Suppose that $\mathbf{u}$ is in $Q_n^{j,0}$, and $\mathbf{v}$ is in $Q_n^{j,1}$. By Theorem 6.1, we have a shortest path P^* between $\mathbf{u}$ and $\mathbf{v}$ in $Q_n - F$ such that P^* crosses the dimension j exactly once. Thus, P^* can be represented as $\langle \mathbf{u}, P_0, \mathbf{x}, (\mathbf{x})^j, P_1, \mathbf{v}\rangle$, where P_0 is a shortest path joining $\mathbf{u}$ to some node $\mathbf{x}$ in $Q_n^{j,0} - F(Q_n^{j,0})$, and P_1 is a shortest path joining $(\mathbf{x})^j$ to $\mathbf{v}$ in $Q_n^{j,1} - F(Q_n^{j,1})$. See Figure 6.2(e,f) for illustration.

Subcase I.1.3.1: Suppose that $\ell(P_0) > 0$ and $\ell(P_1) > 0$. By Theorem 6.2, $Q_n^{j,1} - F(Q_n^{j,1})$ contains a path T_1 of length l_1 between $(\mathbf{x})^j$ and $\mathbf{v}$ for each l_1 satisfying $\ell(P_1) \leq l_1 \leq 2^{n-1} - 1$ and $2|(l_1 - \ell(P_1))$. Suppose that $\ell(P_0) = 1$. It follows from Theorem 6.4 that $Q_n^{j,0} - F(Q_n^{j,0})$ contains a path T_0 of odd length l_0 between $\mathbf{u}$ and $\mathbf{x}$ for any odd integer l_0 in the range from 1 to $2^{n-1} - 1$ excluding 3. Suppose that $\ell(P_0) > 1$. By the inductive hypothesis, $Q_n^{j,0} - F(Q_n^{j,0})$ contains a path T_0 of length l_0 between $\mathbf{u}$ and $\mathbf{x}$ for each l_0 satisfying $\ell(P_0) \leq l_0 \leq 2^{n-1} - 1$ and $2|(l_0 - \ell(P_0))$. As a result, $\langle \mathbf{u}, T_0, \mathbf{x}, (\mathbf{x})^j, T_1, \mathbf{v} \rangle$ is a path of odd length $l_0 + l_1 + 1$, in the range from d^* to $2^n - 3$.

Subcase I.1.3.2: Suppose that $\ell(P_0) = 0$ or $\ell(P_1) = 0$. Since $d^* = d_{Q_n - F}(\mathbf{u}, \mathbf{u}) > 1$, we have $\mathbf{u} \neq \mathbf{x}$ or $\mathbf{v} \neq (\mathbf{x})^j$. With symmetry, we assume that $\ell(P_0) = 0$. By the inductive hypothesis, $Q_n^{j,1} - F(Q_n^{j,1})$ contains a path T_1 of even length l_1 between $(\mathbf{x})^j$ and $\mathbf{v}$ for each even integer l_1 from $\ell(P_1)$ to $2^{n-1} - 2$. As a result, $\langle \mathbf{u} = \mathbf{x}, (\mathbf{x})^j, T_1, \mathbf{v} \rangle$ is a path of odd length $l_1 + 1$ in the range from $\ell(P_1) + 1 = d^*$ to $2^{n-1} - 1$.

The paths of odd lengths in the range from $2^{n-1} + 1$ to $2^n - 1$ are constructed as follows. Since $|V_1(Q_n^{j,0})| = 2^{n-2} > 2n - 5$ for $n \geq 4$, we can choose a node $\mathbf{y}$ from $V_1(Q_n^{j,0})$ such that $(\mathbf{y}, (\mathbf{y})^j)$ is fault-free. Let R_0 be a path joining $\mathbf{u}$ to $\mathbf{y}$ in $Q_n^{j,0} - F(Q_n^{j,0})$, and R_1 be a path joining $(\mathbf{y})^j$ to $\mathbf{v}$ in $Q_n^{j,1} - F(Q_n^{j,1})$. Similar to **Subcase I.1.3.1**, $H = \langle \mathbf{u}, R_0, \mathbf{y}, (\mathbf{y})^j, R_1, \mathbf{v} \rangle$ is a path of any odd length in the range from $d' = d_{Q_n^{j,0} - F(Q_n^{j,0})}(\mathbf{u}, \mathbf{y}) + d_{Q_n^{j,1} - F(Q_n^{j,1})}((\mathbf{y})^j, \mathbf{v}) + 1$ to $2^n - 1$. By Theorem 5.3, we have $d' \leq (n+1) + (n-1) + 1 \leq 2^{n-1} + 1$ for $n \geq 4$. That is, H can be a path of any odd length in the range from $2^{n-1} + 1$ to $2^n - 1$.

Subcase I.2: Suppose that $|F(Q_n^{j,0})| = 2n - 6$ or $|F(Q_n^{j,1})| = 2n - 6$. Without loss of generality, we assume that $|F(Q_n^{j,0})| = 2n - 6$. Thus, $Q_n^{j,1}$ is fault-free. By procedure Partition(Q_n, F, $\mathbf{u}$, $\mathbf{v}$), the faulty links are distributed as shown in Figure 6.1.

Subcase I.2.1: Suppose that both $\mathbf{u}$ and $\mathbf{v}$ are in $Q_n^{j,0}$. Let $(\mathbf{w}, \mathbf{b})$ be a faulty link of $Q_n^{j,0}$ such that both $(\mathbf{w}, (\mathbf{w})^j)$ and $(\mathbf{b}, (\mathbf{b})^j)$ are fault-free. For convenience, let $F_0 = F(Q_n^{j,0}) - \{(\mathbf{w}, \mathbf{b})\}$. By the inductive hypothesis, $Q_n^{j,0} - F_0$ has a path P_l of odd length l between $\mathbf{u}$ and $\mathbf{v}$ for any odd integer l in the range from $d_{Q_n^{j,0} - F_0}(\mathbf{u}, \mathbf{v})$ to $2^{n-1} - 1$. If $(\mathbf{w}, \mathbf{b})$ is on P_l, we write P_l as $\langle \mathbf{u}, P_l', \mathbf{w}, \mathbf{b}, P_l'', \mathbf{v} \rangle$ and define $\widetilde{P_l} = \langle \mathbf{u}, P_l', \mathbf{w}, (\mathbf{w})^j, (\mathbf{b})^j, \mathbf{b}, P_l'', \mathbf{v} \rangle$. Otherwise, P_l can be written as $\langle \mathbf{u}, P_l', \mathbf{x}, \mathbf{y}, P_l'', \mathbf{v} \rangle$, where $(\mathbf{x}, \mathbf{y})$ is a link on P_l such that both $(\mathbf{x}, (\mathbf{x})^j)$ and $(\mathbf{y}, (\mathbf{y})^j)$ are fault-free. Similarly, we define $\widetilde{P_l} = \langle \mathbf{u}, P_l', \mathbf{x}, (\mathbf{x})^j, (\mathbf{y})^j, \mathbf{y}, P_l'', \mathbf{v} \rangle$. Then $\widetilde{P_l}$ is a path of length $l+2$. By Proposition 5.3, we have $d^* = d_{Q_n - F}(\mathbf{u}, \mathbf{v}) \leq h(\mathbf{u}, \mathbf{v}) + 4$ and $d_{Q_n^{j,0} - F_0}(\mathbf{u}, \mathbf{v}) \leq h(\mathbf{u}, \mathbf{v}) + 4$. First, if $d^* = h(\mathbf{u}, \mathbf{v})$ or $d^* = h(\mathbf{u}, \mathbf{v}) + 4$, then we have $d^* = d_{Q_n^{j,0} - F_0}(\mathbf{u}, \mathbf{v})$, and thus l ranges from d^* to $2^{n-1} - 1$. Next, if $d^* = h(\mathbf{u}, \mathbf{v}) + 2 = d_{Q_n^{j,0} - F_0}(\mathbf{u}, \mathbf{v})$, then l ranges from d^* to $2^{n-1} - 1$. Finally, if $d^* = h(\mathbf{u}, \mathbf{v}) + 2$ and $d_{Q_n^{j,0} - F_0}(\mathbf{u}, \mathbf{v}) = h(\mathbf{u}, \mathbf{v}) + 4$, then l ranges from $d^* + 2$ to $2^{n-1} - 1$. For the final case, a shortest path between $\mathbf{u}$ and $\mathbf{v}$ in $Q_n - F$ can be constructed by a breadth-first search. In summary, the paths of odd lengths from $d^* + 2$ to $2^{n-1} + 1$ are constructed.

By Theorem 6.2, $Q_n^{j,1}$ contains a path T_1 of length l_1 between $(\mathbf{w})^j$ and $(\mathbf{b})^j$ for each odd integer l_1 from 1 to $2^{n-1}-1$. Similarly, $Q_n^{j,1}$ contains a path R_1 of length l_1 between $(\mathbf{x})^j$ and $(\mathbf{y})^j$ for each odd integer l_1 from 1 to $2^{n-1}-1$. Thus, $\langle u, P'_{2^{n-1}-1}, \mathbf{w}, (\mathbf{w})^j, T_1, (\mathbf{b})^j, \mathbf{b}, P''_{2^{n-1}-1}, \mathbf{v}\rangle$ (or $\langle \mathbf{u}, P'_{2^{n-1}-1}, \mathbf{x}, (\mathbf{x})^j, R_1, (\mathbf{y})^j, \mathbf{y}, P''_{2^{n-1}-1}, \mathbf{v}\rangle$) is a path of length $2^{n-1}+l_1$, in the range from $2^{n-1}+1$ to 2^n-1.

Subcase I.2.2: Suppose that both $\mathbf{u}$ and $\mathbf{v}$ are in $Q_n^{j,1}$. Let $(\mathbf{w}, (\mathbf{w})^i)$ be a faulty link in $Q_n^{j,0}$ such that both $(\mathbf{w}, (\mathbf{w})^j)$ and $((\mathbf{w})^i, ((\mathbf{w})^i)^j)$ are fault-free. Since $d^* = d_{Q_n - F}(\mathbf{u}, \mathbf{v}) > 1$, we assume that $(\mathbf{w})^j$ is different from $\mathbf{u}$ and $\mathbf{v}$. Moreover, since $n \geq 4$, we assume that $t \in \{0, 1, \ldots, n-1\} - \{j, i\}$. Let $X = \{((\mathbf{w})^j, ((\mathbf{w})^j)^k) \mid k \notin \{i, j, t\}\}$. Since $|X| = n-3$, our inductive hypothesis ensures that $Q_n^{j,1} - X$ contains a path T_1 of odd length l_1 between $\mathbf{u}$ and $\mathbf{v}$ for any odd integer l_1 satisfying $d^* \leq l_1 \leq 2^{n-1}-1$. Let $\overline{T}_1$ denote a path of length $2^{n-1}-1$ between $\mathbf{u}$ and $\mathbf{v}$ in $Q_n^{j,1} - X$. It is noted that $((\mathbf{w})^j, ((\mathbf{w})^j)^i)$ is on $\overline{T}_1$. Hence, $\overline{T}_1$ can be represented as $\langle \mathbf{u}, T'_1, (\mathbf{w})^j, ((\mathbf{w})^j)^i, T''_1, \mathbf{v}\rangle$. By Theorem 6.4, $Q_n^{j,0} - (F(Q_n^{j,0}) - \{(\mathbf{w}, (\mathbf{w})^i)\})$ contains a path T_0 of odd length l_0 between $\mathbf{w}$ and $(\mathbf{w})^i$ for $5 \leq l_0 \leq 2^{n-1}-1$. As a result, $\langle \mathbf{u}, T'_1, (\mathbf{w})^j, \mathbf{w}, T_0, (\mathbf{w})^i, ((\mathbf{w})^j)^i, T''_1, \mathbf{v}\rangle$ is a path of odd length $2^{n-1}+l_0$, in the range from $2^{n-1}+5$ to 2^n-1. See Figure 6.3(a) for illustration.

Let $\overline{T}_0$ denote the longest path between $\mathbf{w}$ and $(\mathbf{w})^i$ in $Q_n^{j,0} - (F(Q_n^{j,0}) - \{(\mathbf{w}, (\mathbf{w})^i)\})$. Moreover, let $A = \{(\overline{T}_0(k), \overline{T}_0(k+1)) \mid 1 \leq k \leq 2^{n-1}, k \equiv 1 \pmod 2\}$ be a set of disjoint links on $\overline{T}_0$. The paths of lengths $2^{n-1}+1$ and $2^{n-1}+3$ can be obtained as follows:

(a) Since $|A| = \lceil \frac{2^{n-1}-1}{2} \rceil > 3$ for $n \geq 4$, there exists a link $(\mathbf{x}, \mathbf{y})$ of A such that both $F \cap \{(\mathbf{x}, (\mathbf{x})^j), (\mathbf{y}, (\mathbf{y})^j)\} = \emptyset$ and $\{(\mathbf{x})^j, (\mathbf{y})^j\} \cap \{\mathbf{u}, \mathbf{v}\} = \emptyset$ are satisfied. Without loss of generality, we assume that $\mathbf{x} \in V_0(Q_n)$. By Lemma 6.1, there exist two node-disjoint paths P_1 and P_2 in $Q_n^{j,1}$ such that (i) P_1 joins $\mathbf{u}$ to $(\mathbf{x})^j$, (ii) P_2 joins $(\mathbf{y})^j$ to $\mathbf{v}$, and (iii) $V(P_1) \cup V(P_2) = V(Q_n^{j,1})$. As a result, $\langle \mathbf{u}, P_1, (\mathbf{x})^j, \mathbf{x}, \mathbf{y}, (\mathbf{y})^j, P_2, \mathbf{v}\rangle$ is a path of length $2^{n-1}+1$. See Figure 6.3(b) for illustration.

(b) We write $\overline{T}_0$ as $\langle \mathbf{w} = \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{2^{n-1}-1} = (\mathbf{w})^i\rangle$. Then we can choose a pair of nodes from $\{\{\mathbf{x}_0, \mathbf{x}_3\}, \{\mathbf{x}_1, \mathbf{x}_4\}, \{\mathbf{x}_2, \mathbf{x}_5\}\}$, namely $\{\mathbf{x}_k, \mathbf{x}_{k+3}\}$, such that both $F \cap \{(\mathbf{x}_k, (\mathbf{x}_k)^j), (\mathbf{x}_{k+3}, (\mathbf{x}_{k+3})^j)\} = \emptyset$ and $|\{(\mathbf{x}_k)^j, (\mathbf{x}_{k+3})^j\} \cap \{\mathbf{u}, \mathbf{v}\}| \leq 1$ are satisfied.

(b.1) Suppose that $\mathbf{x}_k \in V_0(Q_n)$. If $|\{(\mathbf{x}_k)^j, (\mathbf{x}_{k+3})^j\} \cap \{\mathbf{u}, \mathbf{v}\}| = 0$, Lemma 6.1 ensures that $Q_n^{j,1}$ has two node-disjoint paths P_1 and P_2 such that (i) P_1 joins $\mathbf{u}$ to $(\mathbf{x}_k)^j$, (ii) P_2 joins $(\mathbf{x}_{k+3})^j$ to $\mathbf{v}$, and (iii) $V(P_1) \cup V(P_2) = V(Q_n^{j,1})$. Hence, $\langle \mathbf{u}, P_1, (\mathbf{x}_k)^j, \mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{x}_{k+2}, \mathbf{x}_{k+3}, (\mathbf{x}_{k+3})^j, P_2, \mathbf{v}\rangle$ is a path of length $2^{n-1}+3$. If $|\{(\mathbf{x}_k)^j, (\mathbf{x}_{k+3})^j\} \cap \{\mathbf{u}, \mathbf{v}\}| = 1$, we assume that $(\mathbf{x}_k)^j = \mathbf{v}$. By Theorem 4.3, $Q_n^{j,1} - \{\mathbf{v}\}$ has a hamiltonian path H_1 joining $\mathbf{u}$ to $(\mathbf{x}_{k+3})^j$. Then $\langle \mathbf{u}, H_1, (\mathbf{x}_{k+3})^j, \mathbf{x}_{k+3}, \mathbf{x}_{k+2}, \mathbf{x}_{k+1}, \mathbf{x}_k, (\mathbf{x}_k)^j = \mathbf{v}\rangle$ is a path of length $2^{n-1}+3$. See Figure 6.3(c).

(b.2) Suppose that $\mathbf{x}_k \in V_1(Q_n)$. The required paths can be obtained similarly.

Subcase I.2.3: Suppose that $\mathbf{u}$ is in $Q_n^{j,0}$, and $\mathbf{v}$ is in $Q_n^{j,1}$. If $(\mathbf{u}, (\mathbf{u})^j)$ is fault-free, the shortest path between $\mathbf{u}$ and $\mathbf{v}$ can be of the form $\langle \mathbf{u}, (\mathbf{u})^j, P_1, \mathbf{v}\rangle$, where P_1 is a shortest path joining $(\mathbf{u})^j$ to $\mathbf{v}$ in $Q_n^{j,1}$. By the inductive hypothesis, $Q_n^{j,1}$ contains a path T_1 of even length

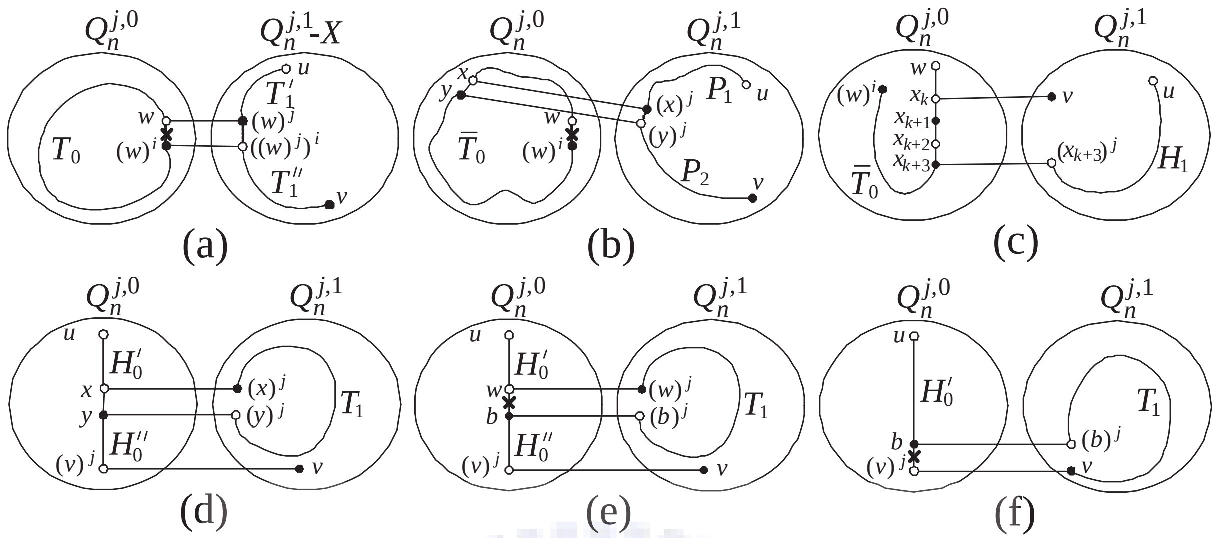

Figure 6.3: Illustration for Subcase I.2.

l_1 between $(\mathbf{u})^j$ and $\mathbf{v}$ for any even integer l_1 from $d_{Q_n^{j,1}}((\mathbf{u})^j, \mathbf{v}) = d^* - 1$ to $2^{n-1} - 2$. Then $\langle \mathbf{u}, (\mathbf{u})^j, T_1, \mathbf{v}\rangle$ is a path of odd length $l_1 + 1$ in the range from d^* to $2^{n-1} - 1$. On the other hand, if $(\mathbf{u}, (\mathbf{u})^j)$ is faulty, we choose a neighbor of $\mathbf{u}$, namely $\mathbf{x}$, in $Q_n^{j,0} - F(Q_n^{j,0})$. Obviously, we have either $h((\mathbf{x})^j, \mathbf{v}) = h(\mathbf{u}, \mathbf{v}) - 2$ or $h((\mathbf{x})^j, \mathbf{v}) = h(\mathbf{u}, \mathbf{v})$. Let R_1 be a shortest path joining $(\mathbf{x})^j$ to $\mathbf{v}$ in $Q_n^{j,1}$. Then $\langle \mathbf{u}, \mathbf{x}, (\mathbf{x})^j, R_1, \mathbf{v}\rangle$ is a path of length $h(\mathbf{u}, \mathbf{v})$ or $h(\mathbf{u}, \mathbf{v}) + 2$. Thus, we have $d^* \leq h(\mathbf{u}, \mathbf{v}) + 2$. By Theorem 6.2, $Q_n^{j,1}$ has a path T_1 of length l_1 between $(\mathbf{x})^j$ and $\mathbf{v}$ for any odd integer l_1 from $h((\mathbf{x})^j, \mathbf{v})$ to $2^{n-1} - 1$. Then $\langle \mathbf{u}, \mathbf{x}, (\mathbf{x})^j, T_1, \mathbf{v}\rangle$ is a path of odd length $l_1 + 2$ in the range from $d^* + 2$ to $2^{n-1} + 1$.

The paths of lengths greater than $2^{n-1} - 1$ can be obtained as follows. Since $|F(Q_n^{j,0})| = 2n - 6$, the j-partition determined by Partition(Q_n, F, $\mathbf{u}$, $\mathbf{v}$) guarantees that link $(\mathbf{v}, (\mathbf{v})^j)$ is fault-free if $h(\mathbf{u}, \mathbf{v})$ is odd. (See (2.2) in Section 6.1). Let $(\mathbf{w}, \mathbf{b})$ be a faulty link in $Q_n^{j,0}$ such that both $(\mathbf{w}, (\mathbf{w})^j)$ and $(\mathbf{b}, (\mathbf{b})^j)$ are fault-free. By the inductive hypothesis, $Q_n^{j,0} - (F(Q_n^{j,0}) - \{(\mathbf{w}, \mathbf{b})\})$ contains a path H_0 of length $2^{n-1} - 2$ between $\mathbf{u}$ to $(\mathbf{v})^j$. Three subcases are distinguished.

Subcase I.2.3.1: Suppose that $(\mathbf{w}, \mathbf{b})$ is not located on H_0. See Figure 6.3(d). We choose a link $(\mathbf{x}, \mathbf{y})$ on H_0 such that $(\mathbf{x}, (\mathbf{x})^j)$ and $(\mathbf{y}, (\mathbf{y})^j)$ are fault-free, and $((\mathbf{x})^j, (\mathbf{y})^j)$ is not incident with $\mathbf{v}$. Thus, H_0 can be represented as $\langle \mathbf{u}, H_0', \mathbf{x}, \mathbf{y}, H_0'', (\mathbf{v})^j\rangle$. By Lemma 6.2, $Q_n^{j,1} - \{\mathbf{v}\}$ contains a path T_1 of odd length l_1 between $(\mathbf{x})^j$ and $(\mathbf{y})^j$ for any odd integer l_1 from 1 to $2^{n-1} - 3$. Consequently, $\langle \mathbf{u}, H_0', \mathbf{x}, (\mathbf{x})^j, T_1, (\mathbf{y})^j, \mathbf{y}, H_0'', (\mathbf{v})^j, \mathbf{v}\rangle$ is a path of odd length $2^{n-1} + l_1$, in the range from $2^{n-1} + 1$ to $2^n - 3$.

Subcase I.2.3.2: Suppose that $(\mathbf{w}, \mathbf{b})$ is located on H_0, and $(\mathbf{w}, \mathbf{b})$ is not incident with $(\mathbf{v})^j$. See Figure 6.3(e). Thus, H_0 can be represented as $\langle \mathbf{u}, H_0', \mathbf{w}, \mathbf{b}, H_0'', (\mathbf{v})^j\rangle$. By Lemma 6.2, $Q_n^{j,1} - \{\mathbf{v}\}$ contains a path T_1 of odd length l_1 between $(\mathbf{w})^j$ and $(\mathbf{b})^j$ for

$1 \leq l_1 \leq 2^{n-1} - 3$. Hence, $\langle \mathbf{u}, H_0', \mathbf{w}, (\mathbf{w})^j, T_1, (\mathbf{b})^j, \mathbf{b}, H_0'', (\mathbf{v})^j, \mathbf{v} \rangle$ is a path of odd length $2^{n-1} + l_1$, in the range $2^{n-1} + 1$ to $2^n - 3$.

Subcase I.2.3.3: Suppose that $(\mathbf{w}, \mathbf{b})$ is located on H_0, and $(\mathbf{w}, \mathbf{b})$ is incident with $(\mathbf{v})^j$. See Figure 6.3(f). Let $\mathbf{w} = (\mathbf{v})^j$. Thus, H_0 can be represented as $\langle \mathbf{u}, H_0', \mathbf{b}, \mathbf{w} = (\mathbf{v})^j \rangle$. By Theorem 6.2, $Q_n^{j,1}$ contains a path T_1 of odd length l_1 between $(\mathbf{b})^j$ and $\mathbf{v}$ for any odd integer l_1 satisfying $1 \leq l_1 \leq 2^{n-1} - 1$. Then $\langle \mathbf{u}, H_0', \mathbf{b}, (\mathbf{b})^j, T_1, \mathbf{v} \rangle$ is a path of odd length $2^{n-1} + l_1 - 2$, in the range from $2^{n-1} - 1$ to $2^n - 3$.

Case II: Suppose that $\mathbf{u}$ and $\mathbf{v}$ belong to the same partite set of $Q_n - F$. Thus, the distance d^* between $\mathbf{u}$ and $\mathbf{v}$ is even. Without loss of generality, we assume that $\mathbf{u}, \mathbf{v} \in V_0(Q_n)$. By Theorem 6.3, $Q_n - F$ is strongly hamiltonian laceable. Moreover, a shortest path between $\mathbf{u}$ and $\mathbf{v}$ can be obtained by a breadth-first search. Hence, we concentrate on the paths of even lengths in the range from $d^* + 2$ to $2^n - 4$.

Subcase II.1: Suppose that $|F(Q_n^{j,0})| \leq 2n - 7$ and $|F(Q_n^{j,1})| \leq 2n - 7$. Without loss of generality, we assume that $|F(Q_n^{j,0})| \geq |F(Q_n^{j,1})|$. Thus, $|F(Q_n^{j,1})| \leq n - 3$.

Subcase II.1.1: Suppose that both $\mathbf{u}$ and $\mathbf{v}$ are in $Q_n^{j,0}$. By the inductive hypothesis, $Q_n^{j,0} - F(Q_n^{j,0})$ has a path H_0 of length $2^{n-1} - 2$ between $\mathbf{u}$ and $\mathbf{v}$. Let $A = \{(H_0(i), H_0(i+1)) \mid 1 \leq i \leq 2^{n-1} - 1, i \equiv 1 \pmod 2\}$ be a set of disjoint links on H_0. First, suppose that $|F(Q_n^{j,0})| > 0$. Since $|A| = \lceil \frac{2^{n-1}-2}{2} \rceil > 2n - 5 - |F(Q_n^{j,0})|$ for $n \geq 4$, there exists a link $(\mathbf{w}, \mathbf{b})$ of A such that $(\mathbf{w}, (\mathbf{w})^j)$, $(\mathbf{b}, (\mathbf{b})^j)$, and $((\mathbf{w})^j, (\mathbf{b})^j)$ are all fault-free. Next, suppose that $|F(Q_n^{j,0})| = 0$ and $n \geq 5$. Since $|A| = \lceil \frac{2^{n-1}-2}{2} \rceil > 2n - 5$, there still exists a link $(\mathbf{w}, \mathbf{b})$ of A such that $(\mathbf{w}, (\mathbf{w})^j)$, $(\mathbf{b}, (\mathbf{b})^j)$, and $((\mathbf{w})^j, (\mathbf{b})^j)$ are all fault-free. Finally, suppose that $|F(Q_n^{j,0})| = 0$ and $n = 4$. If there does not exist any node $\mathbf{z}$ of $V_1(Q_4^{j,0})$ such that $(\mathbf{z}, (\mathbf{z})^j)$ is faulty, there must exist a link $(\mathbf{w}, \mathbf{b})$ on H_0 such that $(\mathbf{w}, (\mathbf{w})^j)$, $(\mathbf{b}, (\mathbf{b})^j)$, and $((\mathbf{w})^j, (\mathbf{b})^j)$ are all fault-free. If there exists a node $\mathbf{z}$ of $V_1(Q_4^{j,0})$ such that $(\mathbf{z}, (\mathbf{z})^j)$ is faulty, then it follows from Theorem 4.3 that $Q_4^{j,0} - \{\mathbf{z}\}$ has a hamiltonian path, still namely H_0, between $\mathbf{u}$ and $\mathbf{v}$. Obviously, there also exists a link $(\mathbf{w}, \mathbf{b})$ on H_0 such that $(\mathbf{w}, (\mathbf{w})^j)$, $(\mathbf{b}, (\mathbf{b})^j)$, and $((\mathbf{w})^j, (\mathbf{b})^j)$ are all fault-free. In summary, H_0 can be written as $\langle \mathbf{u}, H_0', \mathbf{w}, \mathbf{b}, H_0'', \mathbf{v} \rangle$. Since $|F(Q_n^{j,1})| \leq n - 3$, it follows from Theorem 6.2 that $Q_n^{j,1} - F(Q_n^{j,1})$ contains a path H_1 of odd length l_1 between $(\mathbf{w})^j$ and $(\mathbf{b})^j$ for any odd integer l_1 satisfying $1 \leq l_1 \leq 2^{n-1} - 1$. As a result, $\langle \mathbf{u}, H_0', \mathbf{w}, (\mathbf{w})^j, H_1, (\mathbf{b})^j, \mathbf{b}, H_0'', \mathbf{v} \rangle$ is a path of even length in the range from 2^{n-1} to $2^n - 2$.

The paths of lengths less than 2^{n-1} are obtained as follows. By Proposition 5.3, we have $d^* = d_{Q_n - F}(\mathbf{u}, \mathbf{v}) \leq h(\mathbf{u}, \mathbf{v}) + 4$ and $d_{Q_n^{j,0} - F(Q_n^{j,0})}(\mathbf{u}, \mathbf{v}) \leq h(\mathbf{u}, \mathbf{v}) + 4$. By inductive hypothesis, $Q_n^{j,0} - F(Q_n^{j,0})$ has a path T_0 of length l_0 between $\mathbf{u}$ and $\mathbf{v}$ for any even length from $d_{Q_n^{j,0} - F(Q_n^{j,0})}(\mathbf{u}, \mathbf{v})$ to $2^{n-1} - 2$. If $d^* = h(\mathbf{u}, \mathbf{v})$ or $d^* = h(\mathbf{u}, \mathbf{v}) + 4$, then $d_{Q_n^{j,0} - F(Q_n^{j,0})}(\mathbf{u}, \mathbf{v}) = d^*$. If $d^* = h(\mathbf{u}, \mathbf{v}) + 2$, then $d_{Q_n^{j,0} - F(Q_n^{j,0})}(\mathbf{u}, \mathbf{v}) \leq d^* + 2$.

Subcase II.1.2: Suppose that both $\mathbf{u}$ and $\mathbf{v}$ are in $Q_n^{j,1}$. Since $|F(Q_n^{j,1})| \leq n - 3$, it follows from Lemma 5.1 that $d^* \leq h(\mathbf{u}, \mathbf{u}) + 2$. Thus, $Q_n - F$ has a shortest path between $\mathbf{u}$ and $\mathbf{v}$ that does not cross the dimension j. By the inductive hypothesis, $Q_n^{j,1} - F(Q_n^{j,1})$ contains a path T_1 of length l_1 between $\mathbf{u}$ and $\mathbf{v}$ for any even integer l_1 satisfying $d^* \leq l_1 \leq 2^{n-1} - 2$. Let $\overline{T}_1$ be a path of length $2^{n-1} - 2$ between $\mathbf{u}$ and $\mathbf{v}$ in $Q_n^{j,1} - F(Q_n^{j,1})$. Moreover, let $A = \{(\overline{T}_1(i), \overline{T}_1(i+1)) \mid 1 \leq i \leq 2^{n-1} - 1, i \equiv 1 \pmod 2\}$ be a set of disjoint links on $\overline{T}_1$. First, suppose that $|F(Q_n^{j,1})| > 0$. Since $|A| = \lceil \frac{2^{n-1}-2}{2} \rceil > 2n - 5 - |F(Q_n^{j,1})|$ for $n \geq 4$, there exists a link $(\mathbf{w}, \mathbf{b}) \in A$ such that $(\mathbf{w}, (\mathbf{w})^j)$, $(\mathbf{b}, (\mathbf{b})^j)$, and $((\mathbf{w})^j, (\mathbf{b})^j)$ are all fault-free. Next, suppose that $|F(Q_n^{j,1})| = 0$ and $n \geq 5$. Since $|A| = \lceil \frac{2^{n-1}-2}{2} \rceil > 2n - 5$, there still exists a link $(\mathbf{w}, \mathbf{b}) \in A$ such that $(\mathbf{w}, (\mathbf{w})^j)$, $(\mathbf{b}, (\mathbf{b})^j)$ and $((\mathbf{w})^j, (\mathbf{b})^j)$ are all fault-free. Finally, suppose that $|F(Q_n^{j,1})| = 0$ and $n = 4$. If there does not exist any node $\mathbf{z}$ of $V_1(Q_4^{j,1})$ such that $(\mathbf{z}, (\mathbf{z})^j)$ is faulty, there exists a link $(\mathbf{w}, \mathbf{b})$ on $\overline{T}_1$ such that $(\mathbf{w}, (\mathbf{w})^j)$, $(\mathbf{b}, (\mathbf{b})^j)$ and $((\mathbf{w})^j, (\mathbf{b})^j)$ are all fault-free. If there exists a node $\mathbf{z}$ of $V_1(Q_4^{j,1})$ such that $(\mathbf{z}, (\mathbf{z})^j)$ is faulty, Theorem 4.3 ensures that $Q_4^{j,1} - \{\mathbf{z}\}$ has a hamiltonian path, still namely $\overline{T}_1$, between $\mathbf{u}$ and $\mathbf{v}$. Obviously, there also exists a link $(\mathbf{w}, \mathbf{b})$ on $\overline{T}_1$ such that $(\mathbf{w}, (\mathbf{w})^j)$, $(\mathbf{b}, (\mathbf{b})^j)$ and $((\mathbf{w})^j, (\mathbf{b})^j)$ are all fault-free. In summary, $\overline{T}_1$ can be written as $\langle \mathbf{u}, T_1', \mathbf{w}, \mathbf{b}, T_1'', \mathbf{v} \rangle$. Since $|F(Q_n^{j,0})| \leq 2n - 7$, it follows from Theorem 6.4 that $Q_n^{j,0} - F(Q_n^{j,0})$ contains a path T_0 of length l_0 between $(\mathbf{w})^j$ and $(\mathbf{b})^j$ for any odd integer l_0 from 1 to $2^{n-1} - 1$ excluding 3. As a result, $\langle \mathbf{u}, T_1', \mathbf{w}, (\mathbf{w})^j, T_0, (\mathbf{b})^j, \mathbf{b}, T_1'', \mathbf{v} \rangle$ is a path of any even length in the range from 2^{n-1} to $2^n - 2$, excluding $2^{n-1} + 2$.

The path of length $2^{n-1} + 2$ is discussed as follows. When $n = 4$, $|F(Q_n^{j,0})| \leq 1$. Thus, there exists an integer k of $\{0, 1, 2, 3\} - \{j, dim((\mathbf{w}, \mathbf{b}))\}$ such that $((\mathbf{w})^j, ((\mathbf{w})^j)^k)$, $((\mathbf{b})^j, ((\mathbf{b})^j)^k)$, and $(((\mathbf{w})^j)^k, ((\mathbf{b})^j)^k)$ are all fault-free. Hence, $\langle \mathbf{u}, T_1', \mathbf{w}, (\mathbf{w})^j, ((\mathbf{w})^j)^k, ((\mathbf{b})^j)^k, (\mathbf{b})^j, \mathbf{b}, T_1'', \mathbf{v} \rangle$ is a path of length 10. When $n \geq 5$, we have $|A| - |F| = \lceil \frac{2^{n-1}-2}{2} \rceil - (2n - 5) \geq 2$. Thus, there is another link $(\mathbf{x}, \mathbf{y})$ of A, other than $(\mathbf{w}, \mathbf{b})$, such that $(\mathbf{x}, (\mathbf{x})^j)$, $(\mathbf{y}, (\mathbf{y})^j)$, and $((\mathbf{x})^j, (\mathbf{y})^j)$ are all fault-free. Without loss of generality, $\overline{T}_1$ can be written as $\langle \mathbf{u}, R_1', \mathbf{w}, \mathbf{b}, R_1'', \mathbf{x}, \mathbf{y}, R_1''', \mathbf{v} \rangle$. Hence, $\langle \mathbf{u}, R_1', \mathbf{w}, (\mathbf{w})^j, (\mathbf{b})^j, \mathbf{b}, R_1'', \mathbf{x}, (\mathbf{x})^j, (\mathbf{y})^j, \mathbf{y}, R_1''', \mathbf{v} \rangle$ is a path of length $2^{n-1} + 2$.

Subcase II.1.3: Suppose that $\mathbf{u}$ is in $Q_n^{j,0}$ and $\mathbf{v}$ is in $Q_n^{j,1}$. By Theorem 6.1, there exists a shortest path P^* between $\mathbf{u}$ and $\mathbf{v}$ in $Q_n - F$ such that P^* crosses the dimension j exactly once. Thus, P^* can be written as $\langle \mathbf{u}, P_0, \mathbf{x}, (\mathbf{x})^j, P_1, \mathbf{v} \rangle$, where P_0 is a shortest path joining $\mathbf{u}$ to some node $\mathbf{x}$ in $Q_n^{j,0} - F(Q_n^{j,0})$ and P_1 is a shortest path joining $(\mathbf{x})^j$ to $\mathbf{v}$ in $Q_n^{j,1} - F(Q_n^{j,1})$.

Subcase II.1.3.1: Suppose that $\ell(P_0) > 0$ and $\ell(P_1) > 0$. By Theorem 6.2, $Q_n^{j,1} - F(Q_n^{j,1})$ has a path T_1 of length l_1 between $(\mathbf{x})^j$ and $\mathbf{v}$ for each l_1 satisfying $\ell(P_1) \leq l_1 \leq 2^{n-1} - 1$ and $2 | (l_1 - \ell(P_1))$. Suppose that $\ell(P_0) = 1$. By Theorem 6.4, $Q_n^{j,0} - F(Q_n^{j,0})$ has a path T_0 of length l_0 between $\mathbf{u}$ and $\mathbf{x}$ for any odd integer l_0 from 1 to $2^{n-1} - 1$ excluding 3. Suppose that $\ell(P_0) > 1$. By the inductive hypothesis, $Q_n^{j,0} - F(Q_n^{j,0})$ has a path T_0 of length l_0 between $\mathbf{u}$ and $\mathbf{x}$ for each l_0 satisfying $\ell(P_0) \leq l_0 \leq 2^{n-1} - 1$ and $2 | (l_0 - \ell(P_0))$. Hence, $\langle \mathbf{u}, T_0, \mathbf{x}, (\mathbf{x})^j, T_1, \mathbf{v} \rangle$ is a path of even length $l_0 + l_1 + 1$ in the range from d^* to $2^n - 2$.

Subcase II.1.3.2: Suppose that $\ell(P_0) = 0$ or $\ell(P_1) = 0$. With symmetry, we assume $\mathbf{u} = \mathbf{x}$. By the inductive hypothesis, $Q_n^{j,1} - F(Q_n^{j,1})$ contains a path T_1 of length l_1 between $(\mathbf{u})^j$ and $\mathbf{v}$ for any odd integer l_1 form $\ell(P_1)$ to $2^{n-1} - 1$. Then $\langle \mathbf{u}, (\mathbf{u})^j, T_1, \mathbf{v} \rangle$ is a path of even length $l_1 + 1$ in the range from $\ell(P_1) + 1 = d^*$ to 2^{n-1}.

The paths of lengths greater than 2^{n-1} are constructed as follows. Since $|V(Q_n^{j,0}) - \{\mathbf{u}\}| - (2n-5) > 1$ for $n \geq 4$, we can choose a node $\mathbf{y}$ from $V(Q_n^{j,0}) - \{\mathbf{u}\}$ such that $(\mathbf{y}, (\mathbf{y})^j)$ is fault-free and $(\mathbf{y})^j$ is not $\mathbf{v}$. Let R_0 be a path joining $\mathbf{u}$ to $\mathbf{y}$ in $Q_n^{j,0} - F(Q_n^{j,0})$ and R_1 be a path joining $(\mathbf{y})^j$ to $\mathbf{v}$ in $Q_n^{j,1} - F(Q_n^{j,1})$. Similar to **Subcase II.1.3.1**, $H = \langle \mathbf{u}, R_0, \mathbf{y}, (\mathbf{y})^j, R_1, \mathbf{v} \rangle$ is a path of even length in the range from $d' = d_{Q_n^{j,0} - F(Q_n^{j,0})}(\mathbf{u}, \mathbf{y}) + d_{Q_n^{j,1} - F(Q_n^{j,1})}((\mathbf{y})^j, \mathbf{v}) + 1$ to $2^n - 2$. By Theorem 5.3, we have $d' \leq (n+1) + (n-1) + 1 \leq 2^{n-1} + 2$ for $n \geq 4$. Therefore, H is a path of even length in the range from $2^{n-1} + 2$ to $2^n - 2$.

Subcase II.2: Suppose that $|F(Q_n^{j,0})| \leq 2n - 6$ or $|F(Q_n^{j,1})| \leq 2n - 6$. Without loss of generality, we assume that $|F(Q_n^{j,0})| = 2n - 6$. Thus, $Q_n^{j,1}$ is fault-free. It is noticed that the faulty links are distributed as shown in Figure 6.1.

Subcase II.2.1: Suppose that both $\mathbf{u}$ and $\mathbf{v}$ are in $Q_n^{j,0}$. Let $(\mathbf{w}, \mathbf{b})$ be a faulty link of $Q_n^{j,0}$ such that both $(\mathbf{w}, (\mathbf{w})^j)$ and $(\mathbf{b}, (\mathbf{b})^j)$ are fault-free. Let $F_0 = F(Q_n^{j,0}) - \{(\mathbf{w}, \mathbf{b})\}$. By the inductive hypothesis, $Q_n^{j,0} - F_0$ has a path P_l of length l between $\mathbf{u}$ and $\mathbf{v}$ for any even integer l from $d_{Q_n^{j,0} - F_0}(\mathbf{u}, \mathbf{v})$ to $2^{n-1} - 2$. If $(\mathbf{w}, \mathbf{b})$ is on P_l, we write P_l as $\langle \mathbf{u}, P_l', \mathbf{w}, \mathbf{b}, P_l'', \mathbf{v} \rangle$ and define $\widetilde{P}_l = \langle \mathbf{u}, P_l', \mathbf{w}, (\mathbf{w})^j, (\mathbf{b})^j, \mathbf{b}, P_l'', \mathbf{v} \rangle$. Otherwise, P_l can be written as $\langle \mathbf{u}, P_l', \mathbf{x}, \mathbf{y}, P_l'', \mathbf{v} \rangle$, where $(\mathbf{x}, \mathbf{y})$ is a link on P_l such that both $(\mathbf{x}, (\mathbf{x})^j)$ and $(\mathbf{y}, (\mathbf{y})^j)$ are fault-free. Similarly, we define $\widetilde{P}_l = \langle \mathbf{u}, P_l', \mathbf{x}, (\mathbf{x})^j, (\mathbf{y})^j, \mathbf{y}, P_l'', \mathbf{v} \rangle$. Then $\widetilde{P}_l$ is a path of length $l+2$. By Proposition 5.3, we have $d^* = d_{Q_n - F}(\mathbf{u}, \mathbf{v}) \leq h(\mathbf{u}, \mathbf{v}) + 4$ and $d_{Q_n^{j,0} - F_0}(\mathbf{u}, \mathbf{v}) \leq h(\mathbf{u}, \mathbf{v}) + 4$. If $d_{Q_n^{j,0} - F_0}(\mathbf{u}, \mathbf{v}) = d^*$, then path $\widetilde{P}_l$ is the desired path. Otherwise, if $d_{Q_n^{j,0} - F_0}(\mathbf{u}, \mathbf{v}) = d^* + 2$, then $\widetilde{P}_l$ is a path of even length in the range from $d^* + 4$ to 2^{n-1}. It is noticed that a shortest path between $\mathbf{u}$ and $\mathbf{v}$ in $Q_n - F$ can be constructed based on a breadth-first search.

By Theorem 6.2, $Q_n^{j,1}$ contains a path T_1 of length l_1 between $(\mathbf{w})^j$ and $(\mathbf{b})^j$ or a path R_1 of odd length l_1 between $(\mathbf{x})^j$ and $(\mathbf{y})^j$ for any odd integer l_1 from 1 to $2^{n-1} - 1$. Thus, $\langle \mathbf{u}, P'_{2^{n-1}-2}, \mathbf{w}, (\mathbf{w})^j, T_1, (\mathbf{b})^j, \mathbf{b}, P''_{2^{n-1}-2}, \mathbf{v} \rangle$ (or $\langle \mathbf{u}, P'_{2^{n-1}-2}, \mathbf{x}, (\mathbf{x})^j, R_1, (\mathbf{y})^j, \mathbf{y}, P''_{2^{n-1}-2}, \mathbf{v} \rangle$) is a path of even length in the range from 2^{n-1} to $2^n - 2$.

Subcase II.2.2: Suppose that both $\mathbf{u}$ and $\mathbf{v}$ are in $Q_n^{j,1}$. Let $(\mathbf{w}, (\mathbf{w})^i)$ be a faulty link of $Q_n^{j,0}$ such that both $(\mathbf{w}, (\mathbf{w})^j)$ and $((\mathbf{w})^i, ((\mathbf{w})^i)^j)$ are fault-free. Since $n \geq 4$, we assume that $t \in \{0, 1, \ldots, n-1\} - \{j, i\}$. Moreover, we assume that $\mathbf{w} \in V_0(Q_n^{j,0})$. Let $X = \{((\mathbf{x})^j, ((\mathbf{x})^j)^k) \mid k \notin \{i, j, t\}\}$. Since $|X| = n - 3$, our inductive hypothesis ensures that $Q_n^{j,1} - X$ contains a path T_1 of even length l_1 between $\mathbf{u}$ and $\mathbf{v}$ for $d^* \leq l_1 \leq 2^{n-1} - 2$. Let $\overline{T}_1$ denote the longest path between $\mathbf{u}$ and $\mathbf{v}$ in $Q_n^{j,1} - X$. It is noted that $((\mathbf{w})^j, ((\mathbf{w})^j)^i)$ is on $\overline{T}_1$. Hence, $\overline{T}_1$ can be represented as $\langle \mathbf{u}, T_1', (\mathbf{w})^j, ((\mathbf{w})^j)^i, T_1'', \mathbf{v} \rangle$. By the inductive hypothesis, $Q_n^{j,0} - (F(Q_n^{j,0}) - \{(\mathbf{w}, (\mathbf{w})^i)\})$ contains a path T_0 of odd length l_0 between $\mathbf{w}$ to $(\mathbf{w})^i$ for $5 \leq l_0 \leq 2^{n-1} - 1$. As a result, $\langle \mathbf{u}, T_1', (\mathbf{w})^j, \mathbf{w}, T_0, (\mathbf{w})^i, ((\mathbf{w})^j)^i, T_1'', \mathbf{v} \rangle$ is a path of even length $2^{n-1} + l_0 - 1$, in the range from $2^{n-1} + 4$ to $2^n - 2$.

Let $A = \{(\overline{T}_1(k), \overline{T}_1(k+1)) \mid 1 \leq k \leq 2^{n-1} - 1, k \equiv 1 \pmod 2\}$ be a set of disjoint links on $\overline{T}_1$. Then the paths of lengths 2^{n-1} and $2^{n-1} + 2$ can be obtained as follows. When $n = 4$, we suppose that $\{p, q, j, i\} = \{0, 1, 2, 3\}$. Since $(\mathbf{w}, (\mathbf{w})^i)$ is faulty, we have either $\{(\mathbf{w}, (\mathbf{w})^p), ((\mathbf{w})^p, ((\mathbf{w})^p)^i), ((\mathbf{w})^p)^i, (\mathbf{w})^i)\} \cap F = \emptyset$ or $\{(\mathbf{w}, (\mathbf{w})^q), ((\mathbf{w})^q, ((\mathbf{w})^q)^i), ((\mathbf{w})^q)^i,$

$(\mathbf{w})^i, (\mathbf{w})^q)^i)\} \cap F = \emptyset$. Without loss of generality, we assume $\{(\mathbf{w}, (\mathbf{w})^p), ((\mathbf{w})^p, ((\mathbf{w})^p)^i), ((\mathbf{w})^i, (\mathbf{w})^p)^i)\} \cap F = \emptyset$. Obviously, $\langle \mathbf{u}, T_1', (\mathbf{w})^j, \mathbf{w}, (\mathbf{w})^p, ((\mathbf{w})^p)^i, (\mathbf{w})^i, ((\mathbf{w})^j)^i, T_1'', \mathbf{v}\rangle$ is a path of length $2^{n-1}+2$. Moreover, since $|A| - |F| = \lceil \frac{2^{n-1}-1}{2} \rceil - (2n-5) = 1$ for $n = 4$, there exists one link $(\mathbf{x}, \mathbf{y}) \in A$ such that $(\mathbf{x}, (\mathbf{x})^j)$, $(\mathbf{y}, (\mathbf{y})^j)$, and $((\mathbf{x})^j, (\mathbf{y})^j)$ is fault-free. Hence, $\overline{T}_1$ can be represented as $\langle \mathbf{u}, R_1, \mathbf{x}, \mathbf{y}, R_2, \mathbf{v}\rangle$. Obviously, $\langle \mathbf{u}, R_1, \mathbf{x}, (\mathbf{x})^j, (\mathbf{y})^j, \mathbf{y}, R_2, \mathbf{v}\rangle$ is a path of length 2^{n-1}. When $n \geq 5$, we have $|A| - |F| = \lceil \frac{2^{n-1}-2}{2} \rceil - (2n-5) \geq 2$. Thus, there are two links $(\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2) \in A$ such that $\{(\mathbf{x}_k, (\mathbf{x}_k)^j), (\mathbf{y}_k, (\mathbf{y}_k)^j), ((\mathbf{x}_k)^j, (\mathbf{y}_k)^j) \mid k = 1, 2\} \cap F = \emptyset$. Hence, $\overline{T}_1$ can be represented as $\langle \mathbf{u}, R_1, \mathbf{x}_1, \mathbf{y}_1, R_2, \mathbf{x}_2, \mathbf{y}_2, R_3, v\rangle$. Obviously, $\langle \mathbf{u}, R_1, \mathbf{x}_1, (\mathbf{x}_1)^j, (\mathbf{y}_1)^j, \mathbf{y}_1, R_2, \mathbf{x}_2, \mathbf{y}_2, R_3, \mathbf{v}\rangle$ and $\langle \mathbf{u}, R_1, \mathbf{x}_1, (\mathbf{x}_1)^j, (\mathbf{y}_1)^j, \mathbf{y}_1, R_2, \mathbf{x}_2, (\mathbf{x}_2)^j, (\mathbf{y}_2)^j, \mathbf{y}_2, R_3, \mathbf{v}\rangle$ are paths of length 2^{n-1} and of length $2^{n-1}+2$, respectively.

Subcase II.2.3: Suppose that $\mathbf{u}$ is in $Q_n^{j,0}$ and $\mathbf{v}$ is in $Q_n^{j,1}$. If $(\mathbf{u}, (\mathbf{u})^j)$ is fault-free, the shortest path between $\mathbf{u}$ and $\mathbf{v}$ can be of the form $\langle \mathbf{u}, (\mathbf{u})^j, P_1, \mathbf{v}\rangle$, where P_1 is a shortest path joining $(\mathbf{u})^j$ to $\mathbf{v}$ in $Q_n^{j,1}$. By the inductive hypothesis, $Q_n^{j,1}$ contains a path T_1 of odd length l_1 between $(\mathbf{u})^j$ and $\mathbf{v}$ for $d^* - 1 \leq l_1 \leq 2^{n-1} - 1$. Then $\langle \mathbf{u}, (\mathbf{u})^j, T_1, \mathbf{v}\rangle$ is a path of even length in the range from d^* to 2^{n-1}. If $(\mathbf{u}, (\mathbf{u})^j)$ is faulty, we choose a neighbor of $\mathbf{u}$ in $Q_n^{j,0} - F(Q_n^{j,0})$, namely $\mathbf{x}$, such that $(\mathbf{x})^j \neq v$. Obviously, we have either $h((\mathbf{x})^j, \mathbf{v}) = h(\mathbf{u}, \mathbf{v}) - 2$ or $h((\mathbf{x})^j, \mathbf{v}) = h(\mathbf{u}, \mathbf{v})$. Let R_1 be a shortest path joining $(\mathbf{x})^j$ to $\mathbf{v}$ in $Q_n^{j,1}$. Then $\langle \mathbf{u}, \mathbf{x}, (\mathbf{x})^j, R_1, \mathbf{v}\rangle$ is a path of length $h(\mathbf{u}, \mathbf{v})$ or $h(\mathbf{u}, \mathbf{v}) + 2$. By Theorem 6.2, $Q_n^{j,1}$ contains a path T_1 of even length l_1 between $(\mathbf{x})^j$ and $\mathbf{v}$ for any even integer l_1 from $h((\mathbf{x})^j, \mathbf{v})$ to $2^{n-1} - 2$. Then $\langle \mathbf{u}, \mathbf{x}, (\mathbf{x})^j, T_1, \mathbf{v}\rangle$ is a path of even length in the range from $d^* + 2$ to 2^{n-1}.

The paths of lengths greater than 2^{n-1} are obtained as follows. Let $(\mathbf{w}, \mathbf{b})$ be a faulty link in $Q_n^{j,0}$ such that both $(\mathbf{w}, (\mathbf{w})^j)$ and $(\mathbf{b}, (\mathbf{b})^j)$ are fault-free. Depending on whether $(\mathbf{v}, (\mathbf{v})^j)$ is faulty, we distinguish two subcases.

Subcase II.2.3.1: Suppose that $(\mathbf{v}, (\mathbf{v})^j)$ is fault-free. By the inductive hypothesis, $Q_n^{j,0} - (F(Q_n^{j,0}) - \{(\mathbf{w}, \mathbf{b})\})$ contains a path H_0 of length $2^{n-1} - 1$ between $\mathbf{u}$ to $(\mathbf{v})^j$.

Subcase II.2.3.1.a: Suppose that $(\mathbf{w}, \mathbf{b})$ is not located on H_0. We choose a link $(\mathbf{x}, \mathbf{y})$ on H_0 such that $(\mathbf{x}, (\mathbf{x})^j)$ and $(\mathbf{y}, (\mathbf{y})^j)$ are fault-free and $((\mathbf{x})^j, (\mathbf{y})^j)$ is not incident with $\mathbf{v}$. Thus, H_0 can be represented as $\langle \mathbf{u}, H_0', \mathbf{x}, \mathbf{y}, H_0'', (\mathbf{v})^j\rangle$. By Lemma 6.2, $Q_n^{j,1} - \{\mathbf{v}\}$ contains a path T_1 of odd length l_1 between $(\mathbf{x})^j$ and $(\mathbf{y})^j$ for any odd integer l_1 from 1 to $2^{n-1} - 3$. Consequently, $\langle \mathbf{u}, H_0', \mathbf{x}, (\mathbf{x})^j, T_1, (\mathbf{y})^j, \mathbf{y}, H_0'', (\mathbf{v})^j, \mathbf{v}\rangle$ is a path of even length $2^{n-1} + l_1 + 1$, in the range from $2^{n-1} + 2$ to $2^n - 2$.

Subcase II.2.3.1.b: Suppose that $(\mathbf{w}, \mathbf{b})$ is located on H_0 and $(\mathbf{w}, \mathbf{b})$ is not incident with $(\mathbf{v})^j$. Thus, H_0 can be represented as $\langle \mathbf{u}, H_0', \mathbf{w}, \mathbf{b}, H_0'', (\mathbf{v})^j\rangle$. By Lemma 6.2, $Q_n^{j,1} - \{\mathbf{v}\}$ contains a path T_1 of odd length l_1 between $(\mathbf{w})^j$ and $(\mathbf{b})^j$ for any odd integer l_1 from 1 to $2^{n-1} - 3$. Then $\langle \mathbf{u}, H_0', \mathbf{w}, (\mathbf{w})^j, T_1, (\mathbf{b})^j, \mathbf{b}, H_0'', (\mathbf{v})^j, \mathbf{v}\rangle$ is a path of even length $2^{n-1} + l_1 + 1$, in the range from $2^{n-1} + 2$ to $2^n - 2$.

Subcase II.2.3.1.c: Suppose that $(\mathbf{w}, \mathbf{b})$ is on H_0 and $(\mathbf{w}, \mathbf{b})$ is incident with $(\mathbf{v})^j$. Let $\mathbf{b} = (\mathbf{v})^j$. Thus, H_0 can be written as $\langle \mathbf{u}, H_0', \mathbf{w}, \mathbf{b} = (\mathbf{v})^j\rangle$. By Theorem 6.2, $Q_n^{j,1}$ has a path

T_1 of odd length l_1 between $(\mathbf{w})^j$ and $\mathbf{v}$ for $1 \le l_1 \le 2^{n-1} - 1$. Thus, $\langle \mathbf{u}, H'_0, \mathbf{w}, (\mathbf{w})^j, T_1, \mathbf{v} \rangle$ is a path of even length $2^{n-1} + l_1 - 1$, in the range from 2^{n-1} to $2^n - 2$.

Subcase II.2.3.2: Suppose that $(\mathbf{v}, (\mathbf{v})^j)$ is faulty. According to procedure $\mathsf{Partition}(Q_n$, F, $\mathbf{u}$, $\mathbf{v})$, this subcase occurs only when $n = 4$ and there is a unique node $\mathbf{z}$ of $V_1(Q_4)$ such that both $(\mathbf{z}, \mathbf{u})$ and $(\mathbf{z}, \mathbf{v})$ are faulty links. In addition, each faulty link corresponds to a unique dimension. By transitivity, we assume that $\mathbf{z} = 0001$, $\mathbf{u} = 0101$, and $\mathbf{v} = 1001$. Then the paths obtained by brute force are listed in Table 6.3. □

Table 6.3: The paths of lengths 10, 12, and 14 between $\mathbf{u} = 0101$ and $\mathbf{v} = 1001$ in $Q_4 - \{e_f, (0001, 0101), (0001, 1001)\}$.

$e_f \in \{(0000, 0010), (0010, 0011)\}$	$\langle \mathbf{u} = 0101, 0100, 0110, 0111, 0011, 0001, 0000, 1000, 1100, 1101, 1001 = \mathbf{v} \rangle$ $\langle \mathbf{u} = 0101, 0100, 0110, 0111, 0011, 0001, 0000, 1000, 1100, 1110, 1111, 1101, 1001 = \mathbf{v} \rangle$ $\langle \mathbf{u} = 0101, 0100, 0110, 0111, 0011, 0001, 0000, 1000, 1100, 1110, 1010, 1011, 1111, 1101, 1001 = \mathbf{v} \rangle$
$e_f = (0100, 0110)$	$\langle \mathbf{u} = 0101, 0111, 0110, 0010, 0011, 0001, 0000, 1000, 1100, 1101, 1001 = \mathbf{v} \rangle$ $\langle \mathbf{u} = 0101, 0111, 0110, 0010, 0011, 0001, 0000, 1000, 1100, 1110, 1111, 1101, 1001 = \mathbf{v} \rangle$ $\langle \mathbf{u} = 0101, 0111, 0110, 0010, 0011, 0001, 0000, 1000, 1100, 1110, 1010, 1011, 1111, 1101, 1001 = \mathbf{v} \rangle$
$e_f = (0110, 0111)$	$\langle \mathbf{u} = 0101, 0111, 0011, 0010, 0110, 0100, 0000, 1000, 1100, 1101, 1001 = \mathbf{v} \rangle$ $\langle \mathbf{u} = 0101, 0111, 0011, 0010, 0110, 0100, 0000, 1000, 1100, 1110, 1111, 1101, 1001 = \mathbf{v} \rangle$ $\langle \mathbf{u} = 0101, 0111, 0011, 0010, 0110, 0100, 0000, 1000, 1100, 1110, 1010, 1011, 1111, 1101, 1001 = \mathbf{v} \rangle$

Chapter 7

Long Paths in Faulty Hypercubes with Conditional Node-faults

In contrast to the preceding chapter in which only link-faults are taken into account, we address only node-faults in this chapter. Hence, a network will be called conditionally faulty if and only if its every node has at least two fault-free neighbors.

This chapter is aimed to show that a conditionally faulty n-cube, with $f \leq 2n-5$ faulty nodes, contains a fault-free path of length at least $2^n - 2f - 1$ (respectively, $2^n - 2f - 2$) between any two fault-free nodes of odd (respectively, even) distance. Why do we concern only $2n-5$ faulty nodes? Consider a 4-cube with four faulty nodes, 0000, 0011, 1100, and 1111, as shown in Figure 7.1, in which every node has at least two fault-free neighbors. We can see that the length of the longest path between nodes 0110 and 1001 is $4 < 2^4 - 2 \cdot 4 - 2$. This is the reason why we concentrate only on $f \leq 2n-5$ faulty nodes.

It is sufficient to assume that every node should have at least two fault-free neighbors while a long path is constructed between every pair of fault-free nodes. Consider the scenario that $\mathbf{u}$ is a fault-free node with only one fault-free neighbor, namely $\mathbf{v}$. Then the longest path between $\mathbf{u}$ and $\mathbf{v}$ happens to be of length 1. To avoid such a degenerate situation, it is necessary that, for any pair $\mathbf{u}, \mathbf{v}$ of adjacent nodes, $\mathbf{u}$ has some fault-free neighbor other than $\mathbf{v}$, and vice versa. On the other hand, it is also statistically reasonable to require that every node needs to have at least two fault-free neighbors. Suppose, with a random fault model, the probabilities of node failures are identical and independent. Let $P_N(n)$ denote the probability that every node of the n-cube Q_n, containing $2n-5$ faulty nodes, is adjacent to at least two fault-free neighbors. As discussed in Section 1.3, we have $P_N(3) = 1$, $P_N(4) = 1 - \frac{2^4 \times \binom{4}{3}}{\binom{2^4}{3}} = \frac{31}{35}$, and $P_N(n) = 1 - \frac{2^n \times \binom{2^n - n}{n-5} + 2^n \times \binom{n}{n-1}\binom{2^n - n}{n-4}}{\binom{2^n}{2n-5}}$ for each $n \geq 5$. Clearly $P_N(n)$ approaches to 1 as n increases.

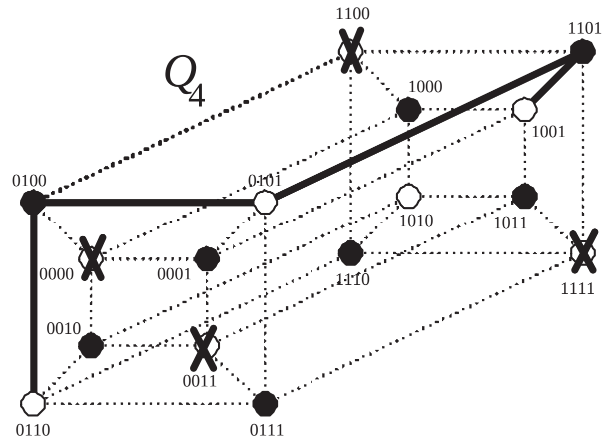

Figure 7.1: A conditionally faulty Q_4 with four faulty nodes. Every faulty node is marked by an "X" symbol. The length of the longest path between nodes 0110 and 1001 is 4.

7.1 Partition of an n-cube with conditional node-faults

Here we will show that a conditionally faulty n-cube can be partitioned into two conditionally faulty subcubes if it has $2n-5$ or less faulty nodes. Recall that $F(G)$ denotes the set of all faulty elements in a network G. Let u be a node of G. For convenience, we use $N_G^F(u)$ to denote the set of all faulty neighbors of u; i.e., $N_G^F(u) = N_G(u) \cap F(G)$.

Suppose Q_n, $n \geq 4$, is conditionally faulty with $f \leq 2n-5$ faulty nodes. Moreover, suppose $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three nodes of this faulty n-cube, and each of them has only two fault-free neighbors. Then we discuss how the faulty nodes will be distributed conditionally. For simplification, let $U = N_{Q_n}^F(\mathbf{u})$, $V = N_{Q_n}^F(\mathbf{v})$, and $W = N_{Q_n}^F(\mathbf{w})$.

If $|V \cap W| = 0$, then we have $f \geq |V \cup W| = |V| + |W| = 2n-4$, contradicting the requirement that $f \leq 2n-5$. Therefore, $|V \cap W| \geq 1$ needs to be satisfied. Similarly, we also have $|U \cap V| \geq 1$ and $|U \cap W| \geq 1$. Since any two nodes of an n-cube can have utmost two common neighbors, we obtain that $|V \cap W|, |U \cap V|, |U \cap W| \in \{1, 2\}$. We first consider the case that at least one of $|V \cap W|$, $|U \cap V|$, and $|U \cap W|$ is equal to 1. Without loss of generality, we suppose $|V \cap W| = 1$.

I. Firstly, we concern the case that $|V \cap W| = |U \cap V| = |U \cap W| = 1$. If $|U \cap V \cap W| \geq 1$, we have $2n-5 \geq f \geq |U \cup V \cup W| = 3(n-2) - (1+1+1) + 1 = 3n-8$; i.e., $n \leq 3$. Since $n \geq 4$, we only concern $|U \cap V \cap W| = 0$. Then we have $2n-5 \geq f \geq |U \cup V \cup W| \geq 3(n-2) - (1+1+1) = 3n-9$; i.e., $n \leq 4$. Figure 7.2(a) depicts a faulty 4-cube with $|V \cap W| = |U \cap V| = |U \cap W| = 1$ and $|U \cap V \cap W| = 0$. Figure 7.2(b) is a cube-styled layout isomorphic to Figure 7.2(a). We can examine Figure 7.2(a) in a top-down viewpoint. Since hypercube is node-transitive, we can assume that $\mathbf{u} = t_1$. By link-transitivity, we assume that t_4 and t_5 are faulty neighbors of $\mathbf{u}$. Since $|U \cap V| = 1$, we obtain $\mathbf{v} \in \{t_7, t_8, t_9, t_{10}\}$. Without loss of generality, we assume that $\mathbf{v} = t_{10}$. Since

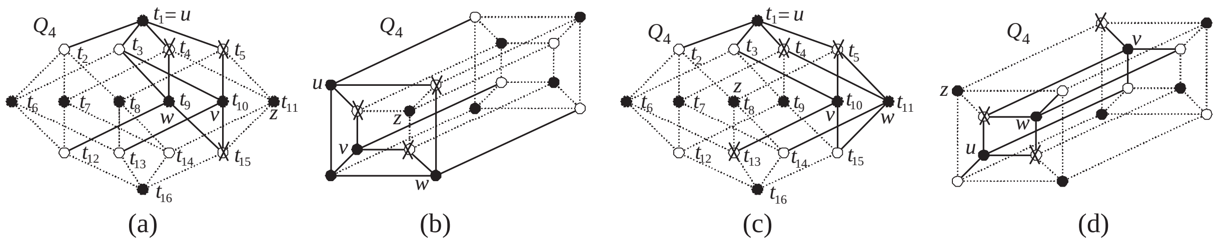

Figure 7.2: Every faulty node is marked by an "X" symbol. (a) The Q_4 with $|N^F_{Q_4}(\mathbf{u}) \cap N^F_{Q_4}(\mathbf{v})| = |N^F_{Q_4}(\mathbf{v}) \cap N^F_{Q_4}(\mathbf{w})| = |N^F_{Q_4}(\mathbf{u}) \cap N^F_{Q_4}(\mathbf{w})| = 1$; (b) a layout isomorphic to (a); (c) the Q_4 with $|N^F_{Q_4}(\mathbf{u}) \cap N^F_{Q_4}(\mathbf{v})| = |N^F_{Q_4}(\mathbf{v}) \cap N^F_{Q_4}(\mathbf{w})| = 1$ and $|N^F_{Q_4}(\mathbf{u}) \cap N^F_{Q_4}(\mathbf{w})| = 2$; (d) a layout isomorphic to (c).

$|U \cap W| = |V \cap W| = 1$ and $|U \cap V \cap W| = 0$, we see that $\mathbf{w} = t_9$ and $V \cap W = \{t_{15}\}$. As a consequence, this happens to be the only possibility. However, node t_{11} has only one fault-free neighbor. Thus it is not conditionally faulty.

II. Secondly, we consider the case that $|V \cap W| = |U \cap V| = 1$ and $|U \cap W| = 2$. By the definition of hypercube, we see that $|N_{Q_n}(\mathbf{u}) \cap N_{Q_n}(\mathbf{v}) \cap N_{Q_n}(\mathbf{w})| \leq 1$. Obviously, we have $|U \cap V \cap W| \leq |N_{Q_n}(\mathbf{u}) \cap N_{Q_n}(\mathbf{v}) \cap N_{Q_n}(\mathbf{w})|$. In particular, we claim that $|U \cap V \cap W| = 1$. Suppose, by contradiction, that $|U \cap V \cap W| = 0$. Then we have $U \cap V \cap W = (U \cap V) \cap (U \cap W) = \emptyset$. Since $U \cap V \neq \emptyset$ and $U \cap W \neq \emptyset$, we conclude that $V \cap W = \emptyset$. That is, the assumption of $|U \cap V \cap W| = 0$ leads to a contradiction between $|V \cap W| = 1$ and $V \cap W = \emptyset$. As a result, $|U \cap V \cap W|$ is equal to 1. Accordingly, we have $2n - 5 \geq f \geq |U \cup V \cup W| = 3(n-2) - (1+1+2) + 1 = 3n - 9$; i.e., $n \leq 4$. See Figure 7.2(c) for illustration. For clarity, Figure 7.2(d) is an isomorphic layout of Figure 7.2(c). Similarly, we can examine Figure 7.2(c) in a top-down viewpoint. By node-transitivity, we assume that $\mathbf{u} = t_1$. By link-transitivity, we assume that t_4 and t_5 are faulty neighbors of $\mathbf{u}$. Since $|U \cap W| = 2$, we have $\mathbf{w} = t_{11}$. Since $|V \cap W| = |U \cap V| = 1$ and $|U \cap V \cap W| = 1$, we obtain $\mathbf{v} \in \{t_7, t_8, t_9, t_{10}\}$. Without loss of generality, we assume that $\mathbf{v} = t_{10}$. Then this turns out to be the only possibility. It is noticed that node t_8 has only two fault-free neighbors.

III. Next, we concern the case that $|V \cap W| = 1$ and $|U \cap V| = |U \cap W| = 2$. Similarly, we have $|U \cap V \cap W| = 1$. Since $(U \cap V) \cup (U \cap W) \subseteq U$, we have $|(U \cap V) \cup (U \cap W)| \leq |U|$. However, we have a contradiction that $|(U \cap V) \cup (U \cap W)| = |U \cap V| + |U \cap W| - |U \cap V \cap W| = 2 + 2 - 1 = 3 > n - 2 = |U|$ if $n \leq 4$. In what follows, we suppose that $n \geq 5$. As a consequence, we have $2n - 5 \geq f \geq |U \cup V \cup W| = 3(n-2) - (1+2+2) + 1 = 3n - 10$; i.e., $n = 5$. See Figure 7.3(a). Again, we examine Figure 7.3(a) in a top-down viewpoint. By node-transitivity, we assume that $\mathbf{u} = t_1$. By link-transitivity, we assume that t_4, t_5, and t_6 are faulty neighbors of $\mathbf{u}$. Since $|U \cap V| = |U \cap W| = 2$, we have $\{\mathbf{v}, \mathbf{w}\} \subset \{t_{14}, t_{15}, t_{16}\}$. Without loss of generality, we assume that $\mathbf{v} = t_{14}$ and $\mathbf{w} = t_{16}$. Since $|V \cap W| = 1$, we have $t_{26} \notin V \cup W$. Moreover, we have $2n - 5 \geq f \geq |V \cup W| = |V| + |W| - |V \cap W| = (n-2) + (n-2) - 1 = 2n - 5$; that is, $f = 2n - 5$ and

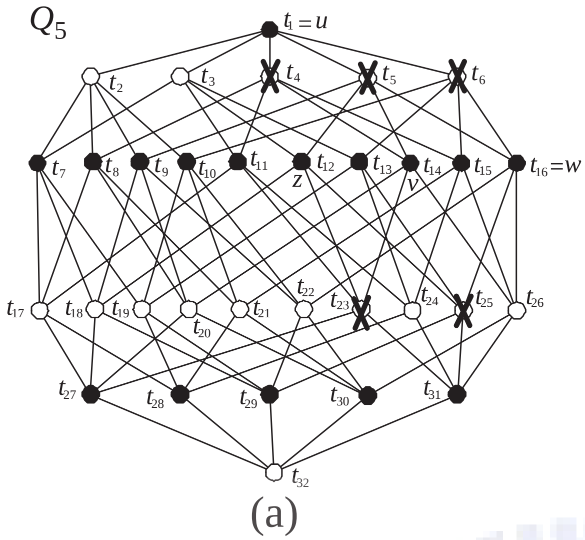

Figure 7.3: Every faulty node is marked by an "X" symbol. Each of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{z}$ has only two fault-free neighbors. (a) The Q_5 with $|N^F_{Q_5}(\mathbf{v}) \cap N^F_{Q_5}(\mathbf{w})| = 1$ and $|N^F_{Q_5}(\mathbf{u}) \cap N^F_{Q_5}(\mathbf{v})| = |N^F_{Q_5}(\mathbf{u}) \cap N^F_{Q_5}(\mathbf{w})| = 2$; (b) the Q_5 with $|N^F_{Q_5}(\mathbf{u}) \cap N^F_{Q_5}(\mathbf{v})| = |N^F_{Q_5}(\mathbf{v}) \cap N^F_{Q_5}(\mathbf{w})| = |N^F_{Q_5}(\mathbf{u}) \cap N^F_{Q_5}(\mathbf{w})| = 2$.

$U \subseteq V \cup W$. Then we have either $t_{20} \in V$ or $t_{23} \in V$. Without loss of generality, we assume that $t_{23} \in V$. Similarly, we can assume that $t_{25} \in W$. As a result, this is the only possibility. It is noted that node $t_{12} = \mathbf{z}$ has three faulty neighbors, and $|N^F_{Q_5}(\mathbf{x})| \leq 2$ for each $\mathbf{x} \in V(Q_5) - \{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$.

Now we consider the case that $|V \cap W| = |U \cap V| = |U \cap W| = 2$. Again, we have $|U \cap V \cap W| = 1$. Since $|(U \cap V) \cup (U \cap W)| \leq |U|$, we still have a contradiction that $|(U \cap V) \cup (U \cap W)| = |U \cap V| + |U \cap W| - |U \cap V \cap W| = 2 + 2 - 1 = 3 > n - 2 = |U|$ if $n \leq 4$. In what follows, we suppose $n \geq 5$. Then we have $2n - 5 \geq f \geq |U \cup V \cup W| = 3(n-2) - (2+2+2) + 1 = 3n - 11$; i.e., $n \in \{5, 6\}$. Note that $|U \cup V \cup W| = 4$ if $n = 5$ and $|U \cup V \cup W| = 7$ if $n = 6$. See Figure 7.3(b) and Figure 7.4(a,b). In Figure 7.3(b), it is not difficult to see that $|N^F_{Q_5}(\mathbf{x})| \leq 2$ for each $\mathbf{x} \in V(Q_5) - \{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$. We explain Figure 7.4 as follows. By node-transitivity, we assume that $\mathbf{u} = t_1$. By link-transitivity, we assume that t_4, t_5, t_6, and t_7 are faulty neighbors of $\mathbf{u}$. Since $|U \cap V| = |U \cap W| = 2$, we deduce that $\{\mathbf{v}, \mathbf{w}\} \subset \{t_i \mid 17 \leq i \leq 22\}$. Since $|U \cap V \cap W| = 1$, we can assume that $\mathbf{v} = t_{20}$ and $\mathbf{w} = t_{22}$. Then we have $|V \cap \{t_{30}, t_{36}, t_{39}, t_{42}\}| = 2$ and $|W \cap \{t_{32}, t_{38}, t_{41}, t_{42}\}| = 2$. Since $|V \cap W| = 2$, we have $V \cap W = \{t_6, t_{42}\}$. If $t_{39} \in V$ and $t_{41} \in W$, then node t_{18} happens to have only two fault-free neighbors (see Figure 7.4(a)); otherwise, we have $|N^F_{Q_6}(\mathbf{x})| \leq 3$ for each $\mathbf{x} \in V(Q_6) - \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ (see Figure 7.4(b), in which nodes t_{36} and t_{41}, for example, are faulty). Hence these figures cover all possibilities.

According to the analysis presented earlier, a conditionally faulty n-cube with $f \leq 2n-5$

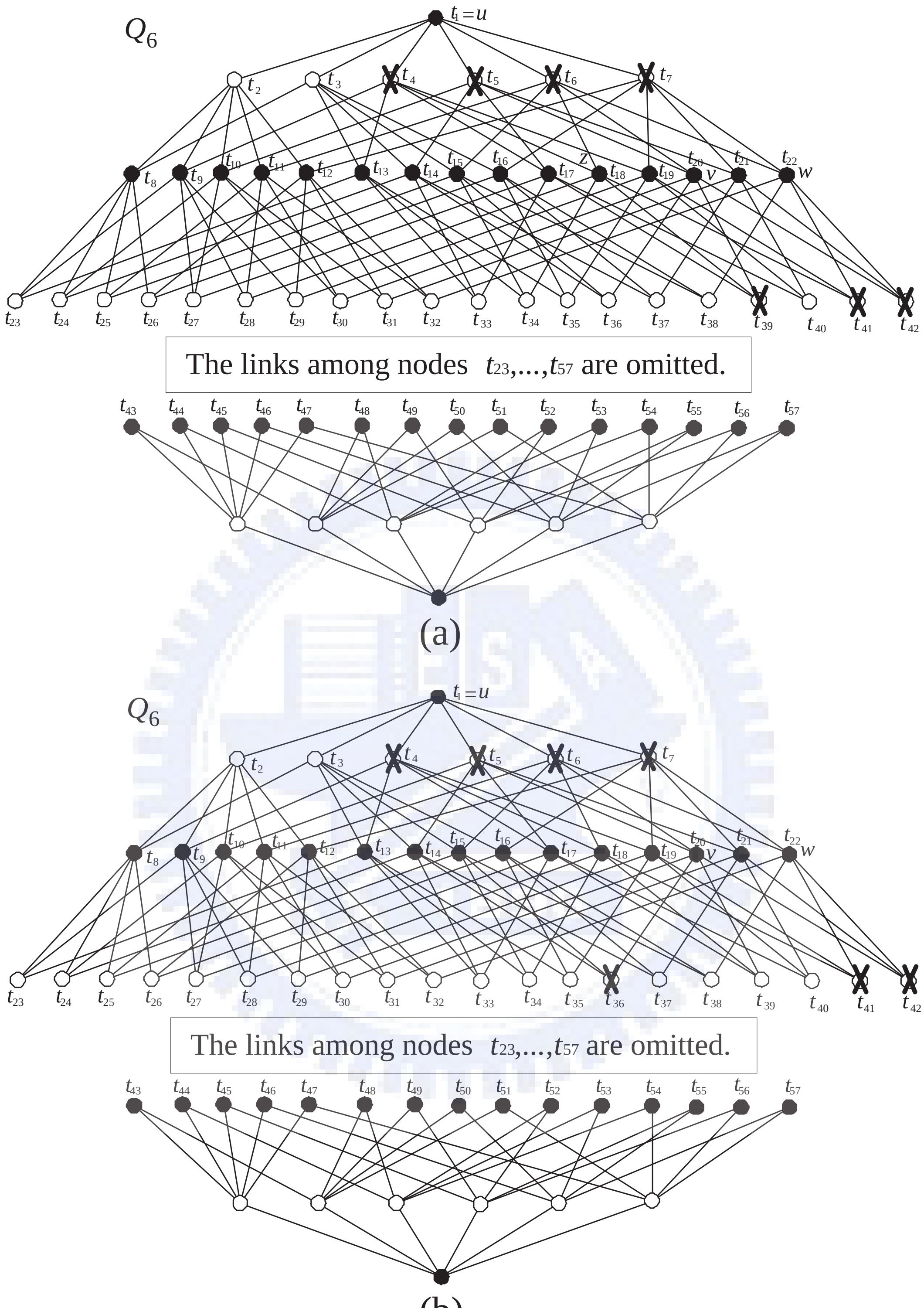

Figure 7.4: Every faulty node is marked by an “X” symbol. The Q_6 with $|N^F_{Q_6}(\mathbf{u}) \cap N^F_{Q_6}(\mathbf{v})| = |N^F_{Q_6}(\mathbf{v}) \cap N^F_{Q_6}(\mathbf{w})| = |N^F_{Q_6}(\mathbf{u}) \cap N^F_{Q_6}(\mathbf{w})| = 2$. (a) $|N^F_{Q_6}(\mathbf{u})| = |N^F_{Q_6}(\mathbf{v})| = |N^F_{Q_6}(\mathbf{w})| = |N^F_{Q_6}(\mathbf{z})| = 4$ and $|N^F_{Q_6}(\mathbf{x})| \leq 3$ for $\mathbf{x} \in V(Q_6) - \{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$; (b) $|N^F_{Q_6}(\mathbf{u})| = |N^F_{Q_6}(\mathbf{v})| = |N^F_{Q_6}(\mathbf{w})| = 4$ and $|N^F_{Q_6}(\mathbf{x})| \leq 3$ for $\mathbf{x} \in V(Q_6) - \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

faulty nodes is likely to contain three or four nodes, every of which has only two fault-free neighbors. Since $2n-5 \leq n-2$ for $n \leq 3$, we concentrate only on the case that $n \geq 4$. To summarize, we have the following two lemmas.

Lemma 7.1. *Suppose that an n-cube Q_n ($n \geq 4$) is conditionally faulty with $f \leq 2n-5$ faulty nodes. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z} \in V(Q_n)$ such that $|N^F_{Q_n}(\mathbf{u})| = |N^F_{Q_n}(\mathbf{v})| = |N^F_{Q_n}(\mathbf{w})| = |N^F_{Q_n}(\mathbf{z})| = n-2$ and $|N^F_{Q_n}(\mathbf{x})| \leq n-3$ for every $\mathbf{x} \in V(Q_n)-\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$. Then the faulty nodes are distributed as illustrated in Figure 7.2(c), Figure 7.3(a,b), and Figure 7.4(a). In Figure 7.2(c) and Figure 7.3(a), no dimensions can be used to partition Q_n in such a way that both resulting subcubes are conditionally faulty. In Figure 7.3(b) and Figure 7.4(a), there exists some dimension j of $\{0, 1, \ldots, n-1\}$ such that both $Q^{j,0}_n$ and $Q^{j,1}_n$ are conditionally faulty with $2n-7$ or less faulty nodes.*

Proof. In Figure 7.2(c) and Figure 7.3(a), we check, by brute force, that either $Q^{k,0}_n$ or $Q^{k,1}_n$ contains a node with only one fault-free neighbor for each $k \in \{0, 1, \ldots, n-1\}$; that is, there does not exist any dimension to partition Q_n such that both $(n-1)$-cubes are conditionally faulty. In Figure 7.3(b) and Figure 7.4(a), let j be any integer of $\{0, 1, \ldots, n-1\}$ such that $(\mathbf{u})^j$ is faulty. Then both $Q^{j,0}_n$ and $Q^{j,1}_n$ are conditionally faulty with $2n-7$ or less faulty nodes. □

Lemma 7.2. *Suppose that an n-cube Q_n ($n \geq 4$) is conditionally faulty with $f \leq 2n-5$ faulty nodes. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V(Q_n)$ such that $|N^F_{Q_n}(\mathbf{u})| = |N^F_{Q_n}(\mathbf{v})| = |N^F_{Q_n}(\mathbf{w})| = n-2$ and $|N^F_{Q_n}(\mathbf{x})| \leq n-3$ for every $\mathbf{x} \in V(Q_n) - \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. Then the faulty nodes are distributed as illustrated in Figure 7.4(b). Moreover, there exists some dimension j of $\{0, 1, \ldots, n-1\}$ such that both $Q^{j,0}_n$ and $Q^{j,1}_n$ are conditionally faulty with $2n-7$ or less faulty nodes.*

Proof. Let $j \in \{0, 1, \ldots, n-1\}$ such that $(\mathbf{u})^j \in N^F_{Q_n}(\mathbf{u}) \cap N^F_{Q_n}(\mathbf{v}) \cap N^F_{Q_n}(\mathbf{w})$. Then both $Q^{j,0}_n$ and $Q^{j,1}_n$ are conditionally faulty with $2n-7$ or less faulty nodes. □

Lemma 7.3. *Suppose that an n-cube Q_n ($n \geq 4$) is conditionally faulty with $f \leq 2n-5$ faulty nodes. Let $\mathbf{u}$ and $\mathbf{v}$ be two nodes of Q_n such that $|N^F_{Q_n}(\mathbf{u})| = |N^F_{Q_n}(\mathbf{v})| = n-2$ and $|N^F_{Q_n}(\mathbf{x})| \leq n-3$ for every $\mathbf{x} \in V(Q_n) - \{\mathbf{u}, \mathbf{v}\}$. Then there exists some dimension k of $\{0, 1, \ldots, n-1\}$ such that both $Q^{k,0}_n$ and $Q^{k,1}_n$ are conditionally faulty. When $n \geq 5$, both $Q^{k,0}_n$ and $Q^{k,1}_n$ contain $2n-7$ or less faulty nodes.*

Proof. Since $|N^F_{Q_n}(\mathbf{u})| = |N^F_{Q_n}(\mathbf{v})| = n-2$ and $f \leq 2n-5$, we have $|N^F_{Q_n}(\mathbf{u}) \cap N^F_{Q_n}(\mathbf{v})| \geq 1$. Since any two nodes of Q_n can have utmost two common neighbors, we consider the following two cases.

Case 1: Suppose that $|N^F_{Q_n}(\mathbf{u}) \cap N^F_{Q_n}(\mathbf{v})| = 2$. Let i and j be two integers such that $\{(\mathbf{u})^i, (\mathbf{u})^j\} = N^F_{Q_n}(\mathbf{u}) \cap N^F_{Q_n}(\mathbf{v})$. Obviously, we have $(\mathbf{u})^i = (\mathbf{v})^j$ and $(\mathbf{u})^j = (\mathbf{v})^i$. Then we can partition Q_n along dimension $k \in \{i, j\}$. As a result, both $Q^{k,0}_n$ and $Q^{k,1}_n$ contain at least $n-3$ faulty nodes. See Figure 7.5(a).

Case 2: Suppose that $|N^F_{Q_n}(\mathbf{u}) \cap N^F_{Q_n}(\mathbf{v})| = 1$. We claim first that this case holds only for $n \geq 5$. By contradiction, we suppose $n = 4$. Let p and q be two integers such that both

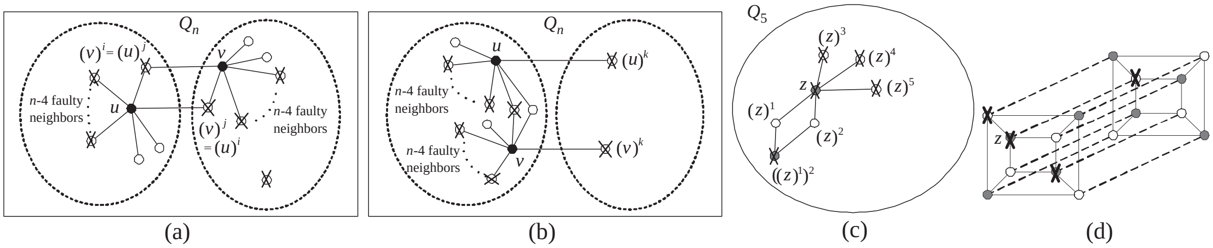

Figure 7.5: Every faulty node is marked by an "X" symbol. (a,b) $|N^F_{Q_n}(\mathbf{u})| = |N^F_{Q_n}(\mathbf{v})| = n-2$ and $|N^F_{Q_n}(\mathbf{x})| \leq n-3$ for $\mathbf{x} \in V(Q_n) - \{\mathbf{u}, \mathbf{v}\}$; (c) a faulty node distribution on Q_5; (d) a conditionally faulty 4-cube with four faulty nodes.

$(\mathbf{u})^p$ and $(\mathbf{u})^q$ are faulty. Since $|N^F_{Q_n}(\mathbf{u}) \cap N^F_{Q_n}(\mathbf{v})| = 1$, we have $\mathbf{v} \neq ((\mathbf{u})^p)^q$. Thus node $((\mathbf{u})^p)^q$ happens to have only two fault-free neighbors, which contradicts the assumption that $|N^F_{Q_n}(\mathbf{x})| \leq n-3$ for every $\mathbf{x} \in V(Q_n) - \{\mathbf{u}, \mathbf{v}\}$.

Let i and j be two integers such that $\{(\mathbf{u})^i\} = \{(\mathbf{v})^j\} = N^F_{Q_n}(\mathbf{u}) \cap N^F_{Q_n}(\mathbf{v})$. Since $|N^F_{Q_n}(\mathbf{u}) - \{(\mathbf{u})^i\}| + |N^F_{Q_n}(\mathbf{v}) - \{(\mathbf{v})^j\}| = 2(n-3) > n-2 = |\{0, 1, \ldots, n-1\} - \{i, j\}|$ for $n \geq 5$, there exists some dimension k of $\{0, 1, \ldots, n-1\} - \{i, j\}$ such that both $(\mathbf{u})^k$ and $(\mathbf{v})^k$ are faulty. As a result, either $Q^{k,0}_n$ or $Q^{k,1}_n$ contains exactly two faulty nodes. See Figure 7.5(b).

In either case, both $Q^{k,0}_n$ and $Q^{k,1}_n$ are conditionally faulty. □

Lemma 7.4. *Suppose that an n-cube Q_n ($n \geq 4$) is conditionally faulty with $f \leq 2n-5$ faulty nodes. Let $\mathbf{z}$ be a unique node with exactly $n-2$ faulty neighbors. Then there exists some dimension j of $\{0, 1, \ldots, n-1\}$ such that both $Q^{j,0}_n$ and $Q^{j,1}_n$ are conditionally faulty. Except for the case depicted in Figure 7.5(c), both $Q^{j,0}_n$ and $Q^{j,1}_n$ contain $2n-7$ or less faulty nodes if $n \geq 5$.*

Proof. Since Q_n is node-transitive, we assume $\mathbf{z} = 0^n$. Since Q_n is also link-transitive, we assume that $(\mathbf{z})^0$ and $(\mathbf{z})^1$ are fault-free. Because $\mathbf{z}$ is a unique node with exactly $n-2$ faulty neighbors, we have $|N^F_{Q_n}(\mathbf{x})| \leq n-3$ for $\mathbf{x} \in V(Q_n) - \{\mathbf{z}\}$. For every $k \in \{2, 3, \ldots, n-1\}$, we have $N^F_{Q^{k,0}_n}(\mathbf{x}) \subseteq N^F_{Q_n}(\mathbf{x})$ and $N^F_{Q^{k,1}_n}(\mathbf{y}) \subseteq N^F_{Q_n}(\mathbf{y})$ for $\mathbf{x} \in V(Q^{k,0}_n) - \{\mathbf{z}\}$ and $\mathbf{y} \in V(Q^{k,1}_n)$. Thus we obtain $|N^F_{Q^{k,0}_n}(\mathbf{x})| \leq |N^F_{Q_n}(\mathbf{x})| \leq n-3$ and $|N^F_{Q^{k,1}_n}(\mathbf{y})| \leq |N^F_{Q_n}(\mathbf{y})| \leq n-3$ for $\mathbf{x} \in V(Q^{k,0}_n) - \{\mathbf{z}\}$ and $\mathbf{y} \in V(Q^{k,1}_n)$. In addition, we have $|N^F_{Q^{k,0}_n}(\mathbf{z})| = (n-2) - 1 = n-3$ for every $k \in \{2, 3, \ldots, n-1\}$. Let j be an integer of $\{2, 3, \ldots, n-1\}$. Then both $Q^{j,0}_n$ and $Q^{j,1}_n$ are conditionally faulty.

Suppose $f \leq 2n-6$. We see that, for any $j \in \{2, 3, \ldots, n-1\}$, both $Q^{j,0}_n$ and $Q^{j,1}_n$ contain $2n-7$ or less faulty nodes.

Suppose $f = 2n-5$. We assume, by contraposition, that either $Q^{j,0}_n$ or $Q^{j,1}_n$ contains $2n-6$ faulty nodes for any $j \in \{2, 3, \ldots, n-1\}$. Then, for any $\mathbf{x}$ of $F(Q_n) - \{(\mathbf{z})^k \mid 2 \leq k \leq n-1\}$, we have $(\mathbf{x})_j = (\mathbf{z})_j$ for every $j \in \{2, 3, \ldots, n-1\}$. Hence we have $F(Q_n) - \{(\mathbf{z})^k \mid 2 \le k \le n-1\} \subseteq \{\mathbf{z}, ((\mathbf{z})^0)^1\}$. Since $|F(Q_n) - \{(\mathbf{z})^k \mid 2 \le k \le n-1\}| = f - (n-2) = n - 3 \le 2 = |\{\mathbf{z}, ((\mathbf{z})^0)^1\}|$, we derive that $n \le 5$. That is, if $n \ge 6$, there exists some dimension j of $\{2, 3, \ldots, n-1\}$ such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty with $2n-7$ or less faulty nodes. Since $|F(Q_n) - \{(\mathbf{z})^k \mid 2 \le k \le n-1\}| = 2$ for $n = 5$, nodes $\mathbf{z}$ and $((\mathbf{z})^0)^1$ are faulty; that is, $F(Q_5) = \{\mathbf{z}, (\mathbf{z})^2, (\mathbf{z})^3, (\mathbf{z})^4, ((\mathbf{z})^0)^1\}$, as shown in Figure 7.5(c). Therefore, Figure 7.5(c) happens to be the only possibility that either $Q_n^{j,0}$ or $Q_n^{j,1}$ contains $2n-6$ faulty nodes for every $j \in \{2, 3, \ldots, n-1\}$. □

Lemma 7.5. *Suppose that an n-cube Q_n ($n \ge 4$) contains $f \le 2n-5$ faulty nodes such that every node has at least three fault-free neighbors. Then there exists some dimension j of $\{0, 1, \ldots, n-1\}$ such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty. For $n \ge 5$, both $Q_n^{j,0}$ and $Q_n^{j,1}$ contain $2n-7$ or less faulty nodes.*

Proof. Since every node has at least three fault-free neighbors, every $(n-1)$-dimensional subcube of Q_n is conditionally faulty. First, we consider the case that $f \le 2n-6$. Let $\mathbf{u}$ and $\mathbf{v}$ be two distinct faulty nodes, and let $j \in \{0, 1, \ldots, n-1\}$ such that $(\mathbf{u})_j \ne (\mathbf{v})_j$. Then both $Q_n^{j,0}$ and $Q_n^{j,1}$ contain $2n-7$ or less faulty nodes.

Now we consider the case that $f = 2n-5$. For $n \ge 5$, we claim that there exists some dimension j of $\{0, 1, \ldots, n-1\}$ such that $|F(Q_n^{j,0})| \le 2n-7$ and $|F(Q_n^{j,1})| \le 2n-7$. For $0 \le k \le n-1$, we define that $q_k = 1$ if $(\mathbf{u})_k = (\mathbf{v})_k$ for every two distinct faulty nodes $\mathbf{u}, \mathbf{v} \in F(Q_n)$, and $q_k = 0$ otherwise. Let $q = \sum_{k=1}^{n} q_k$. Clearly, all faulty nodes are located in either $Q_n^{k,0}$ or $Q_n^{k,1}$ if $q_k = 1$. For convenience, let $\{0 \le k \le n-1 \mid q_k = 0\} = \{i_1, \ldots, i_{n-q}\}$. Then both $Q_n^{j,0}$ and $Q_n^{j,1}$ contain at least one faulty node for $j \in \{i_1, \ldots, i_{n-q}\}$.

Suppose, by contradiction, either $Q_n^{j,0}$ or $Q_n^{j,1}$ contains only one faulty node for every $j \in \{i_1, \ldots, i_{n-q}\}$. For $\mathbf{v} \in F(Q_n)$, let $A(\mathbf{v}) = \{0 \le k \le n-1 \mid F(Q_n^{k,0}) = \{\mathbf{v}\}$ or $F(Q_n^{k,1}) = \{\mathbf{v}\}\}$. Since Q_n is node-transitive, we assume that $\mathbf{e} = 0^n$ is a faulty node such that $|A(\mathbf{e})|$ achieves the maximum of set $\{|A(\mathbf{v})| \mid \mathbf{v} \in F(Q_n)\}$. For convenience, let $p = |A(\mathbf{e})|$. Obviously, we have $1 \le p \le n-q$. Moreover, let $A(\mathbf{e}) = \{i_1, \ldots, i_p\}$. For $\mathbf{v} \in F(Q_n) - \{\mathbf{e}\}$, we see that $(\mathbf{v})_k = 1$ for each $k \in \{i_1, \ldots, i_p\}$. Let $B(k) = \{\mathbf{v} \in F(Q_n) - \{\mathbf{e}\} \mid (\mathbf{v})_k \ne (\mathbf{e})_k\}$ for $k \in \{i_{p+1}, \ldots, i_{n-q}\}$. Since we assumed, by contradiction, that either $Q_n^{j,0}$ or $Q_n^{j,1}$ has only one faulty node for each $j \in \{i_1, \ldots, i_{n-q}\}$, we have $|B(j)| = 1$ for each $j \in \{i_{p+1}, \ldots, i_{n-q}\}$. Since Q_n is link-transitive, we assume that $\{i_1, \ldots, i_p\} = \{0, \ldots, p-1\}$ and $\{i_{p+1}, \ldots, i_{n-q}\} = \{p, \ldots, n-q-1\}$. Then we have $(F(Q_n) - \{\mathbf{e}\}) - \bigcup_{k \in \{i_{p+1}, \ldots, i_{n-q}\}} B(k) \subseteq \{0^{n-p}1^p\}$. Accordingly, we derive that $1 = |\{0^{n-p}1^p\}| \ge |(F(Q_n) - \{\mathbf{e}\}) - \bigcup_{k \in \{i_{p+1}, \ldots, i_{n-q}\}} B(k)| \ge |F(Q_n)| - |\{\mathbf{e}\}| - \sum_{k \in \{i_{p+1}, \ldots, i_{n-q}\}} |B(k)| = (2n-5) - 1 - (n-q-p)$; that is, $p + q \le 7 - n$.

Recall that $p \ge 1$ and $q \ge 0$. Thus, we have $n \in \{5, 6\}$. Now we can identify all faulty nodes according to the values of p, q, and n.

Case 1: Suppose $(n, q, p) = (5, 0, 1)$. Since $p = 1$, we have $(\mathbf{v})_0 = 1$ for each $\mathbf{v} \in F(Q_5) - \{\mathbf{e}\}$ and $|B(j)| = 1$ for each $j \in \{1, 2, 3, 4\}$. Thus we have $F(Q_5) = \{00000, 00011, 00101, 01001, 10001\}$. Clearly, node 00001 has five faulty neighbors.

Case 2: Suppose $(n, q, p) = (5, 0, 2)$. Similarly, we have $F(Q_5) = \{00000, 00111, 01011, 10011, 00011\}$. Then node 00011 has three faulty neighbors.

Case 3: Suppose $(n, q, p) = (5, 1, 1)$. We have $F(Q_5) = \{00000, 00011, 00101, 01001, 00001\}$. Again, node 00001 has four faulty neighbors.

Case 4: Suppose $(n, q, p) = (6, 0, 1)$. We have $F(Q_6) = \{000000, 000011, 000101, 001001, 010001, 100001, 000001\}$. Thus, node 000001 has six faulty neighbors.

In short, node $0^{n-p}1^p$ has at least $n-2$ faulty neighbors, which contradicts the requirement that every node has at least three fault-free neighbors. Hence there exists some dimension j of $\{0, 1, \ldots, n-1\}$ such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty with $2n-7$ or less faulty nodes. □

Suppose that Q_n is conditionally faulty with utmost $2n-5$ faulty nodes. Let $F = F(Q_n)$. For $n \geq 5$, we propose a procedure *PARTITION*(Q_n, F) to determine j-partition of Q_n according to the following rules:

(1) Suppose that at least three nodes of Q_n have exactly $n-2$ faulty neighbors, respectively. If Q_n has its faulty nodes distributed as shown in Figure 7.3(a), it will be partitioned along dimension $j = dim((t_1, t_5))$. Then one resulting subcube has its faulty nodes distributed as in Figure 7.2(b). Otherwise, Lemma 7.1 and Lemma 7.2 ensure that Q_n can be partitioned along some dimension j such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty with $2n-7$ or less faulty nodes.

(2) Suppose that there exist exactly two nodes of Q_n with $n-2$ faulty neighbors, respectively. By Lemma 7.3, there exists some dimension j of $\{0, 1, \ldots, n-1\}$ such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty with $2n-7$ or less faulty nodes.

(3) Suppose that there is only one node of Q_n with exactly $n-2$ faulty neighbors. Denote it by $\mathbf{z}$. If the faulty nodes are distributed as in Figure 7.5(c), we partition Q_n along any dimension $j \in \{i \mid (\mathbf{z})^i \text{ is faulty}\}$. Then one resulting subcube turns out to have $2n-6$ faulty nodes, distributed as in Figure 7.5(d). Otherwise, we can apply Lemma 7.4 to choose a dimension j of $\{0, 1, \ldots, n-1\}$ such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty with $2n-7$ or less faulty nodes.

(4) Suppose that every node of Q_n has at least three fault-free neighbors. Obviously, every $(n-1)$-cube is conditionally faulty. By Lemma 7.5, there exists some dimension j of $\{0, 1, \ldots, n-1\}$ such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ contain $2n-7$ or less faulty nodes.

The following corollary summarizes what is obtained by procedure *PARTITION*(Q_n, F). Also, it is a summary of Lemmas 7.1−7.5.

Corollary 7.1. *Suppose that an n-cube Q_n ($n \geq 5$) is conditionally faulty with $f \leq 2n-5$ faulty nodes. Except for the cases illustrated in Figure 7.2(c), Figure 7.3(a), and Figure 7.5(c), there exists some dimension j of $\{0,1,\ldots,n-1\}$ such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty with $2n-7$ or less faulty nodes.*

7.2 Long path embedding in faulty hypercubes

The following theorem was proved by Fu [23].

Theorem 7.1. *[23] Suppose that $n \geq 3$. Let $\mathbf{u}$ and $\mathbf{v}$ denote two arbitrary fault-free nodes of an n-cube with $f \leq n-2$ faulty nodes. If $h(\mathbf{u},\mathbf{v})$ is odd (or even), then there exists a fault-free path of length at least $2^n - 2f - 1$ (or $2^n - 2f - 2$) between $\mathbf{u}$ and $\mathbf{v}$.*

To improve the above result, we need the following lemma.

Lemma 7.6. *Let $z \in V(Q_4)$, $\{i,j,p,q\} = \{0,1,2,3\}$, and $F = \{(\mathbf{z})^i, (\mathbf{z})^j, (\mathbf{z})^p\}$. Suppose that $\mathbf{s}$ and $\mathbf{t}$ are any two nodes of $Q_4 - F$ such that $\{\mathbf{s},\mathbf{t}\} \neq \{\mathbf{z},(\mathbf{z})^q\}$. Then $Q_4 - F$ has a path of length at least 9 or 8 between $\mathbf{s}$ and $\mathbf{t}$ if $h(\mathbf{s},\mathbf{t})$ is odd or even, respectively.*

Proof. By symmetry, let $\mathbf{z} = 0000$, $i=0$, $j=1$, $p=2$, and $q=3$. We partition Q_4 into $Q_4^{3,0}$ and $Q_4^{3,1}$. Then $Q_4^{3,1}$ is fault-free and $\mathbf{z} \in V_0(Q_4^{3,0})$.

Case 1: Both $\mathbf{s}$ and $\mathbf{t}$ are in $Q_4^{3,0} - F$. Since $Q_4^{3,1}$ is fault-free, Theorem 4.2 ensures that $Q_4^{3,1}$ contains a path P of length 7 (respectively, 6) between $(\mathbf{s})^3$ and $(\mathbf{t})^3$ if $h(\mathbf{s},\mathbf{t})$ is odd (respectively, even). Thus, $\langle \mathbf{s}, (\mathbf{s})^3, P, (\mathbf{t})^3, \mathbf{t} \rangle$ is a fault-free path of length 9 (respectively, 8) between $\mathbf{s}$ and $\mathbf{t}$ if $h(\mathbf{s},\mathbf{t})$ is odd (respectively, even).

Case 2: Both $\mathbf{s}$ and $\mathbf{t}$ are in $Q_4^{3,1}$. If $h(\mathbf{s},\mathbf{t})$ is odd, Theorem 4.2 ensures that $Q_4^{3,1} - \{(1101,1111)\}$ contains a path P of length 7 between $\mathbf{s}$ and $\mathbf{t}$. Clearly, path P does not pass through $(1101,1111)$. Since it spans $Q_4^{3,1}$, we have $1111 \in V(P)$. Accordingly, link $(1110,1111)$ or $(1011,1111)$ is on P. Thus P can be written as $\langle \mathbf{s}, R_1, 1110, 1111, R_2, \mathbf{t} \rangle$ or $\langle \mathbf{s}, T_1, 1011, 1111, T_2, \mathbf{t} \rangle$. As a result, $\langle \mathbf{s}, R_1, 1110, 0110, 0111, 1111, R_2, \mathbf{t} \rangle$ or $\langle \mathbf{s}, T_1, 1011, 0011, 0111, 1111, T_2, \mathbf{t} \rangle$ is a path of length 9 between $\mathbf{s}$ and $\mathbf{t}$. On the other hand, if $h(\mathbf{s},\mathbf{t})$ is even, then we consider two cases as follows. Suppose first that $\mathbf{s},\mathbf{t} \in V_0(Q_4^{3,1})$. By Theorem 4.2, $Q_4^{3,1} - \{(1101,1111)\}$ contains a path P of length 6 between $\mathbf{s}$ and $\mathbf{t}$. Again, link $(1110,1111)$ or $(1011,1111)$ is on P, and thus the desired path can be constructed as above. Suppose that $\mathbf{s},\mathbf{t} \in V_1(Q_4^{3,1})$. By Theorem 4.3, $Q_4^{3,1} - \{1001\}$ contains a path P of length 6 between $\mathbf{s}$ and $\mathbf{t}$. Obviously, link $(1110,1111)$, $(1101,1111)$, or $(1011,1111)$ is on P. Hence the desired path can be constructed similarly.

Case 3: Suppose that $\mathbf{s}$ is in $Q_4^{3,0} - F$ and $\mathbf{t}$ is in $Q_4^{3,1}$. First, we consider the case that $\mathbf{s} \neq \mathbf{z}$. If $\mathbf{s} \in V_0(Q_4)$, then $\mathbf{s}$ is adjacent to node 0111. Clearly, there exists some node $\mathbf{v}$ of $\{0110, 0101, 0011\} - \{\mathbf{s}\}$ such that $(\mathbf{v})^3 \neq \mathbf{t}$. By Theorem 4.2, $Q_4^{3,1}$ has a path P of length 6 or 7 between $(\mathbf{v})^3$ and $\mathbf{t}$ if $h(\mathbf{s},\mathbf{t})$ is odd or even, respectively. Then $\langle \mathbf{s}, 0111, \mathbf{v}, (\mathbf{v})^3, P, \mathbf{t} \rangle$ is a fault-free path of length 9 or 10 if $h(\mathbf{s},\mathbf{t})$ is odd or even, respectively. If $\mathbf{s} \in V_1(Q_4)$, then we have $\mathbf{s} = 0111$. Obviously, there exists some node $\mathbf{u}$ of $\{0110, 0101, 0011\}$ such that $(\mathbf{u})^3 \neq \mathbf{t}$. Similarly, $Q_4^{3,1}$ has a path T of length 7 (respectively, 6) between $(\mathbf{u})^3$ and $\mathbf{t}$ if $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even). Then $\langle \mathbf{s}, \mathbf{u}, (\mathbf{u})^3, T, \mathbf{t} \rangle$ is a fault-free path of length 9 (respectively, 8) if $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even).

Next, we consider the case that $\mathbf{s} = \mathbf{z}$. If $h(\mathbf{s}, \mathbf{t})$ is even, it follows from Theorem 4.2 that $Q_4^{3,1}$ has a path H of length 7 between $(\mathbf{s})^3 = (\mathbf{z})^3$ and $\mathbf{t}$. Then $\langle \mathbf{s} = \mathbf{z}, (\mathbf{z})^3, H, \mathbf{t} \rangle$ is a fault-free path of length 8. If $h(\mathbf{s}, \mathbf{t})$ is odd, Theorem 4.3 ensures that $Q_4^{3,1} - \{1100\}$ has a path R of length 6 between $(\mathbf{z})^3$ and $\mathbf{t}$. Clearly, node 1111 is on R. Accordingly, link $(1111, 1110)$, $(1111, 1101)$, or $(1111, 1011)$ is on R. For example, path R can be written as $\langle (\mathbf{z})^3, R_1, 1111, 1110, R_2, \mathbf{t} \rangle$ if $(1111, 1110) \in E(R)$. Then $\langle \mathbf{s} = \mathbf{z}, (\mathbf{z})^3, R_1, 1111, 0111, 0110, 1110, R_2, \mathbf{t} \rangle$ is a fault-free path of length 9 between $\mathbf{s}$ and $\mathbf{t}$. □

Lemma 7.7. *Suppose that Q_3 is conditionally faulty with $f \leq 2$ faulty nodes. Let $\mathbf{s}$ and $\mathbf{t}$ denote any two fault-free nodes of Q_3. Then Q_3 contains a fault-free path of length at least $7 - 2f$ (respectively, $6 - 2f$) between $\mathbf{s}$ and $\mathbf{t}$ if $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even).*

Proof. If $f < 2$, this result follows from Theorem 7.1. Thus we only consider the case that $f = 2$. For convenience, let $F = F(Q_3)$. Since Q_3 is node-transitive, we assume that node 000 is faulty. To require that every node of Q_3 has at least two fault-free neighbors, the other faulty node must be one of $\{001, 010, 100, 111\}$.

Case 1: One of $\{001, 010, 100\}$ is faulty. Obviously, each of $\{001, 010, 100\}$ is adjacent to 000. Since Q_3 is link-transitive, we assume that $001 \in F$; that is, $F = \{000, 001\}$. Then we partition Q_3 into $Q_3^{1,0}$ and $Q_3^{1,1}$. Hence we have $F \subseteq V(Q_3^{1,0})$. See Figure 7.6(a).

Subcase 1.1: Both $\mathbf{s}$ and $\mathbf{t}$ are in $Q_3^{1,0} - F$. Without loss of generality, we assume that $\mathbf{s} = 101$ and $\mathbf{t} = 100$. Obviously, $\langle \mathbf{s} = 101, 111, 110, 100 = \mathbf{t} \rangle$ is a fault-free path of length $3 = 7 - 2 \cdot 2$.

Subcase 1.2: Both $\mathbf{s}$ and $\mathbf{t}$ are in $Q_3^{1,1}$. If $h(\mathbf{s}, \mathbf{t})$ is odd, then $Q_3^{1,1}$ contains a path of length 3 between $\mathbf{s}$ and $\mathbf{t}$. Otherwise, $Q_3^{1,1}$ contains a path of length 2 between $\mathbf{s}$ and $\mathbf{t}$.

Subcase 1.3: Suppose that $\mathbf{s}$ is in $Q_3^{1,0} - F$ and $\mathbf{t}$ is in $Q_3^{1,1}$. Without loss of generality, we assume $\mathbf{s} = 101$ and list the required path in Table 7.1.

Case 2: Node 111 is faulty. See Figure 7.6(b) for illustration.

Subcase 2.1: Both $\mathbf{s}$ and $\mathbf{t}$ are in $Q_3^{1,0} - \{000\}$. For every possible combination of $\mathbf{s}$ and $\mathbf{t}$, we list the required paths in Table 7.1.

Subcase 2.2: Both $\mathbf{s}$ and $\mathbf{t}$ are in $Q_3^{1,1} - \{111\}$. This subcase is symmetric to Subcase 2.1.

Table 7.1: The required paths for Lemma 7.7 and Lemma 7.8.

Subcase 1.3 of Lemma 7.7		
$\mathbf{s} = 101$	$\mathbf{t} = 010$	$\langle \mathbf{s} = 101, 100, 110, 010 = \mathbf{t} \rangle$
	$\mathbf{t} = 011$	$\langle \mathbf{s} = 101, 111, 011 = \mathbf{t} \rangle$
	$\mathbf{t} = 110$	$\langle \mathbf{s} = 101, 100, 110 = \mathbf{t} \rangle$
	$\mathbf{t} = 111$	$\langle \mathbf{s} = 101, 100, 110, 111 = \mathbf{t} \rangle$
Subcase 2.1 of Lemma 7.7		
$\mathbf{s} = 101$	$\mathbf{t} = 001$	$\langle \mathbf{s} = 101, 100, 110, 010, 011, 001 = \mathbf{t} \rangle$
	$\mathbf{t} = 100$	$\langle \mathbf{s} = 101, 001, 011, 010, 110, 100 = \mathbf{t} \rangle$
$\mathbf{s} = 001$	$\mathbf{t} = 100$	$\langle \mathbf{s} = 001, 011, 010, 110, 100 = \mathbf{t} \rangle$
Subcase 2.3 of Lemma 7.7		
$\mathbf{s} = 001$	$\mathbf{t} = 010$	$\langle \mathbf{s} = 001, 011, 010 = \mathbf{t} \rangle$
	$\mathbf{t} = 011$	$\langle \mathbf{s} = 001, 101, 100, 110, 010, 011 = \mathbf{t} \rangle$
	$\mathbf{t} = 110$	$\langle \mathbf{s} = 001, 011, 010, 110 = \mathbf{t} \rangle$
$\mathbf{s} = 100$	$\mathbf{t} = 010$	$\langle \mathbf{s} = 100, 110, 010 = \mathbf{t} \rangle$
	$\mathbf{t} = 011$	$\langle \mathbf{s} = 100, 110, 010, 011 = \mathbf{t} \rangle$
	$\mathbf{t} = 110$	$\langle \mathbf{s} = 100, 101, 001, 011, 010, 110 = \mathbf{t} \rangle$
$\mathbf{s} = 101$	$\mathbf{t} = 010$	$\langle \mathbf{s} = 101, 100, 110, 010 = \mathbf{t} \rangle$
	$\mathbf{t} = 011$	$\langle \mathbf{s} = 101, 001, 011 = \mathbf{t} \rangle$
	$\mathbf{t} = 110$	$\langle \mathbf{s} = 101, 100, 110 = \mathbf{t} \rangle$
Lemma 7.8		
$\mathbf{b}_1 = 001$	$\mathbf{b}_2 = 010$	$\langle \mathbf{b}_1 = 001, 101, 100, 110, 010 = \mathbf{b}_2 \rangle$
	$\mathbf{b}_2 = 100$	$\langle \mathbf{b}_1 = 001, 101, 111, 110, 100 = \mathbf{b}_2 \rangle$
	$\mathbf{b}_2 = 111$	$\langle \mathbf{b}_1 = 001, 101, 100, 110, 111 = \mathbf{b}_2 \rangle$
$\mathbf{b}_1 = 010$	$\mathbf{b}_2 = 100$	$\langle \mathbf{b}_1 = 010, 110, 111, 101, 100 = \mathbf{b}_2 \rangle$
	$\mathbf{b}_2 = 111$	$\langle \mathbf{b}_1 = 010, 110, 100, 101, 111 = \mathbf{b}_2 \rangle$

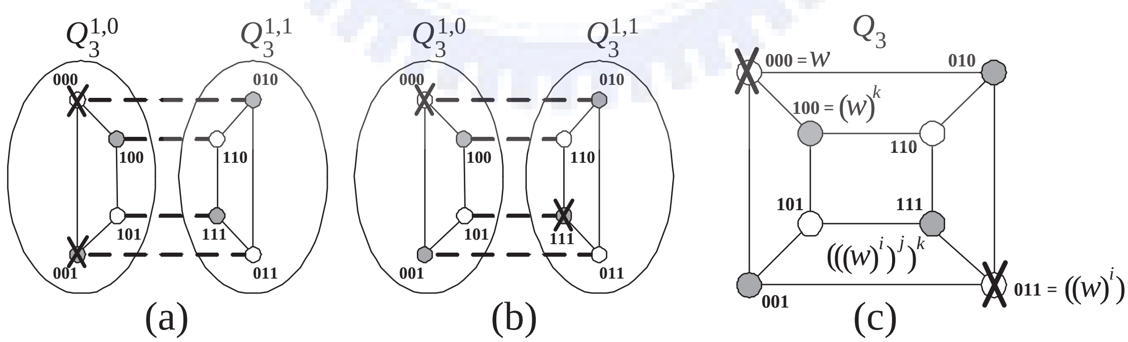

Figure 7.6: (a,b) Illustrations for Lemma 7.7; (c) the distribution of faulty nodes indicated in Lemma 7.8.

Subcase 2.3: Suppose that $\mathbf{s}$ is in $Q_3^{1,0} - \{000\}$ and $\mathbf{t}$ is in $Q_3^{1,1} - \{111\}$. For every possible combination of $\mathbf{s}$ and $\mathbf{t}$, we list the required paths in Table 7.1.

In summary, $Q_3 - F$ contains a path of length at least $7 - 2f$ (respectively, $6 - 2f$) between $\mathbf{s}$ and $\mathbf{t}$ if $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even). □

Lemma 7.8. *Let $\mathbf{w} \in V_0(Q_3)$ and $\{i, j, k\} = \{0, 1, 2\}$. Suppose that $\mathbf{b}_1$ and $\mathbf{b}_2$ are two arbitrary nodes of $V_1(Q_3)$. Then $Q_3 - \{\mathbf{w}, ((\mathbf{w})^i)^j\}$ contains a path of length 4 between $\mathbf{b}_1$ and $\mathbf{b}_2$ if and only if $\{\mathbf{b}_1, \mathbf{b}_2\} \neq \{(\mathbf{w})^k, (((\mathbf{w})^i)^j)^k\}$.*

Proof. Since Q_3 is node-transitive and link-transitive, we assume that $\mathbf{w} = 000$, $i = 0$, $j = 1$, and $k = 2$. See Figure 7.6(c). Then we list all the required paths in Table 7.1. □

Theorem 7.2. *Let F be a set of $f \leq 3$ faulty nodes in Q_4 such that every node of Q_4 has at least two fault-free neighbors. Suppose that $\mathbf{s}$ and $\mathbf{t}$ are two arbitrary nodes of $Q_4 - F$. Then $Q_4 - F$ contains a path of length at least $15 - 2f$ (respectively, $14 - 2f$) between $\mathbf{s}$ and $\mathbf{t}$ if $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even).*

Proof. If $f < 3$, this result follows from Theorem 7.1. Thus we concentrate only on the case that $f = 3$. By Lemmas 7.1–7.5, Figure 7.2(c) happens to be a unique case that a conditionally faulty Q_4 with three faulty nodes cannot be partitioned along any dimension in such a way that both subcubes are conditionally faulty. On this occasion, we partition Q_4 along an arbitrary dimension j; otherwise, there exists some dimension j such that both $Q_4^{j,0}$ and $Q_4^{j,1}$ are conditionally faulty.

Case 1: Both $Q_4^{j,0}$ and $Q_4^{j,1}$ are conditionally faulty. For convenience, let $F_0 = F(Q_4^{j,0})$ and $F_1 = F(Q_4^{j,1})$. Without loss of generality, we assume that $f_0 = |F_0| = 2$ and $f_1 = |F_1| = 1$. Moreover, we assume $\mathbf{s} \in V_0(Q_4 - F)$.

Subcase 1.1: Both $\mathbf{s}$ and $\mathbf{t}$ are in $Q_4^{j,0}$. By Lemma 7.7, $Q_4^{j,0} - F_0$ contains a path H_0 of length at least $3 = 7 - 2f_0$ (respectively, $2 = 6 - 2f_0$) between $\mathbf{s}$ and $\mathbf{t}$ if $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even). Obviously, H_0 can be written as $\langle \mathbf{s} = \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, H_0', \mathbf{t} \rangle$. If $(\mathbf{x}_1)^j$ is faulty, then $(\mathbf{x}_0)^j$ and $(\mathbf{x}_2)^j$ are fault-free. By Theorem 4.3, $Q_4^{j,1}$ is hyper-hamiltonian laceable. Thus $Q_4^{j,1} - \{(\mathbf{x}_1)^j\}$ has a hamiltonian path H_1 between $(\mathbf{x}_0)^j$ and $(\mathbf{x}_2)^j$. As a result, $\langle \mathbf{s} = \mathbf{x}_0, (\mathbf{x}_0)^j, H_1, (\mathbf{x}_2)^j, \mathbf{x}_2, H_0', \mathbf{t} \rangle$ is a fault-free path of length at least $15 - 2f$ (respectively, $14 - 2f$) when $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even). If $(\mathbf{x}_1)^j$ is fault-free, then $(\mathbf{x}_0)^j$ or $(\mathbf{x}_2)^j$ is fault-free. Suppose, for example, that $(\mathbf{x}_0)^j$ is fault-free. By Lemma 7.7, $Q_4^{j,1} - F_1$ has a fault-free path H_1 of length at least $7 - 2f_1$ between $(\mathbf{x}_0)^j$ and $(\mathbf{x}_1)^j$. As a result, $\langle \mathbf{s} = \mathbf{x}_0, (\mathbf{x}_0)^j, H_1, (\mathbf{x}_1)^j, \mathbf{x}_1, \mathbf{x}_2, H_0', t \rangle$ is a fault-free path of length at least $15 - 2f$ (respectively, $14 - 2f$) when $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even).

Subcase 1.2: Both $\mathbf{s}$ and $\mathbf{t}$ are in $Q_4^{j,1}$. First, we consider the case that $h(\mathbf{s}, \mathbf{t})$ is odd. By Lemma 7.7, $Q_4^{j,1} - F_1$ contains a path T_1 of length at least $5 = 7 - 2f_1$ between $\mathbf{s}$ and $\mathbf{t}$. Let $A = \{(T_1(i), T_1(i+1)) \mid 1 \leq i \leq 5 \text{ and } i \equiv 1 \pmod 2\}$ be a set of disjoint links on T_1. Since $|A| = 3 > f_0$, there exists an odd integer $\hat{\imath}$, $1 \leq \hat{\imath} \leq 5$, such that both $(T_1(\hat{\imath}))^j$ and

$(T_1(\hat{\imath}+1))^j$ are fault-free. Let $\mathbf{w} = T_1(\hat{\imath})$ and $\mathbf{b} = T_1(\hat{\imath}+1)$. Accordingly, T_1 can be written as $\langle \mathbf{s}, T_1', \mathbf{w}, \mathbf{b}, T_1'', \mathbf{t} \rangle$. By Lemma 7.7, $Q_4^{j,0} - F_0$ has a path T_0 of length at least $7 - 2f_0$ between $(\mathbf{w})^j$ and $(\mathbf{b})^j$. As a result, $\langle \mathbf{s}, T_1', \mathbf{w}, (\mathbf{w})^j, T_0, (\mathbf{b})^j, \mathbf{b}, T_1'', \mathbf{t} \rangle$ is a fault-free path of length at least $15 - 2f$ between $\mathbf{s}$ and $\mathbf{t}$.

Next, we consider the case that $h(\mathbf{s}, \mathbf{t})$ is even. Hence we have $\mathbf{t} \in V_0(Q_4 - F)$. Let $\mathbf{u}$ denote the faulty node in $Q_4^{j,1}$. Then we distinguish the following two subcases.

Subcase 1.2.1: Suppose that $\mathbf{u} \in V_1(Q_4^{j,1})$. By Theorem 4.3, $Q_4^{j,1}$ is hyper-hamiltonian laceable. Thus $Q_4^{j,1} - \{\mathbf{u}\}$ has a hamiltonian path H_1 from $\mathbf{s}$ to $\mathbf{t}$. Obviously, the length of H_1 is equal to 6. Let $B = \{(H_1(i), H_1(i+1)) \mid 1 \le i \le 6 \text{ and } i \equiv 1 \pmod 2\}$ be a set of disjoint links on T_1. Since $|B| = 3 > f_0$, there exists an odd integer $\hat{\imath}$, $1 \le \hat{\imath} \le 6$, such that both $(H_1(\hat{\imath}))^j$ and $(H_1(\hat{\imath}+1))^j$ are fault-free. Let $\mathbf{w} = H_1(\hat{\imath})$ and $\mathbf{b} = H_1(\hat{\imath}+1)$. Thus H_1 can be written as $\langle \mathbf{s}, H_1', \mathbf{w}, \mathbf{b}, H_1'', \mathbf{t} \rangle$. By Lemma 7.7, $Q_4^{j,0} - F_0$ has a path H_0 of length at least $7 - 2f_0$ between $(\mathbf{w})^j$ and $(\mathbf{b})^j$. As a result, $\langle \mathbf{s}, H_1', \mathbf{w}, (\mathbf{w})^j, H_0, (\mathbf{b})^j, \mathbf{b}, H_1'', \mathbf{t} \rangle$ is a fault-free path of length at least $14 - 2f_0 > 14 - 2f$ between $\mathbf{s}$ and $\mathbf{t}$.

Subcase 1.2.2: Suppose that $\mathbf{u} \in V_0(Q_4^{j,1})$. Since $h(\mathbf{s}, \mathbf{t})$ is even, it follows from Lemma 7.7 that $Q_4^{j,1} - F_1$ has a path T_1 of length at least $6 - 2f_1 = 4$ between $\mathbf{s}$ and $\mathbf{t}$. If there exists a link $(\mathbf{w}, \mathbf{b})$ on T_1 such that both $(\mathbf{w})^j$ and $(\mathbf{b})^j$ are fault-free, then a path of length at least $14 - 2f$ can be constructed in a way similar to that described in Subcase 1.2.1. Otherwise, we have $F_0 \cap \{(T_1(i))^j, (T_1(i+1))^j\} \ne \emptyset$ for every i. Then we claim that both $(T_1(2))^j$ and $(T_1(4))^j$ are faulty. Since $f_0 = 2$, we see that $|F_0 \cap \{(T_1(1))^j, (T_1(2))^j, (T_1(3))^j\}| = 1$ and $|F_0 \cap \{(T_1(3))^j, (T_1(4))^j, (T_1(5))^j\}| = 1$. Then we have $F_0 \cap \{(T_1(1))^j, (T_1(2))^j, (T_1(3))^j\} = (F_0 \cap \{(T_1(1))^j, (T_1(2))^j\}) \cap (F_0 \cap \{(T_1(2))^j, (T_1(3))^j\}) = \{(T_1(2))^j\}$. Similarly, we have $F_0 \cap \{(T_1(3))^j, (T_1(4))^j, (T_1(5))^j\} = \{(T_1(4))^j\}$. That is, $F_0 = \{(T_1(2))^j, (T_1(4))^j\}$. By Lemma 7.8, $Q_4^{j,0} - F_0$ contains either a path T_0 of length 4 between $(T_1(1))^j$ and $(T_1(3))^j$ or a path R_0 of length 4 between $(T_1(3))^j$ and $(T_1(5))^j$. As a result, $\langle \mathbf{s} = T_1(1), (T_1(1))^j, T_0, (T_1(3))^j, T_1(3), T_1(4), T_1(5) = \mathbf{t} \rangle$ or $\langle \mathbf{s} = T_1(1), T_1(2), T_1(3), (T_1(3))^j, R_0, (T_1(5))^j, T_1(5) = \mathbf{t} \rangle$ is a fault-free path of length $8 = 14 - 2f$.

Subcase 1.3: Suppose that $\mathbf{s}$ is in $Q_4^{j,0}$ and $\mathbf{t}$ is in $Q_4^{j,1}$. Since $f_0 = 2$, we have $|V_1(Q_4^{j,0}) - F_0| \ge 2 = |F_1 \cup \{\mathbf{t}\}|$ and $|V(Q_4^{j,0}) - (F_0 \cup \{\mathbf{s}\})| = 5 > |F_1 \cup \{\mathbf{t}\}|$. If $h(\mathbf{s}, \mathbf{t})$ is odd, we choose a node $\mathbf{x}$ of $V_1(Q_4^{j,0}) - F_0$ such that $(\mathbf{x})^j$ is fault-free; otherwise, we choose a node $\mathbf{x}$ of $V(Q_4^{j,0}) - (F_0 \cup \{\mathbf{s}\})$ such that $(\mathbf{x})^j \notin F_1 \cup \{\mathbf{t}\}$. By Lemma 7.7, $Q_4^{j,0} - F_0$ contains a path H_0 of length at least $7 - 2f_0$ (respectively, $6 - 2f_0$) between $\mathbf{s}$ and $\mathbf{x}$ when $h(\mathbf{s}, \mathbf{x})$ is odd (respectively, even). Similarly, $Q_4^{j,1} - F_1$ contains a path H_1 of length at least $7 - 2f_1$ (respectively, $6 - 2f_1$) between $(\mathbf{x})^j$ and $\mathbf{t}$ when $h((\mathbf{x})^j, \mathbf{t})$ is odd (respectively, even). As a result, $\langle \mathbf{s}, H_0, \mathbf{x}, (\mathbf{x})^j, H_1, \mathbf{t} \rangle$ is a fault-free path of length at least $15 - 2f$ (respectively, $14 - 2f$) if $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even).

Case 2: Suppose Q_4 has its faulty nodes distributed as in Figure 7.2(c). To be precise, we assume $F = \{0000, 0011, 1100\}$. Then we partition Q_4 into $Q_4^{3,0}$ and $Q_4^{3,1}$. It is noticed that $Q_4^{3,0}$ is not conditionally faulty.

Subcase 2.1: Both $\mathbf{s}$ and $\mathbf{t}$ are in $Q_4^{3,0} - \{0000, 0011\}$. By Theorem 7.1, $Q_4^{3,0} - \{0000\}$ has a path T_0 of length at least 5 (respectively, 4) between $\mathbf{s}$ and $\mathbf{t}$ if $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even).

We consider first that $h(\mathbf{s}, \mathbf{t})$ is odd. Thus the length of path T_0 is greater than or equal to 5. Then T_0 passes through every node of $V_0(Q_4^{3,0}) - \{0000\}$. In particular, the faulty node 0011 is on T_0. Hence T_0 can be written as $\langle \mathbf{s}, T_0', \mathbf{x}, 0011, \mathbf{y}, T_0'', \mathbf{t}\rangle$. Since $h(0011, 1100) = 4$, both $(\mathbf{x})^3$ and $(\mathbf{y})^3$ are fault-free. Since $h((\mathbf{x})^3, (\mathbf{y})^3)$ is even, Theorem 7.1 ensures that $Q_4^{3,1} - \{1100\}$ has a path T_1 of length at least 4 between $(\mathbf{x})^3$ and $(\mathbf{y})^3$. As a result, $\langle \mathbf{s}, T_0', \mathbf{x}, (\mathbf{x})^3, T_1, (\mathbf{y})^3, \mathbf{y}, T_0'', \mathbf{t}\rangle$ is a fault-free path of length at least $9 = 15 - 2f$.

Next, we consider the case that $h(\mathbf{s}, \mathbf{t})$ is even. We distinguish whether the faulty node 0011 is on T_0. If node 0011 is on T_0, then a path of length at least 8 can be constructed to join $\mathbf{s}$ and $\mathbf{t}$ in a way similar to that described earlier. Otherwise, there exists a link $(\mathbf{w}, \mathbf{b})$ on T_0 such that both $(\mathbf{w})^3$ and $(\mathbf{b})^3$ are fault-free. Hence T_0 can be written as $\langle \mathrm{s}, R_0', \mathbf{w}, \mathbf{b}, R_0'', \mathbf{t}\rangle$. By Theorem 7.1, $Q_4^{3,1} - \{1100\}$ has a path T_1 of length at least 5 between $(\mathbf{w})^3$ and $(\mathbf{b})^3$. Then $\langle \mathbf{s}, R_0', \mathbf{w}, (\mathbf{w})^3, T_1, (\mathbf{b})^3, \mathbf{b}, R_0'', \mathbf{t}\rangle$ turns out to be a fault-free path of length at least $10 > 14 - 2f$.

Subcase 2.2: Suppose that $\mathbf{s}$ is in $Q_4^{3,0} - \{0000, 0011\}$ and $\mathbf{t}$ is in $Q_4^{3,1} - \{1100\}$. By Theorem 7.1, $Q_4^{3,0} - \{0000\}$ has a path T_0 of length at least 5 (respectively, 4) between nodes $\mathbf{s}$ and 0011 if $h(\mathbf{s}, 0011)$ is odd (respectively, even). Accordingly, we write T_0 as $\langle \mathbf{s}, T_0', \mathbf{x}, \mathbf{y}, 0011\rangle$. Since $h(0011, 1100) = 4$, both $(\mathbf{x})^3$ and $(\mathbf{y})^3$ is fault-free. On the one hand, we assume $(\mathbf{y})^3 \neq \mathbf{t}$. By Theorem 7.1, $Q_4^{3,1} - \{1100\}$ has a path T_1 of length at least 5 (respectively, 4) between $(\mathbf{y})^3$ and $\mathbf{t}$ if $h((\mathbf{y})^3, \mathbf{t})$ is odd (respectively, even). As a result, $\langle \mathbf{s}, T_0', \mathbf{x}, \mathbf{y}, (\mathbf{y})^3, T_1, \mathbf{t}\rangle$ is a fault-free path of length at least $9 = 15 - 2f$ (respectively, $8 = 14 - 2f$) if $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even). On the other hand, if $(\mathbf{y})^3 = \mathbf{t}$, then Theorem 7.1 ensures that $Q_4^{3,1} - \{1100\}$ has a path R_1 of length at least 5 between $(\mathbf{x})^3$ and $(\mathbf{y})^3$. Then $\langle \mathbf{s}, T_0', \mathbf{x}, (\mathbf{x})^3, R_1, (\mathbf{y})^3 = \mathbf{t}\rangle$ turns out to be a fault-free path of length at least $9 = 15 - 2f$ (respectively, $8 = 14 - 2f$) if $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even).

Subcase 2.3: Both $\mathbf{s}$ and $\mathbf{t}$ are in $Q_4^{3,1} - \{1100\}$. We list the required paths obtained by brute force in Table 7.2.

Therefore the proof is completed. □

With Theorem 7.2 and Lemma 7.6, we will be able to prove the next theorem.

Theorem 7.3. *Let F be a set of f faulty nodes in Q_n $(n \geq 1)$ such that every node of Q_n has at least two fault-free neighbors. Suppose $f = 0$ if $n \in \{1, 2\}$, and $f \leq 2n - 5$ if $n \geq 3$. Let $\mathbf{s}$ and $\mathbf{t}$ be two arbitrary nodes of $Q_n - F$. Then $Q_n - F$ contains a path of length at least $2^n - 2f - 1$ (respectively, $2^n - 2f - 2$) between $\mathbf{s}$ and $\mathbf{t}$ if $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even).*

Proof. The result is trivial for $n \in \{1, 2\}$. When $n \in \{3, 4\}$, the result follows from Theorem 7.1 or Theorem 7.2, respectively. In what follows we consider the case that $n \geq 5$. Except

Table 7.2: The required paths in Subcase 2.3 of Theorem 7.2.

$\mathbf{s} = 1101$	$\mathbf{t} = 1110$	$\langle \mathbf{s} = 1101, 1001, 0001, 0101, 0100, 0110, 0010, 1010, 1110 = \mathbf{t}\rangle$
	$\mathbf{t} = 1111$	$\langle \mathbf{s} = 1101, 1001, 0001, 0101, 0100, 0110, 0010, 1010, 1110, 1111 = \mathbf{t}\rangle$
	$\mathbf{t} = 1000$	$\langle \mathbf{s} = 1101, 0101, 0001, 1001, 1011, 1111, 1110, 1010, 1000 = \mathbf{t}\rangle$
	$\mathbf{t} = 1001$	$\langle \mathbf{s} = 1101, 0101, 0100, 0110, 1110, 1111, 1011, 1010, 1000, 1001 = \mathbf{t}\rangle$
	$\mathbf{t} = 1010$	$\langle \mathbf{s} = 1101, 0101, 0100, 0110, 1110, 1111, 1011, 1001, 1000, 1010 = \mathbf{t}\rangle$
	$\mathbf{t} = 1011$	$\langle \mathbf{s} = 1101, 0101, 0001, 1001, 1000, 1010, 1110, 1111, 1011 = \mathbf{t}\rangle$
$\mathbf{s} = 1110$	$\mathbf{t} = 1111$	$\langle \mathbf{s} = 1110, 1010, 1000, 1001, 1101, 0101, 0100, 0110, 0111, 1111 = \mathbf{t}\rangle$
	$\mathbf{t} = 1000$	$\langle \mathbf{s} = 1110, 0110, 0100, 0101, 0001, 1001, 1011, 1010, 1000 = \mathbf{t}\rangle$
	$\mathbf{t} = 1001$	$\langle \mathbf{s} = 1110, 0110, 0100, 0101, 1101, 1111, 1011, 1010, 1000, 1001 = \mathbf{t}\rangle$
	$\mathbf{t} = 1010$	$\langle \mathbf{s} = 1110, 0110, 0100, 0101, 0001, 1001, 1101, 1111, 1011, 1010 = \mathbf{t}\rangle$
	$\mathbf{t} = 1011$	$\langle \mathbf{s} = 1110, 0110, 0100, 0101, 0001, 1001, 1101, 1111, 1011 = \mathbf{t}\rangle$
$\mathbf{s} = 1111$	$\mathbf{t} = 1000$	$\langle \mathbf{s} = 1111, 0111, 0110, 0100, 0101, 0001, 1001, 1011, 1010, 1000 = \mathbf{t}\rangle$
	$\mathbf{t} = 1001$	$\langle \mathbf{s} = 1111, 0111, 0101, 0100, 0110, 0010, 1010, 1000, 1001 = \mathbf{t}\rangle$
	$\mathbf{t} = 1010$	$\langle \mathbf{s} = 1111, 0111, 0110, 0100, 0101, 1101, 1001, 1000, 1010 = \mathbf{t}\rangle$
	$\mathbf{t} = 1011$	$\langle \mathbf{s} = 1111, 0111, 0101, 0100, 0110, 0010, 1010, 1000, 1001, 1011 = \mathbf{t}\rangle$
$\mathbf{s} = 1000$	$\mathbf{t} = 1001$	$\langle \mathbf{s} = 1000, 1010, 1110, 0110, 0100, 0101, 1101, 1111, 1011, 1001 = \mathbf{t}\rangle$
	$\mathbf{t} = 1010$	$\langle \mathbf{s} = 1000, 1001, 1101, 0101, 0100, 0110, 1110, 1111, 1011, 1010 = \mathbf{t}\rangle$
	$\mathbf{t} = 1011$	$\langle \mathbf{s} = 1000, 1001, 1101, 0101, 0100, 0110, 1110, 1111, 1011 = \mathbf{t}\rangle$
$\mathbf{s} = 1001$	$\mathbf{t} = 1010$	$\langle \mathbf{s} = 1001, 1011, 1111, 0111, 0101, 0100, 0110, 1110, 1010 = \mathbf{t}\rangle$
	$\mathbf{t} = 1011$	$\langle \mathbf{s} = 1001, 1000, 1010, 1110, 0110, 0100, 0101, 1101, 1111, 1011 = \mathbf{t}\rangle$
$\mathbf{s} = 1010$	$\mathbf{t} = 1011$	$\langle \mathbf{s} = 1010, 1000, 1001, 1101, 0101, 0100, 0110, 0111, 1111, 1011 = \mathbf{t}\rangle$

for the faulty node distribution illustrated in Figure 7.3(a), procedure $PARTITION(Q_n,\, F)$ returns j-partition of Q_n such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty. If Q_5 has its faulty nodes distributed as in Figure 7.3(a), then $PARTITION(Q_5,\, F)$ returns j-partition of Q_5 such that one subcube has its faulty nodes distributed as in Figure 7.2(b). Accordingly, the proof can be justified by the induction on n. Our inductive hypothesis is that the result holds for Q_{n-1}. For convenience, let $F_0 = F(Q_n^{j,0})$ and $F_1 = F(Q_n^{j,1})$. Moreover, let $f_0 = |F_0|$ and $f_1 = |F_1|$. Without loss of generality, we assume that $s \in V_0(Q_n - F)$.

Case 1: Suppose $f_0 \leq 2n - 7$ and $f_1 \leq 2n - 7$. Without loss of generality, we assume that $f_0 \leq f_1$. In particular, for the case illustrated in Figure 7.3(a), $Q_5^{j,0}$ is conditionally faulty with $f_0 = 2$ faulty nodes, and $Q_5^{j,1}$ is not conditionally faulty with $f_1 = 3$ faulty nodes distributed as in Figure 7.2(b).

Subcase 1.1: Both $\mathbf{s}$ and $\mathbf{t}$ are in $Q_n^{j,0}$. By inductive hypothesis, $Q_n^{j,0} - F_0$ contains a path H_0 of length L at least $2^{n-1} - 2f_0 - 1$ (respectively, $2^{n-1} - 2f_0 - 2$) between $\mathbf{s}$ and $\mathbf{t}$ if $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even). Clearly, we have $|\{\mathbf{v} \in V(Q_n^{j,1}) \mid |N^F_{Q_n^{j,1}}(\mathbf{v})| \geq n - 2\}| \leq 1$. Let $A = \{(H_0(i), H_0(i+1)) \mid 1 \leq i \leq L \text{ and } i \equiv 1 \pmod 2\}$ be a set of disjoint links on H_0. Since $|A| = \lceil \frac{L}{2} \rceil > f_1 + 1 \geq |F_1 \cup \{v \in V(Q_n^{j,1}) \mid |N^F_{Q_n^{j,1}}(v)| \geq n - 2\}|$ for $n \geq 5$, there exists an odd integer $\hat{\imath}$, $1 \leq \hat{\imath} \leq L$, such that $|F_1 \cap \{(H_0(\hat{\imath}))^j, (H_0(\hat{\imath}+1))^j\}| = 0$, $|N^F_{Q_n^{j,1}}((H_0(\hat{\imath}))^j)| \leq n - 3$, and $|N^F_{Q_n^{j,1}}((H_0(\hat{\imath}+1))^j)| \leq n - 3$ are satisfied. Let $\mathbf{x} = H_0(\hat{\imath})$ and $\mathbf{y} = H_0(\hat{\imath}+1)$. Hence path H_0 can be written as $\langle \mathbf{s}, H_0', \mathbf{x}, \mathbf{y}, H_0'', \mathbf{t}\rangle$.

If $Q_n^{j,1}$ is conditionally faulty, our inductive hypothesis asserts that $Q_n^{j,1} - F_1$ has a path

H_1 of length at least $2^{n-1} - 2f_1 - 1$ between $(\mathbf{x})^j$ and $(\mathbf{y})^j$. Otherwise, the faulty nodes of $Q_n^{j,1}$ are distributed as in Figure 7.2(b). Since both $(\mathbf{x})^j$ and $(\mathbf{y})^j$ have two or more fault-free neighbors in $Q_n^{j,1}$, Lemma 7.6 ensures that $Q_n^{j,1}$ has a fault-free path H_1 of length at least $2^{n-1} - 2f_1 - 1$ between $(\mathbf{x})^j$ and $(\mathbf{y})^j$. Then $\langle \mathbf{s}, H_0', \mathbf{x}, (\mathbf{x})^j, H_1, (\mathbf{y})^j, \mathbf{y}, H_0'', \mathbf{t}\rangle$ is a fault-free path of length at least $2^n - 2f - 1$ (respectively, $2^n - 2f - 2$) between $\mathbf{s}$ and $\mathbf{t}$ if $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even). See Figure 7.7(a).

Subcase 1.2: Both $\mathbf{s}$ and $\mathbf{t}$ are in $Q_n^{j,1}$. We consider first that the faulty nodes of $Q_5^{j,1}$ are distributed as depicted in Figure 7.2(b). Let $\mathbf{z}$ denote the node with only one fault-free neighbor $\mathbf{r}$ in $Q_5^{j,1}$. Note that $f_0 = 2$ and $f_1 = 3$.

Suppose $\{\mathbf{s}, \mathbf{t}\} = \{\mathbf{z}, \mathbf{r}\}$. Then a long path between $\mathbf{s}$ and $\mathbf{t}$ is constructed as follows. On the one hand, we assume that $\mathbf{s} = \mathbf{z}$ and $\mathbf{t} = \mathbf{r}$. Since $|V_0(Q_5^{j,0}) - F_0| \geq |V_0(Q_5^{j,0})| - |F_0| = 2^4 - 2 > 4 = |F_1 \cup \{\mathbf{t}\}|$, there exists some fault-free node $\mathbf{x}$ of $V_0(Q_5^{j,0})$ such that $(\mathbf{x})^j \notin F_1 \cup \{\mathbf{t}\}$. By inductive hypothesis, $Q_5^{j,0} - F_0$ has a path H_0 of length at least $2^4 - 2f_0 - 1$ between $(\mathbf{s})^j$ and $\mathbf{x}$. By Lemma 7.6, $Q_5^{j,1} - F_1$ has a path H_1 of length at least $2^4 - 2f_1 - 2$ between $(\mathbf{x})^j$ and $\mathbf{t}$. As a result, $\langle \mathbf{s}, (\mathbf{s})^j, H_0, \mathbf{x}, (\mathbf{x})^j, H_1, \mathbf{t}\rangle$ is a fault-free path of length at least $2^5 - 2f - 1$ (see Figure 7.7(b)). On the other hand, we assume that $\mathbf{t} = \mathbf{z}$ and $\mathbf{s} = \mathbf{r}$. Since $|V_1(Q_5^{j,0}) - F_0| \geq |V_1(Q_5^{j,0})| - |F_0| = 2^4 - 2 > 4 = |F_1 \cup \{\mathbf{s}\}|$, there exists some fault-free node $\mathbf{x}$ of $V_1(Q_5^{j,0})$ such that $(\mathbf{x})^j \notin F_1 \cup \{\mathbf{s}\}$. Again, the inductive hypothesis asserts that $Q_5^{j,0}$ has a fault-free path H_0 of length at least $2^4 - 2f_0 - 1$ between $\mathbf{x}$ and $(\mathbf{t})^j$; Lemma 7.6 asserts that $Q^{j,1}$ has a fault-free path H_1 of length at least $2^4 - 2f_1 - 2$ between $\mathbf{s}$ and $(\mathbf{x})^j$. Then $\langle \mathbf{s}, H_1, (\mathbf{x})^j, \mathbf{x}, H_0, (\mathbf{t})^j, \mathbf{t}\rangle$ is a fault-free path of length at least $2^5 - 2f - 1$ (see Figure 7.7(c)).

Suppose $\{\mathbf{s}, \mathbf{t}\} \neq \{\mathbf{z}, \mathbf{r}\}$. Then Lemma 7.6 asserts that $Q_5^{j,1} - F_1$ contains a path H_1 of length L at least $2^4 - 2f_1 - 1$ (respectively, $2^4 - 2f_1 - 2$) between $\mathbf{s}$ and $\mathbf{t}$ if $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even). Let $A = \{(H_1(i), H_1(i+1)) \mid 1 \leq i \leq L \text{ and } i \equiv 1 \pmod 2\}$ be a set of disjoint links. Since $|A| = \lceil \frac{L}{2} \rceil > 2 = f_0$, there exists an odd integer $\hat{\imath}$, $1 \leq \hat{\imath} \leq L$, such that $F_0 \cap \{(H_1(\hat{\imath}))^j, (H_1(\hat{\imath}+1))^j\} = \emptyset$. Let $\mathbf{x} = H_1(\hat{\imath})$ and $\mathbf{y} = H_1(\hat{\imath}+1)$. Accordingly, path H_1 can be written as $\langle \mathbf{s}, H_1', \mathbf{x}, \mathbf{y}, H_1'', \mathbf{t}\rangle$. Again, the inductive hypothesis asserts that $Q_5^{j,0} - F_0$ has a path H_0 of length at least $2^4 - 2f_0 - 1$ between $(\mathbf{x})^j$ and $(\mathbf{y})^j$. Then $\langle \mathbf{s}, H_1', \mathbf{x}, (\mathbf{x})^j, H_0, (\mathbf{y})^j, \mathbf{y}, H_1'', \mathbf{t}\rangle$ is a fault-free path of length at least $2^5 - 2f - 1$ or $2^5 - 2f - 2$ if $h(\mathbf{s}, \mathbf{t})$ is odd or even, respectively. See Figure 7.7(d).

Now we consider the case that faulty nodes of $Q_5^{j,1}$ are not distributed as depicted in Figure 7.2(b), or $n \geq 6$. Then $Q_n^{j,1}$ is conditionally faulty. By inductive hypothesis, $Q_n^{j,1} - F_1$ has a path H_1 of length L at least $2^{n-1} - 2f_1 - 1$ (respectively, $2^{n-1} - 2f_1 - 2$) between $\mathbf{s}$ and $\mathbf{t}$ if $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even). Similarly, let $A = \{(H_1(i), H_1(i+1)) \mid 1 \leq i \leq L \text{ and } i \equiv 1 \pmod 2\}$ be a set of disjoint links. Since $|A| = \lceil \frac{L}{2} \rceil > f_0$ for $n \geq 5$, there is a link $(\mathbf{x}, \mathbf{y})$ of A such that $F_0 \cap \{(\mathbf{x})^j, (\mathbf{y})^j\} = \emptyset$. Accordingly, path H_1 can be written as $\langle \mathbf{s}, H_1', \mathbf{x}, \mathbf{y}, H_1'', \mathbf{t}\rangle$. By inductive hypothesis, $Q_n^{j,0} - F_0$ has a path H_0 of length at least $2^{n-1} - 2f_0 - 1$ between $(\mathbf{x})^j$ and $(\mathbf{y})^j$. Again, $\langle \mathbf{s}, H_1', \mathbf{x}, (\mathbf{x})^j, H_0, (\mathbf{y})^j, \mathbf{y}, H_1'', \mathbf{t}\rangle$ is a fault-free path of length at least $2^n - 2f - 1$ or $2^n - 2f - 2$ if $h(\mathbf{s}, \mathbf{t})$ is odd or even, respectively. See Figure 7.7(d).

Subcase 1.3: Suppose that $\mathbf{s}$ is in $Q_n^{j,0}$ and $\mathbf{t}$ is in $Q_n^{j,1}$. Note that $|\{\mathbf{x} \in V(Q_n^{j,1}) \mid$

$|N^F_{Q_n^{j,1}}(\mathbf{x})| \geq n-2\}| \leq 1$. On the one hand, we consider the case that node $\mathbf{t}$ has only one fault-free neighbor, denoted by $\mathbf{r}$, in $Q_n^{j,1}$. On this occasion, n is equal to 5. Since $|V_1(Q_n^{j,0}) - F_0| \geq 2^{n-2} - f_0 > f_1 + 2 = |F_1 \cup \{\mathbf{t}, \mathbf{r}\}|$ for $n = 5$, there exists a fault-free node $\mathbf{b}$ of $V_1(Q_n^{j,0}) - F_0$ such that $(\mathbf{b})^j \notin F_1 \cup \{\mathbf{t}, \mathbf{r}\}$. On the other hand, we consider the case that node $\mathbf{t}$ has at least two fault-free neighbors in $Q_n^{j,1}$. Since $|V_1(Q_n^{j,0}) - F_0| \geq 2^{n-2} - f_0 > f_1 + 2 \geq |F_1| + |\{\mathbf{t}\}| + |\{\mathbf{x} \in V(Q_n^{j,1}) \mid |N^F_{Q_n^{j,1}}(\mathbf{x})| \geq n-2\}| \geq |F_1 \cup \{\mathbf{t}\} \cup \{\mathbf{x} \in V(Q_n^{j,1}) \mid |N^F_{Q_n^{j,1}}(\mathbf{x})| \geq n-2\}|$ for $n \geq 5$, there exists a fault-free node $\mathbf{b}$ of $V_1(Q_n^{j,0}) - F_0$ such that $(\mathbf{b})^j \notin F_1 \cup \{\mathbf{t}\} \cup \{\mathbf{x} \in V(Q_n^{j,1}) \mid |N^F_{Q_n^{j,1}}(\mathbf{x})| \geq n-2\}$.

By inductive hypothesis, $Q_n^{j,0} - F_0$ has a path H_0 of length at least $2^{n-1} - 2f_0 - 1$ between $\mathbf{s}$ and $\mathbf{b}$. If the faulty nodes of $Q_n^{j,1}$ are distributed as illustrated in Figure 7.2(b), Lemma 7.6 asserts that $Q_n^{j,1} - F_1$ has a path H_1 of length at least $2^{n-1} - 2f_1 - 1$ (respectively, $2^{n-1} - 2f_1 - 2$) between $(\mathbf{b})^j$ and $\mathbf{t}$ if $h((\mathbf{b})^j, \mathbf{t})$ is odd (respectively, even); otherwise, the inductive hypothesis asserts that $Q_n^{j,1} - F_1$ has a path H_1 of length at least $2^{n-1} - 2f_1 - 1$ (respectively, $2^{n-1} - 2f_1 - 2$) between $(\mathbf{b})^j$ and $\mathbf{t}$ if $h((\mathbf{b})^j, \mathbf{t})$ is odd (respectively, even). Then $\langle \mathbf{s}, H_0, \mathbf{b}, (\mathbf{b})^j, H_1, \mathbf{t} \rangle$ is a fault-free path of length at least $2^n - 2f - 1$ (respectively, $2^n - 2f - 2$) between $\mathbf{s}$ and $\mathbf{t}$ if $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even). See Figure 7.7(e).

Case 2: Suppose either $f_0 = 2n - 6$ or $f_1 = 2n - 6$. By Lemmas 7.1–7.5, we know that this case may occur while $n = 5$. More precisely, the faulty nodes happen to be distributed as illustrated in Figure 7.5(c) where $\mathbf{z}$ is itself a faulty node with three faulty neighbors. Without loss of generality, we assume that $f_0 = 4$; thus, $(\mathbf{z})^j$ is a unique faulty node in $Q_5^{j,1}$.

Subcase 2.1: Both $\mathbf{s}$ and $\mathbf{t}$ are in $Q_5^{j,0}$. By inductive hypothesis, $Q_5^{j,0} - (F_0 - \{\mathbf{z}\})$ contains a path H_0 of length L at least $9 = 2^4 - 2 \cdot 3 - 1$ (respectively, $8 = 2^4 - 2 \cdot 3 - 2$) between $\mathbf{s}$ and $\mathbf{t}$ if $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even).

First, we consider the case that node $\mathbf{z}$ is not on H_0. Let $A = \{(H_0(i), H_0(i+1)) \mid 1 \leq i \leq L \text{ and } i \equiv 1 \pmod 2\}$ be a set of disjoint links on H_0. Since $|A| = \lceil \frac{L}{2} \rceil > 1 = f_1$, there exists an odd integer $\hat{\imath}$, $1 \leq \hat{\imath} \leq L$, such that both $(H_0(\hat{\imath}))^j$ and $(H_0(\hat{\imath}+1))^j$ are fault-free. Let $\mathbf{x} = H_0(\hat{\imath})$ and $\mathbf{y} = H_0(\hat{\imath}+1)$. Hence path H_0 can be written as $\langle \mathbf{s}, H_0', \mathbf{x}, \mathbf{y}, H_0'', \mathbf{t} \rangle$. It follows from inductive hypothesis that $Q_5^{j,1} - \{(\mathbf{z})^j\}$ has a path H_1 of length at least $13 = 2^4 - 2 \cdot 1 - 1$ between $(\mathbf{x})^j$ and $(\mathbf{y})^j$. Then $\langle \mathbf{s}, H_0', \mathbf{x}, (\mathbf{x})^j, H_1, (\mathbf{y})^j, \mathbf{y}, H_0'', \mathbf{t} \rangle$ is a fault-free path of length at least $23 > 2^5 - 2 \cdot 5 - 1$ (respectively, $22 > 2^5 - 2 \cdot 5 - 2$) between $\mathbf{s}$ and $\mathbf{t}$ if $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even).

Now we consider the case that node $\mathbf{z}$ is on H_0. Since the length of H_0 is at least 9, we can write H_0 as $\langle \mathbf{s}, H_0', \mathbf{x}, \mathbf{z}, \mathbf{y}, H_0'', \mathbf{t} \rangle$. Clearly, $(\mathbf{x})^j$ and $(\mathbf{y})^j$ are fault-free nodes in the same partite set of $Q_5^{j,1}$. By Theorem 4.3, $Q_5^{j,1}$ is hyper-hamiltonian laceable; thus $Q_5^{j,1} - \{(\mathbf{z})^j\}$ has a path H_1 of length 14 between $(\mathbf{x})^j$ and $(\mathbf{y})^j$. Then $\langle \mathbf{s}, H_0', \mathbf{x}, (\mathbf{x})^j, H_1, (\mathbf{y})^j, \mathbf{y}, H_0'', \mathbf{t} \rangle$ is a fault-free path of length at least $23 > 2^5 - 2 \cdot 5 - 1$ (respectively, $22 > 2^5 - 2 \cdot 5 - 2$) between $\mathbf{s}$ and $\mathbf{t}$ if $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even).

Subcase 2.2: Both $\mathbf{s}$ and $\mathbf{t}$ are in $Q_5^{j,1}$. For the sake of clarity, we distinguish whether

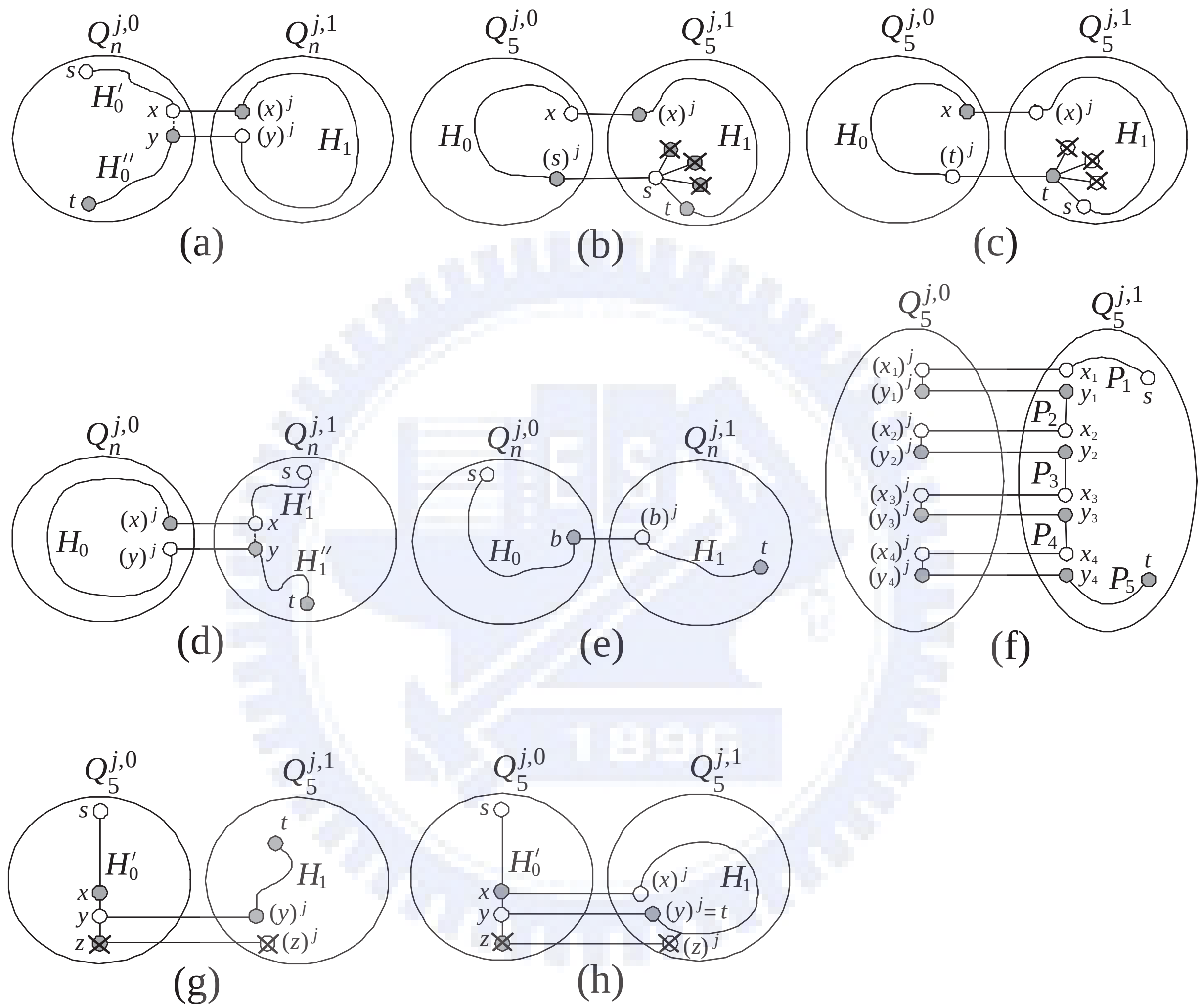

Figure 7.7: Illustration for Theorem 7.3.

$h(\mathbf{s}, \mathbf{t})$ is odd or even.

Suppose that $h(\mathbf{s}, \mathbf{t})$ is odd. By inductive hypothesis, $Q_5^{j,1} - \{(\mathbf{z})^j\}$ contains a path H_1 of length L at least 13 between $\mathbf{s}$ and $\mathbf{t}$. Obviously, we have $(\mathbf{z})^j \notin V(H_1)$. Consequently, $(\mathbf{v})^j \neq \mathbf{z}$ for any $\mathbf{v} \in V(H_1)$. Let $A = \{(H_1(i), H_1(i+1)) \mid 1 \leq i \leq L \text{ and } i \equiv 1 \pmod 2\}$ be a set of disjoint links on H_1. Since $|A| - |F_0 - \{z\}| = \lceil \frac{L}{2} \rceil - (f_0 - 1) \geq 7 - (4-1) = 4$, there exist four links of A, namely $(\mathbf{x}_1, \mathbf{y}_1)$, $(\mathbf{x}_2, \mathbf{y}_2)$, $(\mathbf{x}_3, \mathbf{y}_3)$, and $(\mathbf{x}_4, \mathbf{y}_4)$, such that $(\mathbf{x}_i)^j$ and $(\mathbf{y}_i)^j$ are fault-free for all $i \in \{1, 2, 3, 4\}$. Thus path H_1 can be written as $\langle \mathbf{s}, P_1, \mathbf{x}_1, \mathbf{y}_1, P_2, \mathbf{x}_2, \mathbf{y}_2, P_3, \mathbf{x}_3, \mathbf{y}_3, P_4, \mathbf{x}_4, \mathbf{y}_4, P_5, \mathbf{t}\rangle$. Then $\langle \mathbf{s}, P_1, \mathbf{x}_1, (\mathbf{x}_1)^j, (\mathbf{y}_1)^j, \mathbf{y}_1, P_2, \mathbf{x}_2, (\mathbf{x}_2)^j, (\mathbf{y}_2)^j, \mathbf{y}_2, P_3, \mathbf{x}_3, (\mathbf{x}_3)^j, (\mathbf{y}_3)^j, \mathbf{y}_3, P_4, \mathbf{x}_4, (\mathbf{x}_4)^j, (\mathbf{y}_4)^j, \mathbf{y}_4, P_5, \mathbf{t}\rangle$ is a fault-free path of length at least $21 = 2^5 - 2 \cdot 5 - 1$ between $\mathbf{s}$ and $\mathbf{t}$. See Figure 7.7(f).

Suppose that $h(\mathbf{s}, \mathbf{t})$ is even. If $\mathbf{s}$ and $(\mathbf{z})^j$ belong to the different partite sets of $Q_5^{j,1}$, Theorem 4.3 asserts that $Q_5^{j,1} - \{(\mathbf{z})^j\}$ has a path H_1 of length 14 between $\mathbf{s}$ and $\mathbf{t}$. Similar to the case that $h(\mathbf{s}, \mathbf{t})$ is odd, there exist four disjoint links on H_1, namely $(\mathbf{x}_1, \mathbf{y}_1)$, $(\mathbf{x}_2, \mathbf{y}_2)$, $(\mathbf{x}_3, \mathbf{y}_3)$, and $(\mathbf{x}_4, \mathbf{y}_4)$, such that $(\mathbf{x}_i)^j$ and $(\mathbf{y}_i)^j$ are fault-free for all $i \in \{1, 2, 3, 4\}$. Accordingly, we can write $H_1 = \langle \mathbf{s}, P_1, \mathbf{x}_1, \mathbf{y}_1, P_2, \mathbf{x}_2, \mathbf{y}_2, P_3, \mathbf{x}_3, \mathbf{y}_3, P_4, \mathbf{x}_4, \mathbf{y}_4, P_5, \mathbf{t}\rangle$. Then $\langle \mathbf{s}, P_1, \mathbf{x}_1, (\mathbf{x}_1)^j, (\mathbf{y}_1)^j, \mathbf{y}_1, P_2, \mathbf{x}_2, (\mathbf{x}_2)^j, (\mathbf{y}_2)^j, \mathbf{y}_2, P_3, \mathbf{x}_3, (\mathbf{x}_3)^j, (\mathbf{y}_3)^j, \mathbf{y}_3, P_4, \mathbf{x}_4, (\mathbf{x}_4)^j, (\mathbf{y}_4)^j, \mathbf{y}_4, P_5, \mathbf{t}\rangle$ is a fault-free path of length at least $22 > 2^5 - 2 \cdot 5 - 2$ between $\mathbf{s}$ and $\mathbf{t}$. If nodes $\mathbf{s}$ and $(\mathbf{z})^j$ belong to the same partite set of $Q_5^{j,1}$, then we construct a fault-free path as follows. Since $Q_5^{j,0}$ is conditionally faulty, we denote by $\mathbf{x}$ any fault-free neighbor of $\mathbf{z}$ in $Q_5^{j,0}$. By inductive hypothesis, $Q_5^{j,0} - (F_0 - \{\mathbf{z}\})$ has a path H_0 of length at least $9 = 2^4 - 2 \cdot 3 - 1$ between $\mathbf{x}$ and $\mathbf{z}$. We can write path H_0 as $\langle \mathbf{x}, H_0', \mathbf{y}, \mathbf{z}\rangle$, where $\mathbf{y}$ is also a fault-free neighbor of $\mathbf{z}$. Without loss of generality, let $j = 4$, $\{\mathbf{x}, \mathbf{y}\} = \{(\mathbf{z})^0, (\mathbf{z})^1\}$, and $X = \{((\mathbf{z})^j, ((\mathbf{z})^j)^2), ((\mathbf{z})^j, ((\mathbf{z})^j)^3)\}$. Since $|X| = 2$, Theorem 4.2 ensures that $Q_5^{j,1} - X$ is strongly hamiltonian laceable; hence it has a path H_1 of length 14 between $\mathbf{s}$ and $\mathbf{t}$. Obviously, both $((\mathbf{z})^j, (\mathbf{x})^j)$ and $((\mathbf{z})^j, (\mathbf{y})^j)$ are on H_1, and we can write H_1 as $\langle \mathrm{s}, H_1', (\mathbf{x})^j, (\mathbf{z})^j, (\mathbf{y})^j, H_1'', \mathbf{t}\rangle$. Then $\langle \mathrm{s}, H_1', (\mathbf{x})^j, \mathrm{x}, H_0', \mathbf{y}, (\mathbf{y})^j, H_1'', \mathbf{t}\rangle$ is a fault-free path of length at least $22 > 2^5 - 2 \cdot 5 - 2$ between $\mathbf{s}$ and $\mathbf{t}$.

Subcase 2.3: Suppose that $\mathbf{s}$ is in $Q_5^{j,0}$ and $\mathbf{t}$ is in $Q_5^{j,1}$. By inductive hypothesis, $Q_5^{j,0} - (F_0 - \{z\})$ has a path H_0 of length at least 9 (respectively, 8) between $\mathbf{s}$ and $\mathbf{z}$ if $h(\mathbf{s}, \mathbf{z})$ is odd (respectively, even). Accordingly, path H_0 can be written as $\langle \mathbf{s}, H_0', \mathbf{x}, \mathbf{y}, \mathbf{z}\rangle$. Since $(\mathbf{z})^j$ is a unique faulty node in $Q_5^{j,1}$, both $(\mathbf{x})^j$ and $(\mathbf{y})^j$ are fault-free.

If $(\mathbf{y})^j \neq \mathbf{t}$, it follows from inductive hypothesis that $Q_5^{j,1} - \{(\mathbf{z})^j\}$ has a path H_1 of length at least 13 (respectively, 12) between $(\mathbf{y})^j$ and $\mathbf{t}$ if $h((\mathbf{y})^j, \mathbf{t})$ is odd (respectively, even). Then $\langle \mathbf{s}, H_0', \mathbf{x}, \mathbf{y}, (\mathbf{y})^j, H_1, \mathbf{t}\rangle$ is a path of length at least $21 = 2^5 - 2 \cdot 5 - 1$ (respectively, $20 = 2^5 - 2 \cdot 5 - 2$) between $\mathbf{s}$ and $\mathbf{t}$ if $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even). See Figure 7.7(g). Otherwise, if $(\mathbf{y})^j = \mathbf{t}$, then our inductive hypothesis asserts that $Q_5^{j,1} - \{(\mathbf{z})^j\}$ has a path H_1 of length at least 13 between $(\mathbf{x})^j$ and $(\mathbf{y})^j$. Then $\langle \mathbf{s}, H_0', \mathbf{x}, (\mathbf{x})^j, H_1, (\mathbf{y})^j = \mathbf{t}\rangle$ is a path of length at least $21 = 2^5 - 2 \cdot 5 - 1$ (respectively, $20 = 2^5 - 2 \cdot 5 - 2$) between $\mathbf{s}$ and $\mathbf{t}$ if $h(\mathbf{s}, \mathbf{t})$ is odd (respectively, even). See Figure 7.7(h).

Therefore the proof is completed. $\square$

Chapter 8

Conclusion and Future Works

Paths and cycles are two network structures extensively used in distributed systems and parallel computation. In this thesis, we introduce some research issues on embedding paths and cycles into interconnection networks.

Firstly, we devote to investigating fault-tolerant hamiltonian connectedness of cycle composition networks. In Chapter 2, we improve the result of Chen et al. [12] by showing that the cycle composition network $G_{\langle 0,1,\ldots,n-1,0\rangle}$ is super fault-tolerant hamiltonian even if it is composed of n 4-regular super fault-tolerant hamiltonian networks $G_0, \ldots, G_{n-1}$, provided that $n \geq 3$. However, we conjecture that this result may not be true if the cycle composition network is constructed on the basis of cubic networks. Therefore such an improvement is of significance because only the remaining case for 3-regular graphs needs to be checked with brute force or by computer.

Secondly, we restrict our attention to the applicability of hamiltonian cycles on interconnection networks. Both Chapter 3 and Chapter 4 are dedicated to exploring how to embed mutually independent hamiltonian cycles onto interconnection networks. In Chapter 3 we show that the binary wrapped butterfly graph $BF(n)$ has 4-mutually independent hamiltonian cycles, beginning from any vertex, for $n \geq 3$. In Chapter 4, we first prove that a faulty n-cube contains $(n-1-f)$-mutually independent hamiltonian cycles, beginning from any vertex, when not more than $f \leq n-2$ faulty edges may occur accidentally. However, we conjecture this result can be further refined; that is, we believe that a faulty n-cube really can be embedded with up to $(n-f)$-mutually independent hamiltonian cycles, beginning from any vertex, when $f \leq n-2$ faulty edges occur. Next, we also prove that a faulty star network S_n has $(n-1-f)$-mutually independent hamiltonian cycles, beginning from any vertex, if only $f \leq n-2$ faulty edges occur accidentally, provided that $n \geq 4$.

Finally, we concern the problem of embedding various paths into conditionally faulty hypercubes. In advance, the fault diameter of the n-cube is computed in Chapter 5. In Chapter 6 we investigate the method for embedding paths of variable lengths into hypercubes, whose every node is assumed to be incident to at least two fault-free links. In Chapter 7 we show that a long path between any two nodes can be embedded into a conditionally faulty hypercube, whose every node is assumed to have at least two fault-free neighbors.

For the purpose of efficient data transmission, one of our future work is directed to explore the feasibility of finding as many mutually independent edge-disjoint hamiltonian cycles as possible. Another future research issue will be dedicated to generalizing the conditional-fault tolerance in the perspective on path embedding. Besides path and cycle embedding, tree embedding is also an important research topic widely addressed in the area of interconnection networks. By definition, a *tree* is a connected graph without cycles. In practice, tree structures are very useful for network communication too. Hence, in our future work, we also plan to design efficient communication algorithms on the basis of tree embedding.

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