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碩士論文

圖與重邊圖結合各式的設計之研究

A Study of Graphs and Multigraphs Associated with Various Designs

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中華民國 九十六 年 六 月

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圖和重邊圖結合各式的設計之研究

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摘　要

Bose 首先提出強正則重邊圖的概念，接著 Neumair 和 Metsch 利用強正則重邊圖的概念進一步地解決準剩餘 2-設計的問題。近來，不完全幾何設計的概念被 van Dam 和 Spence 使用在具有 2 個奇異值的組合設計。我們將 Neumair 和 Metsch 兩篇論文中的定義與結果做整理，並以統一形式呈現在此論文裡，進而舉出一些 2-設計及其對應的強正則重邊圖。藉由這些圖，研究具有 3 或 4 個相異特徵值的連通正則圖之特性。

A Study of Graphs and Multigraphs Associated with Various Designs

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Abstract

The notion of strongly regular multigraphs was first introduced by R. C. Bose, followed by Neumaier for characterizing quasi-residual 2-designs, and further by Metsch for embeddings of residual 2-designs. Recently, the notion of partial geometric designs was also used by van Dam and Spence over combinatorial designs with two singular values. The basic definitions and most results regarding strongly regular multigraphs and partial geometric designs covered in the works of Neumaier and Metsch are given in a unified way in this thesis. The associated multigraphs or graphs of 2-designs are then studied, followed by a few examples of 2-designs and their corresponding strongly regular multigraphs. Motivated by these graphs, connected regular graphs with 3 or 4 distinct eigenvalues are also studied.

Contents

1 Introduction

The notion of *strongly regular multigraphs* (*SR multigraphs*) was first introduced by R. C. Bose in a very cumbersome notation. While characterizing *quasi-residual 2-designs*, Neumaier gave an equivalent definition of strongly regular multigraphs in an elegant and self-contained way [2]. Metsch continued the study of embeddings of residual 2-designs within the framework of strongly regular multigraphs. However, the notations used by them are quite different.

Neumaier showed that the block multigraph of a 2-design of order n is a strongly regular multigraph, together with a partial converse with some constraints over its parameters. Its proof involves 2-designs and its variations, called $1\frac{1}{2}$ - designs (or called *partial geometric designs*), or weak $1\frac{1}{2}$ - designs. Neumaier showed also that a strongly regular multigraph under some numerical constraints is the point multigraph of a unique $1\frac{1}{2}$ - design. Recently, the notion of partial geometric designs was also used by van Dam and Spence [3, 4] over combinatorial designs with two singular values.

Though strongly regular multigraphs and partial geometric designs are the common themes covered in [7, 6], the notations used by Neumaier and Metsch are quite different. We expect that these notions will keep playing essential roles in the future. The basic definitions and most results regarding strongly regular multigraphs and partial geometric designs covered in [7, 6] are given in a unified way in section 2. The block multigraphs or associated block graphs, and the point multigraphs of 2-designs are studied in section 3, followed by a few examples of 2-designs and their corresponding strongly regular multigraphs. Motivated by these graphs, connected regular graphs with 3 or 4 distinct eigenvalues are studied in section 4.

2 Basic Definitions and preliminary

2.1 Graphs and multigraphs

Definition 2.1. A *graph* is a triple consisting of a vertex set $V(\Gamma)$, an edge set $E(\Gamma)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its endpoints. Two nonloops are parallel when they have the same ends; graphs that contain them are called *multigraphs*. Equivalently, a *multigraph* Γ consists of a nonempty set V of vertices and a multiset E of edges.

For a *multigraph* Γ and x, $y \in V = V(\Gamma)$, let m_{xx}:=0 , and m_{xy}:=the number of edges joining x and y.

Definition 2.2. The eigenvalues of an adjacency matrix $A(\Gamma) = A$ of a connected graph are called *eigenvalues* of the graph Γ, denoted by $Spec(\Gamma) = ({\theta_0}^{m_0}, {\theta_1}^{m_1}, {\theta_2}^{m_2}, \ldots, {\theta_s}^{m_s})$ with $\theta_0 > \theta_1 > \theta_2 > \ldots > \theta_s$ and m_i is the multiplicity of the eigenvalue θ_i.

Definition 2.3. For a connected k-regular graph of diameter d with $Spec(\Gamma) = (k^1, {\theta_1}^{m_1}, {\theta_2}^{m_2}, \ldots, {\theta_s}^{m_s})$, then

$$h(x) = \frac{\prod_{1 \le i \le s} (x - \theta_i)}{\prod_{1 \le i \le s} (k - \theta_i)}$$

is called the *Hoffman polynomial* of the graph Γ.

Definition 2.4. A simple graph Γ is *strongly regular* if there are parameters k, λ, μ such that Γ is k-regular, every adjacent pair of vertices have λ common neighbors, and every nonadjacent pair of vertices have *mu* common neighbors, denoted by SR(v, k, λ, μ), where v is the number of vertices in Γ.

Lemma 2.1. If Γ is an SR graph (v, k, λ, μ), then $k(k - \lambda - 1) = \mu(v - k - 1)$.

For a connected graph Γ with an adjacency matrix A, then Γ is a strongly regular graph if and only if

$$AJ = kJ, A^2 = kI + \lambda A + \mu(J - I - A)$$
$$= (\lambda - \mu)A + (k - \mu)I + \mu J.$$

A multigraph version of strongly regular graphs is defined by Bose and Shrikhande 1973; similar to strongly regular graphs, the *matrix expressions* in terms of their adjacency matrices $A = [m_{i,j}]$ for strongly regular multigraphs are given.

Definition 2.5. [1, 6] A multigraph Γ is called a *strongly regular multigraph* (*SR* multigraph) with parameters (m, n, μ, γ, R) for real numbers m, n, μ, γ, R with $n > 0$, if

1. $\sum_{x \in V} m_{ax} = R$ for each $a \in V$;
2. $\sum_{x \in V} m_{ax}m_{bx} = (n - 2m)m_{ab} + m(n - m)\delta_{ab} + \mu$, where $\delta_{aa} = 1$ and $\delta_{ab} = 0$ for $a \neq b$.
3. $\sum_{x \in V} m_{ax}(m_{ax} - 1) = \gamma$ for each $x \in V$.
4. $v\mu = (R + m)(R + m - n)$.

Indeed, condition 4 is a consequence of the fact that
$\sum_{x \in V}(m_{ax})^2 = \sum_{x \in V} m_{ax} + \sum_{x \in V} m_{ax}(m_{ax} - 1)$ and the conditions 1 3 above.

Note that the parameters $\gamma = \sum_{y \in V} m_{xy}(m_{xy} - 1)$, $R = \sum_{y \in V} m_{xy}$ for each vertex x were defined explicitly, though m, n, μ and are not. The parameter $\gamma = \sum_{y \in V} m_{xy}(m_{xy} - 1)$ measures the derivations of the multigraphs from graphs; if $\gamma = \sum_{y \in V} m_{xy}(m_{xy} - 1) = 0$, then either $m_{xy} = 0$ or $m_{xy} = 1$ for $x, y \in V$. The parameter μ in $SRMG(m, n, \mu, \gamma, R)$ is identical with that of μ in $SRG(v, k, \lambda, \mu)$ in case $\gamma = 0$. Some subsets of vertices including *claws*, *maximal cliques* play essential roles in the study of their geometric structures. The parameter m is the constant number of maximal cliques containing a fix vertex in the study of maximal claws under some numerical constraints.

Proposition 2.1. [6] An SR multigraph Γ with parameters $(m,\ n,\ \mu,\ \gamma,\ R)$ with $\gamma = 0$ is an SR graph with parameters

$$(m, n, \mu, \gamma, R) = (\frac{R(R-n+2m-\mu-1)}{\mu} + R + 1, R, n - 2m + \mu, \mu)$$

Proof: Let $A = [m_{xy}]$, it suffices to show that $m_{xy} \geq 0$ for all $x, y \in V(\Gamma)$. Consider the $(a,\ b)$ entry of $a, b \in V(\Gamma)$:

1. $\sum\limits_{x \in V(\Gamma)} = R = k$ for each $a \in V(\Gamma)$.
2. $\sum\limits_{x} m_{ax} m_{bx} = (n-2m)m_{ab} + m(n-m)\delta_{ab} + \mu$,

   when $a = b$

$\sum\limits_{x} m_{ax} m_{ax} = \sum\limits_{x} m_{ax} = m(n-m) + \mu = R = k$;

when $a,\ b$ are adjacent,

$\sum\limits_{x} m_{ax} m_{bx} = (n-2m)m_{ab} + \mu = (n-2m) + \mu = \lambda$;

when $a,\ b$ are nonadjacent,

$\sum\limits_{x} m_{ax} m_{bx} = \mu$.

Since $k(k-\lambda-1) = \mu(v-k-1)$ for SR graphs, we have

$v = \dfrac{k(k-\lambda-1)}{\mu} + k + 1 = \dfrac{R(R-n+2m-\mu-1)}{\mu} + R + 1.$ ∎

The following lemma will be used in the proof of an inequality below.

Lemma 2.2. [7] Let A be an integral symmetric matrix with zero diagonal satisfying $AJ = RJ$, $A^2 = (n-2m)A + m(n-m)I + \mu J$. If

$n \geq max\{2m-4, 2m-1+\mu+\gamma\}$

where $\gamma = m(n-m) + \mu - R$, then A is the adjacency matrix of a SR multigraph $(m,\ n,\ \mu,\ \gamma,\ R)$.

Proof: Since $A^2 = (n-2m)A + m(n-m)I + \mu J$, then

$$\sum_x m_{ax} m_{bx} = (n-2m)m_{ab} + m(n-m)\delta_{ab} + \mu,$$
$$\sum_x m_{ax}(m_{ax}-1) = \gamma = m(n-m) + \mu - R.$$

For $a \neq b$, we have:

$$\begin{aligned} 2m_{ab}(m_{ab}-1) &\leq \sum_x (m_{ax}+m_{bx})(m_{ax}+m_{bx}-1) \\ &= \sum_x m_{ax}(m_{ax}-1) + 2\sum_x m_{ax} + m_{bx} - \sum_x m_{bx}(m_{bx}-1) \\ &= \gamma + 2((n-2m)m_{ab} + \mu) + \gamma. \end{aligned}$$

it follows that

$$m_{ax}(m_{ax}-1) \leq ((n-2m)m_{ab} + \mu) + \gamma,$$

and

$$(m_{ab}+1)(m_{ab}+2) \leq (n-2m+4)m_{ab} + \mu + \gamma + 2 \geq 0.$$

Since

$$n \geq max\{2m-4, 2m-1+\mu+\gamma\},$$

then

$$(n-2m+4)m_{ab} + \mu + \gamma + 2 \geq (\mu+\gamma+3)m_{ab} + \mu + \gamma + 2 \geq 0, \text{ so}$$

$m_{ab} \geq 0$. Hence A is the adjacency matrix of an SR multigraph $(m,\ n,\ \mu,\ \gamma,\ R)$.

Some other combinatorial interpretations for the parameters can be found in the following lemma.

Lemma 2.3. [7] For an SR multigraph $(m,\ n,\ \mu,\ \gamma,\ R)$

1. $m - n \leq m_{ab} \leq m$ whenever $a \neq b$.

2. $m \geq 1$, with equality if and only if it is the disjoint union of complete graphs.

3. If there are nonadjacent vertices, then $n \geq m$.

4. $\mu \geq (R+m)(m-n)$, with equality if and only if $m_{ab} = m-n$ for all $a \neq b$.

5. $\mu\gamma \leq (n-2m+\mu)(m(n-m)+\mu)$, with equality if Γ contains no triangles.

6. $\mu \geq 2m-n$.

7. If $n \leq 2m+4$, then $\gamma < 2m(n-m)+n-2m-1+\mu$.

Proof:

1. $$2{m_{ab}}^2 = {m_{ab}}^2 + {m_{ab}}^2 \leq \sum_{x=a \ or \ b} (m_{ax}-m_{bx})^2 + \sum_{x \neq a \ or \ b} (m_{ax}-m_{bx})^2$$
$$= \sum_x {m_{ax}}^2 - 2\sum_x m_{ax}m_{bx} + \sum_x {m_{bx}}^2$$
$$= 2m(n-m) - 2(n-2m)m_{ab}$$

$\Rightarrow {m_{ab}}^2 + (n-2m)m_{ab} - m(n-m) \leq 0$

$\Rightarrow (m_{ab}-m)(m_{ab}+(n-m)) \leq 0$.

Then $m-n \leq m_{ab} \leq m$.

2. For adjacent a, b, $1 \leq m_{ab} \leq m \Rightarrow 1 \leq m$ by 1 above.

If $m_{ab} = 1$, since $2{m_{ab}}^2 \leq \sum_{x=a \ or \ b} (m_{ax}-m_{bx})^2 + \sum_{x \neq a \ or \ b} (m_{ax}-m_{bx})^2$, then $m_{ab} = 1 = m_{ba}$ and $m_{ax} = m_{bx}$, i.e. a and b are joined to exactly the same points. This implies that it is the disjoint union of complete graphs.

3. For nonadjacent a, b, then $m-n \leq m_{ab} = 0$ and hance $m \leq n$ by (1).

4. $\gamma = m(n-m)+\mu-R \geq 0$

$\Rightarrow \mu \geq R+m(m-n) \geq R(m-n)+m(m-n)$ (since $m-n \leq 0$)

$\Rightarrow \mu \geq (R+m)(m-n)$

If $m_{ab} = m-n$ for distinct a, b, then

$$\sum_x m_{ax}m_{bx} = (v-2)(m-n)^2 = (n-2m)(m-n)+\mu$$

$\Rightarrow (\frac{(R+m)(R+m-n)}{\mu} - 2)(m-n)^2 - (n-2m)(m-n) = \mu$ (since $v\mu = (R+m)(R+m-n)$)

$\Rightarrow \mu^2 + (n-2m)(m-n)\mu - (m-n)^2((R+m)(R+m-n) - 2\mu) = 0$

$\Rightarrow [\mu - (R+m)(m-n)] \cdot [\mu + (R+m-n)(m-n)] = 0$

$\mu = (R+m)(m-n)$ or $\mu = -(R+m-n)(m-n)$.

Since $\mu \geq (R+m)(m-n)$, we assume $\mu = (R+m)(m-n) + s$ for some $s \geq 0$,

$$\begin{aligned} 0 \leq \sum_{x \neq a} (m_{ax} - \frac{R}{v-1})^2 &= \sum_{x \neq a} m_a x^2 - 2\frac{R}{v-1} \sum_{x \neq a} m_{ax} + \sum_{x \neq a} (\frac{R}{v-1})^2 \\ &= m(n-m) + \mu - \frac{R^2}{v-1} \\ &= s - R(m-n) - \frac{R^2}{v-1} \\ &= \frac{s(nR-s)}{\mu(v-1)} \end{aligned}$$

When $s = 0$, then $m_{ax} = \frac{R}{v-1} = \frac{R}{\frac{R}{m-n}} = m-n$ for each $x \neq a$. 5. For a fixed point a, the number of triangles containing a is

$$\begin{aligned} \sum_x \sum_y m_{ax} m_{xy} m_{ya} &= \sum_x m_{ax} (\sum_y m_{xy} m_{ya}) \\ &= \sum_x m_{ax}((n-2m)m_{ax} + \mu) \\ &= (n-2m)(m(n-m) + \mu) + \mu(m(n-m) + \mu - \gamma) \\ &= (n-2m+\mu)(m(n-m)+\mu) - \gamma\mu \geq 0, \end{aligned}$$

it follows that $\gamma\mu \leq (n-2m+\mu)(m(n-m)+\mu)$.

When $\gamma\mu = (n-2m+\mu)(m(n-m)+\mu)$, then $\sum_x \sum_y m_{ax} m_{xy} m_{ya} = 0$, i.e. the number of triangles is 0.

6. Since $(n-2m+\mu)(m(n-m)+\mu) - \gamma\mu \geq 0$ by (5), and $\gamma \geq 0$, then $(m(n-m)+\mu) \geq 0$

and $(n-2m+\mu) \geq 0$, hence $\mu \geq 2m-n$.

7. Let Γ be an SR multigraph $(m,\ n,\ \mu,\ \gamma,\ R)$ with an adjacency matrix M, then

$MJ = RJ \ldots (*)$

$M^2 = (n-2m)M + m(n-m)I + \mu J \ldots (**)$

Let $M' = -M$,then

$M'J = -MJ = -RJ = R'J$, and

$(M')^2 = (-M)^2 = M^2$.

Hence,

$(n'-2m')(-M) + m'(n'-m')I + \mu' J = (n-2m)M + m(n-m)I + \mu J.$

Compare the coefficients of the above two equations, we have

$m' = n-m,\ n' = n,\ \mu' = \mu,\ R' = R,$

$\gamma' = m'(n'-m') + \mu' - R' = 2m(n-m) + 2\mu - \gamma.$

By lemma 2.2 [7], it follows that if $n \leq 2m+4$, then $\gamma < 2m(n-m)+n-2m-1+\mu$, as required. ∎

Similar to SR graphs, the *matrix expressions* in terms of their adjacency matrices $A = [m_{ij}]$ for strongly regular multigraphs are given below.

Lemma 2.4. [7] Let A be an adjacency matrix of a multigraph Γ of order v, then the following are equivalent:

1. Γ is an SR multigraph $(m,\ n,\ \mu,\ \gamma,\ R)$,

2. $AJ = RJ$ and $A^2 = (n-2m)A + m(n-m)I + \mu J$
$$= (m(n-m)+\mu)I + (n-2m+\mu)A + \mu(J-I-A)$$
for some real numbers $R,\ m,\ n,\ \mu$ with $n > 0$.

Moreover, $v = \dfrac{(R+m)(R+m-n)}{\mu}$.

Proof: Let A be an adjacency matrix of an SR multigraph Γ with vertex set X.

1. Since $\sum_x m_{ax} = R$ for each $a \in X$, hence $AJ = RJ$.
2. For $a, b \in X$,

$$\sum_x m_{ax} m_{bx} = (n-2m)m_{ab} + m(n-m)\delta_{ab} + \mu$$

gives

$$A^2 = (n-2m)A + m(n-m)I + \mu J \ldots (*)$$

Multiplying both sides of (*) by J, and $J^2 = vJ$ with $v = |X|$, then

$$A^2 J = (n-2m)AJ + m(n-m)IJ + \mu J^2, \text{ and}$$

$$R^2 J = R(n-2m)J + m(n-m)J + \mu v J,$$

It follows that

$$R^2 = R(n-2m) + m(n-m) + \mu v, \text{ and}$$

$$\mu v = R^2 - (n-2m) - m(n-m) = (R+m)(R+m-n),$$

hence $v = \dfrac{(R+m)(R+m-n)}{\mu}$.

Conversely, $\sum_x m_{ax}(m_{ax} - 1) = \sum_x {m_{ax}}^2 - \sum_x m_{ax} = m(n-m) + \mu - R$ is a constant, denoted by γ, and hence $R + \gamma = m(n-m) + \mu$. ∎

Lemma 2.5. [7, 6] Suppose Γ is an SR multigraph $(m,\ n,\ \mu,\ \gamma,\ R)$, then

1. There are unique $k > 1$, $r > 0$ and $t, c \geq 0$ such that

   $$(m, n, \mu, \gamma, R) = (r, k + r + c - 1 - t, rt, rc, r(k-1)),$$

   and $(r,\ k,\ t,\ c)$ is called the *geometric parameters* of this multigraph.
2. The number of vertices of an SR graph with parameters $(m,\ n,\ \mu,\ \gamma,\ R)$, or with geometric parameter $(r,\ k,\ t,\ c)$, is

   $$v = (R+m)(R+m-n)/\mu = r((r-1)(k-1) + t - c)/t.$$

The parameter μ in SR multigraph$(m,\ n,\ \mu,\ \gamma,\ R)$ is identical with that of μ in SR graph$(v,\ k,\ \lambda,\ \mu)$ in case $\gamma = 0$. Some subsets of vertices including *claws*, *maximal* *cliques* play essential roles in the study of their geometric structures. The parameter m is the constant number of maximal cliques containing a fix vertex in the study of maximal claws under some numerical constraints.

A *clique* of a multigraph is a set of pairwise adjacent points; a clique which cannot be extended to a larger clique is called *maximal clique*. In an SR multigraph (m, n, μ, γ, R), motivated by the following lemma, a maximal clique C with $|C| > (n/2) + \mu + 1 - m$ is called a grand clique.

Lemma 2.6. [7] In an SR multigraph (m, n, μ, γ, R), an edge of multiplicity1 is in at most one grand clique.

Proof: Let ab be an edge of multiplicity 1 contained in two distinct grand cliques C and C'. Since C and C' are maximal, there is $x \in C'$ such that $C \cup \{x\}$ is not a clique, and hence there is $y \in C$ with $m_{xy} = 0$.

1. The points $z \in C \cap C'$ are adjacent to x and y and hence $|C \cap C'| \leq \sum_z m_{xz} m_{yz} = \mu$.
2. The points $z \in |C \cup C'| - \{a, b\}$ are adjacent to both a and b, whence

$$|C \cup C'| - 2 \leq \sum_z m_{az} m_{bz} = n - 2m + \mu.$$

Hence $|C| + |C'| = |C \cap C'| + |C \cup C'| \leq n + 2(\mu + 1 - m)$, this contradicts the fact that both $|C|$ and $|C'|$ are grand cliques. ∎

Theorem 2.1. [7] If C is a clique of an SR multigraph(m, n, μ, γ, R) with $\mu > 0$, then

$$|C|(R + m - \mu) \leq (n + 1 - m)(R + m).$$

Equality holds if and only if

1. every edge contained in C has multiplicity 1, and

2. for $x \notin C$, there are a constant number α of edges containing x and intersecting C; in this case, $\alpha = |C| + m - 1 - n$.

Proof: Let C be a clique with $|C| = c$ points. Define $\alpha_x = \sum_{x \in C} m_{ax}$. Then, for $x \notin C$, α is the number of edges containing x and intersecting C. We compute the expression

$$N(\alpha) = \sum_{x \notin C} (\alpha_x - \alpha)^2 + \sum_{x \in C} (\alpha_x - \alpha + m - n)(\alpha_x - \alpha + m).$$

Since

1. $\sum_x 1 = v$,
2. $\sum_x \alpha_x = \sum_{a \in C} \sum_x m_{ax} = cR$,
3. $$\begin{aligned} \sum_x {\alpha_x}^2 &= \sum_{a,\ b \in C} \sum_x m_{ax} m_{bx} \\ &= (n - 2m) \sum_{x \in C} \alpha_x + m(n - m)c + \mu c^2, \end{aligned}$$

Whence

$$\begin{aligned} N(\alpha) &= \sum_x (\alpha_x - \alpha)^2 + (2m - n) \sum_{x \in C} \alpha_x + (m(m - n) - \alpha(2m - n))|C| \\ &= \mu c^2 - c\alpha(2R + 2m - n) + \alpha^2 v \\ &= \mu^{-1}(c\mu - \alpha(R + m))(c\mu - \alpha(R + m - n)). \end{aligned}$$

In particular, for $\alpha = c\mu/(R + m)$, $N(\alpha) = 0$, and we conclude from

$$N(\alpha) = \sum_{x \notin C} (\alpha_x - \alpha)^2 + \sum_{x \in C} (\alpha_x - \alpha + m - n)(\alpha_x - \alpha + m)$$

that $\alpha_x \leq \alpha + n - m$ for all $x \in C$ since otherwise $N(\alpha)$ would be strictly positive. But, for $x \in C$, $\alpha_x = \sum_{a \in C} m_{ax} \geq c - 1$ since C is a clique. Hence $c - 1 \leq \alpha + n - m$ which leads to $|C|(R + m - \mu) \leq (n + 1 - m)(R + m)$.

If equality holds then $\alpha_x = c - 1 = \alpha + n - m$ for all $x \in C$. Hence, C contains only edges of multiplicity 1. Moreover, $N(\alpha) = 0$ implies that $\alpha_x = \alpha = c+m-1-n$ for all $x \notin C$.

Conversely, if C contains only edges of multiplicity 1, and $\alpha_x = \alpha'$ for all $x \notin C$(for some α') then $\alpha_x = c - 1$ for all $x \in C$, and we obtain from 1~3. ∎

2.2 Designs

Definition 2.6. Let X be a set of v points and $B \subseteq \binom{X}{k}$ such that any two elements of X lie in exactly λ blocks, then $(X,\ B)$ is called 2-$(v,\ k,\ \lambda)$*design*

Definition 2.7. A 2-$(v,\ k,\ \lambda)$ design has exactly b blocks, and every point occurs in exactly r blocks.

1. A 2-$(v,\ k,\ \lambda)$design is called a *symmetric design* if $b = v$(or, equivalently, $r = k$ or $\lambda(v-1) = k^2 - k$).

2. A 2-$(v,\ k,\ \lambda)$design is called a *quasi symmetric* if the cardinality of the intersection of two distinct blocks takes only two distinct values.

Note that for a 2-$(v,\ k,\ \lambda)$ design,

$r = \lambda(v-1)/(k-1)$, $b = \lambda v(v-1)/k(k-1)$ and

$v = k + \dfrac{n(k-1)}{\lambda}$ where $n = r - \lambda$ is the *order*, and

$v = 1 + \dfrac{k(k-1)}{\lambda}$ for *symmetric designs.*

Definition 2.8. [7]

1. The *block multigraph* of a 2-design $\pi = (X, \beta)$ is the multigraph defined over the set of blocks, and two distinct vertices (blocks) A, B are connected by $m_{AB} = |A \cap B|$ edges.

2. The *point multigraph*(*collinearity graph*) of a 2-design $\pi = (X, \beta)$ is the multigraph defined over the set of points, and two distinct vertices (points) x, y are connected by m_{xy} edges if they are contained in m_{xy} blocks.

3. The *block graph* of a quasi-symmetric 2-design with sizes x, y of intersections of blocks is defined over the set of blocks, and two distinct vertices (blocks) A, B are adjacent if and only if their intersection has cardinality y.

Theorem 2.2. [7] The block multigraph of a 2-(v, k, λ) design of order $n = r - \lambda$ is an SR multigraph (m, n, μ, γ, R) with

$$(m, n, \mu, \gamma, R) = (k, n, k^2\lambda, k(k-1)(\lambda-1), k(n+\lambda-1)).$$

A partial converse is given in the following theorem with some constraints over its parameters:

Theorem 2.3. [7] Every SR multigraph with parameters

$$(m, n, \mu, \gamma, R) = (k, n, k^2\lambda, k(k-1)(\lambda-1), k(n+\lambda-1))$$

for positive integers n, $k(\neq 1)$, λ, and

$$n > max\{k(k-1)\lambda^2 - (k-1)^2\lambda, 2(k-1)(k^2\lambda + k\lambda - 2\lambda + 1), \frac{1}{2}(k^2-1)(k^2\lambda - k + 2)\}$$

is isomorphic to the block multigraph of a 2-(v, k, λ) design with $v = k + \dfrac{(r-\lambda)(k-1)}{\lambda}$.

2.3 $1\frac{1}{2}$ -designs(or called partial geometric designs)

It is well known that the *block graphs* of *quasi-symmetric* 2-*designs* are strongly regular. This leads to the question whether strongly regular multigraphs can be associated with some designs of various types? The notion of *partial geometric design* with parameters (r, k, t, c) was introduced as a generalization of a *partial geometry* with parameters $(r,\ k,\ t)$(with $c=0$ above). As a generalization of strongly regular graphs(SR graph), a partial geometric design with parameters $(r,\ k,\ t,\ c)$ gives rise in a natural manner to a strongly regular multigraph(SR multigraph) whose parameters depend on r, k, t and c.

The notion of $1\frac{1}{2}$ -*designs*(called *partial geometric designs* by R.C. Bose 1976). Note that 2-designs, transversal designs, semiregular partially balanced incomplete block designs, partial geometries, and polar spaces are examples of $1\frac{1}{2}$ -designs. The *block multigraphs* of $1\frac{1}{2}$ -designs, and dually, the *point multigraphs* of weak $1\frac{1}{2}$ -designs still are strongly regular, and by investigating closely the properties of cliques and claws in a multigraph, general characterization theorems which specialize to Theorem 2.3([7]). The matrix techniques were used by Neumaier in order to get the relations among the five parameters, and then to derive the essential relations between SR multigraph and $1\frac{1}{2}$ -design.

Definition 2.9. [7] An incidence structure with an incidence matrix A is

1. a *weak 2-design* if $AJ = rJ$, $AA^T = nI + \lambda J$ and
2. a *weak* $1\frac{1}{2}$ -*design* if $AJ = rJ$, and $AA^TA = nA + \lambda JA$.

A class of incidence structure lies between 1-designs (regular) and 2-designs is considered. For an incidence structures, let

$m_{xx} = 0$, and

m_{xy}= the number of blocks containing points x and y. ([6])

Definition 2.10. [7, 6] A $1\frac{1}{2}$ -dsign (or called *partial geometric design*) with parameter $(r,\ k,\ t,\ c)$ is an incidence structure $I = (X, B)$ such that

1. each point lies on r blocks of $\boldsymbol{B}$;

2. each block consists of k points in X;

3. for a point x and a block B

   a. $t = \sum_{y \in B} m_{xy} \geq 1$ is a constant if $x \notin B$;

   b. $c = \sum_{y \in B-x} (m_{xy} - 1)$ is a constant if $x \in B$.

A *partial geometry* is a partial geometry design with parameters (r, k, t, 0) with $c = 0$, i.e., a *semilinear* incidence structure such that

1. each point lies on r blocks of B;

2. each block consists of k points in X;

3. for a point x and a block b with $x \notin B$, there are exactly t blocks contain x meeting b.

Let A be the incidence matrix of the incidence structure under consideration, and (x, B) is a pair of point and block, let $c = \sum_{y \in B-x} (m_{xy} - 1)$ for $x \in B$, and $t = \sum_{y \in B} m_{xy}$ for $x \notin B$.

If $x \notin B$, then

$$\begin{aligned} AA^T A(x, B) &= \sum_{all\ C} \sum_{all\ y} A(x, C)A(y, C)A(y, B) \\ &= \sum_{y \in B} A(x, C)A^T(C, y) \\ &= \sum_{y \in B} m_{xy} \quad \text{(say } t\text{)}. \end{aligned}$$

If $x \in B$, then

$$
\begin{aligned}
AA^TA(x,B) &= \sum_{all\ C}\sum_{all\ y} A(x,C)A(y,C)A(y,B) \\
&= 1 + (k-1) + (r-1) + \sum_{y\in B-x}(m_{xy}-1) \\
&= r + k - 1 + c.
\end{aligned}
$$

Remark[6]:

If $x \notin B$, then

$\alpha(x,B) = \sum_{C\neq B}\sum_{y\neq x} A(x,C)A^T(C,y)A(y,B) = \sum_{y\in B} m_{xy} = \alpha$ (i.e., t);

If $x \in B$, then

$\alpha(x,B) = \sum_{C\neq B}\sum_{y\neq x} A(x,C)A^T(C,y)A(y,B) = \sum_{y\in B-x}(m_{xy}-1) = n+\alpha-(r+k-1) = \beta$ (i.e., c).

Lemma 2.7. [7, 6] For a binary matrix A, the following are equivalent:

1. A is the incidence matrix of a $1\frac{1}{2}$-design with parameters $(r,\ k,\ t,\ c)$and $t \geq 1$.

2. $AJ = rJ,\ JA = kJ$ and

   $AA^TA = (r+k-1+c)A + t(J-A) = (r+k-1+c-t)A + tJ$(i.e., $nA + \alpha J$ in[7]) for some integers $r,\ k,\ t,\ c$ with $t \geq 1$.

We restate the conditions for SR graphs and SR multigraphs in terms of their adjacency matrices as following:

$$\begin{aligned}
AJ &= kJ, \\
A^2 &= kI + \lambda A + \mu(J - I - A) \\
&= (\lambda - \mu)A + (k - \mu)I + \mu J.
\end{aligned}$$

$$\begin{aligned}
AJ &= RJ, \\
A^2 &= (n - 2m)A + m(n - m)I + \mu J \\
&= (m(n - m) + \mu)I + (n - 2m + \mu)A + \mu(J - I - A).
\end{aligned}$$

$$\begin{aligned}
AJ &= rJ,\ \ JA = kJ, \\
AA^TA &= (r + k - 1 + c)A + t(J - A) = (r + k - 1 + c - t)A + tJ.
\end{aligned}$$

Lemma 2.8. [7]

1. Each 2-(v, k, λ) design is a $1\frac{1}{2}$-design with parameters
   $(r, k, t, c) = (\dfrac{\lambda(v-1)}{k-1}, k, k\lambda, (k-1)(\lambda-1))$.
2. Each $1\frac{1}{2}$-design with parameters (r, k, t, c) satisfying $(t + 1 - c - k)k = t$ is a 2-(v, k, λ) design with
   $(v, \lambda) = (1 + \dfrac{r(k-1)}{\lambda}, t + 1 - c - k)$.

Proof: 1. For a 2-(v, k, λ) design (X, B), each block consists of k points in X and each points lies on $r = \dfrac{\lambda(v-1)}{k-1}$ blocks of B. For a point x and a block b,

if $x \notin b$, then $t = \sum_{y \in b} m_{xy} = k \cdot \lambda \geq 1$;

if $x \in b$, then

$$c = \sum_{y \in b - x} (m_{xy} - 1) = (\sum_{y \in b - x} m_{xy}) - (k-1) = (k-1)\lambda - (k-1) = (k-1)(\lambda - 1).$$

Hence it is a $1\frac{1}{2}$ -design with parameters

$$(r,k,t,c) = (\frac{\lambda(v-1)}{k-1}, k, k\lambda, (k-1)(\lambda-1)).$$

2. Let A be an incidence matrix of a $1\frac{1}{2}$ -design with parameters (r, k, t, c) satisfying $(t+1-c-k)k = t$. To show $AA^T = kI + \lambda(J-I)$, consider $X = AA^T - kI - \lambda(J-I)$, then show that X^2 is the zero matrix, and hence $X = 0$ as required. ∎

Theorem 2.4. [6] The collinearity graph of a partial geometry with parameters $(r, k, t, c = 0)$ is an SR graph (v, K, λ, μ) with

$$(v, K, \lambda, \mu) = (r(\frac{(r-1)(k-1)}{t} + 1), r(k-1), (k-2) + (r-1)(t-1), rt).$$

Proof:

$$v = \frac{r((r-1)(k-1)+t-c)}{t} = \frac{r((r-1)(k-1)+t)}{t} = r\cdot(\frac{(r-1)(k-1)}{t} + 1),$$

$K = r(k-1)$ since each point lies on r blocks of B.

If x, y are in the same block, there are $(k-2)$ points in the block containing x, y. Fix x, there are $(r-1)$ blocks containing x but not containing y. Since y is not in those $(r-1)$ blocks, then $\sum_{y\in b-x} m_{xy} = t-1$ for each b of those $(r-1)$ blocks. Hence, there are $\lambda = (k-2) + (r-1)\cdot(t-1)$ points in the block containing x and y . Let x, y be in the different block, there are r blocks containing x but not y, then $\sum_{y\in b} m_{xy} = t$ for each b of those r blocks. There are $\mu = r\cdot t$ points in the same blocks containing x and y. ∎

Theorem 2.5. [6] The collinearity graph of a $1\frac{1}{2}$ -design with parameters (r, k, t, c) is an SR multigraph with parameters

$$(m, n, \mu, \lambda, R) = (r, k+r+c-1-t, rt, rc, r(k-1)).$$

Proof: Let A be an incidence matrix of a $1\frac{1}{2}$ -design with parameters (r, k, t, c), then $M = AA^T - rI$ is an adjacency matrix of the cor responding collinearity graph.

Since $AJ = rJ$, $JA = kJ$, and $AA^TA = (r + k - 1 + c - t)A + tJ$(i.e., $nA + tJ$), We have

$$MJ = AA^TJ - rIJ = rkJ - rJ = r(k-1)J,$$

$$AA^TAA^T = (AA^T)^2 = (r + k - 1 + c - t)AA^T + tJA^T = nAA^T + trJ$$

$$\Rightarrow (M + rI)^2 = n(M + rI) + trJ$$

$$\Rightarrow M^2 = (n - 2r)M + r(n - r)I + trJ.$$

If M is an adjacency matrix of an SR multigraph with parameters (m, n, μ, γ, R), then

$MJ = RJ$,

$M^2 = (n - 2m)M + m(n - m)I + \mu J.$

Compare the coefficients, then

$m = r$, $n = n = (r + k + c - 1 - t)$, $\mu = rt$,

$\gamma = m(n - m) + \mu - R = r(k + c - 1 - t) + rt - r(k - 1) = rc$, $R = r(k - 1)$. ∎

The above lemma shows that the collinearity graph of a $1\frac{1}{2}$ -design is an SR multigraph. Following this trend, we are interested in those strongly regular multigraphs which are the collinearity graph of $1\frac{1}{2}$ -designs? Theorem 2.6 provides sufficient numerical constrains to guarantee the uniqueness of such $1\frac{1}{2}$ -designs. There is no example of SR multigraphs meeting those numerical constraints found in the papers of Bose [1], Neumaier and Metsch [6]. A class of SR multigraphs associated with the distance regular graphs $Alt(n, q)$ was considered by Huang [5].

Theorem 2.6. [7] If Γ is an SR multigraph (m, n, μ, γ, R) with integral $m \geq 2$, integral $\mu \equiv 0 \bmod m$, $\mu > 0$, and

$$n > \max\{m-1+\frac{(\mu+m)\gamma}{m^2}, 2(m-1)(\mu+1-m)+2\gamma, \frac{m(m-1))}{2}(\mu+1)+m\frac{\gamma}{2}+m-1\}$$

then Γ is the point multigraph of a unique $1\frac{1}{2}$ -design, with parameters

$$(r, k, t, c) = (m, \frac{R}{m} + 1, \frac{\mu}{m}, \frac{\gamma}{m}).$$

The above bound

$$n > \max\{m-1+\frac{(\mu+m)\gamma}{m^2}, 2(m-1)(\mu+1-m)+2\gamma, \frac{m(m-1))}{2}(\mu+1)+m\frac{\gamma}{2}+m-1\}$$

was simproved by Metsch as shown below.

Theorem 2.7. [6]

Suppose that Γ is an SR multigraph whose parameters

$$(m, n, \mu, \gamma, R) = (r, k + r + c - 1 - t, rt, rc, r(k - 1))$$

with integers $r \geq 3$ and $t \geq 1$, and real numbers $k > 0$ and $c \geq 0$. If

$$k > (\frac{8}{\sqrt{3}}r + r + 5)rt \approx 5, 6r^2t,\ k > (c + 1)t, \text{ and } r(c + r - 1) \leq (r - 1)t,$$

then Γ is the collinearity graph of a $1\frac{1}{2}$ -design with parameters (r, k, t, c).

A construction method for cliques in multigraphs was proposed by Metsch [6], this method generalized the ideas used in improving the well-known completion theorem for nets of Bruck. The bound for k in the above 2 improves previous bounds given by Bose et al. [1] and Neumaier [7], however note that the condition $r(c+r-1) \leq (r-1)t$ did not occur in [7].

Corollary 2.1. [6] Suppose the parameters (m, n, μ, γ, R) of an SR multigraph can be written in the form

$$(m, n, \mu, \gamma, R) = (k, r - \lambda, k^2\lambda, k(k - 1), k(r - 1))$$

for some integers $k \geq 3$, r, and λ. If

$$r > (\frac{8}{\sqrt{3}}k + k + 5)k^2\lambda \approx 5, 6K^3\lambda, \text{ and } r > k(k - 1)\lambda^2 - k(k - 2)\lambda,$$

then Γ is the block-multigraph of a 2-(v, k, λ) design with point degree r.

Corollary 2.2. [7] Two distinct blocks A and B of a 2-(v, k, λ) design intersect in at least $k - r + \lambda$ points.

Theorem 2.8. [1, 6] A quasi-residual 2-(w, n, λ) design **B** is embeddable iff the following three conditions are satisfied:

1. Any distinct blocks A and B intersect $\mu_{AB} \leq \lambda$ points,

2. The multigraph Γ on the blocks, with $m_{AB} = \lambda - \mu_{AB}$ edges between A and B, is a strongly regular multigraph $SR(m, n, \mu, \gamma, R)$, where
   $m = \lambda$, n, $\mu = \lambda^2(\lambda - 1)$, $\gamma = \lambda(\lambda - 1)(\lambda - 2)$, $R = \lambda(n + \lambda - 2)$,

3. Γ is isomorphic to the block multigraph of a 2-($n + \lambda$, λ, $\lambda - 1$) design **B'**.

Theorem 2.9. [7] Let **B** be a quasi-residual 2-(w, n, λ) design with

$n \geq 2\lambda^3 - 4\lambda^2 + 4\lambda - 1$.

Then two distinct blocks intersect in at most λ points, and property 2 of Theorem 2.7 holds.

Proof: Since **B** is quasi-residual, $r = n + \lambda$, $b = r(r-1)/\lambda$. Hence, the incidence matrix A of **B** satisfies $AJ = nJ$, $JA = (n + \lambda)J$, $AA^T = nI + \lambda J$. By straightforward calculation, the matrix $M = (n - \lambda)I + \lambda J - AA^T$ satisfies $MJ = RJ$, $M^2 = (n - 2m)M + m(n - m)I + \mu J$, $n > 0$ with $m = \lambda$, n, $\mu = \lambda^2(\lambda - 1)$, $R = \lambda(n + \lambda - 2)$. Hence, with $\gamma = \lambda(\lambda - 1)(\lambda - 2)$, Lemma 2.2 applies. Therefore M is the adjacency matrix of a $SR(m, n, \mu, \gamma, R)$, i.e., 2 of Theorem 2.7 holds. In particular, the offdiagonal entries $\lambda - \mu_{AB}$ of M are nonnegative, i.e., two distinct blocks A and B intersect in $\mu_{AB} \leq \lambda$ points. ∎

The next two results are preliminary conditions for an SR multigraph to be the point multigraph of a weak $1\frac{1}{2}$ -design, respective a $1\frac{1}{2}$ -design.

Theorem 2.10. [7] An SR multigraph(m, n, μ, γ, R) is the point multigraph of a weak $1\frac{1}{2}$ -design if and only if there is a collection $\sum$ of cliques such that every point is in exactly m cliques of $\sum$, and every edge ab of multiplicity m_{ab} is in exactly m_{ab} cliques of $\sum$. In this case the blocks are the cliques of $\sum$, and the weak $1\frac{1}{2}$ -design has parameters (v, m, r, λ) with

$$(v, m, r, \lambda) = (\frac{(R+m)(R+m-n)}{\mu}, n, m, \frac{\mu}{R+m}).$$

Proof: Let g be an SR(m, n, μ, γ, R). If g is the point multigraph of a weak $1\frac{1}{2}$ -design **B** then the blocks of **B** are cliques in g, and $\sum$=**B** satisfies the Conditions of the theorem.

Conversely, if $\sum$ is a collection of cliques with the stated properties, then define a design **B** with $\sum$ as set of blocks and natural incidence. If A is the incidence matrix of **B**, then the assumed properties can be stated in terms of A and the adjacency matrix M of g as $AJ = mJ$, $AA^T = M + mI$.

With $\lambda = \mu/(R+m)$, the property that g is a $SR(m, n, \mu, \gamma, R)$ means $MJ = RJ$, $(M+mI)(M+(m-n)I-\lambda J) = 0$. Hence $X = (AA^T - nI - \lambda J)A$ satisfies $XX^T = 0$, whence $X = 0$. Therefore, $AA^TA = nA + \lambda JA$, and by $v\mu = (R+m)(R+m-n)$, **B** is a weak $1\frac{1}{2}$ -design with parameters as required. ∎

3 The graphs and multigraphs associated with some designs

Theorem 3.1. The block graph of a symmetric design is the complete graph K_b, and the adjacency matrix of the block multigarph of a symmetric design is $\lambda(J-I)$.

Proof: Since (X, B) is a symmetric design, any two blocks have λ common points in any two blocks are adjacent. ∎

Theorem 3.2. Let (X, B) be a quasi-symmetric $2-(v, k, \lambda)$ design with sizes x and y of intersections of blocks, then

1. the block graph is a SR graph (v', R, λ', μ) with $(v', R, \lambda', \mu) = (b, R, (\theta_1+\theta_2)+\frac{f(R)}{b}, \frac{f(R)}{b})$, where $R = \frac{k(r-1)-x(b-1)}{(y-x)} = -\theta_1\theta_2 + \frac{f(R)}{b}$,
   $\theta_1 = \frac{r-\lambda-k+x}{(y-x)}$, $\theta_2 = \frac{x-k}{y-x}$, are three distinct eigenvalues of A,
   $f(R) = (R-\theta_1)(R-\theta_2)$, $b = \frac{\lambda v(v-1)}{k(k-1)}$, and
2. its block multigraph is a SR multigraph (m, n, μ, γ, R) with $(m, n, \mu, \gamma, R) = (k, r-\lambda, \lambda k^2, k(k-1)(\lambda-1), k(r-1))$.

Proof: To prove 1, let N be the $v \times b$ ncidence matrix of the design and A be the adjacency matrix of its block graph Γ. We have (using the parameters v, k, b, r, λ of the 2 - design):

$$NN^T = (r-\lambda)I + \lambda J,\ N^TN = kI + yA + x(J-I-A).$$

We know that both NN^T and N^TN have all-one eigenvectors j with eigenvalue kr. Also, we know that NN^T has only the eigenvalue $r-\lambda$ on j^T, with multiplicity $v-1$. Therefore N^TN has this same eigenvalue, with the same multiplicity, and the eigenvalue 0 with multiplicity $b-v$. Since $x \neq y$, A is a linear combination of I, J, and N^TN. Therefore A has eigenvector j and only two eigenvalues on the space j^T. They are $(r-\lambda-k+x)/(y-x)$ with multiplicity $v-1$ and $(x-k)/(y-x)$ with multiplicity $b-v$. By our observation above, Γ is an SR graph.

$$A = \frac{1}{(y-x)}N^TN - \frac{(k-x)}{(y-x)}I - \frac{x}{(y-x)}J \quad (*)$$

Multiplying both sides of (*) by J, and $J^2 = bJ$, then

$$AJ = RJ = \frac{kr}{(y-x)}J - \frac{(k-x)}{(y-x)}J - \frac{xb}{(y-x)}J$$
$$\Rightarrow R = \frac{kr-(k-x)-xb}{(y-x)} = \frac{k(r-1)-x(b-1)}{(y-x)}$$

Since R, $\theta_1 = \dfrac{r-\lambda-k+x}{(y-x)}$, $\theta_2 = \dfrac{x-k}{(y-x)}$ are three distinct eigenvalues of A

Let $f(x) = (x-\theta_1)(x-\theta_2) = x^2-(\theta_1+\theta_2)x+\theta_1\theta_2$

Then $A^2 = (\theta_1+\theta_2)A - \theta_1\theta_2 I + \dfrac{f(R)}{b}J$, where $b = \dfrac{\lambda v(v-1)}{k(k-1)}$

Thus $\lambda' = (\theta_1+\theta_2) + \dfrac{f(R)}{b}$, $\mu = \dfrac{f(R)}{b}$.

To prove 2, let $M = N^TN - kI = yA + x(J-I-A)$ be a adjacency matrix of a block multigraph of the design

$$\begin{aligned} MJ &= (N^TN-kI)J \\ &= k(r-1)J \\ &= RJ \\ \Rightarrow R &= k(r-1) \\ M^2 &= (N^TN-kI)^2 \\ &= N^TNN^TN - 2kN^TN + k^2I \\ &= (r-\lambda-2k)(N^TN-kI) + k(r-\lambda-k)I + \lambda k^2 J \\ &= (r-\lambda-2k)M + k(r-\lambda-k)I + \lambda k^2 J \end{aligned}$$

Compare the coefficients with Lemma 2.4

$m = k$, $n = r - \lambda$, $\mu = \lambda k^2$, $\gamma = m(n-m) + \mu - R = k(k-1)(\lambda - 1)$. ∎

Theorem 3.3. [7] The block multigraph of a 2-(v, k, λ) design is an SR multigraph with $(m, n, \mu, \gamma, R) = (k, r - \lambda, \lambda k^2, k(k-1)(\lambda - 1), k(r-1))$.

Proof: Similarly to Theorem 3.2 (2). ∎

Remark: when are the above multigraphs simple graphs? Are they designs with some interests?

Seven examples of 2-designs together with the related graphs and multigraphs are given below:

2-$(v,\ k,\ \lambda)$	SR$(m,\ n,\ \mu,\ \gamma,\ R)$
2 - (9, 3, 1)	SR(3, 3, 9, 0, 9)
2 - (6, 3, 2)	SR(3, 3, 18, 6, 12)
2 - (8, 4, 3)	SR(4, 4, 48, 24, 24)
2 - (10, 4, 2)	SR(4, 4, 32, 12, 20)
2 - (16, 4, 1)	SR(4, 4, 16, 0, 16)
2 - (16, 6, 2)	SR(6, 4, 72, 30, 30)
2 - (16, 6, 3)	SR(6, 6, 108, 60, 48)

Example 1: 2 - (6, 3, 2) design

Let $X = \{1, 2, 3, 4, 5, 6\}$, then (X, B) is a 2 - (6, 3, 2) design where $B = \{B_x | 1 \leq x \leq 10\}$,

$B_1 = \{1, 2, 3\}$, $B_2 = \{1, 2, 4\}$, $B_3 = \{1, 3, 5\}$, $B_4 = \{1, 4, 6\}$, $B_5 = \{1, 5, 6\}$,

$B_6 = \{2, 3, 6\}$, $B_7 = \{2, 4, 5\}$, $B_8 = \{2, 5, 6\}$, $B_9 = \{3, 4, 5\}$, $B_{10} = \{3, 4, 6\}$.

Note that $|B_i \cap B_j| = 1$ or 2 for distinct $1 \leq i, j \leq 10$, that is it is quasi-symmetric.

Note also that $r = \dfrac{\lambda(v-1)}{(k-1)} = \dfrac{2 \cdot (6-1)}{(3-1)} = 5$.

The block multigraph of Γ the above 2 - (6, 3, 2) design of order $n = r - \lambda = 3$ is an SR multigraph $(m, n, \mu, \gamma, R) = (k, r - \lambda, \lambda k^2, k(k-1)(\lambda - 1), k(r-1)) =$

$(3, 3, 18, 6, 12)$. The block graph of the 2 - (6, 3, 2) design is the Petersen graph(see Figure 1).

Example 2: 2 - (8, 4, 3) design

Let $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$, then (X, B) is a 2 - (8, 4, 3) design where $B = \{B_x | 1 \leq x \leq 14\}$,

$B_1 = \{0, 1, 2, 3\}$, $B_2 = \{0, 1, 2, 4\}$, $B_3 = \{0, 1, 5, 6\}$, $B_4 = \{0, 2, 5, 7\}$, $B_5 = \{0, 3, 4, 5\}$, $B_6 = \{0, 3, 6, 7\}$, $B_7 = \{0, 4, 6, 7\}$, $B_8 = \{1, 2, 6, 7\}$, $B_9 = \{1, 3, 4, 6\}$, $B_{10} = \{1, 3, 5, 7\}$, $B_{11} = \{1, 4, 5, 7\}$, $B_{12} = \{2, 3, 4, 7\}$, $B_{13} = \{2, 3, 5, 6\}$, $B_{14} = \{2, 4, 5, 6\}$.

Note that $r = \dfrac{3 \cdot (8-1)}{(4-1)} = 7$. The block multigraph Γ of 2 - (8, 4, 3) design of order $n = 4$ is an SR multigraph (4, 4, 48, 24, 24).

Example 3: 2 - (9, 3, 1) design

Let $X = \{1, 2, 3, \ldots, 9\}$, and those 9 elements are arranged in a 3×3 array as show below:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}$$

Then (X, B) is a 2 - (9, 3, 1) design where $B = \{B_x | 1 \leq x \leq 12\}$, and

slope 0, $B_1 = \{1, 2, 3\}$, $B_2 = \{4, 5, 6\}$, $B_3 = \{7, 8, 9\}$,

slope ∞, $B_4 = \{1, 4, 7\}$, $B_5 = \{2, 5, 8\}$, $B_6 = \{3, 6, 9\}$,

slope 1, $B_7 = \{1, 6, 8\}$, $B_8 = \{2, 4, 9\}$, $B_9 = \{3, 5, 7\}$,

slope -1, $B_{10} = \{1, 5, 9\}$, $B_{11} = \{2, 6, 7\}$, $B_{12} = \{3, 4, 8\}$.

The block multigraph Γ of the above 2 - (9, 3, 1) design of order $n = 3$ is an SR multigraph (3, 3, 9, 0, 9).

Since $\gamma = 0$, Γ is an SR graph with parameters

$$(v, k, \lambda, \mu) = (\frac{R(R-n+2m-\mu-1)}{\mu} + R + 1, R, n-2m+\mu, \mu) = (12, 9, 6, 9).$$

Example 4: a 2 - (10, 4, 2) design

Let $X = \{0, 1, 2, \ldots, 9\}$ and $B = \{B_x | 1 \leq x \leq 15\}$, where

$B_1 = \{0, 1, 2, 3\}$, $B_2 = \{0, 1, 4, 5\}$, $B_3 = \{0, 2, 4, 6\}$, $B_4 = \{0, 3, 7, 8\}$,

$B_5 = \{0, 5, 7, 9\}$, $B_6 = \{0, 6, 8, 9\}$, $B_7 = \{1, 2, 7, 8\}$, $B_8 = \{1, 3, 6, 9\}$,

$B_9 = \{1, 4, 7, 9\}$, $B_{10} = \{1, 5, 6, 8\}$, $B_{11} = \{2, 3, 5, 9\}$, $B_{12} = \{2, 4, 8, 9\}$,

$B_{13} = \{2, 5, 6, 7\}$, $B_{14} = \{3, 4, 5, 8\}$, $B_{15} = \{3, 4, 6, 7\}$.

Note that $|B_i \cap B_j| = 1$ or 2 for distinct $1 \leq i, j \leq 15$, and hence it is a quasi-symmetric design; note also that $r = 6$. The block multigraph Γ of 2 - (10, 4, 2) design of order $n = 4$ is an SR multigraph (4, 4, 32, 12, 20).

Example 5: a 2 - (16, 4, 1) design

Let $X = \{0, 1, 2, \ldots, 9, a, b, \ldots, f\}$, then (X, B) is a 2 - (16, 4, 1) design where $B = \{B_x | 1 \leq x \leq 20\}$,

$B_1 = \{0, 1, 2, 3\}$, $B_2 = \{0, 4, 5, 6\}$, $B_3 = \{0, 7, 8, 9\}$, $B_4 = \{0, a, b, c\}$,

$B_5 = \{0, d, e, f\}$, $B_6 = \{1, 4, 7, a\}$, $B_7 = \{1, 5, b, d\}$, $B_8 = \{1, 6, 8, e\}$,

$B_9 = \{1, 9, c, f\}$, $B_{10} = \{2, 4, c, e\}$, $B_{11} = \{2, 5, 7, f\}$, $B_{12} = \{2, 6, 9, b\}$,

$B_{13} = \{2, 8, a, d\}$, $B_{14} = \{3, 4, 9, d\}$, $B_{15} = \{3, 5, 8, c\}$, $B_{16} = \{3, 6, a, f\}$,

$B_{17} = \{3, 7, b, e\}$, $B_{18} = \{4, 8, b, f\}$, $B_{19} = \{5, 9, a, e\}$, $B_{20} = \{6, 7, c, d\}$,

Note that $|B_i \cap B_j| = 0$ or 1 for distinct $1 \leq i, j \leq 20$, and hence it is a quasi-symmetric design. $r = 5$. The block multigraph Γ of the above 2 - (16, 4, 1) design of order $n = 4$ is an SR multigraph (4, 4, 16, 0, 16).

Since $\gamma = 0$, that is Γ is an SR graph (7, 16, 12, 16).

Example 6: a 2-(16, 6, 2) design

Let $X = \{0, 1, 2, \dots, 15\}$, and those 16 elements are arranged in a 4×4 array A as shown below:

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{array}$$

For each x, $0 \le x \le 15$, we define a block B_x consisting of the elements in the same row or column of A as x, excluding x. Then (X, B) is a 2-(16, 6, 2) design where $B = \{B_x | 0 \le x \le 15\}$. More precisely, 2 - (16, 6, 2) design

$B_0 = \{1, 2, 3, 4, 8, 12\}$, $B_1 = \{0, 2, 3, 5, 9, 13\}$, $B_2 = \{0, 1, 3, 6, 10, 14\}$, $B_3 = \{0, 1, 2, 7, 11, 15\}$,

$B_4 = \{5, 6, 7, 0, 8, 12\}$, $B_5 = \{4, 6, 7, 1, 9, 13\}$, $B_6 = \{4, 5, 7, 2, 10, 14\}$, $B_7 = \{4, 5, 6, 3, 11, 15\}$,

$B_8 = \{9, 10, 11, 0, 4, 12\}$, $B_9 = \{8, 10, 11, 1, 5, 13\}$, $B_{10} = \{8, 9, 11, 2, 6, 14\}$,

$B_{11} = \{8, 9, 10, 3, 7, 15\}$, $B_{12} = \{13, 14, 15, 0, 4, 8\}$, $B_{13} = \{12, 14, 15, 1, 5, 9\}$,

$B_{14} = \{12, 13, 15, 2, 6, 10\}$, $B_{15} = \{12, 13, 14, 3, 7, 11\}$.

The block multigraph Γ of the above 2 - (16, 6, 2) design of order $n = 4$ is an SR multigraph (6, 4, 72, 30, 30). Moreover, since it is symmetric, the adjacency matrix A of Γ is

$$A = \begin{pmatrix} 0 & & 2 \\ & \ddots & \\ 2 & & 0 \end{pmatrix}_{16 \times 16} = 2J - 2I$$

Example 7: a 2-(16, 6, 3) design

Let $X = \{1, 2, \dots, 24\}$, then (X, B) is a 2 - (16, 6, 3) design with $B = \{B_x | 0 \le x \le 24\}$, where

(i) $B_1 = \{1, 2, 5, 6, 9, 10\}$, $B_2 = \{1, 3, 5, 7, 9, 11\}$, $B_3 = \{1, 4, 5, 8, 9, 12\}$;

$B_4 = \{3, 4, 7, 8, 11, 12\}$, $B_5 = \{2, 4, 6, 8, 10, 12\}$, $B_6 = \{2, 3, 6, 7, 10, 11\}$;

(ii) $B_7 = \{1, 2, 7, 8, 15, 16\}$, $B_8 = \{1, 3, 6, 8, 14, 16\}$, $B_9 = \{1, 4, 6, 7, 14, 15\}$;

$B_{10} = \{3, 4, 5, 6, 13, 14\}$, $B_{11} = \{2, 4, 5, 7, 13, 15\}$, $B_{12} = \{2, 3, 5, 8, 13, 16\}$;

(iii) $B_{13} = \{1, 2, 11, 12, 13, 14\}$, $B_{14} = \{1, 3, 10, 12, 13, 15\}$, $B_{15} = \{1, 4, 10, 11, 13, 16\}$;

$B_{16} = \{3, 4, 9, 10, 15, 16\}$, $B_{17} = \{2, 4, 9, 11, 14, 16\}$, $B_{18} = \{2, 3, 9, 12, 14, 15\}$;

(iv) $B_{19} = \{5, 6, 11, 12, 15, 16\}$, $B_{20} = \{5, 7, 10, 12, 14, 16\}$, $B_{21} = \{5, 8, 10, 11, 14, 15\}$;

$B_{22} = \{7, 8, 9, 10, 13, 14\}$, $B_{23} = \{6, 8, 9, 11, 13, 15\}$, $B_{24} = \{6, 7, 9, 12, 13, 16\}$.

The 6 blocks $\{B_1, B_2, B_3, B_4, B_5, B_6\}$ in case (i) satisfying the conditions that

1. $|B_i \cap B_j| = \begin{cases} 0 & \text{if } |i-j| = 3 \\ 3 & \text{if } |i-j| \neq 3 \end{cases}$ for $i \neq j$, $i, j \in \{1, 2, \ldots, 6\}$
2. Each block B_i of (i) meet 2 points $|B_i \cap B_j| = 2$ for B_k is any block of (ii), (iii), or (iv).

Similarly conclusion hold for cases (ii), (iii), and (iv).

Then very vertex (i.e., a block) lies on $3 \times 4 + 2 \times 18 = 48$ edges, the block multigraph of the above 2 - (16, 6, 3) design of order is an SR multigraph (6, 6, 108, 60, 48).

4 Regular graphs of 3 or 4 distinct eigenvalues

Lemma 4.1. Let Γ be a graph which is not complete or empty, with adjacency matrix A Then Γ is an SR graph if and only if A^2 is a linear combination of A, I and J.

Proof: The ij - entry of A^2 is equal to the number of walks of length two from i to j in Γ. If Γ is an SR graph with parameters k, λ, μ according as i and j are equal, adjacent or distinct and non - adjacent, hence $A^2 = kI + \lambda I + \mu(J - I - A)$. Conversely, if A^2 is a linear combination of A, I and J. $A^2 = m_1A + m_2I + m_3A$ this number are $(m_2 + m_3)$, $(m_1 + m_3)$, m_3 according as i and j are equal, adjacent or distinct and non-adjacent. Hence Γ is an SR graph. ∎

Lemma 4.2. If Γ is a connected graph with diameter d then $A(\Gamma)$ has at least $d+1$ distinct eigenvalues, or equivalently if Γ is a graph with $d+1$ distinct eigenvalues, the the diameter of Γ is at most d.

Proof: Suppose A=$A(\Gamma)$ has distinct eigenvalues $\theta_0, \theta_1, \theta_2, \cdots, \theta_m$ where $m < d$. Then $m(x) = \prod_{i=0}^{m}(x-\theta_i)$ is the minimal polynomial of A, and hence $A^{d-(m+1)\cdot m(A)} = 0$, then we have $A^d = C_{d-1}A^{d-1} + C_{d-2}A^{d-2} + \cdots + C_1A + C_0I$ for some $C_i \in \mathbb{R}$. For two vertices $x, y \in V(\Gamma)$ with $\partial(x, y) = d$, the xy position in the above equation and $0 \neq (A^d)_{xy} = C_{d-1}(A^{d-1})_{xy} + C_{d-2}(A^{d-2})_{xy} + \cdots + C_1A_{xy} + C_0I_{xy} = 0$, a contradiction; so $A(\Gamma)$ has at least $d + 1$ distinct eigenvalues. ∎

Theorem 4.1. Let Γ be a connected k-regular graph with s distinct eigenvalues,

1. if $s = 2$, then Γ is complete graph.

2. if $s = 3$, with distinct eigenvalues $k > \theta_1 > \theta_2$, then Γ is an SR graph.

Proof: To prove 1, by above lemma, the diameter $d(\Gamma)$ of Γ is 0 or 1. Since Γ is connected, $d(\Gamma) \neq 0$, hence $d(\Gamma) = 1$, and Γ is a complete graph.

To prove 2, let A be an adjacency matrix of Γ, and

$f(x) = (x - \theta_1)(x - \theta_2) = x^2 - (\theta_1 + \theta_2)x + \theta_1\theta_2$,

Then $A^2 = (\theta_1 + \theta_2)A - \theta_1\theta_2 I + \frac{f(k)}{n}J$. Hence Γ is an SR graph with $(v, k, \lambda, \mu) = (v, -\theta_1\theta_2 + \frac{f(k)}{v}, \theta_1 + \theta_2 + \frac{f(k)}{v}, \mu)$. ∎

Theorem 4.2. Let Γ be a connected k-regular graph with 4 distinct eigenvalues $k > \theta_1 > \theta_2 > \theta_3$, then

1. Γ is walk regular;
2. the diameter of Γ is 3 if the number of vertices of the graph is more than $\frac{\alpha + \sqrt{\alpha^2 - 4\beta}}{2}$ or less than $\frac{\alpha - \sqrt{\alpha^2 - 4\beta}}{2}$, where
   $\alpha = 1 + k^2 - (\theta_1 + \theta_2 + \theta_3)k - \theta_1\theta_2\theta_3$,and
   $\beta = (k - \theta_1)(k - \theta_2)(k - \theta_3)$.

Proof: To prove 1, let Γ be a regular graph with 4 distinct eigenvalues, and A be an adjacency matrix of Γ.

$$\begin{aligned} f(x) &= (x - \theta_1)(x - \theta_2)(x - \theta_3) \\ &= x^3 - (\theta_1 + \theta_2 + \theta_3)x^2 + (\theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1)x - \theta_1\theta_2\theta_3 \end{aligned}$$

Let $\alpha_0 = \theta_1+\theta_2+\theta_3$, $\alpha_1 = \theta_1\theta_2+\theta_2\theta_3+\theta_3\theta_1$, $\alpha_2 = \theta_1\theta_2\theta_3$ and $\beta = (k-\theta_1)(k-\theta_2)(k-\theta_3)$, then $A^3 - \alpha_0 A^2 + \alpha_1 A - \alpha_2 I = \frac{\beta}{n}J$ (*), where $n = |V(\Gamma)|$, and hence $A^3 = \alpha_0 A^2 - \alpha_1 A + \alpha_2 I + \frac{\beta}{n}J$.

Since the diagonal entries of A^2, A, I, J are constant, the diagonal entries of A^3 are constant $\alpha_0 k + \alpha_2 + \frac{\beta}{n}$. Multiplying both sides of (*) by A will give a recursive formula for the diagonal entries of $A^4, A^5, A^6, \cdots$, etc are constant. Hence all regular graphs with 4 distinct eigenvalues are walk regular.

To prove 2, let A be an adjacency matrix of Γ, and

$$\begin{aligned} f(x) &= (x-\theta_1)(x-\theta_2)(x-\theta_3) \\ &= x^3 - (\theta_1+\theta_2+\theta_3)x^2 + (\theta_1\theta_2+\theta_2\theta_3+\theta_3\theta_1) - \theta_1\theta_2\theta_3 \end{aligned}$$

Then

$$A^3 - (\theta_1+\theta_2+\theta_3)A^2 + (\theta_1\theta_2+\theta_2\theta_3+\theta_3\theta_1)A - \theta_1\theta_2\theta_3 I = \frac{\beta}{n}J(*)$$

and hence the diameter of Γ is at most 3.

We will claim that $k_3(x) > 0$ for each $x \in V(\Gamma)$ under the numerical constraints. For $x \in V(\Gamma)$, we first evaluate A^3_{xx}:

$$\begin{aligned} A^3_{xx} &= \sum_{y\in\Gamma_1(x)} A^2_{xx} \text{ by definition, and} \\ A^3_{xx} &= \frac{f(k)}{n} + (\theta_1+\theta_2+\theta_3)A^2_{xx} - (\theta_1\theta_2+\theta_2\theta_3+\theta_3\theta_1)A_{xx} + \theta_1\theta_2\theta_3 I_{xx} \\ &= \frac{f(k)}{n} + (\theta_1+\theta_2+\theta_3)k + \theta_1\theta_2\theta_3 \text{ by (*).} \end{aligned}$$

Hence,

$$\begin{aligned} &\sum_{y\in V(\Gamma)} A^2_{xy} = (A^2J)_{xx} = k^2, \\ &\sum_{y\in\Gamma_1(x)} A_{xy} = A^2_{xx} = k, \\ &\sum_{y\in\Gamma_1(x)} A^2_{xy} = A^3_{xx} = \frac{\beta}{n} + (\theta_1+\theta_2+\theta_3)k + \theta_1\theta_2\theta_3, \end{aligned}$$

and then

$$\sum_{y\in\Gamma_2(x)} A_{xy}^2 = \sum_{y\in V(\Gamma)} A_{xy}^2 - A_{xx}^2 - \sum_{y\in\Gamma_1(x)} A_{xy}^2 - \sum_{y\in\Gamma_3(x)} A_{xy}^2$$
$$= k^2 - k - \frac{\beta}{n} - (\theta_1+\theta_2+\theta_3)k - \theta_1\theta_2\theta_3 \geq k_2(x).$$

It follows that

$$1+k+k_2(x) \leq 1+k^2-k-\frac{\beta}{n}-(\theta_1+\theta_2+\theta_3)k-\theta_1\theta_2\theta_3$$
$$= \alpha - \frac{\beta}{n},$$

whenever $n > \dfrac{\alpha+\sqrt{\alpha^2-4\beta}}{2}$ or $n < \dfrac{\alpha-\sqrt{\alpha^2-4\beta}}{2}$, as required. ∎

Theorem 4.3. [2] Let Γ be a connected k-regular graph on v vertices with Spec(Γ)=($k^1, \theta_1^{m_1}, \theta_2^{m_2}, \theta_3^{m_3}$). Then

1. $m_1 = m_2 = m_3 = (v-1)/3$ and $k = (v-1)/3$ or $2(v-1)/3$, or

2. Γ has two or four integral eigenvalues.

Moreover, if Γ has exactly two integral eigenvalues, then the other two have the same mulitplicities and are of the form $\frac{1}{2}(a \pm \sqrt{b})$ for $a, b \in \mathbb{Z}$.

Theorem 4.4. Let Γ be a connected regular graph on v vertices with four distinct eigenvalues, say $Spec(\Gamma) = (k^1, \theta_1^{m_1}, \theta_2^{m_2}, \theta_3^{m_3})$.

Let $\lambda = (k^3 + m_1{\theta_1}^3 + m_2{\theta_2}^3) + m_3{\theta_3}^3/vk$. Then Γ is distance-regular if and only if for every vertex x the number of vertices $k_2(x)$ at distance two from x is

$k_2(x) = \dfrac{k(k-1-\lambda)^2}{(k-\lambda)(\lambda-k-(\theta_1+\theta_2+\theta_3)-(\theta_1\theta_2+\theta_2\theta_3+\theta_3\theta_1)+\theta_1\theta_2\theta_3)}.$

It was conjectured by van Dam [2] that the proposition was also true without the conditions for $k_2(x)$, i.e., that for every connected regular graph with four distinct eigenvalues we have that the number of vertices k_2 at distance two from a given vertex is at least $k_2(x)$.

The following are examples of some connected regular graphs with 4 distinct eigenvalues and with diameter 2, all of them are walk-regular, though some of them are not distance regular.

Example 1: $\Gamma = \overline{C_6}$, the complement of C_6(see Figure 2)

$$A(\overline{C_6}) = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

the characteristic polynomial of $A(\overline{C_6})$ is $f(x) = x^2(x-1)(x-3)(x+2)^2$, and $Spec(\overline{C_6}) = (3^1, 1^1, 0^2, -2^2)$, this graph is walk regular.

Example 2: $\Gamma = \overline{Q_3}$, the complement of Q_3(see Figure 3)

$$A(\overline{Q_3}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

the characteristic polynomial of $A(\overline{Q_3})$ is $f(x) = x^3(x-2)(x-4)(x+2)^3$, and $Spec(\overline{Q_3}) = (4^1, 2^1, 0^3, -2^3)$, this graph is walk regular.

Example 3: $\Gamma = \overline{2C_4}$, the complement of $2C_4$(see Figure 4)

$$A(\overline{2C_4}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

the characteristic polynomial of $A(\overline{2C_4})$ is $f(x) = (x-5)(x-1)^2(x+1)^4(x+3)$, and $Spec(\overline{2C_4}) = (5^1, 1^2, -1^4, -3^1)$, this graph is walk regular.

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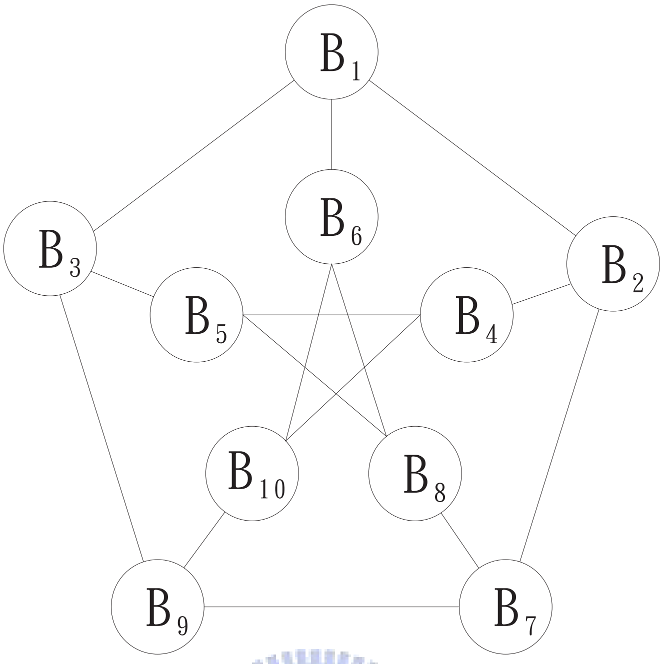

Figure 1

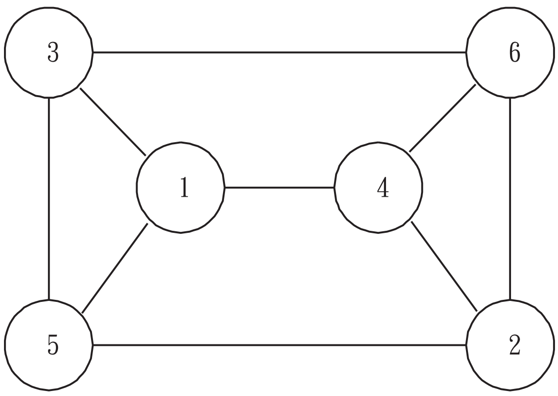

Figure 2

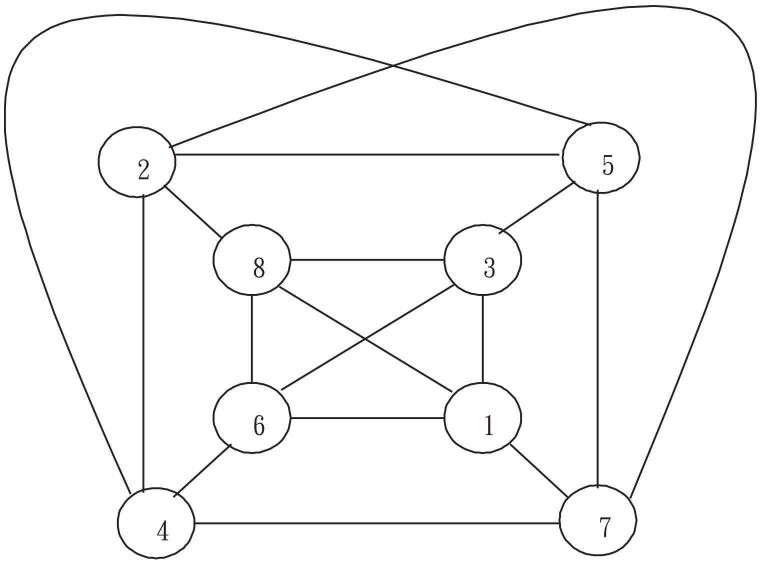

Figure 3

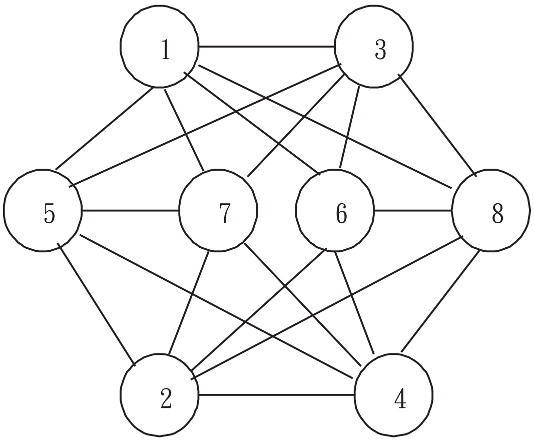

Figure 4