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3-bounded property in a triangle-free distance-regular graph☆

Yeh-jong Pan[1], Chih-wen Weng[1]

Department of Applied Mathematics, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu, Taiwan 300, ROC

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Abstract

Let Γ denote a distance-regular graph with classical parameters (D, b, α, β) and $D \geq 3$. Assume the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. We show that Γ is 3-bounded in the sense of the article [C. Weng, D-bounded distance-regular graphs, European Journal of Combinatorics 18 (1997) 211–229].
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1. Introduction

Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geq 3$ and distance function ∂. Recall that a sequence x, y, z of vertices of Γ is *geodetic* whenever

$$\partial(x, y) + \partial(y, z) = \partial(x, z).$$

A sequence x, y, z of vertices of Γ is *weak-geodetic* whenever

$$\partial(x, y) + \partial(y, z) \leq \partial(x, z) + 1.$$

Definition 1.1. A subset $\Omega \subseteq X$ is *weak-geodetically closed* if for any weak-geodetic sequence x, y, z of Γ,

$$x, z \in \Omega \Longrightarrow y \in \Omega.$$

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E-mail addresses: yjp.9222803@nctu.edu.tw (Y.-j. Pan), weng@math.nctu.edu.tw (C.-w. Weng).
[1] Fax: +886 3 5724679.

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Weak-geodetically closed subgraphs are called *strongly closed subgraphs* in [8]. We refer the reader to [7,3,5,9,12,4] for information on weak-geodetically closed subgraphs.

Definition 1.2. Γ is said to be i-*bounded* whenever for all $x, y \in X$ with $\partial(x, y) \leq i$, there is a regular weak-geodetically closed subgraph of diameter $\partial(x, y)$ which contains x, y.

The properties of D-bounded distance-regular graphs were studied in [13], and these properties were used in the classification of classical distance-regular graphs of negative type [14]. Before stating our main result we give one more definition.

By a *parallelogram of length* i, we mean a 4-tuple $xyzw$ consisting of vertices of Γ such that $\partial(x, y) = \partial(z, w) = 1$, $\partial(x, z) = i$, and $\partial(x, w) = \partial(y, w) = \partial(y, z) = i - 1$.

It was proved that if $a_1 = 0$, $a_2 \neq 0$ and Γ contains no parallelograms of length 3, then Γ is 2-bounded [12, Proposition 6.7], [9, Theorem 1.1]. The following theorem is our main result.

Theorem 1.3. *Let Γ denote a distance-regular graph with classical parameters (D, b, α, β) and $D \geq 3$. Assume the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then Γ is 3-bounded.*

Note that if Γ has classical parameters (D, b, α, β) with $D \geq 3$, $a_1 = 0$ and $a_2 \neq 0$, then Γ contains no parallelograms of any length. See [6, Theorem 1.1] or Theorem 3.3 in this article.

2. Preliminaries

In this section we review some definitions, basic concepts and some previous results concerning distance-regular graphs. See Bannai and Ito [1] or Terwilliger [10] for more background information.

Let $\Gamma = (X, R)$ denote a finite undirected, connected graph without loops or multiple edges with vertex set X, edge set R, distance function ∂, and diameter $D := \max\{\partial(x, y) \mid x, y \in X\}$. By a *pentagon*, we mean a 5-tuple $x_1x_2x_3x_4x_5$ consisting of vertices in Γ such that $\partial(x_i, x_{i+1}) = 1$ for $1 \leq i \leq 4$ and $\partial(x_5, x_1) = 1$.

For a vertex $x \in X$ and an integer $0 \leq i \leq D$, set $\Gamma_i(x) := \{z \in X \mid \partial(x, z) = i\}$. The *valency* $k(x)$ of a vertex $x \in X$ is the cardinality of $\Gamma_1(x)$. The graph Γ is called *regular* (with *valency* k) if each vertex in X has valency k.

A graph Γ is said to be *distance-regular* whenever for all integers $0 \leq h, i, j \leq D$, and all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\{z \in X \mid z \in \Gamma_i(x) \cap \Gamma_j(y)\}|$$

is independent of x, y. The constants p^h_{ij} are known as the *intersection numbers* of Γ.

Let $\Gamma = (X, R)$ be a distance-regular graph. For two vertices $x, y \in X$, with $\partial(x, y) = i$, set

$$B(x, y) := \Gamma_1(x) \cap \Gamma_{i+1}(y),$$
$$C(x, y) := \Gamma_1(x) \cap \Gamma_{i-1}(y),$$
$$A(x, y) := \Gamma_1(x) \cap \Gamma_i(y).$$

Note that

$$|B(x, y)| = p^i_{1\ i+1},$$
$$|C(x, y)| = p^i_{1\ i-1},$$
$$|A(x, y)| = p^i_{1\ i}$$

are independent of x, y.

For convenience, set $c_i := p^i_{1\ i-1}$ for $1 \le i \le D$, $a_i := p^i_{1\ i}$ for $0 \le i \le D$, $b_i := p^i_{1\ i+1}$ for $0 \le i \le D-1$ and put $b_D := 0$, $c_0 := 0$, $k := b_0$. Note that k is the valency of Γ. It is immediate from the definition of p^h_{ij} that $b_i \ne 0$ for $0 \le i \le D-1$ and $c_i \ne 0$ for $1 \le i \le D$. Moreover

$$k = a_i + b_i + c_i \quad \text{for } 0 \le i \le D. \tag{2.1}$$

From now on we assume that $\Gamma = (X, R)$ is distance-regular with diameter $D \ge 3$. Recall that a sequence x, y, z of vertices of Γ is weak-geodetic whenever

$$\partial(x, y) + \partial(y, z) \le \partial(x, z) + 1.$$

Definition 2.1. Let Ω be a subset of X, and pick any vertex $x \in \Omega$. Ω is said to be *weak-geodetically closed with respect to* x whenever, for all $z \in \Omega$ and for all $y \in X$,

$$x, y, z \text{ are weak-geodetic} \Longrightarrow y \in \Omega. \tag{2.2}$$

Note that Ω is weak-geodetically closed with respect to a vertex $x \in \Omega$ if and only if

$$C(z, x) \subseteq \Omega \quad \text{and} \quad A(z, x) \subseteq \Omega \qquad \text{for all } z \in \Omega$$

[12, Lemma 2.3]. Also Ω is weak-geodetically closed if and only if for any vertex $x \in \Omega$, Ω is weak-geodetically closed with respect to x. We list a few results which will be used later in this paper.

Theorem 2.2 (*[12, Theorem 4.6]*). *Let Γ be a distance-regular graph with diameter $D \ge 3$. Let Ω be a regular subgraph of Γ with valency γ and set $d := \min\{i \mid \gamma \le c_i + a_i\}$. Then the following* (i), (ii) *are equivalent.*

(i) *Ω is weak-geodetically closed with respect to at least one vertex $x \in \Omega$.*
(ii) *Ω is weak-geodetically closed with diameter d.*

In this case $\gamma = c_d + a_d$.

Lemma 2.3 (*[9, Lemma 2.6]*). *Let Γ be a distance-regular graph with diameter* 2, *and let x be a vertex of Γ. Suppose $a_2 \ne 0$. Then the subgraph induced on $\Gamma_2(x)$ is connected of diameter at most* 3.

Theorem 2.4 (*[12, Proposition 6.7], [9, Theorem 1.1]*). *Let Γ be a distance-regular graph with diameter $D \ge 3$. Suppose $a_1 = 0$, $a_2 \ne 0$ and Γ contains no parallelograms of length* 3. *Then Γ is* 2*-bounded.*

Theorem 2.5 (*[12, Lemma 6.9], [9, Lemma 4.1]*). *Let Γ be a distance-regular graph with diameter $D \ge 3$. Suppose $a_1 = 0$, $a_2 \ne 0$ and Γ contains no parallelograms of any length. Let x be a vertex of Γ, and let Ω be a weak-geodetically closed subgraph of Γ with diameter* 2. *Suppose that there exists an integer i and a vertex $u \in \Omega \cap \Gamma_{i-1}(x)$, and suppose $\Omega \cap \Gamma_{i+1}(x) \ne \emptyset$. Then for all $t \in \Omega$, we have $\partial(x, t) = i - 1 + \partial(u, t)$.*

3. Q-polynomial properties

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$. Let $\mathbb{R}$ denote the real number field. Let $\mathrm{Mat}_X(\mathbb{R})$ denote the algebra of all the matrices over $\mathbb{R}$ with the rows and columns indexed by the elements of X. For $0 \leq i \leq D$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{R})$ defined by the rule

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x, y) = i; \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \quad \text{for } x, y \in X.$$

We call A_i the *distance matrices* of Γ. We have

$$\begin{aligned} &A_0 = I, \\ &A_i^t = A_i \quad \text{for } 0 \leq i \leq D \text{ where } A_i^t \text{ means the transpose of } A_i, \\ &A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h \quad \text{for } 0 \leq i, j \leq D. \end{aligned}$$

Let M denote the subspace of $\mathrm{Mat}_X(\mathbb{R})$ spanned by $A_0, A_1, \ldots, A_D$. Then M is a commutative subalgebra of $\mathrm{Mat}_X(\mathbb{R})$, and is known as the *Bose–Mesner algebra* of Γ. By [2, p. 59, 64], M has a second basis $E_0, E_1, \ldots, E_D$ such that

$$\begin{aligned} &E_0 = |X|^{-1} J \quad \text{where } J = \text{all 1's matrix}, \\ &E_i E_j = \delta_{ij} E_i \quad \text{for } 0 \leq i, j \leq D, \\ &E_0 + E_1 + \cdots + E_D = I, \\ &E_i^t = E_i \quad \text{for } 0 \leq i \leq D. \end{aligned} \tag{3.1}$$

The $E_0, E_1, \ldots, E_D$ are known as the *primitive idempotents* of Γ, and E_0 is known as the *trivial* idempotent. Let E denote any primitive idempotent of Γ. Then we have

$$E = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i \tag{3.2}$$

for some $\theta_0^*, \theta_1^*, \ldots, \theta_D^* \in \mathbb{R}$, called the *dual eigenvalues* associated with E.

Set $V = \mathbb{R}^{|X|}$ (column vectors), and view the coordinates of V as being indexed by X. Then the Bose–Mesner algebra M acts on V by left multiplication. We call V the *standard module* of Γ. For each vertex $x \in X$, set

$$\hat{x} = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^t, \tag{3.3}$$

where the 1 is in coordinate x. Also, let $\langle , \rangle$ denote the dot product

$$\langle u, v \rangle = u^t v \quad \text{for } u, v \in V. \tag{3.4}$$

Then referring to the primitive idempotent E in (3.2), we compute from (3.1)–(3.4) that for x, $y \in X$,

$$\langle E\hat{x}, E\hat{y} \rangle = |X|^{-1} \theta_i^*, \tag{3.5}$$

where $i = \partial(x, y)$.

Let $\circ$ denote the entrywise multiplication in $\mathrm{Mat}_X(\mathbb{R})$. Then

$$A_i \circ A_j = \delta_{ij} A_i \quad \text{for } 0 \le i, j \le D,$$

so M is closed under $\circ$. Thus there exists $q_{ij}^k \in \mathbb{R}$ for $0 \le i, j, k \le D$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{k=0}^{D} q_{ij}^k E_k \quad \text{for } 0 \le i, j \le D.$$

Γ is said to be *Q-polynomial* with respect to the given ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents if for all integers $0 \le h, i, j \le D$, $q_{ij}^h = 0$ (resp. $q_{ij}^h \ne 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two. Let E denote any primitive idempotent of Γ. Then Γ is said to be Q-polynomial with respect to E whenever there exists an ordering $E_0, E_1 = E, \ldots, E_D$ of the primitive idempotents of Γ, with respect to which Γ is Q-polynomial. If Γ is Q-polynomial with respect to E, then the associated dual eigenvalues are distinct [10, p. 384].

The following theorem about the Q-polynomial property will be used in this paper.

Theorem 3.1 (*[11, Theorem 3.3]*). *Assume Γ is Q-polynomial with respect to a primitive idempotent E, and let $\theta_0^*, \ldots, \theta_D^*$ denote the corresponding dual eigenvalues. Then for all integers $1 \le h \le D$, $0 \le i, j \le D$ and for all $x, y \in X$ such that $\partial(x, y) = h$,*

$$\sum_{\substack{z \in X \\ \partial(x,z)=i \\ \partial(y,z)=j}} E\hat{z} - \sum_{\substack{z' \in X \\ \partial(x,z')=j \\ \partial(y,z')=i}} E\hat{z'} = p_{ij}^h \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_h^*} (E\hat{x} - E\hat{y}). \tag{3.6}$$

Γ is said to have *classical parameters* (D, b, α, β) whenever the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) \quad \text{for } 0 \le i \le D, \tag{3.7}$$

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right) \quad \text{for } 0 \le i \le D, \tag{3.8}$$

where

$$\begin{bmatrix} i \\ 1 \end{bmatrix} := 1 + b + b^2 + \cdots + b^{i-1}. \tag{3.9}$$

The following theorem characterizes the distance-regular graphs with classical parameters in an algebraic way.

Theorem 3.2 (*[11, Theorem 4.2]*). *Let Γ denote a distance-regular graph with diameter $D \ge 3$. Choose $b \in \mathbb{R} \setminus \{0, -1\}$, and let $[\ ]$ be as in (3.9). Then the following* (i)–(ii) *are equivalent.*

(i) *Γ is Q-polynomial with associated dual eigenvalues $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ satisfying*

$$\theta_i^* - \theta_0^* = (\theta_1^* - \theta_0^*) \begin{bmatrix} i \\ 1 \end{bmatrix} b^{1-i} \quad \textit{for } 1 \le i \le D. \tag{3.10}$$

(ii) *Γ has classical parameters (D, b, α, β) for some real constants α, β.*

The following theorem characterizes the distance-regular graphs with classical parameters and $a_1 = 0$, $a_2 \neq 0$ in a combinatorial way.

Theorem 3.3 (*[6, Theorem 1.1]*). *Let Γ denote a distance-regular graph with diameter $D \geq 3$ and intersection numbers $a_1 = 0$, $a_2 \neq 0$. Then the following* (i)–(iii) *are equivalent.*

(i) *Γ is Q-polynomial and contains no parallelograms of length* 3.
(ii) *Γ is Q-polynomial and contains no parallelograms of any length i for $3 \leq i \leq D$.*
(iii) *Γ has classical parameters (D, b, α, β) for some real constants b, α, β with $b < -1$.*

4. Proof of main theorem

Assume $\Gamma = (X, R)$ is a distance-regular graph with classical parameters (D, b, α, β) and $D \geq 3$. Suppose the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then Γ contains no parallelograms of any length by Theorem 3.3. We first give a definition.

Definition 4.1. For any vertex $x \in X$ and any subset $C \subseteq X$, define

$$[x, C] := \{v \in X \mid \text{there exists } z \in C, \text{ such that } \partial(x, v) + \partial(v, z) = \partial(x, z)\}.$$

Throughout this section, fix two vertices $x, y \in X$ with $\partial(x, y) = 3$. Set

$$C := \{z \in \Gamma_3(x) \mid B(x, y) = B(x, z)\}$$

and

$$\Delta = [x, C]. \tag{4.1}$$

We shall prove that Δ is a regular weak-geodetically closed subgraph of diameter 3. Note that the diameter of Δ is at least 3. If $D = 3$ then $C = \Gamma_3(x)$ and $\Delta = \Gamma$ is clearly a regular weak-geodetically closed graph. Thereafter we assume $D \geq 4$. By referring to Theorem 2.2, we shall prove that Δ is weak-geodetically closed with respect to x, and the subgraph induced on Δ is regular with valency $a_3 + c_3$.

Lemma 4.2. *For all adjacent vertices $z, z' \in \Gamma_i(x)$, where $i \leq D$, we have $B(x, z) = B(x, z')$.*

Proof. By symmetry, it suffices to show that $B(x, z) \subseteq B(x, z')$. Suppose there exists $w \in B(x, z) \setminus B(x, z')$. Then $\partial(w, z') \neq i + 1$. Note that $\partial(w, z') \leq \partial(w, x) + \partial(x, z') = 1 + i$ and $\partial(w, z') \geq \partial(w, z) - \partial(z, z') = i$. This implies $\partial(w, z') = i$ and $wxz'z$ forms a parallelogram of length $i + 1$, a contradiction. □

We know that Γ is 2-bounded by Theorem 2.4. For two vertices z, s in Γ with $\partial(z, s) = 2$, let $\Omega(z, s)$ denote the regular weak-geodetically closed subgraph containing z, s of diameter 2.

Lemma 4.3. *Suppose $stuzw$ is a pentagon in Γ, where $s, u \in \Gamma_3(x)$ and $z \in \Gamma_2(x)$. Pick $v \in B(x, u)$. Then $\partial(v, s) \neq 2$.*

Proof. Suppose $\partial(v, s) = 2$. Note $\partial(z, s) \neq 1$, since $a_1 = 0$. Note that $z, w, s, t, u \in \Omega(z, s)$. Then $s \in \Omega(z, s) \cap \Gamma_2(v)$ and $u \in \Omega(z, s) \cap \Gamma_4(v) \neq \emptyset$. Hence $\partial(v, z) = \partial(v, s) + \partial(s, z) = 2 + 2 = 4$ by Theorem 2.5. A contradiction occurs since $\partial(v, x) = 1$ and $\partial(x, z) = 2$. □

Lemma 4.4. *Suppose $stuzw$ is a pentagon in Γ, where $s, u \in \Gamma_3(x)$ and $z \in \Gamma_2(x)$. Then $B(x, s) = B(x, u)$.*

Proof. Since $|B(x,s)| = |B(x,u)| = b_3$, it suffices to show $B(x,u) \subseteq B(x,s)$. By Lemma 4.3,

$$B(x,u) \subseteq \Gamma_3(s) \cup \Gamma_4(s).$$

Suppose

$$|B(x,u) \cap \Gamma_3(s)| = m,$$
$$|B(x,u) \cap \Gamma_4(s)| = n.$$

Then

$$m + n = b_3. \tag{4.2}$$

By Theorem 3.1,

$$\sum_{r \in B(x,u)} E\hat{r} - \sum_{r' \in B(u,x)} E\hat{r'} = b_3 \frac{\theta_1^* - \theta_4^*}{\theta_0^* - \theta_3^*}(E\hat{x} - E\hat{u}). \tag{4.3}$$

Observe $B(u,x) \subseteq \Gamma_3(s)$; otherwise $\Omega(u,s) \cap B(u,x) \neq \emptyset$ and this leads $\partial(x,s) = 4$ by Theorem 2.5, a contradiction. Taking the inner product of s with both sides of (4.3) and evaluating the result using (3.5), we have

$$m\theta_3^* + n\theta_4^* - b_3\theta_3^* = b_3 \frac{\theta_1^* - \theta_4^*}{\theta_0^* - \theta_3^*}(\theta_3^* - \theta_2^*). \tag{4.4}$$

Solve (4.2) and (4.4) to obtain

$$n = b_3 \frac{(\theta_2^* - \theta_3^*)}{(\theta_3^* - \theta_4^*)} \frac{(\theta_1^* - \theta_4^*)}{(\theta_0^* - \theta_3^*)}. \tag{4.5}$$

Simplifying (4.5) using (3.10), we have $n = b_3$ and then $m = 0$ by (4.2). This implies $B(x,u) \subseteq B(x,s)$ and ends the proof. □

Lemma 4.5. *Let $z, u \in \Delta$. Suppose $stuzw$ is a pentagon in Γ, where $z, w \in \Gamma_2(x)$ and $u \in \Gamma_3(x)$. Then $w \in \Delta$.*

Proof. Observe $\Omega(z,s) \cap \Gamma_1(x) = \emptyset$ and $\Omega(z,s) \cap \Gamma_4(x) = \emptyset$ by Theorem 2.5. Hence $s, t \in \Gamma_2(x) \cup \Gamma_3(x)$. Observe $s \in \Gamma_3(x)$; otherwise $w, s \in \Omega(x,z)$, and this implies $u \in \Omega(x,z)$, a contradiction to the diameter of $\Omega(x,z)$ being 2. Hence $B(x,s) = B(x,u)$ by Lemma 4.4. Then $s \in C$ and $w \in \Delta$ by construction. □

Lemma 4.6. *The subgraph Δ is weak-geodetically closed with respect to x.*

Proof. Clearly $C(z,x) \subseteq \Delta$ for any $z \in \Delta$. It suffices to show $A(z,x) \subseteq \Delta$ for any $z \in \Delta$. Suppose $z \in \Delta$. We discuss this case by case in the following. The case $\partial(x,z) = 1$ is trivial since $a_1 = 0$. For the case $\partial(x,z) = 3$, we have $B(x,y) = B(x,z) = B(x,w)$ for any $w \in A(z,x)$ by definition of Δ and Lemma 4.2. This implies $A(z,x) \subseteq \Delta$ by the construction of Δ. For the remaining case $\partial(x,z) = 2$, fix $w \in A(z,x)$ and we shall prove $w \in \Delta$. There exists $u \in C$ such that $z \in C(u,x)$. Observe that $\partial(w,u) = 2$ since $a_1 = 0$. Choose $s \in A(w,u)$ and $t \in C(u,s)$. Then $stuzw$ is a pentagon in Γ. The result comes immediately by Lemma 4.5. □

Proof of Theorem 1.3. By Theorem 2.2 and Lemma 4.6, it suffices to show that Δ defined in (4.1) is regular with valency a_3+c_3. Clearly from the construction and Lemma 4.6, $|\Gamma_1(z) \cap \Delta| =$

$a_3 + c_3$ for any $z \in C$. First we show that $|\Gamma_1(x) \cap \Delta| = a_3 + c_3$. Note that $y \in \Delta \cap \Gamma_3(x)$ by construction of Δ. For any $z \in C(x, y) \cup A(x, y)$,

$$\partial(x, z) + \partial(z, y) \le \partial(x, y) + 1.$$

This implies $z \in \Delta$ by Definition 2.1 and Lemma 4.6. Hence $C(x, y) \cup A(x, y) \subseteq \Delta$. Suppose $B(x, y) \cap \Delta \neq \emptyset$. Choose $t \in B(x, y) \cap \Delta$. Then there exists $y' \in \Gamma_3(x) \cap \Delta$ such that $t \in C(x, y')$. Note that $B(x, y) = B(x, y')$. This leads to a contradiction to $t \in C(x, y')$. Hence $B(x, y) \cap \Delta = \emptyset$ and $\Gamma_1(x) \cap \Delta = C(x, y) \cup A(x, y)$. Then we have $|\Gamma_1(x) \cap \Delta| = a_3 + c_3$.

Since each vertex in Δ appears in a sequence of vertices $x = x_0, x_1, x_2, x_3$ in Δ, where $\partial(x, x_j) = j$ and $\partial(x_{j-1}, x_j) = 1$ for $1 \le j \le 3$, it suffices to show

$$|\Gamma_1(x_i) \cap \Delta| = a_3 + c_3 \tag{4.6}$$

for $1 \le i \le 2$. For each integer $0 \le i \le 2$, we show

$$|\Gamma_1(x_i) \setminus \Delta| \le |\Gamma_1(x_{i+1}) \setminus \Delta|$$

by the 2-way counting of the number of the pairs (s, z) for $s \in \Gamma_1(x_i) \setminus \Delta$, $z \in \Gamma_1(x_{i+1}) \setminus \Delta$ and $\partial(s, z) = 2$. For a fixed $z \in \Gamma_1(x_{i+1}) \setminus \Delta$, we have $\partial(x, z) = i+2$ by Lemma 4.6, so $\partial(x_i, z) = 2$ and $s \in A(x_i, z)$. Hence the number of such pairs (s, z) is at most $|\Gamma_1(x_{i+1}) \setminus \Delta| a_2$.

On the other hand, we show that this number is exactly $|\Gamma_1(x_i) \setminus \Delta| a_2$. Fix an $s \in \Gamma_1(x_i) \setminus \Delta$. Observe $\partial(x, s) = i + 1$ by Lemma 4.6. Observe $\partial(x_{i+1}, s) = 2$ since $a_1 = 0$. Pick any $z \in A(x_{i+1}, s)$. We shall prove $z \notin \Delta$. Suppose $z \in \Delta$ in the arguments below and choose any $w \in C(s, z)$.

Case 1: $i = 0$.

Observe $\partial(x, z) = 2$, $\partial(x, s) = 1$ and $\partial(x, w) = 2$. This will force $s \in \Delta$ by Lemma 4.6, a contradiction.

Case 2: $i = 1$.

Observe $\partial(x, z) = 3$; otherwise $z \in \Omega(x, x_2)$ and this implies $s \in \Omega(x, x_2) \subseteq \Delta$ by Lemmas 2.3 and 4.6, a contradiction. This also implies $s \in \Delta$ by Definition 2.1 and Lemma 4.6, a contradiction.

Case 3: $i = 2$.

Observe $\partial(x, z) = 2$ or 3. Suppose $\partial(x, z) = 2$. Then $B(x, x_3) = B(x, s)$ by Lemma 4.4 (with $x_3 = u$, $x_2 = t$). Hence $s \in \Delta$, a contradiction. So $z \in \Gamma_3(x)$. Note that $\partial(x, w) \neq 2, 3$; otherwise $s \in \Delta$ by Lemmas 4.4 and 4.6 respectively. Hence $\partial(x, w) = 4$. Then by applying $\Omega = \Omega(x_2, w)$ in Theorem 2.5 we have $\partial(x_2, z) = 1$, a contradiction to $a_1 = 0$.

From the above counting, we have

$$|\Gamma_1(x_i) \setminus \Delta| a_2 \le |\Gamma_1(x_{i+1}) \setminus \Delta| a_2 \tag{4.7}$$

for $0 \le i \le 2$. Eliminating a_2 from (4.7), we find

$$|\Gamma_1(x_i) \setminus \Delta| \le |\Gamma_1(x_{i+1}) \setminus \Delta|, \tag{4.8}$$

or equivalently

$$|\Gamma_1(x_i) \cap \Delta| \ge |\Gamma_1(x_{i+1}) \cap \Delta| \tag{4.9}$$

for $0 \le i \le 2$. We already know that $|\Gamma_1(x_0) \cap \Delta| = |\Gamma_1(x_3) \cap \Delta| = a_3 + c_3$. Hence (4.6) follows from (4.9). $\square$

Remark 4.7. The 4-bounded property seems to be much harder to prove. We expect the 3-bounded property to be enough for classifying all the distance-regular graphs with classical parameters, $a_1 = 0$ and $a_2 \neq 0$.

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