Report of NSC Project on “Analysis of Instant Trend of The Price of A Stock by Solving The Model of The Markov Chains in Random Environments”

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Contents

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1. Introduction

The Markov chains in random environments have been pursued for some time. Nawrotzki [1] established a general theory of this topic. Cogburn [2]-[5] developed a theory in a wider context by making use of some more powerful tools such as the theory of Hopf Markov chains. Orey [6] reviewed the works of this field. The objects of all of the above works are Markov chains in random environments, and the set of time parameter is discrete. We extend it to the sets of parameters of the environments being continuous. We use the model of the Markov chains in random environments to analyze the instant trend of the price of a stock. We assume first the price of a stock to be a geometric Brownian motion. Conditional on the Brownian motion, the trend follows a two-state Markov chains. We find the finite time distributions and the limiting distributions of the stochastic transition probabilities of the Markov chains. Extension of this model is also explored.

2. Research Purpose

The purpose of our research lies in the behavior of the instant trend of the price of a stock. The instant trend of a stock includes two terms, upward and downward. Suppose the price of a stock varies according to a geometric Brownian motion [7], and suppose the appearance of the instant trend depends on the price of that stock. Thus we define the state 0 to be upward and the state 1 to be downward. Upon observations on the price up to the present, the instant trend is assumed to have Markov property. The stochastic infinitesimal matrix of the process is

$$Q(t)=\begin{bmatrix}-\exp(A_t) & \exp(A_t)\\ \exp(A_t) & -\exp(A_t)\end{bmatrix}$$

where A_t is a standard Brownian motion. The transition matrix $P(s,t)$ is therefore random as well. We find the finite time distributions and the limiting distributions of the stochastic transition probabilities. Extensions of our model are also considered as the following

(i) A_t is a general Brownian motion.

(ii) A_t is a Brownian motion with drift.

(iii) $Q(t)=\begin{bmatrix}-\exp(A_t) & \exp(A_t)\\ \exp(B_t) & -\exp(B_t)\end{bmatrix}$ where A_t and B_t are two independent Brownian motions.

3. Literature Review

Nawrotzki [1] established a general theory of this topic. Cogburn [2]-[5] developed a theory in a wider context by making use of some more powerful tools such as the theory of Hopf Markov chains. Orey [6] reviewed the works of this field.

4. Research Method

Consider a continuous time nonhomogeneous Markov chain $X(t)$ with state space $\{1,2,\cdots\}$ and time-dependent infinitesimal matrix $Q(t)=\left(q_{ij}(t)\right)$. Let $P_{ij}(s,t)=P(X(t)=j \mid X(s)=i)$ be the transition probability from state *i* at epoch *s* to *j* at epoch *t*. The well known system of Kolmogorov forward and backward differential equations written in matrix forms are

(1) $$\begin{aligned}\frac{\partial P(s,t)}{\partial t} &= P(s,t)Q(t)\\ \frac{\partial P(s,t)}{\partial s} &= -Q(s)P(s,t)\end{aligned}$$

where $P(s,t)=(P_{ij}(s,t))$ and $\frac{\partial P(s,t)}{\partial t}=\left(\frac{\partial P_{ij}(s,t)}{\partial t}\right)$. For a given *t*, the solutions to (1) are unique under the condition that $\left|q_{ij}(\tau)\right|<\infty, \forall\, \tau\le t$. However, they are difficult to obtain in analytic form. Yet for some cases we do have ways to write them in nice patterns, first of which is the following trivial proposition.

Proposition 1

The series solution to (1) is

(2) $$\begin{aligned}P(s,t) &= I+\int_s^t Q(x)dx+\int_s^t\int_x^t Q(x)Q(y)dydx+\int_s^t\int_x^t\int_y^t Q(x)Q(y)Q(z)dzdydx+\cdots\\ &= I+\int_s^t Q(x)dx+\int_s^t\int_s^x Q(y)Q(x)dydx+\int_s^t\int_s^x\int_s^y Q(z)Q(y)Q(x)dzdydx+\cdots\end{aligned}$$

where *I* is the identity matrix. Hence, if *Q(t)* ‘s are commutative to each other, that is, *Q(x)Q(y) = Q(y)Q(x)* for all *x* and *y*, then we have

(3) $$P(s,t)=e^{\int_s^t Q(x)dx}.$$

Corollary 1

Suppose $Q(x)=\lambda(x)A$, where $\lambda(x)$ is a real valued function and *A* is a constant matrix. The transition probability matrix becomes $P(s,t)=e^{\int_s^t \lambda(x)dx\, A}$.

Corollary 2

Suppose $Q(x)=\sum\lambda_i(x)A_i$ where $\lambda_i(x)$’s are real valued functions and A_i’s are constant matrices satisfying $A_i^2 = A_i$ and $A_i A_j = 0, \forall i \neq j$. That is, *Q(x)* is diagonalizable with constant eigenspaces but time-dependent eigenvalues. The transition probability matrix becomes $P(s,t) = \sum_{i=1}^{k} e^{\int_s^t \lambda_i(x)dx} A_i$.

If $Q(t)$'s are not commutative to each other, it is also possible to find individual transition probabilities for some cases as the following.

For *n*-state finite Markov chain, we multiply both sides of (2) from the left by $(1,0,\cdots,0)$ and from the right by the matrix *A* where $A = \begin{bmatrix} 1 & 0 & \cdots & \\ 1 & 1 & & \\ \vdots & 0 & \ddots & \\ 1 & \vdots & & 1 \end{bmatrix}$, the inverse of which is easily seen to be $A^{-1} = \begin{bmatrix} 1 & 0 & \cdots & \\ -1 & 1 & & \\ \vdots & 0 & \ddots & \\ -1 & \vdots & & 1 \end{bmatrix}$. Thus, by letting *s* = 0 and writing $P(t) = P(0,t)$, we obtain

$$
\begin{aligned}
&(1, P_{12}(t), \cdots, P_{1n}(t)) = (1,0,\cdots,0)P(t)A \\
&= (1,0,\cdots,0)\left[I + \int_0^t Q(x)dx + \int_0^t \int_x^t Q(x)Q(y)dydx + \cdots \right] A \\
(4) \quad &= (1,0,\cdots,0)\left[I + \int_0^t Q(x)dx + \int_0^t \int_x^t Q(x)AA^{-1}Q(y)dydx + \cdots \right] A \\
&= (1,0,\cdots,0) + \int_0^t (0, q_{12}(x), \cdots, q_{1n}(x)) \left[I + \int_x^t B(y)dy + \int_x^t \int_y^t B(y)B(z)dzdy + \cdots \right] dx
\end{aligned}
$$

where $B(x) = A^{-1}Q(x)A = \begin{bmatrix} 0 & q_{12} & \cdots & q_{1n} \\ \vdots & -q_{12} + q_{22} & \cdots & -q_{1n} + q_{2n} \\ & \vdots & & \vdots \\ 0 & -q_{12} + q_{n2} & \cdots & -q_{1n} + q_{nn} \end{bmatrix}(x)$. Note that the term in the bracket of (4) has the same form as that of (2). Hence, by denoting the submatrix $C(x) = \begin{bmatrix} -q_{12}+q_{22} & \cdots & -q_{1n}+q_{2n} \\ \vdots & & \vdots \\ -q_{12}+q_{n2} & \cdots & -q_{1n}+q_{nn} \end{bmatrix}(x)$, we have

Corollary 3

If $C(x)$ satisfies the conditions of Corollary 1, that is, $C(x) = \lambda(x)A$, then

$$(P_{12}(t),\cdots,P_{1n}(t)) = \int_0^t (q_{12}(x),\cdots,q_{1n}(x))e^{\int_x^t \lambda(y)dy\, A}dx .$$

If $C(x)$ satisfies the conditions of Corollary 2, that is, $C(x) = \sum \lambda_i(x)A_i$ where A_i 's are idempotent and orthogonal to each other, then

$$(P_{12}(t),\cdots,P_{1n}(t)) = \int_0^t (q_{12}(x),\cdots,q_{1n}(x))\left[\sum_{i=1}^k e^{\int_x^t \lambda_i(y)dy} A_i\right]dx .$$

Corollary 4

If $q_{ij} = q_{1j}$ for $i \neq j$, $i > 1$ and $j > 1$, then

$C(x) = diag(-q_{12}+q_{22},\cdots,-q_{1n}+q_{nn})$. Hence $P_{1i}(t) = \int_0^t q_{1i}(x)e^{\int_x^t (-q_{1i}+q_{ii})(y)dy}dx, \quad i \geq 2$.

Corollary 5

A special case is a 2-state Markov chain whose

$C(x) = (-q_{12}+q_{22})(x) = -(q_{12}+q_{21})(x)$, and $P_{12}(t) = \int_0^t q_{12}(x)e^{-\int_x^t (q_{12}+q_{21})(y)dy}dx$. The other three transition probabilities can be obtained by symmetry and complementary.

Apply Corollary 5 we have

$$P_{10}(s,t) = P_{01}(s,t) = \int_s^t \exp(A_x - 2\int_x^t \exp(A_y)dy)dx$$

and $$P_{00}(s,t)=1-P_{01}(s,t)=P_{11}(s,t)\ .$$

Theorem 1

The transition probability $P_{01}(t)\equiv P_{01}(0,t)$ in the case (i) can be reduced to

$$P_{01}(t)=\frac{1}{2}-\frac{1}{2}\exp(-2\int_{0}^{t}\exp(A_x)dx)\ .$$

The term $\int_{0}^{t}\exp(A_x)dx$ is an integrated geometric Brownian motion, and its finite time distribution was obtained in [8]. Also, $P_{01}(t)\rightarrow \frac{1}{2}$ almost surely.

Theorem 2

The transition probability $P_{01}(t)\equiv P_{01}(0,t)$ in the case (ii) can be reduced to

$$P_{01}(t)=\frac{1}{2}-\frac{1}{2}\exp(-2\int_{0}^{t}\exp(A_x+\mu x)dx)$$

where μ is the drift constant.

Theorem 3

The transition probability $P_{01}(t)\equiv P_{01}(0,t)$ in the case (iii) can be reduced to

$$P_{12}(t)=\int_{0}^{t}\exp(A(x))\exp(-\int_{x}^{t}(\exp(A(y))+\exp(B(y))dy)dx\ .$$

5. Results and Discussion

The transition probabilities are now random variables and the distributions can be written in a complicated form in terms of the density given in [8]. Two major achievements are goaled in this project, one is the transition probabilities of nonhomogeneous Markov chains written in a concise form, while the other is the application to the transient trend of the price of a stock, that is, given the stock price now, we can predict the trend of the next instance.

6. Judgements for Research Results

This is a good result on the topic of prediction in a reasonable financial model. We hope we can proceed this research in a similar aspect.

7. References

[1] Nawrotzki, K, Discrete open systems of Markov chains in a random environments, I, II, J. Inform. Process, Cybernet, 1981,17: 569-599,1982, 18: 83-98.

[2] Cogburn, R. Markov chains in random environments, the case of Markovian environments, Ann. Probability, 1980,8:908-916

[3] Cogburn, R. The ergodic theory of Markov chains in random environments, Z. Wahrach. Verw, Gebiete, 1984, 66, 109-128

[4] Cogburn, R. On direct convergence and periodicity for transition probability of Markov chains in random environments, Ann. Probab. 1990, 18, 642-654

[5] Cogburn, R. On the central limit theorem for Markov chains in random environments, Ann. Probab. 1991, 19, 587-604

[6] Orey, S. Markov chains with stochastically stationary transition probabilities, Ann. Probab. 1991, 19, 907-928

[7] Ross, S. Stochastic Processes, 2nd Ed. 1996, Wiley

[8] Dufresne, D. The integral of geometric Brownian motions, Adv. Appl. Prob.2001, 33,1, 223-241

[9] Durrett, R. Probability, Theory and Examples, 2nd Ed. 1995, Duxbury