（一） 計畫中文摘要

有限馬可夫鏈的切割現象是一個劇烈的相變行為。假設 K 是馬可夫鏈的轉置矩陣、μ是初始分佈、π是穩定分佈。令 $f(m)=\|\mu K^m-\pi\|_{TV}$ 為馬可夫鏈在時間 m 與穩定分佈間的全變量。所謂的(全變量)切割現象就是指距離函數 f 的相變：首先 f 會維持在幾乎是最大值(1)一段時間，接著函數值在極短的時間內遞減的很小，最後會指數收斂至 0。該距離函數產生劇烈相變的時間就是(全變量)切割時間。在馬可夫過程的計量分析裡，最令人驚訝的發現就是大多數的模型都有切割現象的相變。第一個被觀察到的例子就是 Diaconis 和 Shahshahani[4]的隨機位移(random transposition)洗牌法。在他們的文章裡，群的表現論(group representation)是第一次被運用到機率的研究。

在這個專題計畫裡，我們將考慮可逆馬可夫過程的 L^2 切割。根據古典算子理論，可逆馬可夫過程的機率分佈和穩定分佈之間的 L^2 距離是可以表示成一個特徵值和特徵向量的函數。我們的目的就是藉由對這個距離函數的研究，找出 L^2 切割存在性的判定方法，並推導 L^2 切割時間的公式。在建立一般性的理論後，我們將比較連續時間型和離散時間型的 L^2 切割時間，並探討幾個機率上典型的例子。

關鍵詞：可逆馬可夫過程，L^2 切割現象，L^2 切割時間

（二）計畫英文摘要

Chains presenting a cutoff show a sharp phase transition as follows. Let K be the transition matrix of a finite Markov chain with stationary distribution π and initial distribution μ. The (total variation) cutoff is such a phenomenon that the distance $\|\mu K^m-\pi\|_{TV}$ holds at almost its maximum for a while, then goes down in a relatively short time to a small value and converges to 0 exponentially fast. One of the most striking observations in the quantitative study of Markov chains is that many models present such a phase transition. The first example presenting such a phenomenon is the random transposition model studied by Diaconis and Shahshahani using the group representation.

In this project, we shall focus on the L^2 cutoff for reversible Markov processes. It is known that the L^2 distance can be represented using the spectral information (eigenvalues and eigenfunctions). Our aim is to derive a criterion on the existence of the L^2 cutoff and generate a formula on the L^2 mixing time by exploring the L^2 distance function. After that, comparisons of discrete time and continuous time cases should be made and typical examples will be studied.

Keywords: reversible Markov processes, L^2 cutoff, L^2 mixing time.

（三）報告內容

In the past year (2007-2008), Laurent Saloff-Coste and I worked very hard on finding the conditions on the L^2-cutoff for families of reversible Markov processes with arbitrary initial distributions. We successfully generalized some results on the L^2-mixing time of symmetric random walks on finite groups in [1]. Different from what has been developed there, we studied the L^2-cutoff phenomenon on families of reversible Markov processes using their spectral information. Applying the theory of spectral decomposition, the L^2-distince between the distributions of Markov processes and their stationarity can be expressed as a Laplace transform with respect to the spectral measure. In detail, let $(\Omega,\mathscr{B})$ be a measurable space and P_t be a Markov semigroup on $L^2(\Omega,\pi)$ with infinitesimal generator A, where π is an invariant probability measure of P_t. Assume the reversibility of P_t, that is, A is self-adjoint, and let E_λ be the resolution of the identity for –A. If the initial distribution μ has an L^2 density f with respect to π, then

$$\|\mu P_t-\pi\|_2^2=\left\|\frac{d(\mu P_t)}{d\pi}-1\right\|_2^2=\int_{[0,\infty)}e^{-2\lambda t}d\langle E_\lambda f,f\rangle$$

Based on this observation, we derive a series of criteria on the existence of L^2-cutoff and formulas of L^2-cutoff time.

Consider a family of Markov processes $(\Omega_n, P_{n,t}, \pi_n)_{n\geq 1}$ with initial distributions μ_n and set

$$f_n(t)=\|\mu_n P_{n,t}-\pi_n\|_2$$

This family is said to present an L^2-cutoff if there exists a sequence of positive numbers t_n such that

$$\lim_{n\to\infty} f_n(at_n)=\begin{cases}0 & \forall a\in(1,\infty)\\ \infty & \forall a\in(0,1)\end{cases}$$

In the above setting, the sequence t_n is called an L^2-cutoff time. This definition describes the phenomenon that a sharp phase transition on the L^2 distance happens around the time t_n. The sequence t_n is in fact closely related to the finite time behavior of Markov processes. To see this, we let $T(f,\varepsilon)$ be the ε-L^2-mixing time defined by

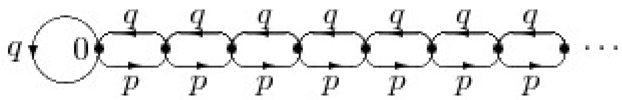
$$T(f,\varepsilon)=\inf\{t>0: f(t)\leq\varepsilon\},\quad f(t)=\|\mu P_t-\pi\|_2$$

According to this definition, the family $(\Omega_n, P_{n,t}, \pi_n)_{n\geq 1}$ has an L^2-cutoff if and only if

$$T(f_n,\varepsilon)\sim T(f_n,\delta)\qquad \forall \varepsilon,\delta\in(0,\infty)$$

where two sequences of positive numbers a_n,b_n have the relation $a_n\sim b_n$ if the ratio a_n/b_n converges to 1 as n tends to infinity. If the above asymptote holds, then the cutoff sequence can be chosen to be T(f_n,ε) for any ε>0. A detailed discussion on variants of cutoffs is available in [2].

In the notion of cutoff phenomenon introduced above, we cite one of our main results in [3] as following. Consider the constant rate birth and death chain on $\Omega_n \equiv \{0,1,2,\ldots\}$ with birth rate p and death rate q=1-p with p<1/2. See the following figure.



Then, the family of continuous-time birth-and-death chains as above with starting states x_n has an L^2-cutoff if and only if x_n tends to infinity. Moreover, if there is a cutoff, then

$$t_n = \frac{\log q - \log p}{2(1-2\sqrt{pq})} x_n$$

is a cutoff time sequence. Theoretical results and further complicated examples are collected in [3].

Reference

[1] Guan-Yu Chen. *The cut-off phenomenon for finite Markov chains.* Ph.D. thesis, Cornell University, 2006.

[2] Guan-Yu Chen and Laurent Saloff-Coste. The cutoff phenomenon for ergodic Markov processes. Electronic Journal of Probability, 13 (2008), 26--78.

[3] Guan-Yu Chen and Laurent Saloff-Coste. On the L^2-cutoff for normal Markov processes. In preparation.