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Apéry limits and special values of L-functions

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Abstract

We describe a general method to determine the Apéry limits of a differential equation that has a modular-function origin. As a by-product of our analysis, we discover a family of identities involving the special values of L-functions associated with modular forms. The proof of these identities is independent of differential equations and Apéry limits.

Keywords: Special values of L-functions; Apéry limits; Modular forms

1. Introduction

In 1978, R. Apéry proved that $\zeta(3)=\sum_{n=1}^\infty n^{-3}$ is an irrational number by constructing two sequences

$$a_n=\sum_{k=0}^n\binom{n}{k}^2\binom{n+k}{k}^2,$$
$$b_n=\sum_{k=0}^n\binom{n}{k}^2\binom{n+k}{k}^2\left\{\sum_{m=1}^n\frac{1}{m^3}+\sum_{m=1}^k\frac{(-1)^{m-1}}{2m^3\binom{n}{m}\binom{n+m}{m}}\right\},$$

and then showing that b_n/a_n converges to $\zeta(3)$ fast enough to ensure irrationality of $\zeta(3)$ (see [5]). Another remarkable discovery of Apéry is that a_n and b_n satisfy the recursive relation

$$(n+2)^3u_{n+2}-\left(34n^3+153n^2+231n+117\right)u_{n+1}+(n+1)^3u_n=0\quad(u_n=a_n\text{ or }b_n).$$

Thus, if we set $A(t)=\sum_{n=0}^\infty a_nt^n$ and $B(t)=\sum_{n=0}^\infty b_nt^n$, then the functions $A(t)$ and $B(t)$ satisfy the differential equations

$$\left(1-34t+t^2\right)\theta^3A+\left(3t^2-51t\right)\theta^2A+\left(3t^2-27t\right)\theta A+\left(t^2-5t\right)A=0\tag{1}$$

and

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$$\bigl(1-34t+t^2\bigr)\theta^3 B+\bigl(3t^2-51t\bigr)\theta^2 B+\bigl(3t^2-27t\bigr)\theta B+\bigl(t^2-5t\bigr)B=6t,$$

where θ denotes the differential operator td/dt.

Apéry also had an analogous result for $\zeta(2)=\pi^2/6$. He showed that if $\{a_n\}$ and $\{b_n\}$ are sequences of rational numbers satisfying the recursive relation

$$(n+2)^2u_{n+2}-\bigl(11n^2+33n+25\bigr)u_{n+1}-(n+1)^2u_n=0 \quad (u_n=a_n \text{ or } b_n)$$

with the initial values

$$a_{-1}=0, \qquad a_0=1, \qquad b_0=0, \qquad b_1=5,$$

then b_n/a_n converges to $\zeta(2)$. Again, due to the recursive relation, the generating functions $A(t)=\sum a_nt^n$ and $B(t)=\sum b_nt^n$ satisfy

$$\bigl(1-11t-t^2\bigr)\theta^2 A-\bigl(11t+2t^2\bigr)\theta A-\bigl(3t+t^2\bigr)A=0 \tag{2}$$

and

$$\bigl(1-11t-t^2\bigr)\theta^2 B-\bigl(11t+2t^2\bigr)\theta B-\bigl(3t+t^2\bigr)B=5t,$$

respectively.

A common feature of Apéry's two examples is the existence of a differential equation $LA(t)=0$ with regular singularities whose local exponents at $t=0$ are all 0 such that if $A(t)=1+\sum a_nt^n$ is the unique holomorphic solution at $t=0$ and $B(t)=t+\sum b_nt^n$ is the unique holomorphic solution of the inhomogeneous differential equation $LB(t)=t$ at $t=0$, then the ratios b_n/a_n converge to a special value of the Riemann zeta function. Inspired by these two examples, Zudilin et al. [1,8,9] considered the Apéry limits of a differential equation. The general setting is described as follows. Let $Lf(t)=0$ be a linear differential equation with polynomial coefficients and regular singularities. Assume that the local exponents of L at $t=0$ are all 0 so that the monodromy around $t=0$ is maximally unipotent. In other words, the differential operator L takes the form

$$\theta^k+tP_1(\theta)+t^2P_2(\theta)+\cdots+t^dP_d(\theta), \quad \theta=td/dt,$$

where P_j are polynomials of degree $\leqslant k$. Let $A(t)=1+\sum a_nt^n$ be the unique holomorphic solution at $t=0$. Then the sequence $\{a_n\}$ satisfy a $(d+1)$-term recursive relation

$$(n+d)^ka_{n+d}+P_1(n+d-1)a_{n+d-1}+\cdots+P_d(n)a_n=0$$

with initial values $a_{-d+1}=\cdots=a_{-1}=0$ and $a_0=1$. Now for each integer j from 1 to $d-1$, we let $B(t)=t^j+\sum_{n=j+1}^\infty b_nt^n$ be a solution of the inhomogeneous differential equation $LB(t)=j^kt^j$. (Note that when $j\geqslant d$, there may not exist a solution of $LB(t)=t^j$ that is holomorphic at $t=0$.) The coefficients b_n also satisfy the recursive relation

$$(n+d)^kb_{n+d}+P_1(n+d-1)b_{n+d-1}+\cdots+P_d(n)b_n=0$$

with initial values $b_{j-d+1}=\cdots=b_{j-1}=0$ and $b_j=1$. Then the jth *Apéry limit* is defined to be the limit of b_n/a_n. For example, in [8], Zudilin gave a sixth order differential equation whose Apéry limit is the Catalan number $\sum_{n=0}^\infty(-1)^n/(2n+1)^2$ and a fifth order differential equation whose Apéry limit is $\zeta(4)$. In [9], he also found a third order differential equations whose first and second Apéry limits give simultaneous approximations to $\log 2$ and $\zeta(2)$. The paper also contains an example of a sixth order differential equation that gives simultaneous approximation to $\zeta(2)$ and $\zeta(3)$. More recently, Almkvist et al. [1] consider the Apéry limits of fourth order differential equations of Calabi–Yau type. The numerical computation finds that the recognized limits are all rational combinations of values of zeta functions or Dirichlet series associated with odd characters modulo 3 or 4.

In practice, one would be interested only in the cases where the coefficients a_n are integers having nice properties, such as having a closed form in terms of binomial coefficients. When such differential equations have order 2 or 3, they are often related to modular forms. This is because in these cases the monodromy groups happen to be nice arithmetic subgroups of $SL(2,\mathbb{R})$. Therefore, it is possible to use the theory of modular forms and modular functions to determine the Apéry limits for these cases. In fact, this has been done by Beukers [2] earlier. However, we remark that in [2] the main goal is to give a modular-function interpretation of Apéry's irrationality results, so the argument is not readily applicable to general situations. Thus, the main purpose of the present article is to present a method that works for more general differential equations. In Section 2 we will briefly review Beukers' argument, and then describe our general approach. In Sections 3 and 4, we specialize our method to the cases where the nonzero singularity closest to the origin corresponds to an elliptic point or a cusp of the underlying modular curve. The examples we work out include cases (e), (h), and (β) of [1]. These are some of the examples where [1] fails to determine their Apéry limits. We also give two examples in which the Apéry limits are values of L-functions associated with cusp forms.

As a by-product, our analysis in one particular example leads us to a family of identities involving the special values of L-functions associated with modular forms of weight 3. (See Lemma 5 in Section 5.) A typical example is

$$\sum_{n\equiv 1 \bmod 12}\frac{c_n}{n^2}=\frac{2+\sqrt{3}}{3}\sum_{n=1}^{\infty}\frac{c_n}{n^2}$$

for the cusp form $f(\tau)=\eta(2\tau)^3\eta(6\tau)^3=\sum_{n=1}^{\infty}c_ne^{2\pi in\tau}$ of weight 3. The proof of these identities is independent of differential equations and Apéry limits. We expect that our argument can be extended to modular forms of higher weight, but we do not attempt to do so since this will be too far astray from the main topic of the paper.

After finishing this paper, Zudilin kindly informed of us that in an unpublished manuscript [7], Zagier also has described an approach to the determination of Apéry limits very close to ours (presumably in the same spirit as [2]), but we do not know to what extent the two papers overlap.

2. Modular-function approach

Recall that a result of Stiller [4] (see also [6]) states that if $A(\tau)$ is a (meromorphic) modular form of weight k with character χ and $t(\tau)$ is a non-constant modular function on a subgroup Γ of $SL(2,\mathbb{R})$ commensurable with $SL(2,\mathbb{Z})$, then $A,\tau A,\dots,\tau^kA$, as functions of t, are linearly independent solutions of a $(k+1)$st order linear differential equation $LA=0$ with algebraic functions of t as coefficients. We assume that the differential equation $LA=0$ has polynomial coefficients. This is true whenever $t(\tau)$ is a uniformizer (Hauptmodul) of the modular curve $X(\Gamma)$. From now on we assume that Γ is of genus zero so that such a uniformizer exists. We assume that $t(i\infty)=0$ and the Fourier expansion of $A(\tau)$ starts from 1. Then the differential equation satisfied by A and t has local exponents all equal to 0 at $t=0$. When the function $t:\tau\mapsto t(\tau)$ is locally one-to-one and $A(\tau)$ is holomorphic at a point τ_0, the function $A(\tau)$ is a well-defined analytic function of t near $t_0=t(\tau_0)$. Conversely, if either $t:\tau\mapsto t(\tau)$ is not locally one-to-one at $\tau=\tau_0$ or $A(\tau_0)=\infty$, then t_0 is a singularity of the differential equation. Thus, singular points $t_0=t(\tau_0)$ of the differential equation $LA(t)=0$ can occur at four types of points where

(1) t_0 corresponds to an elliptic point of the modular curve $X(\Gamma)$, or
(2) t_0 corresponds to a cusp of $X(\Gamma)$, or
(3) $A(t_0)=\infty$, or
(4) t_0 is a branch point of the covering map $X(\Gamma)\to\mathbb{P}^1(\mathbb{C})$ given by $h(\tau)\to t(\tau)$, where $h(\tau)$ is a Hauptmodul of Γ.

Examples.

(1) Apéry's differential equation (1) has a modular-function parameterization given by

$$t(\tau)=\left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^{12},\qquad A(\tau)=\frac{(\eta(2\tau)\eta(3\tau))^7}{(\eta(\tau)\eta(6\tau))^5}. \tag{3}$$

They are modular on $\Gamma_0(6)+\omega_6$. There are four singularities 0, $17\pm 12\sqrt{2}$, and ∞. The points $t=0$ and ∞ correspond to the two cusps ∞ and $1/2$. The points $t=17-12\sqrt{2}$ and $17+12\sqrt{2}$ correspond to the elliptic points $\tau=i/\sqrt{6}$ and $\tau=2/5+i/5\sqrt{6}$ of order 2 fixed by $\left(\begin{smallmatrix}0&-1\\6&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}12&-5\\30&-12\end{smallmatrix}\right)$, respectively.

(2) Apéry's differential equation (2) is satisfied by

$$t(\tau)=q\prod_{n=1}^{\infty}\left(1-q^n\right)^{5\left(\frac{n}{5}\right)}, \tag{4}$$

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$$A(\tau) = 1 + \sum_{n=1}^{\infty}\left(\frac{3q^{5n-4}}{1-q^{5n-4}} + \frac{q^{5n-3}}{1-q^{5n-3}} - \frac{q^{5n-2}}{1-q^{5n-2}} - \frac{3q^{5n-1}}{1-q^{5n-1}}\right), \tag{5}$$

where $q = e^{2\pi i\tau}$ and $(\frac{n}{5})$ is the Legendre symbol. The functions are modular on $\Gamma_1(5)$. The singularities $0, \infty, (11-5\sqrt{5})/2, (11+5\sqrt{5})/2$ are the values of $t(\tau)$ at the four cusps ∞, $2/5$, 0, and $1/2$, respectively.

We now briefly review Beukers' argument [2]. Assume that $t_0 = 0, t_1, \ldots, t_m$ are singularities of the differential equation $LA = 0$ with $|t_1| < |t_2| < \cdots$. Let $A(t) = 1 + \sum a_n t^n$ be the t-expansion of A. In general, the radius of convergence of the series is $|t_1|$. If the inhomogeneous differential equation $LB(t) = t$ has a holomorphic solution $B(t) = t + \sum_{n=2}^{\infty} b_n t^n$ near $t = 0$, this series $B(t)$ in general also has a radius of convergence equal to $|t_1|$. Now suppose that there is a constant c such that the series $B(t) - cA(t)$ has a larger radius of convergence, i.e., such that t_1 is no longer a singularity of $B(t) - cA(t)$. This would mean that the sequence $\{b_n/a_n\}$ converges to c. Moreover, the larger the ratio $|t_2/t_1|$ is, the better the rate of convergence is. For example, in Apéry's differential equation for $\zeta(3)$, we have $t_1 = 17 - 12\sqrt{2}$, $t_2 = 17 + 12\sqrt{2}$ and $t_2/t_1 = (17 + 12\sqrt{2})^2 = 1153.999\ldots$. Using the theory of modular forms, Beukers [2] showed that with the choice of constant $c = \zeta(3)/6$, the series $B(t) - cA(t)$ no longer has a singularity at t_1. Then the exceptionally large ratio t_2/t_1 is sufficient to imply the irrationality of $\zeta(3)$. We now explain why the constant c is $\zeta(3)/6$.

Lemma 1. *Let Γ be a discrete subgroup of $SL(2,\mathbb{R})$ commensurable with $SL(2,\mathbb{Z})$. Let $A(\tau)$ be a modular form of weight k and $t(\tau)$ be a non-constant modular function on Γ such that $t(i\infty) = 0$. Let*

$$L:\quad \theta^{k+1} + r_k(t)\theta^k + \cdots + r_0(t)$$

be the differential operator annihilating A. Assume that $g(t)$ is a rational function of t. Then a solution of the inhomogeneous differential equation $LB(t) = g(t)$ is given by

$$B = A\int^q\left(\cdots\left(\int^q\left(\frac{q\,dt/dq}{t}\right)^{k+1}\frac{g(t)}{A}\frac{dq}{q}\right)\cdots\right)\frac{dq}{q}, \tag{6}$$

where the integration is iterated $k+1$ times, and $q = e^{2\pi i\tau}$.

Remark. We did not specify the starting points and the paths of integration in (6) because different choices just give different coefficients d_i in general solutions $d_0A + d_1\tau A + \cdots + d_k\tau^k A + B_0(t)$, where $B_0(t)$ is a fixed solution of $LB(t) = g(t)$. However, when the integrand is a holomorphic modular form $\sum_{n=1}^{\infty} c_n q^n$ that vanishes at $i\infty$, we specify the solution to be

$$B = A\int_0^q\left(\cdots\left(\int_0^q\left(\sum_{n=1}^{\infty} c_n q^n\right)\frac{dq}{q}\right)\cdots\right)\frac{dq}{q} = A\sum_{n=1}^{\infty}\frac{c_n}{n^{k+1}}q^n.$$

In particular, this is what we refer to in the examples in the next two sections.

Proof. The proof uses the standard method of variation. Here we only prove the case $k = 1$; general cases can be proved in the same way.

For convenience, we set

$$G_1 = \frac{q\,dt/dq}{t}, \qquad G_2 = \frac{q\,dA/dq}{A}.$$

Then we have

$$\theta A = A\frac{G_2}{G_1}, \qquad \theta\tau = \frac{1}{2\pi i G_1}. \tag{7}$$

We will look for two functions $p_1(t)$ and $p_2(t)$ such that $B = p_1A + p_2\tau A$ will solve the differential equation $LB(t) = g(t)$. We have $\theta B = A\theta p_1 + \tau A\theta p_2 + p_1\theta A + p_2\theta(\tau A)$. At this point, we make an additional assumption that p_1 and p_2 satisfy

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$$A\theta p_1 + \tau A\theta p_2 = 0$$

so that

$$\theta B = p_1\theta A + p_2\theta(\tau A).$$

Differentiating again, we obtain

$$\theta^2 B = \theta p_1\theta A + \theta p_2\theta(\tau A) + p_1\theta^2 A + p_2\theta^2(\tau A).$$

Since A and τA are solutions of L, we find

$$LB = \theta^2 B + r_1\theta B + r_2 B = \theta p_1\theta A + \theta p_2\theta(\tau A).$$

Therefore, if p_1 and p_2 satisfy

$$\begin{cases} A\theta p_1 + \tau A\theta p_2 = 0, \\ \theta A\theta p_1 + \theta(\tau A)\theta p_2 = g(t), \end{cases}$$

then $B = p_1 A + p_2\tau A$ is a solution of $LB(t) = g(t)$. Solving the linear equations and using the expressions in (7), we find

$$\theta p_1 = -\frac{2\pi i\tau g G_1}{A}, \qquad \theta p_2 = \frac{2\pi i g G_1}{A}.$$

It follows that

$$B(t) = -2\pi i A\int^t \frac{\tau g G_1}{A}\frac{dt}{t} + 2\pi i\tau A\int^t \frac{g G_1}{A}\frac{dt}{t}.$$

Making the change of variable $t \mapsto q$, we obtain

$$B(t) = -2\pi i A\int^q \frac{\tau g G_1^2}{A}\frac{dq}{q} + 2\pi i\tau A\int^q \frac{g G_1^2}{A}\frac{dq}{q}.$$

(Again, although $t \to q$ is a one-to-many mapping, different choices of branches just give different solutions.) Finally, applying integration by parts to the first integral, we conclude that

$$B(t) = A\int^q \left(\int^q \frac{g G_1^2}{A}\frac{dq}{q}\right)\frac{dq}{q}.$$

This proves the lemma for the case $k = 1$. General cases can be proved in the same way. □

Lemma 2. *Let all the notations be given as in Lemma* 1. *The function*

$$\left(\frac{q\,dt/dq}{t}\right)^{k+1}\frac{g(t)}{A}$$

in the above lemma is a modular form of weight $k+2$ on Γ with character $\overline{\chi}$.

Proof. The lemma is an immediate consequence of the well-known property that $(q\,dt/dq)/t$ is a modular form of weight 2 on Γ with trivial character. □

Example. Let $t(\tau)$ and $A(\tau)$ be given as in (3), which satisfy Apéry's differential equation (1) for $\zeta(3)$. Then we have

$$\frac{q\,dt/dq}{t} = \frac{1}{2}\big(E_2(\tau) - 2E_2(2\tau) - 3E_2(3\tau) + 6E_2(6\tau)\big),$$

where $E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} nq^n/(1-q^n)$. By Lemma 1, a solution of the inhomogeneous solution $LB = t$ near $t = 0$ is given by

$$B(t) = A(t) \int_0^q \int_0^q \int_0^q f(\tau) \frac{dq}{q} \frac{dq}{q} \frac{dq}{q},$$

where

$$f(\tau) = \left(\frac{q\, dt/dq}{t} \right)^3 \frac{t}{(1-34t+t^2)A} = \frac{1}{240} \big(E_4(\tau) - 28E_4(2\tau) + 63E_4(3\tau) - 36E_4(6\tau) \big)$$

and $E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} n^3 q^n/(1-q^n)$ is the normalized Eisenstein series of weight 4 on $SL(2,\mathbb{Z})$.

Since the singularity $t = 17 - 12\sqrt{2}$ corresponds to the elliptic point $\tau = i/\sqrt{6}$ fixed by $\left(\begin{smallmatrix} 0 & -1 \\ 6 & 0 \end{smallmatrix}\right)$, the assertion that $B(\tau) - cA(\tau)$, as a function of t, is not singular at $t = 17 - 12\sqrt{2}$ is equivalent to that $B(\tau) - cA(\tau)$ is invariant under the substitution $\tau \to -1/6\tau$. Now we have $A(-1/6\tau) = -6\tau^2 A(\tau)$. Thus, to prove the latter assertion, it suffices to show that the function

$$E(\tau) = \int_0^q \int_0^q \int_0^q f(\tau) \frac{dq}{q} \frac{dq}{q} \frac{dq}{q}$$

satisfies

$$E(-1/6\tau) - \frac{\zeta(3)}{6} = -\frac{1}{6\tau^2} \left(E(\tau) - \frac{\zeta(3)}{6} \right). \tag{8}$$

For this purpose, Beukers invoked the standard method of Mellin transforms. By the Mellin inversion formula, one can write

$$E(-1/6\tau) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(s) L(s+3) \left(\frac{2\pi i}{6\tau} \right)^{-s} ds,$$

where $\Gamma(s)$ is the Gamma function and

$$L(s) = \zeta(s)\zeta(s-3)\big(1 - 28 \cdot 2^{-s} + 63 \cdot 3^{-s} - 36 \cdot 6^{-s}\big)$$

is the L-function associated with the modular form $f(\tau)$. We then check that the integrand has simple poles only at $s = 0$ and $s = -2$. Moving the line of integration to the left of $\operatorname{Re} s = -2$, counting residues, making a change of variable $s \mapsto -2 - s$, using the functional equation

$$\left(\frac{2\pi}{6} \right)^{-s} \Gamma(s) L(s) = -\left(\frac{2\pi}{6} \right)^{s-4} \Gamma(4-s) L(4-s),$$

and then applying the Mellin inversion formula again, we deduce that (8) indeed holds. (See [2, Proposition 1] for more details. See also the examples in Sections 3 and 4.) This basically summarizes Beukers' argument.

As we have remarked earlier, the focus of [2] is to give a modular-function proof of Apéry's irrationality results. Therefore, the differential equations considered there are very well-behaved in some sense. That is, the Apéry limits c of them all have the special property that the functions $B(t) - cA(t)$ no longer have singularities at t_1. However, in many cases, we do not need such a strong property in order for the Apéry limits to exist. For example, consider the differential equation

$$\theta^2 A + 3t\big(18\theta^2 + 18\theta + 7\big)A + 729t^2(\theta+1)^2 A = 0.$$

It has three singularities 0, $-1/27$, and ∞. The local exponents at these points are $\{0,0\}$, $\{-2/3, -1/3\}$, and $\{1,1\}$, respectively. Thus, near $t = -1/27$, the solution $A(t) = 1 + \sum a_n t^n$ has a series expansion

$$A(t) = c_1(t+1/27)^{-2/3} + c_2(t+1/27)^{-1/3} + \cdots$$

for some constants c_1 and c_2, at least one of which is nonzero. On the other hand, we can show that the solution $B(t) = t + \sum b_n t^n$ of

$$\theta^2 B + 3t\big(18\theta^2 + 18\theta + 7\big)B + 729t^2(\theta+1)^2 B = t$$

behaves asymptotically as

$$A(t)\big(d_0 + d_1(t+1/27)^{1/3} + \cdots\big)$$

near $t = -27$. Therefore, we have

$$B(t) - d_0 A(t) = A(t)\big(d_1(t+1/27)^{1/3} + \cdots\big).$$

We then can apply the Tauberian theorems to conclude that

$$\left|\frac{b_n}{a_n} - d_0\right| \ll \frac{1}{n^{1/3}}.$$

To determine the exact value of d_0, we use the theory of modular forms. The detailed computation is carried out in the next section.

In general, if the singular point $t_1 = t(\tau_1)$ closest to the origin corresponds to an elliptic point τ_1 of order n, then the local exponents at t_1 take the form $a_1/n < \cdots < a_{k+1}/n$ for some integers a_i, at least one of which is relatively prime to n. If the $(k+1)$-times iterated integral $E(\tau)$ in Lemma 1 is a holomorphic function of τ at τ_1, then the same argument as that in the previous paragraph will imply that the Apéry limit is equal to $E(\tau_1)$, provided that the differential equation does not have another singularity t_2 of the same modulus as t_1. Likewise, if the singular point t_1 corresponds to a cusp, then the local exponents are all equal to a rational number a. In this case, we have

$$A(t) = (t-t_1)^a\big(c_k \log^k(t-t_1) + c_{k-1}\log^{k-1}(t-t_1) + \cdots\big).$$

Under the same condition on $E(\tau)$, we will also be able to determine the Apéry limit, although the rate of convergence, in general, is $|b_n/a_n - d_0| \ll 1/\log n$, which is extremely slow.

In the following two sections, we will determine the Apéry limits of several examples using the above ideas.

3. Elliptic point cases

In this section, we will explain in more detail how to determine the Apéry limits when the singularity t_1 closest to the origin corresponds to an elliptic point τ_1. Throughout the section, we assume that Γ is a discrete subgroup of $SL(2,\mathbb{R})$ of genus zero commensurable with $SL(2,\mathbb{Z})$, $A(\tau)$ is a (meromorphic) modular form of weight k with character χ on Γ, and $t(\tau)$ is a uniformizer of the modular curve $X(\Gamma)$ so that the differential equation $LA(t) = 0$ satisfied by A and t has polynomial coefficients. We also assume that $A(i\infty) = 1$ and $t(i\infty) = 0$ so that the differential equation has local exponents all equal to 0 at $t = 0$. We let $B(t) = t^j + \cdots$ be the solution of $LB(t) = j^{k+1}t^j$ holomorphic at $t = 0$, where j is a positive integer less than the maximal degree of the coefficients of L. For a function f, with a slight abuse of notations, we write $f(\tau)$ if we consider f as a function of τ, and write $f(t)$ if we consider it as a (multi-valued) function of t.

Let $h(t)$ be the coefficient of θ^{k+1} in the differential equation $LA = 0$. We assume that the integrand

$$f(\tau) = \left(\frac{q\,dt/dq}{t}\right)^{k+1} \frac{t^j}{Ah(t)}$$

inside (6) is a holomorphic modular form so that its L-function converges in some half-plane. Note that, by the assumption that $t(i\infty) = 0$ and $A(i\infty) = 1$, the constant term of the Fourier expansion $\sum_{n=1}^{\infty} c_n q^n$ of $f(\tau)$ is 0. According to the discussion in the previous section, the Apéry limit is equal to the value of the $(k+1)$-times iterated integral

$$E(\tau) = \int_0^q \left(\cdots\left(\int_0^q f(\tau)\frac{dq}{q}\right)\cdots\right)\frac{dq}{q} = \sum_{n=1}^{\infty} \frac{c_n}{n^{k+1}} q^n$$

at τ_1. Now to evaluate $E(\tau_1)$, we express $E(\tau)$ using the Mellin inversion formula

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$$e^{-x} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) x^{-s}\, ds$$

which holds for all $c > 0$ and all complex numbers x with $\operatorname{Re} x > 0$, and then try to deduce information about $E(\tau_1)$ by complex analytic argument. For example, if $f(\tau)$ is an eigenfunction with eigenvalue ϵ of the Atkin–Lehner involution $\omega_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$ for some positive integer N, then by an argument similar to that in the proof of (8), we can show that

$$E(-1/N\tau) = \epsilon^{-1}(\sqrt{N}\tau)^{-k} E(\tau) + (\text{residues}).$$

Then we set $\tau = i/\sqrt{N}$ to get the value of $E(i/\sqrt{N})$ (provided that $i^k \epsilon \neq 1$). When the elliptic element that fixes τ_1 is not $\left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$, the situation is more complicated. Assume that $f(\tau)$ is modular on $\Gamma_0(N)$. If we apply the Mellin inversion formula in a straightforward manner and write

$$E(\tau) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) L(s+k+1, f) \left(\frac{2\pi\tau}{i}\right)^{-s} ds,$$

the complex analytic argument will yield

$$E(\tau) = \frac{\epsilon}{\tau^k} \sum_{n=1}^{\infty} \frac{c_n^*}{n^{k+1}} e^{-2\pi i n/N\tau} + (\text{residues}),$$

where c_n^* denote the Fourier coefficients of $(\sqrt{N}\tau)^{-k-2} f(-1/N\tau)$ (which is a modular form on $\Gamma_0(N)$ since ω_N normalizes $\Gamma_0(N)$). At this point, it is not clear how one should proceed to obtain information about $E(\tau_1)$. In such a situation, we need to apply the Mellin inversion formula in a different way. The key observation is the following simple fact.

Lemma 3. *For a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$ with $c > 0$ and for a pair of functions $g, g^* : \mathbb{H} \to \mathbb{C}$, the relation*

$$g\left(\frac{a\tau + b}{c\tau + d}\right) = \epsilon (c\tau + d)^k g^*(\tau)$$

holds for some constant ϵ and some integer k if and only if

$$g\left(\frac{\tau}{c} + \frac{a}{c}\right) = \epsilon (-\tau)^{-k} g^*\left(-\frac{1}{c\tau} - \frac{d}{c}\right)$$

holds.

Proof. We have

$$\frac{\tau}{c} + \frac{a}{c} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left(-\frac{1}{c\tau} - \frac{d}{c}\right).$$

This proves the lemma. □

Now if $g(\tau) = \sum_{n=0}^{\infty} c_n q^n$ and $g^*(\tau) = \sum_{n=0}^{\infty} c_n^* q^n$ in the above lemma are holomorphic modular forms, then the lemma yields a functional equation between the pair of functions

$$L(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s} e^{2\pi i n a/c}, \qquad L^*(s) = \sum_{n=1}^{\infty} \frac{c_n^*}{n^s} e^{-2\pi i n d/c}.$$

(See Lemma 4 below.) Using the functional equation, we deduce that

$$E\left(-\frac{1}{c\tau} - \frac{d}{c}\right) = \frac{\epsilon}{\tau^k} E\left(\frac{\tau}{c} + \frac{a}{c}\right) + (\text{residues}).$$

We then make a suitable choice of τ such that $-1/c\tau - d/c = \tau/c + a/c = \tau_1$ and an evaluation of $E(\tau_1)$ follows.

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We now prove the functional equation between $L(s)$ and $L^*(s)$ in the following lemma. The lemma may have appeared somewhere in literature. However, failing to locate such a reference, we give a complete proof here. Notice that when $g = g^*$ is a modular form of weight k on $\Gamma_0(N)$ that is an eigenfunction of the Atkin–Lehner involution w_N with eigenvalue ϵ, the lemma gives the familiar functional equation

$$i^k\epsilon\left(\frac{2\pi}{\sqrt{N}}\right)^{-s}\Gamma(s)L(s,g) = \left(\frac{2\pi}{\sqrt{N}}\right)^{s-k}\Gamma(k-s)L(k-s,g)$$

for the L-function $L(s,g)$ of g.

Lemma 4. *Let Γ and Γ^* be two discrete subgroups of $SL(2,\mathbb{R})$ commensurable with $SL(2,\mathbb{Z})$ such that the cusp ∞ has width 1. Assume that $g(\tau) = \sum_{n=0}^\infty c_n q^n$ and $g^*(\tau) = \sum_{n=0}^\infty c_n^* q^n$ are holomorphic modular forms of weight k on Γ and Γ^*, respectively. Suppose that $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ with $c > 0$ is a matrix in $SL(2,\mathbb{R})$ such that*

$$g\left(\frac{a\tau+b}{c\tau+d}\right) = \epsilon(c\tau+d)^k g^*(\tau) \tag{9}$$

for some constant ϵ. Then the two functions

$$L(s) = \sum_{n=1}^\infty \frac{c_n}{n^s} e^{2\pi i n a/c}, \qquad L^*(s) = \sum_{n=1}^\infty \frac{c_n^*}{n^s} e^{-2\pi i n d/c}$$

satisfy the functional equation

$$\left(\frac{2\pi}{c}\right)^{-s}\Gamma(s)L(s) = i^k\epsilon\left(\frac{2\pi}{c}\right)^{s-k}\Gamma(k-s)L^*(k-s). \tag{10}$$

Proof. For the ease of representation, here we only prove the case where the constant terms c_0 and c_0^* are 0. The case $c_0, c_0^* \neq 0$ can be proved by a minor modification. First of all, setting $\tau = (-d\tau+b)/(c\tau-a)$ in (9), we obtain

$$g^*\left(\frac{-d\tau+b}{c\tau-a}\right) = \epsilon^{-1}(-1)^k(c\tau-a)^k g(\tau),$$

which by Lemma 3 is equivalent to

$$g^*\left(\frac{\tau}{c}-\frac{d}{c}\right) = \epsilon^{-1}\tau^{-k} g\left(-\frac{1}{c\tau}+\frac{a}{c}\right). \tag{11}$$

We now consider the integral

$$\int_0^\infty y^{s-1} g\left(\frac{iy}{c}+\frac{a}{c}\right)dy.$$

We have

$$\begin{aligned}\int_0^\infty y^{s-1} g\left(\frac{iy}{c}+\frac{a}{c}\right)dy &= \sum_{n=1}^\infty c_n e^{2\pi i n a/c}\int_0^\infty y^{s-1}e^{-2\pi n y/c}\,dy \\ &= \left(\frac{2\pi}{c}\right)^{-s}\Gamma(s)\sum_{n=1}^\infty \frac{c_n}{n^s}e^{2\pi i n a/c} = \left(\frac{2\pi}{c}\right)^{-s}\Gamma(s)L(s).\end{aligned}$$

Now we break the integral into two parts as usual, one from 0 to 1 and the other from 1 to ∞. For the integral from 0 to 1, we make a change of variable $y \mapsto 1/y$ and obtain

$$\int_0^1 y^{s-1} g\left(\frac{iy}{c}+\frac{a}{c}\right)dy = \int_1^\infty y^{1-s} g\left(\frac{i}{cy}+\frac{a}{c}\right)\frac{dy}{y^2}.$$

Substituting the relation (11) with $\tau = iy$ into the last expression, we see that

$$\int_0^1 y^{s-1} g\left(\frac{iy}{c} + \frac{a}{c}\right) dy = i^k \epsilon \int_1^\infty y^{k-s-1} g^*\left(\frac{iy}{c} - \frac{d}{c}\right) dy$$

and

$$\left(\frac{2\pi}{c}\right)^{-s} \Gamma(s) L(s) = \int_1^\infty y^{s-1} g\left(\frac{iy}{c} + \frac{a}{c}\right) dy + i^k \epsilon \int_1^\infty y^{k-s-1} g^*\left(\frac{iy}{c} - \frac{d}{c}\right) dy.$$

By the same token, we can also show that

$$\left(\frac{2\pi}{c}\right)^{-s} \Gamma(s) L^*(s) = \int_1^\infty y^{s-1} g^*\left(\frac{iy}{c} - \frac{d}{c}\right) dy + \epsilon^{-1} (-i)^k \int_1^\infty y^{k-s-1} g\left(\frac{iy}{c} + \frac{a}{c}\right) dy.$$

Upon the substitution $s \mapsto k - s$ into the last expression, we immediately get the claimed functional equation. This completes the proof. □

We now give two examples to illustrate our method.

Example 1. Consider the differential equation

$$\theta^2 A + 3t\left(18\theta^2 + 18\theta + 7\right)A + 729t^2(\theta + 1)^2 A = 0. \tag{12}$$

This is case (h) that [1] fails to identify the Apéry limit. According to [1], Arne Meurman conjectures that the limit is $2\pi^2/81 - L(2, \chi_3)/2$, where $L(s, \chi_3)$ is the Dirichlet L-function associates with $\chi_3 = (\frac{\cdot}{3})$. Meurman also suggests the following Ramanujan-like formula

$$\sum_{n=1}^\infty \left(\frac{n}{3}\right) \frac{(-e^{-\pi/\sqrt{3}})^n}{n^2(1 - (-e^{-\pi/\sqrt{3}})^n)} = \frac{2\pi^2}{81} - \frac{1}{2} L(2, \chi_3).$$

Here we prove that these conjectures are indeed correct.

We first note that (12) is the differential equation satisfied by

$$t(\tau) = \frac{\eta(3\tau)^{12}}{\eta(\tau)^{12}}, \qquad A(\tau) = \frac{1}{1 + 27t(\tau)} \left(1 + 6 \sum_{n=1}^\infty \left(\frac{n}{3}\right) \frac{q^n}{1 - q^n}\right),$$

where $A(\tau)$ is a modular form of weight 1 on $\Gamma_0(3)$ with character

$$\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{d}{3}\right). \tag{13}$$

The singular points 0, ∞, and $-1/27$ correspond to the cusps ∞, 0, and the elliptic point

$$\tau_0 = \frac{3 + \sqrt{-3}}{6},$$

respectively. Using Lemma 1 we find a solution of

$$\theta^2 B + 3t\left(18\theta^2 + 18\theta + 7\right)B + 729t^2(\theta + 1)^2 B = t$$

is given by

$$B(t) = A(t) \int_0^q \int_0^q \left(\frac{q\,dt/dq}{t}\right)^2 \frac{t}{A(t)(1 + 27t)^2} \frac{dq}{q} \frac{dq}{q} = A(t) \int_0^q \int_0^q \frac{\eta(3\tau)^9}{\eta(\tau)^3} \frac{dq}{q} \frac{dq}{q}.$$

The local exponents of the differential equation at $-1/27$ are $-1/3$ and $-2/3$. Thus, near $t=-1/27$, we have the series expansion

$$A(t)=c_1(t+1/27)^{-2/3}+c_2(t+1/27)^{-1/3}+\cdots$$

for some constants c_i with at least one of c_1 and c_2 being nonzero. Now write

$$E(\tau)=\int_0^q\int_0^q \frac{\eta(3\tau)^9}{\eta(\tau)^3}\frac{dq}{q}\frac{dq}{q}.$$

Regardless of what the constants c_1 and c_2 are, we have

$$B(t)-E(\tau_0)A(t)=A(t)\bigl(E(t)-E(\tau_0)\bigr)=A(t)\bigl(0+d_1(t+1/27)^{1/3}+\cdots\bigr).$$

At this point, the Tauberian theorems (using, for example, [3, Corollary 1.7.3]) already assert that if $A(t)=1+\sum_n a_nt^n$ and $B(t)=t+\sum_n b_nt^n$, then we have

$$\left|\frac{\sum_{n=1}^N b_n}{\sum_{n=1}^N a_n}-E(\tau_0)\right|\ll N^{-1/3}$$

as $N\to\infty$. To get the stronger statement $b_n/a_n\to E(\tau_0)$, we consider the sequences $a_n'=a_n-a_{n-1}$ and $b_n'=b_n-b_{n-1}$, whose generating functions are $(1-t)A(t)$ and $(1-t)B(t)$, respectively. Since these two functions have similar asymptotic behaviors as $A(t)$ and $B(t)$ near $t=-1/27$, the same Tauberian theorem implies that

$$\left|\frac{b_n}{a_n}-E(\tau_0)\right|\ll n^{-1/3}.$$

In particular, the Apéry limit is $E(\tau_0)$. It remains to determine the value of $E(\tau_0)=E((3+\sqrt{-3})/6)$.

The integrand $f(\tau)=\eta(3\tau)^9/\eta(\tau)^3$ in the definition of $E(\tau)$ is in fact the Eisenstein series of weight 3 associated with the cusp 0 with the same character as (13). It can be written as

$$\sum_{n=1}^\infty \frac{n^2(q^n-q^{2n})}{1-q^{3n}}=\sum_{n=1}^\infty n^2\sum_{k=1}^\infty\left(\frac{k}{3}\right)q^{kn}.$$

It follows that

$$E(\tau)=\sum_{n=1}^\infty\sum_{k=1}^\infty\frac{1}{k^2}\left(\frac{k}{3}\right)q^{kn}.$$

Our strategy is to use Lemma 4 to show that

$$E(-1/3\tau+1/3)=\frac{1}{\tau}E(\tau/3+2/3)+r(\tau)$$

for some function $r(\tau)$ involving τ and special values of Dirichlet L-functions. Then we set $\tau=(-1+\sqrt{-3})/2$. With this choice of τ, we have

$$-\frac{1}{3\tau}+\frac13=\frac{\tau}{3}+\frac23=\frac{3+\sqrt{-3}}{6}=\tau_0.$$

From this we can obtain the values of $E(\tau_0)$.

Write the Fourier coefficients of q^n in $f(\tau)$ as c_n. We apply Lemma 4 with $g=g^*=f$,

$$\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}2&-1\\3&-1\end{pmatrix},$$

$\epsilon=g(\chi)=-1$, and define

$$L(s)=\sum_{n=1}^\infty\frac{c_ne^{4\pi in/3}}{n^s},\qquad L^*(s)=\sum_{n=1}^\infty\frac{c_ne^{2\pi in/3}}{n^s}.$$

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By the Mellin inversion formula, we have

$$E(-1/3\tau+1/3)=\sum_{n=1}^{\infty}\frac{c_n e^{2\pi in/3}}{n^2}e^{-2\pi in/3\tau}=\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty}\Gamma(s)L^*(s+2)\left(\frac{2\pi i}{3\tau}\right)^{-s}ds. \tag{14}$$

We then move the line of integration to $\operatorname{Re}s=-5/2$ and make a change of variable $s\mapsto -1-s$. (By the expression of $L(s)$ in terms of $\zeta(s)$ and $L(s,\chi_3)$ given later in (16), we see that this is justified.) We obtain

$$E(-1/3\tau+1/3)=(\text{residues})+\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty}\Gamma(-1-s)L^*(1-s)\left(\frac{2\pi i}{3\tau}\right)^{s+1}ds.$$

Now Lemma 4 implies that

$$\left(\frac{2\pi}{3}\right)^{-s}\Gamma(s)L^*(s)=-i\left(\frac{2\pi}{3}\right)^{s-3}\Gamma(3-s)L(3-s).$$

It follows that

$$\begin{aligned}\Gamma(-1-s)L^*(1-s)=\frac{\Gamma(1-s)L^*(1-s)}{s(s+1)}&=-i\left(\frac{2\pi}{3}\right)^{-1-2s}\frac{\Gamma(s+2)L(s+2)}{s(s+1)}\\&=-i\left(\frac{2\pi}{3}\right)^{-1-2s}\Gamma(s)L(s+2),\end{aligned}$$

and we have

$$E(-1/3\tau+1/3)=(\text{residues})+\frac{1}{2\pi i\tau}\int_{3/2-i\infty}^{3/2+i\infty}\Gamma(s)L(s+2)\left(\frac{2\pi\tau}{3i}\right)^{-s}ds.$$

By the Mellin inversion formula again, the integral in the last expression is precisely $E(\tau/3+2/3)$. That is, we have

$$E(-1/3\tau+1/3)=(\text{residues})+\frac{1}{\tau}E(\tau/3+2/3). \tag{15}$$

It remains to compute the residues.

We have

$$L^*(s)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}n^2\left(\frac{k}{3}\right)\frac{e^{2\pi ink/3}}{n^sk^s}.$$

Partitioning the double sum according to the residue classes modulo 3 and simplifying, we obtain

$$L^*(s)=-\frac{1}{2}\zeta(s-2)\left(1-3^{3-s}\right)L(s,\chi_3)+\frac{\sqrt{3}i}{2}L(s-2,\chi_3)\zeta(s)\left(1-3^{-s}\right). \tag{16}$$

Using the fact that $\zeta(s)$ has zeros at negative even integers and $L(s,\chi_3)$ has zeros at negative odd integers, we see that the integrand

$$\begin{aligned}\Gamma(s)L^*(s+2)\left(\frac{2\pi i}{3\tau}\right)^{-s}&=\Gamma(s)\left(\frac{2\pi i}{3\tau}\right)^{-s}\\&\quad\times\left(-\frac{1}{2}\zeta(s)\left(1-3^{1-s}\right)L(s+2,\chi_3)+\frac{\sqrt{3}i}{2}L(s,\chi_3)\zeta(s+2)\left(1-3^{-2-s}\right)\right)\end{aligned}$$

in (14) has poles only at $s=0$ and $s=-1$. We find that the residue at $s=0$ is

$$\operatorname{Res}_{s=0}=-\frac{1}{2}\zeta(0)(1-3)L(2,\chi_3)+\frac{\sqrt{3}i}{2}L(0,\chi_3)\zeta(2)(1-1/9). \tag{17}$$

With the evaluations $\zeta(0) = -1/2$, $L(0, \chi_3) = 1/3$, and $\zeta(2) = \pi^2/6$, we simplify the expression into

$$-\frac{1}{2}L(2, \chi_3) + \frac{2\sqrt{3}\pi^2 i}{81}.$$

Also, the residue at $s = -1$ is

$$-\left(-\frac{1}{2}\zeta(-1)(1-9)L(1, \chi_3) + \frac{\sqrt{3}i}{2}L'(-1, \chi_3)(1 - 1/3)\right)\frac{2\pi i}{3\tau}.$$

Using the functional equation

$$\left(\frac{\pi}{3}\right)^{-(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)L(s, \chi_3) = \left(\frac{\pi}{3}\right)^{-(2-s)/2}\Gamma\left(\frac{2-s}{2}\right)L(1-s, \chi_3)$$

for $L(s, \chi_3)$, one may deduce that

$$L'(-1, \chi_3) = \frac{3\sqrt{3}}{4\pi}L(2, \chi).$$

Together with the evaluations $\zeta(-1) = -1/12$ and $L(1, \chi_3) = \pi\sqrt{3}/9$, we find that the residue at $s = -1$ can be simplified as

$$\mathrm{Res}_{s=-1} = \frac{2\pi i\sqrt{3}}{81\tau} + \frac{1}{2\tau}L(2, \chi). \tag{18}$$

In summary, the above computation (15), (17), (18) shows that

$$E(-1/3\tau + 1/3) = -\frac{\tau - 1}{2\tau}L(2, \chi) + \frac{\tau + 1}{\tau}\frac{2\pi^2 i\sqrt{3}}{81} + \frac{1}{\tau}E(\tau/3 + 2/3).$$

Setting $\tau = (-1 + \sqrt{-3})/2$, we find

$$E\left(\frac{3 + \sqrt{-3}}{6}\right) = \frac{2\pi^2}{81} - \frac{1}{2}L(2, \chi_3).$$

We summarize the above computation as follows.

Theorem 1. *Let $\{a_n\}$ and $\{b_n\}$ be the sequences satisfying the recursive relation*

$$(n+2)^2u_{n+2} + \left(54n^2 + 162n + 129\right)u_{n+1} + (n+1)^2u_n = 0 \quad (u_n = a_n \text{ or } b_n)$$

with the initial values $a_0 = 1$, $a_1 = -21$, $b_0 = 0$, and $b_1 = 1$. Then we have

$$\left|\frac{b_n}{a_n} - \frac{2\pi^2}{81} + \frac{1}{2}L(2, \chi_3)\right| \ll n^{-1/3}$$

as $n \to \infty$. Moreover, the evaluation

$$\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{(-e^{-\pi/\sqrt{3}})^n}{n^2(1 - (-e^{-\pi/\sqrt{3}})^n)} = \frac{2\pi^2}{81} - \frac{1}{2}L(2, \chi_3)$$

holds.

Example 2. In this example, we will construct a differential equation whose Apéry limits give a good rational approximation to the value $L(2, f)$ at 2 of the L-function associated with the cusp form $f(\tau) = \eta(\tau)^3\eta(7\tau)^3$. The Hauptmodul of the genus-zero group $\Gamma_0(7) + \omega_7$ is given by $\eta(\tau)^4/\eta(7\tau)^4 + 49\eta(7\tau)^4/\eta(\tau)^4$. We set

$$t(\tau) = \left(\frac{\eta(\tau)^4}{\eta(7\tau)^4} + 14 + 49\frac{\eta(7\tau)^4}{\eta(\tau)^4}\right)^{-1} = q - 10q^2 + 49q^3 - 184q^4 + \cdots.$$

The value of $t(\tau)$ at the unique cusp ∞ is obviously 0, and at the elliptic points $i/\sqrt{7}$, $(5+\sqrt{-3})/14$, and $(7+\sqrt{-7})/14$, it takes the values $1/28$, 1, and ∞, respectively. According to Lemma 1 and the discussion at the beginning of this section, we can basically choose any rational function $g(t)$ such that $g(0)=0$ and its numerator has a degree smaller than its denominator and then set

$$A(\tau)=\left(\frac{q\,dt/dq}{t}\right)^2\frac{g(t)}{f(\tau)}.$$

Then the differential equation satisfied by t and A will give us a rational approximation to $L(2,f)$.

Here we choose $g(t)=t/(1-29t+28t^2)$. With this choice, we find

$$A(\tau)=1+2\sum_{n=1}^{\infty}\left(\frac{n}{7}\right)\frac{q^n}{1-q^n}$$

and the differential equation satisfied by t and A is

$$(1-t)^2(1-28t)\theta^2A-14t(1-t)^2\theta A-\bigl(2t+4t^2\bigr)A=0.$$

The solution $B(t)$ of the inhomogeneous differential equation

$$(1-t)^2(1-28t)\theta^2B-14t(1-t)^2\theta B-\bigl(2t+4t^2\bigr)B=t(1-t)$$

has a t-expansion $B(t)=t+45t^2/4+\cdots$. Furthermore, we can show that $B(t)-L(2,f)A(t)$ has no singularity at $t=1/28$. Therefore, we have the following good rational approximation to $L(2,f)$.

Theorem 2. *Let $f(\tau)=\eta(\tau)^3\eta(7\tau)^3$ be the cusp form of weight 3 with character $\chi\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)=\left(\frac{d}{7}\right)$ on $\Gamma_0(7)$ and $L(s,f)$ be its associated L-function. Let $\{a_n\}$ and $\{b_n\}$ be sequences satisfying*

$$(n+3)^2u_{n+3}-\bigl(30n^2+134n+150\bigr)u_{n+2}+\bigl(57n^2+142n+81\bigr)u_{n+1}-\bigl(14n+28n^2\bigr)u_n=0$$

with the initial values

$$a_0=1,\qquad a_1=2,\qquad a_2=24,\qquad b_0=0,\qquad b_1=1,\qquad b_2=45/4.$$

Then we have

$$\left|\frac{b_n}{a_n}-L(2,f)\right|\ll 28^{-n}.$$

4. Cusp cases

Let $t(\tau)$ and $A(\tau)$ be given and assume that the integrand in (6) is a holomorphic modular form that vanishes at $i\infty$ as before. In this section we consider the cases where the singularity t_1 closest to the origin corresponds to a cusp α of the modular curve $X(\Gamma)$. Pick an element in $\sigma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in SL(2,\mathbb{R})$ (preferably lying in the normalizer of Γ in $SL(2,\mathbb{R})$) such that $\sigma\infty=\alpha$. Note that the asymptotic behavior of $A(\tau)$ near α is determined by that of $A(\sigma\tau)$ in a neighborhood of ∞. To be more precise, we have $A(\sigma\tau)=(c\tau+d)^kA^*(\tau)$ for some modular form $A^*(\tau)$ of weight k on the group $\sigma^{-1}\Gamma\sigma$. Also, $t^*(\tau)=t(\sigma\tau)-t_1$ is an algebraic function of t that vanishes at ∞. Then for some positive integer r depending on the order of $A^*(\tau)$ at ∞, $A^*(\tau)^r$ has a power series expansion in t^*. The term $(c\tau+d)^k$ gives a logarithmic factor $d_k\log^k t^*+\cdots+d_0$. Therefore

$$A(\sigma\tau)=\bigl(d_k\log^k t^*+\cdots+d_0\bigr)(t^*)^a\times(\text{a power series in } t^*)$$

for some rational number a. (The number a in fact is equal to the local exponent of the differential equation at t_1.) This gives us the asymptotic behavior of $A(\tau)$ near α.

By the same token, to determine the asymptotic behavior of $B(t)$ near $t=t_1$, we need to consider $B(\sigma\tau)$ in a neighborhood of ∞. Let $f(\tau)$ be the modular form of weight $k+2$ inside the integral in (6). We apply Lemma 4 to $f(\tau)$ and $f^*(\tau)=(c\tau+d)^{-k-2}f(\tau)$, which is a modular form on $\sigma^{-1}\Gamma\sigma$. We then use the functional equation to find the asymptotics of $B(t)$. The actual computation is similar to that for the elliptic point cases. We now give some examples.

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Example 3. Consider the differential equation

$$\theta^2 A - 4t\big(8\theta^2 + 8\theta + 3\big)A + 256t^2(\theta+1)^2 A = 0,$$

which is case (e) in [1]. It is conjectured that the Apéry limit is $L(2,\chi_{-1})/2$, where χ_{-1} is the odd Dirichlet character modulo 4. We now show that this is indeed the case. The differential equation is the one satisfied by

$$t(\tau) = \frac{\eta(\tau)^8\eta(4\tau)^{16}}{\eta(2\tau)^{24}}, \qquad A(\tau) = \frac{\eta(2\tau)^{22}}{\eta(\tau)^{12}\eta(4\tau)^8},$$

where $A(\tau)$ is a meromorphic modular form of weight 1 on $\Gamma_0(4)$ with character

$$\chi\begin{pmatrix} a & b \\ 4c & d\end{pmatrix} = (-1)^c\chi_{-1}(d), \tag{19}$$

and $t(\tau)$ is a modular function on $\Gamma_0(4)$. By Lemma 1,

$$B(\tau) = A(\tau)\int_0^q\int_0^q \frac{\eta(\tau)^4\eta(4\tau)^8}{\eta(2\tau)^6}\frac{dq}{q}\frac{dq}{q}$$

is a solution of the inhomogeneous $\theta^2 B - 4t(8\theta^2+8\theta+3)B + 256t^2(\theta+1)^2 B = t$. The singularities 0, $1/16$, ∞ correspond to the cusps ∞, 0, and $1/2$, respectively, and the local exponents are $\{0,0\}$, $\{-1/2,-1/2\}$, and $\{1,1\}$. Since

$$A(-1/4\tau) = \frac{\tau}{2i}\frac{\eta(2\tau)^{22}}{\eta(\tau)^8\eta(4\tau)^{12}},$$

we know that

$$A(t) = (t-1/16)^{-1/2}\big(c_1\log(t-1/16) + c_2 + \cdots\big)$$

as t approaches $1/16$ for some nonzero constant c_1. (The exact value of c_1 is not important, as it suffices to know that it is not zero.) Let

$$E(\tau) = \int_0^q\int_0^q \frac{\eta(\tau)^4\eta(4\tau)^8}{\eta(2\tau)^6}\frac{dq}{q}\frac{dq}{q}.$$

In the following we will show that

$$E(-1/4\tau) = \frac{1}{2}L(2,\chi_{-1}) - \frac{\pi^2 i}{32\tau} + \frac{i}{8\tau}\int_0^q\int_0^q \frac{\eta(\tau)^8\eta(4\tau)^4}{\eta(2\tau)^6}\frac{dq}{q}\frac{dq}{q}. \tag{20}$$

This would imply that

$$B(t) - \frac{1}{2}L(2,\chi_{-1})A(t) = (t-1/16)^{-1/2}(d_0+\cdots).$$

Then we deduce from the Tauberian theorems that

$$\left|\frac{b_n}{a_n} - \frac{1}{2}L(2,\chi_{-1})\right| \ll \frac{1}{\log n}.$$

(Again, we shall apply the Tauberian theorems to the sequence $a_n' = a_n - a_{n-1}$ and $b_n' = b_n - b_{n-1}$ with generating functions $(1-t)A(t)$ and $(1-t)B(t)$ to get the stronger conclusion.) We now prove (20).

The function $\eta(\tau)^4\eta(4\tau)^8/\eta(2\tau)^6$ is in fact the Eisenstein series of weight 3 associated with the cusp $1/2$ with character given by (19). Its q-expansion can be alternatively written as

$$\frac{\eta(\tau)^4\eta(4\tau)^8}{\eta(2\tau)^6} = \sum_{n=1}^\infty (-1)^{n-1}\frac{n^2q^n}{1+q^{2n}}.$$

Then by the Mellin inversion formula

$$E(-1/4\tau)=\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty}\Gamma(s)L(s+2)\left(\frac{2\pi i}{4\tau}\right)^{-s}ds,$$

where

$$L(s+2)=\sum_{n=1}^{\infty}(-1)^{n-1}n^2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{((2k-1)n)^{s+2}}=\zeta(s)L(s+2,\chi_{-1})\big(1-2^{1-s}\big).$$

Moving the path of integration to $\operatorname{Re}s=-5/2$ and making a change of variable $s\mapsto -1-s$, we obtain

$$\begin{aligned}E(-1/4\tau)&=\frac{1}{2}L(2,\chi_{-1})-\frac{\pi^2 i}{32\tau}\\&\quad+\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty}\Gamma(-1-s)\zeta(-1-s)L(1-s,\chi_{-1})\big(1-2^{2+s}\big)\left(\frac{2\pi i}{4\tau}\right)^{s+1}ds.\end{aligned}$$

Using the functional equations for $\zeta(s)$ and $L(s,\chi_{-1})$ and the Legendre duplication formula for $\Gamma(s)$, we find the integral on the right-hand side is equal to

$$\frac{1}{2\pi i}\frac{i}{2\tau}\int_{3/2-i\infty}^{3/2+i\infty}\Gamma(s)\zeta(s+2)L(s,\chi_{-1})\big(1-2^{-s-2}\big)(-\pi i\tau)^{-s}ds.$$

By the Mellin inversion formula again, this is equal to

$$\frac{i}{8\tau}\int_0^q\int_0^q\sum_{n=1}^{\infty}\left(\frac{-1}{n}\right)\frac{n^2q^{n/2}}{1-q^n}\frac{dq}{q}\frac{dq}{q}=\frac{i}{8\tau}\int_0^q\int_0^q\frac{\eta(\tau)^4\eta(4\tau)^8}{\eta(2\tau)^6}\frac{dq}{q}\frac{dq}{q}.$$

This establishes (20). In summary, what we have shown is the following result.

Theorem 3. *Let $\{a_n\}$ and $\{b_n\}$ be the sequences satisfying*

$$(n+2)^2u_{n+2}-\big(32n^2+96n+76\big)u_{n+1}+(n+1)^2u_n=0\quad(u_n=a_n\text{ or }b_n)$$

with the initial values $a_0=1$, $a_1=12$, $b_0=0$, and $b_1=1$. Then

$$\left|\frac{b_n}{a_n}-\frac{1}{2}L(2,\chi_{-1})\right|\ll\frac{1}{\log n}$$

as $n\to\infty$.

Example 4. Let $f(\tau)=\eta(2\tau)^3\eta(6\tau)^3$ be the cusp form of weight 3 on $\Gamma_0(6)+\omega_3$ with character

$$\chi\begin{pmatrix}a&b\\6c&d\end{pmatrix}=(-1)^c\left(\frac{d}{3}\right),\qquad\chi\begin{pmatrix}3&-1\\6&-3\end{pmatrix}=i.$$

We will find a rational approximation to $L(f,2)$.

Choose

$$t(\tau)=\frac{\eta(6\tau)^5\eta(2\tau)}{\eta(\tau)^5\eta(3\tau)},\qquad A(\tau)=\frac{\eta(\tau)^2\eta(3\tau)^3}{\eta(2\tau)\eta(6\tau)},\tag{21}$$

where $t(\tau)$ is modular on the smaller group $\Gamma_0(6)$. The differential equation satisfied by t and A is

$$(1+8t)^2(1+9t)\theta^2A+9t(1+8t)^2\theta A+2t\big(1+16t+72t^2\big)A=0.$$

The singularities 0, $-1/9$, $-1/8$, and ∞ correspond to the cusps ∞, $1/2$, $1/3$, and 0, respectively. Set $g(t)=t/(1+8t)(1+9t)$ so that

$$\left(\frac{q\,dt/dq}{t}\right)^2\frac{g(t)}{A}=f(\tau).$$

Let $A(t)=1+\sum_{n=1}^\infty a_nt^n$ and $B(t)=t-7t^2+\sum_{n=3}^\infty b_nt^n$ be the solution of

$$(1+8t)^2(1+9t)\theta^2B+9t(1+8t)^2\theta B+2t\bigl(1+16t+72t^2\bigr)B=t+8t^2$$

holomorphic at $t=0$. We now compute the limit of b_n/a_n. For this purpose, we need to determine the behavior of

$$E(\tau)=\int_0^q\int_0^q f(\tau)\frac{dq}{q}\frac{dq}{q}$$

near the cusp $1/2$. Choose

$$\sigma=\frac{1}{\sqrt3}\begin{pmatrix}3&-2\\6&-3\end{pmatrix}\in\Gamma_0(6)+\omega_3,$$

and consider $E(\sigma\tau)$.

Since $f(\sigma\tau)=i(2\sqrt3\tau-\sqrt3)^3f(\tau)$, according to Lemma 4, if $f(\tau)=\sum_{n=1}^\infty c_nq^n$, then the function

$$L(s)=\sum_{n=1}^\infty(-1)^n\frac{c_n}{n^2}=-L(s,f)$$

satisfies

$$\left(\frac{\pi}{\sqrt3}\right)^{-s}\Gamma(s)L(s)=\left(\frac{\pi}{\sqrt3}\right)^{s-3}\Gamma(3-s)L(3-s).$$

Also, we have

$$E\left(-\frac{1}{2\sqrt3\tau}+\frac12\right)=\frac{1}{2\pi i}\int_{3/2-\infty}^{3/2+i\infty}\Gamma(s)L(s+2)\left(\frac{\pi i}{\sqrt3\tau}\right)^{-s}ds.$$

The integrand has simple poles at $s=0$ and $s=-1$ with residues

$$\operatorname{Res}_{s=0}=L(2)=-L(2,f),\qquad \operatorname{Res}_{s=-1}=-L(1)\left(\frac{\pi i}{\sqrt3\tau}\right)=\frac{i}{\tau}L(2,f).$$

Using the functional equation for $L(s)$ and the Mellin inversion formula, we find

$$E\left(-\frac{1}{2\sqrt3\tau}+\frac12\right)+L(2,f)=\frac{i}{\tau}\left(E\left(\frac{\tau}{2\sqrt3}+\frac12\right)+L(2,f)\right).$$

By Lemma 3, this amounts to

$$E(\sigma\tau)+L(2,f)=i(2\sqrt3\tau-\sqrt3)^{-1}\bigl(E(\tau)+L(2,f)\bigr).$$

Since $A(\tau)$ satisfies

$$A(\sigma\tau)=-i(2\sqrt3\tau-\sqrt3)A(\tau),$$

we have

$$A(\sigma\tau)\bigl(E(\sigma\tau)+L(2,f)\bigr)=A(\tau)\bigl(E(\tau)+L(2,f)\bigr).$$

From this we see that the function $A(\tau)(E(\tau)+L(2,f))$, as a function of t, is holomorphic at $t=-1/9$. Therefore, we have the following approximation of $L(2,f)$.

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Theorem 4. *Let $f(\tau)=\eta(2\tau)^3\eta(6\tau)^3$ and $L(s,f)$ be its associated L-function. Let $\{a_n\}$ and $\{b_n\}$ be the sequences satisfying*

$$(n+3)^2u_{n+3}+\bigl(25n^2+109n+120\bigr)u_{n+2}+16\bigl(13n^2+35n+24\bigr)u_{n+1}+144(2n+1)^2u_n=0$$

with the initial values

$$a_0=1,\qquad a_1=-2,\qquad a_2=10,\qquad b_0=0,\qquad b_1=1,\qquad b_2=-7.$$

Then we have

$$\left|\frac{b_n}{a_n}+L(2,f)\right|\ll(8/9)^n.$$

Remark. Observe that the result in the above example shows that the composition $t\mapsto\tau\mapsto F(\tau)=A(\tau)(E(\tau)+L(2,f))$ is a single-valued analytic function at $t=-1/9$. This in turn implies that the function $F(\tau)$ is invariant under the substitution $\tau\mapsto\gamma\tau$ for any γ in the stabilizer subgroup of $1/2$ in $\Gamma_0(6)$. This observation leads us to some interesting identities for the values of L-functions. We will discuss these identities in the next section.

Example 5. Consider the differential equation

$$\theta^3A-8t(2\theta+1)\bigl(2\theta^2+2\theta+1\bigr)A+256t^2(\theta+1)^3A=0.$$

This is equation (β) in [1]. We now determine its Apéry limit. The differential equation is satisfied by

$$t(\tau)=\frac{\eta(\tau)^8\eta(4\tau)^{16}}{\eta(2\tau)^{24}},\qquad A(\tau)=\frac{\eta(2\tau)^{20}}{\eta(\tau)^8\eta(4\tau)^8}.$$

They are modular on $\Gamma_0(4)$. The singularities $0,\ 1/16,\ \infty$ correspond to the cusps $\infty,\ 0,\ 1/2$, respectively. The solution of the inhomogeneous differential equation

$$\theta^3B-8t(2\theta+1)\bigl(2\theta^2+2\theta+1\bigr)B+256t^2(\theta+1)^3B=t$$

is $B(\tau)=A(\tau)E(\tau)$, where

$$E(\tau)=\frac{1}{240}\int_0^q\int_0^q\int_0^q\bigl(E_4(\tau)-17E_4(2\tau)+16E_4(4\tau)\bigr)\frac{dq}{q}\frac{dq}{q}\frac{dq}{q}.$$

We have

$$A(-1/4\tau)=-4\tau^2A(\tau).$$

To determine the Apéry limit, we should study how $E(\tau)$ transforms under the substitution $\tau\mapsto-1/4\tau$. As usual, we apply the Mellin inversion formula and write

$$E(-1/4\tau)=\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty}\Gamma(s)L(s+3)\left(\frac{2\pi i}{4\tau}\right)^{-s}ds,$$

where

$$L(s)=\zeta(s)\zeta(s-3)\bigl(1-17\cdot2^{-s}+16\cdot2^{-2s}\bigr)$$

satisfies

$$\pi^{-s}\Gamma(s)L(s)=\pi^{s-4}\Gamma(4-s)L(4-s).$$

Moving the line of integration to $\operatorname{Re}s=-7/2$, counting the residues, making a change of variable $s\to-2-s$, using the functional equation above, and invoking the Mellin inversion formula, we obtain

$$E(-1/4\tau)=\frac{7}{16}\zeta(3)-\frac{\pi^3i}{64\tau}-\frac{7}{64\tau^2}\zeta(3)+\frac{1}{4\tau^2}E(\tau).$$

From this we deduce the following result.

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Theorem 5. *Let $\{a_n\}$ and $\{b_n\}$ be the sequences satisfying*

$$(n+2)^3 u_{n+2} - 8(2n+3)\big(2n^2+6n+5\big)u_{n+1} + 256n^3 u_n = 0 \quad (u_n = a_n \text{ or } b_n)$$

with the initial values $a_0 = 1$, $a_1 = 8$, $b_0 = 0$, and $b_1 = 1$. Then we have

$$\left|\frac{b_n}{a_n} - \frac{7}{16}\zeta(3)\right| \ll \frac{1}{\log n}$$

as $n \to \infty$.

5. Identities involving L-values

Let the functions $t(\tau)$, $A(\tau)$, $f(\tau)$, and $E(\tau)$ be given as in Example 4. Following the remark at the end of the example, we know that the function $F(\tau) = A(\tau)(E(\tau) + L(2, f))$ satisfies $F(\gamma^n\tau) = F(\tau)$ for all integers n, where $\gamma = \left(\begin{smallmatrix} 7 & -3 \\ 12 & -5 \end{smallmatrix}\right)$ is a generator of the stabilizer subgroup of the cusp $1/2$ in $\Gamma_0(6)$. Now we have $A(\gamma\tau) = (12\tau - 5)A(\tau)$. Thus, the function $E(\tau)$ satisfies $E(\gamma\tau) + L(2, f) = (12\tau - 5)^{-1}(E(\tau) + L(2, f))$. By Lemma 3, this is equivalent to

$$E\left(\frac{\tau}{12} + \frac{7}{12}\right) + L(2, f) = -\tau\left(E\left(-\frac{1}{12\tau} + \frac{5}{12}\right) + L(2, f)\right). \tag{22}$$

On the other hand, we have $f(\gamma\tau) = (12\tau - 5)^3 f(\tau)$. By Lemma 4, letting c_n denote the Fourier coefficients of $f(\tau) = \sum_{n=1}^{\infty} c_n q^n$, the pair of functions

$$L(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^2} e^{14\pi i/12}, \qquad L^*(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^2} e^{10\pi i/12}$$

satisfy

$$\left(\frac{\pi}{6}\right)^{-s} \Gamma(s)L(s) = i^3 \left(\frac{\pi}{6}\right)^{s-3} \Gamma(3-s)L^*(3-s).$$

Then, arguing as before, we have

$$E\left(-\frac{1}{12\tau} + \frac{5}{12}\right) = L^*(2) - \frac{1}{\tau}L(2) - \frac{1}{\tau}E\left(\frac{\tau}{12} + \frac{7}{12}\right). \tag{23}$$

Comparing (22) and (23), we find

$$\sum_{n=1}^{\infty} \frac{c_n}{n^2} e^{14\pi i n/12} = \sum_{n=1}^{\infty} \frac{c_n}{n^2} e^{10\pi i n/12} = -\sum_{n=1}^{\infty} \frac{c_n}{n^2}.$$

Now we observe that the deduction of (22) and (23) is really independent of differential equations and rational approximations. Thus, we expect that identities of this type must exist in a general situation.

Lemma 5. *Let $\Gamma = \Gamma_0(N)$ or $\Gamma_1(N)$. Let $\alpha = p/q \neq \infty$ be a cusp of Γ. Let $f(\tau) = \sum_{n=1}^{\infty} c_n q^n$ be a holomorphic modular form of weight 3 on Γ with character χ. Assume that $f(\tau)$ vanishes at ∞ and α. Then for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c > 0$ in the stabilizer subgroup of α in Γ such that $\chi(\gamma) = 1$, we have*

$$L(2) = L^*(2) = L_\alpha(2),$$

where the L-functions are defined by

$$L(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s} e^{2\pi i n a/c}, \qquad L^*(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s} e^{-2\pi i n d/c}, \qquad L_\alpha(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s} e^{2\pi i n\alpha}$$

for $\operatorname{Re} s > 3$ *and then continued analytically to the whole complex plane.*

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Proof. Let $\mathscr{C}$ be the path consisting of the vertical line from $-d/c+i\infty$ to $-d/c$, the semi-circle from $-d/c$ to α, and the vertical line from α to $\alpha+i\infty$. Consider the integral

$$\int_{\mathscr{C}} (c\tau+d) f(\tau)\, d\tau.$$

Since $-d/c$ is equivalent to ∞ under Γ, by the assumption that $f(\tau)$ vanishes at ∞ and α, the integral is convergent. Therefore, it is equal to 0 because $f(\tau)$ is assumed to be a holomorphic modular form. Now we make a change of variable $\tau \to \gamma^{-1}\tau$ in the integral from $-d/c$ to α. We have

$$\int_{-d/c}^{\alpha} (c\tau+d) f(\tau)\, d\tau = \int_{i\infty}^{\alpha} \bigl(c\gamma^{-1}\tau+d\bigr) f(\gamma\tau) \frac{d\tau}{(-c\tau+a)^2} = \int_{i\infty}^{\alpha} f(\tau)\, d\tau.$$

It follows that

$$\int_{-d/c}^{-d/c+i\infty} (c\tau+d) f(\tau)\, d\tau = \int_{\alpha}^{\alpha+i\infty} (c\tau+d-1) f(\tau)\, d\tau. \tag{24}$$

Now recall that a stabilizer γ of the cusp $\alpha = p/q$ takes the from

$$\begin{pmatrix} 1+pqm & -p^2m \\ q^2m & 1-pqm \end{pmatrix}$$

for some integer m. Hence, (24) can be written as

$$\int_{-d/c}^{-d/c+i\infty} (c\tau+d) f(\tau)\, d\tau = \int_{\alpha}^{\alpha+i\infty} (c\tau-c\alpha) f(\tau)\, d\tau.$$

Finally, the two sides of the above identity are equal to

$$\frac{ic}{4\pi^2} L^*(2), \qquad \frac{ic}{4\pi^2} L_\alpha(2),$$

respectively. This gives $L^*(2) = L_\alpha(2)$. By a similar argument, we can also show that $L(2) = L_\alpha(2)$. This proves the lemma. □

Example 6. Consider the differential equation

$$\bigl(1-11t-t^2\bigr)\theta^2 A - \bigl(11t+2t^2\bigr)\theta A - \bigl(3t+t^2\bigr)A = 0,$$

appearing in Apéry's sequence for $\zeta(2)$, where t and A are given by (4) and (5). Using Lemma 1, we find a solution for the inhomogeneous differential equation

$$\bigl(1-11t-t^2\bigr)\theta^2 B - \bigl(11t+2t^2\bigr)\theta B - \bigl(3t+t^2\bigr)B = t$$

is given by $B(t) = A(t)E(t)$, where

$$E(t) = \int_0^q \int_0^q \biggl(\sum_{n\equiv 1 \bmod 5} -2 \sum_{n\equiv 2 \bmod 5} +2 \sum_{n\equiv 3 \bmod 5} - \sum_{n\equiv 4 \bmod 5} \biggr) \frac{n^2 q^n}{1-q^n} \frac{dq}{q} \frac{dq}{q}. \tag{25}$$

In [2], Beukers mentioned that it can be shown that

$$A\biggl(\frac{\tau}{5\tau+1}\biggr)\biggl(E\biggl(\frac{\tau}{5\tau+1}\biggr) - \frac{\zeta(2)}{5}\biggr) = A(\tau)\biggl(E(\tau) - \frac{\zeta(2)}{5}\biggr),$$

but did not give details. Here, we will prove this transformation formula using Lemma 5.

Since $A(\tau/(5\tau+1)) = (5\tau+1)A(\tau)$, by Lemma 3, it suffices to prove that

$$E\left(-\frac{1}{5\tau}-\frac{1}{5}\right)-\frac{\zeta(2)}{5}=-\frac{1}{\tau}\left(E\left(\frac{\tau}{5}+\frac{1}{5}\right)-\frac{\zeta(2)}{5}\right). \tag{26}$$

Let $f(\tau)=\sum_{n=1} c_n q^n$ be the modular form of weight 3 in the definition of $E(\tau)$ in (25). It satisfies $f(\tau/(5\tau+1)) = (5\tau+1)^3 f(\tau)$. Now, by the usual argument, we find

$$E\left(-\frac{1}{5\tau}-\frac{1}{5}\right)=L^*(2)+\frac{L(2)}{\tau}-\frac{1}{\tau}E\left(\frac{\tau}{5}+\frac{1}{5}\right),$$

where $L(s)=\sum c_n e^{2\pi i n/5}/n^s$ and $L^*=\sum c_n n^{-2\pi i n/5}/n^2$. Now applying Lemma 5 with $\left(\begin{smallmatrix}1&0\\5&1\end{smallmatrix}\right)$ that fixes 0, we have

$$L(2)=L^*(2)=L(2,f)=\sum_{n=1}^{\infty}\frac{c_n}{n^2},$$

which is shown to be $\zeta(2)/5$ in [2]. This gives the desired equality (26).

Corollary 1. *Let $\sum_{n=1}^{\infty} c_n q^n$ be the Fourier expansion of $f(\tau)=\eta(2\tau)^3\eta(6\tau)^3$. Let $L(s,f)=\sum_{n=1}^{\infty} c_n/n^s$ be the L-function associated with f. Then we have*

$$\sum_{n\equiv 1 \bmod 12}\frac{c_n}{n^2}=\frac{2+\sqrt{3}}{3}L(2,f), \qquad \sum_{n\equiv 7 \bmod 12}\frac{c_n}{n^2}=\frac{2-\sqrt{3}}{3}L(2,f).$$

Proof. Obviously, c_n is 0 whenever n is even. Also, using Jacobi's triple product identity, we have

$$\eta(2\tau)^3\eta(6\tau)^3=\left(\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{(2n+1)^2/4}\right)\left(\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{3(2n+1)^2/4}\right).$$

From this we see that $c_n=0$ whenever $n\equiv 5, 11 \bmod 12$. Thus, letting $L_1=\sum_{n\equiv 1 \bmod 12} c_n/n^2$, $L_2=\sum_{n\equiv 7 \bmod 12} c_n/n^2$, and $L_3=\sum_{3|n} c_n/n^2$, we have $L(2,f)=L_1+L_2+L_3$. Moreover, since $c_{3n}=c_3c_n$ for all n, we have $L_3=-L(2,f)/3$. It follows that

$$L_1+L_2=\frac{4}{3}L(2,f). \tag{27}$$

We now apply Lemma 5 with the matrix $\left(\begin{smallmatrix}1&0\\12&1\end{smallmatrix}\right)$, which fixes the cusp 0. Lemma 5 gives

$$e^{2\pi i/12}(L_1-L_2)+\sum_{n=1}^{\infty}\frac{c_{3n}}{(3n)^2}i^n=L(2,f).$$

Comparing the real parts, we find

$$L_1-L_2=\frac{2}{\sqrt{3}}L(2,f). \tag{28}$$

Combining (27) and (28), we obtain the claimed identities. □

Corollary 2. *Let $\sum_{n=1}^{\infty} c_n q^n$ be the Fourier expansion of $f(\tau)=\eta(4\tau)^6$. Let $L(s,f)=\sum_{n=1}^{\infty} c_n/n^s$ be the L-function associated with f. Then we have*

$$\sum_{n\equiv 1 \bmod 16}\frac{c_n}{n^2}-\sum_{n\equiv 9 \bmod 16}\frac{c_n}{n^2}=\cos\frac{\pi}{8}L(2,f),$$
$$\sum_{n\equiv 5 \bmod 16}\frac{c_n}{n^2}-\sum_{n\equiv 13 \bmod 16}\frac{c_n}{n^2}=-\sin\frac{\pi}{8}L(2,f).$$

Proof. Clearly, the coefficients c_n vanishes for all n not congruent to 1 modulo 4. Let L_i, $i = 0, \dots, 3$, denotes $\sum_{n \equiv 4i+1 \bmod 16}^{\infty} c_n/n^2$. Apply Lemma 5 with the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 16 & 1 \end{smallmatrix}\right)$, which fixes the cusps 0. We obtain

$$e^{2\pi i/16}(L_0 - L_2) + e^{10\pi i/16}(L_1 - L_3) = L(2, f).$$

Comparing the real parts and the imaginary parts of the two sides, we find

$$\cos\frac{\pi}{8}(L_0 - L_2) - \sin\frac{\pi}{8}(L_1 - L_3) = L(2, f), \qquad \sin\frac{\pi}{8}(L_0 - L_2) + \cos\frac{\pi}{8}(L_1 - L_3) = 0.$$

From this, we obtain the claimed identities. □

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