行政院國家科學委員會專題研究計畫成果報告
一非線性整數規劃問題的全域最佳化之充分條件
Sufficient Conditions for Global Optimum of A Class of Nonlinear Integer Programs

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一、中英文摘要

傳統非線性整數規劃方法甚少探討全域最佳化的條件，本研究目的在於提出一種特殊形式整數規劃全域最佳化的必要與充分條件。我們首先引介橋變數（Bridge variable**）及其所組成的新函數，接下來我們證明一個全域最佳解就是這新函數的最小解。本文所提出的全域最佳解條件對於特定之非線性整數規劃的求解非常有幫助。**

關鍵詞：橋變數，非線性整數規劃，非線性整數規劃的必要與充分條件

Abstract

Classical nonlinear integer programming methods do not recognize conditions for global optimality. This study proposes a necessary and sufficient condition for global optimality of a special structured integer programs. By denoting a bridge variable as one appears in more than one cross product term in the program, we define a function composed of all bridge variables. Following that we prove that a global optimum is one which has minimal value in this function. The proposed condition is very helpful in finding the global optimum of a nonlinear integer program where many variables are non-bridge variables.

Keywords: Bridge variable , Box-constrained nonlinear integer program , Necessary and sufficient condition for nonlinear integer program.

二、緣由與目的

The conventional optimization tools such as gradients, subgradients, and second order constructions as Hessiane, cannot be expected to yield conditions of optimality for a nonlinear integer program. An exhaustive enumeration method (Vizvari and Yilmaz,1994) requires to evaluate most of feasible points for finding the global optimum of a nonlinear integer program. Several special types of nonlinear integer programs were studied under different assumptions: Various Lagrangean decomposition methods (Floudas,1995) solve a class of nonlinear integer programming problems based on nonlinear duality theory. Some outer approximation algorithms (Duran and Grossmann,1986,Horst and Tuy,1990) solve specific nonlinear integer programs with linear constraints.

This paper proposes a necessary and sufficient condition of global optimality for a special structured Nonlinear Integer Program (NIP). This study classifies all variables as bridge variables and non-bridge variables. A bridge variable is one which appears in more than one cross product terms in a NIP. The special structured program we are interested in is one where many of variables are non-bridge variables. By defining a new function based on bridge variables, our necessary condition of global optimality states that if a point has minimal value of this new function, then this point is a global minimum of a NIP. Our sufficient condition of global optimality states that a global minimum of a NIP should have minimal value of the new function.

For the simplicity of expression, the

nonlinear integer program discussed in this paper is formulated as following box-constrained program. It could be straightforwardly extended into constrained nonlinear programs:

NIP **Min f(X) =** $\sum_{i,k} a_{ik} x_i^{\sim_k} + \sum_{i,j,k} c_{ijk} x_i^{\ \ ik} x_j^{\text{¥}_{jk}}$

subject to $\underline{x_i} \le x_i \le \overline{x_i}$ for i = 1,2,...,n, x_i are integers, $\underline{x_i}$ and $\overline{x_i}$ are respectively the lower and upper bounds, $a_{ik}, c_{ijk}, \sim_k, \ _{ik}, \text{¥}_{jk}$ are real, where X = ($x_1, x_2, ..., x_n$).

Follows is an example of a NIP :

Min f(X) = 2 x_1^2 - x_1^4 + x_1^6 + $x_2^{2.5}$ + x_3^2 -2 $x_1 x_3$ + $x_1^2 x_3$ -2 $x_2 x_4$ + $x_3 x_4$ 1)

subject to 0≤ x_1 ≤4 , 0≤ x_2 ≤3 , 0≤ x_3 ≤2 , 0≤ x_4 ≤1, x_1 , x_2 , x_3 , x_4 are integers.

If solving this problem by an exhaustive enumeration method, the number of integer points required to check is 5x4x3x2 = 120 .

Observing NIP problem in (1) we know that there are four cross product terms composed by following three pairs of variables (x_1, x_3), (x_2, x_4), (x_3, x_4) where x_3 and x_4 appear twice, and x_1 and x_2 appear once. Here x_3 and x_4 are called "bridge variables" since they serve as bridges to link related variables in the cross product terms. For instance, x_3 links (x_1, x_3) with (x_3, x_4) and x_4 links (x_2, x_4) with (x_3, x_4).

This paper is interested in a NIP problem where many of variables are non-bridge variables. We propose a necessary and sufficient condition of global optimality for this special structured NIP problems. The proposed condition is quite useful in finding a global solution of such a problem.

三、結果與討論

Denote Y as a non-bridge set composed of all non-bridge variables, and Z as a bridge set composed by all bridge variables, the decision vector can be expressed as X = {Y, Z}, where Y = ($y_1, y_2, ..., y_m$) and Z = ($z_1, z_2, ..., z_q$).

Then a NIP problem can be rewritten as

$$\text{Min f(X) = f(Y,Z) = } \sum_i f_i(y_i) + \sum_j f_j(z_j) + \sum_{i,j} f_{ij}(y_i, z_j) + \sum_{i,k} f_{ik}(z_i, z_k) \qquad (2)$$

subject to $\underline{y_i} \le y_i \le \overline{y_i}$, $\underline{z_j} \le z_j \le \overline{z_j}$

where f_i is composed by a non-bridge variable, f_j is composed by a bridge variable, f_{ij} is composed by one non-bridge and one bridge variables, f_{jk} is composed by two bridge variables.

A condition of global optimality for a BIP problem is proposed as follows.

Theorem 1： (Necessary condition of global optimality)

A global optimum (Y^*, Z^*) of (2) should satisfy

$f(Y^*, Z^*)$ = Min$\left\{ g(z_1^0, z_2^0, ..., z_q^0) \mid \text{for all } \underline{z_k} \le z_k^0 \le \overline{z_k}, k = 1,2,...,q \right\}$

Theorem 2 (Sufficient condition of global optimality for NIP problem)

If there is a point (Y^*, Z^*) satisfying following conditions

$f(Y^*, Z^*)$ = Min $\left\{ g(z_1^0, z_2^0, ..., z_q^0) \mid \text{for all } \underline{z_k} \le z_k^0 \le \overline{z_k}, k = 1,2,...,q \right\}$,then (Y^*, Z^*) is a global minimum for NIP problem of (2)

We use some numerical examples to illustrate that the condition is useful in solving a NIP problem .

<u>**Example 1**</u> Consider following problem which does not have cross product term :

Minimize $f(x) = x_1^3 - x_2^2 + x_3$

subject to $-2 \le x_i \le 2$, x_i are integers ,

i=1,2,3.

In this example Q = Φ and BS = Φ, The optimal solution (x_1^*, x_2^*, x_3^*) should satisfy $g(x_1^*, x_2^*, x_3^*)$ = Min$\{x_1^3 \ , \ -2 \le x_1 \le 2\}$ + Min$\{-x_2^2 \ , \ -2 \le x_2 \le 2\}$+ Min$\{x_3 \ , \ -2 \le x_3 \le 2\}$ = -8-4-2 = -14, with x_1^* = -2, x_2^* = 2 or –2 , and x_3^* = -2.

Example 2 Consider following three camel NIP problem

Minimize $f(x)$=

$$2x_1^2 - 1.19x_1^4 + \frac{1}{6}x_1^6 - x_1x_2 + .1x_2^2 + 3x_3^2 - x_3^4 + \frac{1}{3}x_3^6 - 2x_2x_3$$

subject to $-2 \le x_1 \le 1$, $-2 \le x_2 \le 2$, $-2 \le x_3 \le 2$; x_1, x_2, x_3 are integers.

Here $Q = \{(x_1, x_2), (x_2, x_3)\}$ and BS = $\{x_2\}$. Following Definition 2, $g(x_2^0)$ can be specified as $g(x_2^0)$= Min $\{f_1(x_1) + f_{12}(x_1, x_2^0) \ , \ -2 \le x_1 \le 2 \ \}$ + Min $\{f_3(x_3) + f_{23}(x_2^0, x_3) \ , \ -2 \le x_3 \le 2 \ \}$+ $f_2(x_2^0)$

By specifying x_2^0 as –2, $g(x_2^0)$ becomes

$g(x_2^0 = -2)$=Min

$\{ \ 2x_1^2 - 1.19x_1^4 + \frac{1}{6}x_1^6 + 2x_1 | -2 \le x_1 \le 2 \ \}$+

Min$\{ \ 3x_3^2 - x_3^4 + \frac{1}{3}x_3^6 + 4x_3 | -2 \le x_3 \le 2 \ \}$+.4

= -3.974

The best solution is $(x_1 = -2, x_2 = -2, x_3 = 0)$ for given x_2 = -2.

Similarly, we have

$g(x_2 = -1)$=-2.274 with best known x_1 = -2 , x_3 =0,

$g(x_2 = 0)$=-.374 with best known x_1 =1 or -2 , x_3 =0,

$g(x_2 = 1)$=-2.274 with best known x_1 = 2 , x_3 =0,

$g(x_2 = 2)$=-1.974 with best known x_1 = 1 , x_3 =0,

the global optimum is then (x_1^*, x_2^*, x_3^*) = $(-2,-2,0)$ with objective value –3.974 .

To find a global minimum of this example by an exhaustive enumeration way , the total number of points required to be checked is 4x5x5=100 . By utilizing the proposed condition of global optimality, the number of required check points becomes 4x5 + 5x5 = 45.

Example 3 Consider a NIP problem , where $Q = \{(y_1, z_1), (y_2, z_2), (z_1, z_2)\}$ and BS=$\{z_1, z_2\}$. The total number of points corresponding to (z_1, z_2) is 3x2 = 6.

A $g(z_1 = 1, z_2 = 1)$ value is computed. Similarly all other 24 $g(z_1^0, z_2^0)$ can be obtained. The global solution of this example is $(y_1^*, y_2^*, z_1^*, z_2^*)$= $(1,1,1,1)$.

Here the total number of points required to be checked is 5x3 + 3x4 + 4x2 = 35.

As described before, if solving this example by an exhaustive enumeration way, the required number of check points is 5x4x3x2=120.

The proposed global optimality condition can also be extended to solve a NIP problem where a cross term contains more than two variables. Consider following example:

Example 4 Solving following integer problem

Minimize

$f(x) = x_1^3 + x_2^2 - x_3^3 + x_1x_3 - x_2x_3x_4$

subject to $-2 \le x_1 \le 4$, $-2 \le x_2 \le 3$,

$-2 \le x_3 \le 2$, $-2 \le x_4 \le 1$

where $Q = \{(x_1, x_3), (x_2, x_3, x_4)\}$ and BS = $\{x_3\}$.

A $g(x_3^0)$ is expressed as

$g(x_3^0)$=Min$\{\ x_1^3 + x_1 x_3^0 |\ -2 \le x_1 \le 4\ \}$+

Min

$\{\ x_2^2 - x_2 x_3^0 x_4 |\ -2 \le x_2 \le 3, -2 \le x_4 \le 1\}$-

$(x_3^0)^3$

where $x_2 x_3^0 x_4$ is a cross term..

To compute a $g(x_3^0)$ requires to check 7+6x4= 31 integer points. The total number required to examine for finding a global minimum is 31x6= 186. It is less than 7x6x5x4= 840 which is the required check number for using an exhaustive way to solve the problem, the obtained global minimum is $(x_1^*, x_2^*, x_3^*, x_4^*)$ = (-2 , 1 , 2 , 2).

This research proposes a necessary and sufficient condition for global optimum of a specific structured nonlinear integer programs. This condition is quite useful for finding the global minimum of a NIP problem where most of variables are non-bridge variables.

四、計畫成果自評

1. **本文找到一非線性整數規劃最佳解的充分條件，此條件陳述如定理一及定理二。此條件對求解非線性整數規劃問題甚有助益。**
2. **但此條件目前只適用於特殊題型，尚無法擴充至一般性整數問題此仍待後續研究。**
3. **本研究結果已撰寫成兩篇論文投稿至** Journal of Global Optimization **及** Journal of Mathematical Analysis and Applications。

五、參考文獻

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