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Willmore 曲面的點態估計之間隙

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Gaps between pointwise estimates of Willmore surfaces

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摘要

假設 M 為 3 維球面上之緊緻 Willmore 曲面。此報告旨導出一點態估計，並討論此點態估計所衍生的問題。主要考慮梯度估計與特徵化的問題。

關鍵詞 Willmore 曲面、球面

Abstract
In this report, we will find a pointwise estimate which improves our previous result. This estimate characterizes the Willmore spheres with nonnegative Gaussian curvature and the flat Willmore tori. For the latter case, we find a gradient estimate and a characterization of the Clifford torus.

Keywords: Willmore surface, Sphere

1. Introduction

Let M be a compact immersed surface in the 3-dimensional unit sphere S^3. Let h_{ij} be the components of the second fundamental form of M, by $H=\sum h_{ii}$ the mean curvature. Let $\phi_{ij}=h_{ij}-\frac{H}{2}\delta_{ij}$ be the trace free tensor and $\Phi=\sum(\phi_{ij})^2$ the square length of ϕ_{ij} . Then the Willmore functional of X is given by

$$W(X)=\int_M \Phi,$$

where the integration is with respect to the area measure of M. This functional is preserved if we move M via conformal transformations of S^3.

The critical points of the Willmore functional are called Willmore surfaces, they satisfy the Euler-Lagrang equation

$$\Delta H+\Phi H=0.$$

The minimal surfaces in the 3 dimensional unit sphere S^3 are Willmore surfaces (see [W]).

One expects that the technique of pointwise estimate for minimal surfaces are also working for Willmore surfaces. However the geometric structure of Willmore surfaces are more complicate than that of minimal surfaces.

Let M be a compact immersed Willmore surface in the 3-dimensional unit sphere. Using an integral inequality, we proved that if $0\le\Phi\le 2+\frac{H^2}{4}$, then M is either totally umbilical or the Clifford torus. This estimate is sharp in the sense that for every given positive ε, there is a compact Willmore surface, which is not the Clifford torus, satisfying $0 \le \Phi \le 2 + \frac{H^2}{4} + \varepsilon$ (see [CH1]). However, M is not necessary to be the Clifford torus when $\Phi \ge 2 + \frac{H^2}{4}$ on M. In fact, we do not know whether the coefficient constant 1/4 related to the mean curvature term is optimal or not. Similar results also work for conformal classes and Willmore surfaces in the n-dimensional sphere (see [CH2]).

In the first part of this report we show the following pointwise estimate:

Theorem Let M be a compact immersed Willmore surface in the 3-dimensional unit sphere. If $0 \le \Phi \le 2 + \frac{H^2}{2}$,then M is either a Willmore sphere with nonnegative Gaussian curvature or a flat Willmore torus. Furthermore, if $\Phi \ge 2 + \frac{H^2}{2}$ on M, M is a flat Willmore torus.

For the first case, using a holomorphic quartic differential, Willmore sphere was classified by Bryant, so call Bryant's sphere (see [B1] and [B2]). For the second case, one would like to know any properties of such a flat Willmore torus.

In the second part of this report we establish a gradient estimate for the mean curvature,

$$|\nabla H|^2 + 2H^2 + \frac{1}{4}H^4 \le c,$$

where c is the maximum value of $2H^2 + \frac{1}{4}H^4$.

This gradient estimate will give a Harnack inequality for the mean cuvature.

Finally, when M is a flat Willmore torus, we show that if after a translation the component of coordinates of the immersion are eigenfunctions, then M is the Clifford torus.

2. The Main Pointwise Estimate

In this section we characterize brief the Willmore spheres and the flat Willmore tori by a pointwise pinching condition. We need the following two Lemmas.

Lemma 1 ([CH1]). Let M be a compact immersed Willmore surface in the 3-dimensional unit sphere. Then

$$\frac{1}{2}\Delta\Phi = \phi_{ijk}{}^2 + \phi_{ij}H_{ij} + \Phi(2 + \frac{H^2}{2} - \Phi).$$

Lemma 2([CH1]). Let M be a compact immersed Willmore surface in the 3-dimensional unit sphere. Then

$$\Phi\phi_{ijk}{}^2 = \frac{|\nabla\Phi|^2}{2} + \Phi\frac{|\nabla H|^2}{2} - \phi_{ij}H_i\Phi_j.$$

Assume that $\Phi > 0$ on M. It follows from Lemmas 1 and 2 that

$$\begin{aligned}\int_M (2 + \frac{H^2}{2} - \Phi) &= \int_M (\frac{1}{2}\frac{\Delta\Phi}{\Phi} - \frac{\phi_{ijk}{}^2}{\Phi} - \frac{\phi_{ij}H_{ij}}{\Phi}) \\ &= \int_M (\frac{1}{2}\Delta\log\Phi - \frac{|\nabla\Phi|^2}{2\Phi^2} + \frac{\phi_{ij}H_i\Phi_j}{\Phi^2} - \frac{\phi_{ij}H_{ij}}{\Phi}) \\ &= \int_M (-\frac{|\nabla H|^2}{2\Phi} + \frac{\phi_{ij}H_i\Phi_j}{\Phi^2} + \frac{\Phi\phi_{ijj} - \phi_{ij}\Phi_i}{\Phi^2}H_i) \\ &= 0.\end{aligned}$$

Lemma 3. Let M be a compact immersed Willmore surface in the 3-dimensional unit sphere. If $\Phi > 0$ on M, then

$$\int_M (2 + \frac{H^2}{2} - \Phi) = 0.$$

We notice that the Gauss equation gives

$$2K = 2 + \frac{H^2}{2} - \Phi.$$

In the case $0 \le \Phi \le 2 + \frac{H^2}{2}$, the classical Gauss-Bonnet formula gives that g = 0 or 1, where g is the genus of M. That is, M is either a Willmore sphere with nonnegative

Gaussian curvature or a Willmore torus. In the latter case, Lemma 3 shows that M must be flat.

It is different to our previous result that if $\Phi \geq 2+\frac{H^2}{2}$ on M, applying Lemma 3 again, M is also a flat Willmore torus. This completes the proof of Theorem .

3. Gradient estimate

In this section we want to find some properties for flat Willmore tori. In this case, the mean curvature H satisfying the semi-linear equation

$$\Delta H + (2+\frac{H^2}{2})H = 0. \qquad (*)$$

First if M is not minimal, we follow the general properties of linear elliptic equations ([Be]) to describe the local behaviour of the zero set of the mean curvature.

(a) The critical points on the zero set of the mean curvature are isolated and finite.

(b) The zero set of the mean curvature consists of a number of C^2–immersed circles.

(c) When the immersed circles meet, they form an equiangular system.

Next we find a gradient estimate for the mean curvature, and hence we obtain a Harnack inequality for the mean curvature. Let $P = |\nabla H|^2 + 2H^2 + \frac{1}{4}H^4 - c$, where c is the maximum value of $2H^2 + \frac{1}{4}H^4$.

Since

$$P_i = 2H_j H_{ji} + 4HH_i + H^3 H_i,$$

for all i, it follows that if $|\nabla H| \neq 0$, then

$$H_{11} = \frac{1}{2|\nabla H|^2}(P_1H_1 - P_2H_2 - 4HH_1^2 - H^3H_1^2),$$

$$H_{12} = H_{21} = \frac{1}{2|\nabla H|^2}(P_1H_2 - P_2H_1 - 4HH_1H_2 - H^3H_1H_2),$$

$$H_{22} = \frac{1}{2|\nabla H|^2}(-P_1H_1 + P_2H_2 - 4HH_2^2 - H^3H_2),$$

and

$$2H_{ij}^2 = \frac{1}{2|\nabla H|^2}(2|\nabla P|^2 + (4H+H^3)^2|\nabla H|^2 - 2(4H+H^3)\nabla P\cdot\nabla H).$$

On the other hand, the equation (*) implies that

$$\Delta P = 2H_{ij}^2 - \frac{1}{2}(4H+H^3)^2.$$

Thus

$$|\nabla H|^2 \Delta P = |\nabla P|^2 - (4H+H^3)\nabla P\cdot\nabla H$$

holds for all points where $|\nabla H| \neq 0.$ However this equation holds on whole M. In fact, $|\nabla P| = 0$ if $|\nabla H| = 0.$ Therefore, P satisfies a degenerate elliptic equation.

Let m be the maximum value of P, and K be the set of all points where P=m. Then K is a nonempty compact subset. If $|\nabla H|(x_1) = 0,$ for some x_1 in M, then

$$P(x) \leq P(x_1) \leq (2H^2 + \frac{1}{4}H^4 - c)(x_1) \leq 0$$

for all x. Thus in this case $P \leq 0$ on M. Now suppose that $|\nabla H| > 0$ on K. We shall get a contradiction. First, we use the connected argument to show K=M. Indeed, for any $x_1 \in K$, let B_1 be a geodesic disk around x_1, outside the cut locus of x_1 and $|\nabla H| > 0$ on B_1. Suppose that $P(x_2)<m$, for some x_2 in B_1. We then construct a auxiliary function of the form

$$Z = e^{-\alpha r^2} - e^{-\alpha r_0^2},$$

where r is the distance function on M starting from x_0. As we choose α large enough, ε small enough and choosing suitable x_0 , the function W=P+ ε Z assumes its maximum in some geodesic disk B_2 , and $\Delta W > 0$ on B_2. Here we have use the Laplacian comparison Theorem because M is flat. This contradiction shows that K = M. The technique used here is essentially that of the maximum principle. We note that if $2H(x)^2 + \frac{1}{4}H(x)^4 = c,$ then $H(x)$ is the maximum or minimum of H, and hence $|\nabla H|(x) = 0$ in both cases, thus $|\nabla H| = 0$

somewhere in K. We conclude that $P \leq 0$ on M. That is, the gradient estimate

$$|\nabla H|^2 + 2H^2 + \frac{1}{4}H^4 \leq c$$

holds on M.

4. A Characterization of Clifford Torus

Let M is a flat Willmore torus. Then there is a lattice Γ (l,a,b) in R^2 generated by $(l,0)$ (a,b) with $a \geq 0, b > 0$ and $a^2 + b^2 \geq l^2$ such that M is isometric to the flat torus $R^2 / \Gamma(l,a,b)$. As one know the eigenfunctions of the Laplacian on $R^2 / \Gamma(l,a,b)$ are given by

$$f_{pq}(x,y) = \cos(2\pi \frac{p}{l}x + 2\pi \frac{1}{b}(q - \frac{p}{l}a)y),$$

$$g_{pq}(x,y) = \sin(2\pi \frac{p}{l}x + 2\pi \frac{1}{b}(q - \frac{p}{l}a)y),$$

where $p > 0$ or $p = 0$ and $q \geq 0$. Using a rotation of the 3-sphere, we may assume the immersion X of M into the 3-sphere is given by

$$(c + \Sigma a_0^1 f_0 + b_0^1 g_0, \Sigma a_0^2 f_0 + b_0^2 g_0, \Sigma a_0^3 f_0 + b_0^3 g_0, \Sigma a_0^4 f_0 + b_0^4 g_0)$$

where $f_0 = f_{pq}$, $g_0 = g_{pq}$ when 0 = (p,q).

Denote by X_1 and X_2 the derivatives of X with respect to x and y, respectively. Since X, X_1 and X_2 are orthonormal, the coefficients a's, b's and c of X satisfy twelve equations. On the other hand, the structure equations of M and the Euler-Lagrange equation give that

$$H = \det \begin{vmatrix} X \\ \Delta X \\ X_1 \\ X_2 \end{vmatrix},$$

$$2\det \begin{vmatrix} X \\ \Delta\Delta X \\ X_1 \\ X_2 \end{vmatrix} - 3\det \begin{vmatrix} X \\ \Delta X \\ (\Delta X)_1 \\ X_2 \end{vmatrix} - 3\det \begin{vmatrix} X \\ \Delta X \\ X_1 \\ (\Delta X)_2 \end{vmatrix} = 0.$$

Now we consider the special case that

$$X = (c + a_1 f_1 + b_1 g_1, a_2 f_2 + b_2 g_2, a_3 f_3 + b_3 g_3, a_4 f_4 + b_4 g_4),$$

where $f_i = f_{p_i q_i}$, $g_i = g_{p_i q_i}$ when $i = (p_i, q_i)$.

In this case, it follows from the orthonormal conditions that $ca_1 = cb_1 = 0$. By classifying the index, there are seven cases. Among these seven cases, there is only one possible case which satisfies the twelve orthonormal equations, the only case is $p_1 = p_2 \neq p_3 = p_4, q_1 = q_2 \neq q_3 = q_4$. Furthermore in this case c = 0, and X is given by

$$(a_1 f_1 \pm a_2 g_1, a_2 f_1 \mp a_1 g_1, a_3 f_3 \pm a_4 g_3, a_4 f_3 \mp a_3 g_3).$$

After orthogonal changing the coordinates of the 4-dimensional Euclidean space, we may assume that

$$X = (c_1 f_1, c_1 g_1, c_2 f_2, c_2 g_2).$$

For such X, the Euler-Lagrange equation implies $\Lambda_1 = \Lambda_2$ and $\lambda_1^1 \lambda_2^2 - \lambda_1^2 \lambda_2^1 = 0$, where

$$\lambda_0^1 = 2\pi \frac{p}{l}, \lambda_0^2 = 2\pi \frac{1}{b}(q - \frac{p}{l}a), \Lambda_0 = (\lambda_0^1)^2 + (\lambda_0^2)^2$$

when 0 = (p,q). Finally, we compute the mean curvature

$$H = \det \begin{vmatrix} X \\ \Delta X \\ X_1 \\ X_2 \end{vmatrix} = (\Lambda_1 - \Lambda_2)(\lambda_1^1 \lambda_2^2 - \lambda_1^2 \lambda_2^1) \frac{1}{(lb)^2} c_1^2 c_2^2 = 0.$$

That is, M is a minimal flat torus, and hence M is the Clifford torus.

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