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奇異差分方程擾動後的混沌動態現象(1/2)

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國科會專題研究計畫期中報告

奇異差分方程擾動後的混沌動態現象 **(1/2)**

Chaotic dynamics of perturbed singular difference equations

計畫編號：NSC 94-2115-M-018-006

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摘要

In this project, we consider solutions of difference equations $\Phi_\lambda(y_n, y_{n+1}, \ldots, y_{n+m}) = 0$, $n \in \mathbb{Z}$, with parameter λ close to those exceptional values λ_0 for which the function Φ depends on two variables: $\Phi_{\lambda_0}(x_0, \ldots, x_m) = \chi(x_{M+N}, x_N)$, where it is assumed that for the equation $\chi(y, x) = 0$ there is a branch $y = \varphi(x)$. We prove that if φ has positive topological entropy $h_{\text{top}}(\varphi)$, then among solutions of the difference equation with λ close enough to λ_0, a closed (in the product topology for the space of bi-infinite solutions) invariant set of solutions on which the restriction of the shift map has positive topological entropy arbitrarily close to $h_{\text{top}}(\varphi)$.

1 Introduction

This paper is a continuation of our study on chaotic behaviors of solution for perturbed singular difference equations in [1]. Therein, we considered difference equations

$$\Phi_\lambda(y_n, y_{n+1}, \ldots, y_{n+m}) = 0, \ n \in \mathbb{Z}, \tag{1}$$

which depend on only one variable at the exceptional value λ_0 of the parameter, i.e.,

$$\Phi_{\lambda_0}(x_0, \dots, x_m) = \varphi(x_N), \tag{2}$$

where $0 \leq N \leq m$. By establishing an uniform version of implicit function theorem for , we showed that solutions for perturbed difference equation (1) with $\lambda \neq \lambda_0$ have chaotic structure whenever the local map φ has at least two simple zeros. In this paper, we allow that difference equation at the exceptional value depend on two variables, i.e.,

$$\Phi_{\lambda_0}(x_0, \dots, x_m) = \chi(x_{M+N}, x_N), \tag{3}$$

where $0 \leq N \leq m$ and $1 \leq M + N \leq m$, and the function $\chi(y, x)$ is assumed to satisfy the following: for the equation $\chi(y, x) = 0$ there is a branch $y = \varphi(x)$, i.e., one has $\chi(\varphi(x), x) = 0$, where φ is a C^2 function with positive topological entropy. Notice that in the case when $M = 1$, the difference equation at the exceptional value of parameter corresponds to the one-dimensional map $x \mapsto \varphi(x)$. Therefore, if M is not equal to 1, we have at this parameter, some kind of generalized one-dimensional transformation which can be regarded as "$1/M$-th" iterate of φ.

By using Mañé's result on eventual hyperbolicity and the concept of inverse limit process, we show the existence of chaotic behaviors of solutions of the perturbed system provided that the local map φ has positive topological entropy. So by comparison with the previous mentioned result from [1], which can be regarded as multidimensional perturbation of zero-dimensional dynamics, this one is in a sense a multidimensional perturbation of generalized one-dimensional map.

Together with the results from [1], our new results can be applied to study on dynamics of several models including the generalized cellular neural networks, the time discrete version of the cellular neural networks, coupled Chua's circuit and discrete version of some partial differential equations. By easy examinations of the local maps, one can show that chaotic structures exhibits in stationary, traveling wave, or spatially-homogeneous solutions of the above systems.

2 Perturbation of one dimensional systems

We shall obtain a result on chaotic behaviors of solutions for perturbed singular difference equations depending on two variables at the exceptional value of parameter, i.e., in the case when

$$\Phi_{\lambda_0}(x_0, x_1, \dots, x_m) = \chi(x_{M+N}, x_N).$$

It is enough for this to take care of only one branch of the equation $\chi(\varphi(x), x) = 0$. So we assume that at the exceptional value of parameter the difference equation (1) reads $y_{M+n} = \varphi(y_N)$, $n \in \mathbb{Z}$, where $\varphi : Q \to [s_1, s_2]$ is a C^2 function on $Q = [s_1, s_2] \backslash V$ and V is a finite union of open intervals in $[s_1, s_2]$. Notice that under our assumption, at λ_0 corresponds to a map $f_{\lambda_0} : Q^M \to [s_1, s_2]^M$ of the form

$$f_{\lambda_0}(x_1, x_2, \dots, x_{M-1}, x_M) = (x_2, x_3, \dots, x_M, \varphi(x_1)), \tag{4}$$

So f_{λ_0} can be regarded as the "$1/M$-th iteration of the one-dimensional map φ. Also, it is easy to see that the iterates $f_{\lambda_0}^{kM}$ of f_{λ_0} are of the form

$$f_{\lambda_0}^{kM}(x_1, x_2, \dots, x_{M-1}, x_M) = (\varphi^k(x_1), \varphi^k(x_2), \dots, \varphi^k(x_M)) \tag{5}$$

and correspond to the difference equation

$$y_{kM+n} - \varphi^k(y_n) = 0, \ n \in \mathbb{Z}. \tag{6}$$

We will prove that a compact invariant hyperbolic set of the one-dimensional map φ can be "continued" by appropriate sets of orbits of difference equation (1) for λ close to λ_0. To this end we use the following result by Mañé.

Lemma 1. *[3] Let I be a compact interval of $\mathbb{R}$ and $g : I \to I$ be a C^2 map. Let U be a neighborhood of the set of critical points of g. Then*

1. *All periodic orbits of g contained in $I \backslash U$ of sufficiently large period are hyperbolic repelling.*

2. *If all the periodic orbits of g contained in $I \backslash U$ are hyperbolic, then there exist $C > 1$ and λ such that $\|Dg^n(x)\| \geq C\lambda^n$, whenever $g^i(x) \in I \backslash (U \cup B_0)$ for all $0 \leq i \leq n-1$, where B_0 is the union of the immediate basins of the periodic attractors of g contained in $I \backslash U$.*

It is also known that for any compact invariant hyperbolic set K of one-dimensional C^1 map h there is an integer k_0 such that for all $x \in K$ one has $\|Dh^{k_0}(x)\| > 1 + \varepsilon$ with some $\varepsilon > 0$. In order to have opportunity to apply the above Mañé's theorem for our map φ, we need to extend φ to a C^2 self map on slightly bigger interval $[s_1 - \varepsilon_0, s_2 + \varepsilon_0]$ and put

$$U = [s_1 - \varepsilon_0, s_1) \cup (s_2, s_2 + \varepsilon_0] \cup V \cup V_0,$$

where V_0 is a small neighborhood of critical and non-hyperbolic periodic points of φ. Denote the extend map by g. By choosing ε_0 and V_0 sufficiently small, we can have $h_{\text{top}}(g)$ arbitrarily close to $h_{\text{top}}(\varphi)$ (see [4]). By using Lemma 1 and the above arguments, we may choose a compact invariant hyperbolic set K for φ and find an integer k_0, and then we can replace φ by φ^{k_0} (replacing also the difference equation at λ_0 by (6)). Then we have that the partial derivative operator $D_2F(\lambda_0, \underline{y})$ for all orbits $\underline{y}$ on K has matrix of the form $\sigma^{k_0 M} \circ (I + \Lambda)$, where σ is the matrix of the shift operator, I is the identity matrix and Λ is a shifted diagonal matrix with entries bigger than $1 + \varepsilon$ in absolute value. So, in order to check (the most delicate) assumption (iv) from Theorem 2.1 of [1] we need the following lemma.

Lemma 2. *Let $A : \ell_\infty \to \ell_\infty$ be a bounded linear operator which can be represented by $A = \sigma^k \circ (I + \Lambda)$, where σ is the matrix of the shift operator, k is an integer, and Λ is an operator associated with matrix $(\Lambda_{ij})_{i,j=-\infty}^{\infty}$ of the form*

$$\Lambda_{ij} = \begin{cases} 0, & \text{if } j \neq i + M, \\ q_i, & \text{if } j = i + M. \end{cases}$$

for some sequence $\{q_i\}_{i=-\infty}^{\infty}$ satisfying $\inf_{i \in \mathbb{Z}} |q_i| := q > 1$. Then A is invertible and $\|A^{-1}\| < \frac{1}{q-1}$.

Summarizing the above arguments, Theorem 2.1 of [1] implies the following.

Theorem 1. *Let $\Phi_\lambda(y_n, y_{n+1}, \ldots, y_{n+m}) = 0$ be a difference equation with parameter $\lambda \in [\lambda_0, \lambda_1]$, and let the function $\Phi_\lambda : Q^{m+1} \to \mathbb{R}$, where $Q = [s_1, s_2] \setminus V$ for some numbers $s_1 < s_2$ and V is a finite union of open intervals in $[s_1, s_2]$ be such that it is C^1 for each λ and continuous in λ, and so are the partial derivatives $\frac{\partial}{\partial i}\Phi_\lambda$, $i = 1, \ldots, m+1$. Assume that Φ_{λ_0} is a function in two variables:*

$$\Phi_{\lambda_0}(x_0, x_1, \ldots, x_m) = \chi(x_{M+N}, x_N),$$

where M, N are integers with $0 \leq N \leq m$, $1 \leq M+N \leq m$. Assume, in addition, that for the equation $\chi(y,x)=0$ there is a branch $y=\varphi(x)$, where $\varphi : Q \to [s_1, s_2]$ is a C^2 function with positive topological entropy.

Then for any $\epsilon > 0$ there exists $\bar{\delta} > 0$ such that for any $\lambda \in [\lambda_0, \lambda_0 + \bar{\delta})$, there is a closed σ-invariant subset Γ_λ of Y_λ, the set of solutions for (1) *in the product topology, such that $h_{top}(\sigma|\Gamma_\lambda) > \frac{1}{M}(h_{top}(\varphi) - \epsilon)$.*

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