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NONEMPTINESS PROBLEMS OF PLANE SQUARE TILING WITH TWO COLORS

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Abstract. This investigation studies nonemptiness problems of plane square tiling. In the edge coloring (or Wang tiles) of a plane, unit squares with colored edges of p colors are arranged side by side such that adjacent tiles have the same colors. Given a set of Wang tiles $\mathcal{B}$, the nonemptiness problem is to determine whether or not $\Sigma(\mathcal{B}) \neq \emptyset$, where $\Sigma(\mathcal{B})$ is the set of all global patterns on $\mathbb{Z}^2$ that can be constructed from the Wang tiles in $\mathcal{B}$.

When $p \geq 5$, the problem is well known to be undecidable. This work proves that when $p = 2$, the problem is decidable. $\mathcal{P}(\mathcal{B})$ is the set of all periodic patterns on $\mathbb{Z}^2$ that can be generated by $\mathcal{B}$. If $\mathcal{P}(\mathcal{B}) \neq \emptyset$, then $\mathcal{B}$ has a subset $\mathcal{B}'$ of minimal cycle generator such that $\mathcal{P}(\mathcal{B}') \neq \emptyset$ and $\mathcal{P}(\mathcal{B}'') = \emptyset$ for $\mathcal{B}'' \subsetneqq \mathcal{B}'$. This study demonstrates that the set of all minimal cycle generators $\mathcal{C}(2)$ contains 38 elements. $\mathcal{N}(2)$ is the set of all maximal noncycle generators: if $\mathcal{B} \in \mathcal{N}(2)$, then $\mathcal{P}(\mathcal{B}) = \emptyset$ and $\widetilde{\mathcal{B}} \supsetneqq \mathcal{B}$ implies $\mathcal{P}(\widetilde{\mathcal{B}}) \neq \emptyset$. $\mathcal{N}(2)$ has eight elements. That $\Sigma(\mathcal{B}) = \emptyset$ for any $\mathcal{B} \in \mathcal{N}(2)$ is proven, implying that if $\Sigma(\mathcal{B}) \neq \emptyset$, then $\mathcal{P}(\mathcal{B}) \neq \emptyset$. The problem is decidable for $p = 2$: $\Sigma(\mathcal{B}) \neq \emptyset$ if and only if $\mathcal{B}$ has a subset of minimal cycle generators. The approach can be applied to corner coloring with a slight modification, and similar results hold.

1. Introduction

The coloring of unit squares on $\mathbb{Z}^2$ has been studied for many years [6]. In 1961, in studying the method of proving theorems by pattern recognition, Wang [12] started to study the square tiling of a plane. The unit squares with colored edges are arranged side by side so that the adjacent tiles have the same color; the tiles cannot be rotated or reflected. Today, such tiles are called Wang tiles or Wang dominos [4, 6].

The 2×2 unit square is denoted by $\mathbb{Z}_{2\times 2}$. Let $\mathcal{S}_p$ be a set of p (≥ 1) colors. The total set of all Wang tiles is denoted by $\Sigma^w_{2\times 2}(p) \equiv \mathcal{S}_p^{\mathbb{Z}_{2\times 2}}$. A set $\mathcal{B}$ of Wang tiles, such that $\mathcal{B} \subset \Sigma^w_{2\times 2}(p)$, is called a basic set (of Wang tiles). Let $\Sigma(\mathcal{B})$ be the set of all global patterns on $\mathbb{Z}^2$ that can be constructed from the Wang tiles in $\mathcal{B}$ and let $\mathcal{P}(\mathcal{B})$ be the set of all periodic patterns on $\mathbb{Z}^2$ that can be constructed from the Wang tiles in $\mathcal{B}$. Clearly, $\mathcal{P}(\mathcal{B}) \subseteq \Sigma(\mathcal{B})$. The nonemptiness problem is to determine

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whether or not $\Sigma(\mathcal{B}) \neq \emptyset$. In [12], Wang conjectured that any set of tiles that can tile a plane can tile the plane periodically, i.e.,

$$\text{if } \Sigma(\mathcal{B}) \neq \emptyset, \quad \text{then } \mathcal{P}(\mathcal{B}) \neq \emptyset. \tag{1.1}$$

However, in 1966, Berger [4] proved that Wang's conjecture was wrong. He presented a set $\mathcal{B}$ of 20426 Wang tiles that could only tile the plane aperiodically:

$$\Sigma(\mathcal{B}) \neq \emptyset \quad \text{and} \quad \mathcal{P}(\mathcal{B}) = \emptyset. \tag{1.2}$$

Later, he reduced the number of tiles to 104. Thereafter, smaller basic sets were found by Knuth, Läuchli, Robinson, Penrose, Ammann, Culik and Kari [5, 6, 7, 10, 11]. Currently, the smallest number of tiles that can tile the plane aperiodically is 13, with five colors: (1.2) holds and then (1.1) fails for $p = 5$ [5].

In studying the multi-dimensional shifts of finite type and tiling dynamics [1, 2, 3], we come across this problem. This work shows that Wang's conjecture (1.1) holds provide $p = 2$: any set of Wang tiles with two colors that can tile a plane can tile the plane periodically.

Statement (1.1) is understood by studying how periodic patterns can be generated from a given basic set. First, the minimal cycle generator is introduced. $\mathcal{B} \subset \Sigma^w_{2\times 2}(p)$ is called a minimal cycle generator if $\mathcal{P}(\mathcal{B}) \neq \emptyset$ and $\mathcal{P}(\mathcal{B}') = \emptyset$ whenever $\mathcal{B}' \subsetneqq \mathcal{B}$. $\mathcal{B} \subset \Sigma^w_{2\times 2}(p)$ is called a maximal noncycle generator if $\mathcal{P}(\mathcal{B}) = \emptyset$ and $\mathcal{P}(\mathcal{B}'') \neq \emptyset$ for any $\mathcal{B}'' \supsetneqq \mathcal{B}$. Given $p \geq 2$, denote the set of all minimal cycle generators by $\mathcal{C}(p)$ and the set of maximal noncycle generators by $\mathcal{N}(p)$. Clearly,

$$\mathcal{C}(p) \cap \mathcal{N}(p) = \emptyset. \tag{1.3}$$

Statement (1.1) follows for $p = 2$ if

$$\Sigma(\mathcal{B}) = \emptyset \quad \text{for any } \mathcal{B} \in \mathcal{N}(2) \tag{1.4}$$

can be shown.

In this study, for $p = 2$, $\mathcal{C}(2)$ and $\mathcal{N}(2)$ can be listed explicitly, and (1.4) is shown to hold. Indeed, $\mathcal{C}(2)$ has 38 members and $\mathcal{N}(2)$ has eight members. Furthermore, under the symmetry group D_4 of $\mathbb{Z}_{2\times 2}$ and the permutation group S_p of colors of horizontal and vertical edges separately, $\mathcal{C}(2)$ can be classified into six classes and $\mathcal{N}(2)$ into only one class.

To prove $\Sigma(\mathcal{B}) = \emptyset$ for any $\mathcal{B} \in \mathcal{N}(2)$, the vertical ordering matrices $\mathbf{Y}_{w;m\times 2}$ of local patterns on $\mathbb{Z}_{m\times 2}$, which was developed in another work [1], are applied. The impossibility of generating an allowable pattern on $\mathbb{Z}_{4\times 4}$ from any $\mathcal{B} \in \mathcal{N}(2)$ is demonstrated. For $p = 3$, (1.4) is still under investigation because the numbers of elements in $\mathcal{C}(3)$ and $\mathcal{N}(3)$ are high. The number of $\mathcal{C}(3)$ exceeds $5 \cdot 10^5$. Under the symmetry groups, $\mathcal{C}(3)$ still contains thousands of classes.

Notably, if (1.1) holds, then the nonemptiness problem can easily be determined by studying $\mathcal{P}(\mathcal{B})$, as in the case $p = 2$. More precisely, $\Sigma(\mathcal{B}) \neq \emptyset$ if and only if $\mathcal{B}$ has a subset of minimal cycle generators.

In corner coloring, the basic set of 44 tiles with six colors that can tile the plane aperiodically without any periodic patterns has been established elsewhere [9]. Hence, (1.2) holds and then (1.1) fails for $p = 6$. The method used to study Wang tiles (edge coloring) can also be applied to study corner coloring. For $p = 2$, no allowable pattern on $\mathbb{Z}_{5\times 5}$ can be generated from any maximal noncycle generator. Hence, (1.4) holds and the nonemptiness problem is decidable for corner coloring, too.

For recent results on Wang tiles (colored edges) and colored corners with their applications to computer graphics, see Lagae and Dutré [8] and the references therein.

The rest of this paper is arranged as follows. Section 2 proves (1.1) for corner coloring when $p = 2$. Section 3 proves (1.1) for Wang tiles when $p = 2$.

2. Corner coloring

This section studies corner coloring and proves that (1.1) holds for $p = 2$. Corner coloring is studied first because the ordering matrix of corner coloring is simpler than that of edge coloring. Some notation must be introduced first. In this section, $\mathbb{Z}_{2\times 2}$ represents the square lattice with vertices $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. Furthermore, for any $(i,j) \in \mathbb{Z}^2$, define $\mathbb{Z}_{2\times 2}(i,j) = \{(i,j),(i,j+1),(i+1,j),(i+1,j+1)\}$.

For given positive integers m and n, the rectangular lattice $\mathbb{Z}_{m\times n}$ is defined by

$$\mathbb{Z}_{m\times n} = \{(i,j) | 0 \leq i \leq m-1 \text{ and } 0 \leq j \leq n-1\}.$$

Denote the set of p colors by $\mathcal{S}_p = \{0,1,\ldots,p-1\}$. The set of all global patterns on $\mathbb{Z}^2$ with colors in $\mathcal{S}_p$ is denoted by

$$\Sigma_p^2 = \mathcal{S}_p^{\mathbb{Z}^2} = \left\{U | U : \mathbb{Z}^2 \to \mathcal{S}_p\right\}.$$

The set of all local patterns on $\mathbb{Z}_{m\times n}$ is defined by

$$\Sigma_{m\times n}(p) = \{U|_{\mathbb{Z}_{m\times n}} : U \in \Sigma_p^2\}.$$

Now, for any given $\mathcal{B}_c \subset \Sigma_{2\times 2}(p)$, $\mathcal{B}_c$ is called a basic set. The set $\Sigma_{m\times n}(\mathcal{B}_c)$ of all patterns on $\mathbb{Z}_{m\times n}$ generated by $\mathcal{B}_c$ is defined by

$$\Sigma_{m\times n}(\mathcal{B}_c) = \left\{U \in \Sigma_{m\times n}(p) : U\, |_{\mathbb{Z}_{2\times 2}(i,j)} \in \mathcal{B}_c \text{ for } 0 \leq i \leq m-2, 0 \leq j \leq n-2\right\},$$

and the set $\Sigma(\mathcal{B}_c)$ of all global patterns on $\mathbb{Z}^2$ generated by $\mathcal{B}_c$ is defined by

$$\Sigma(\mathcal{B}_c) = \left\{U \in \Sigma_p^2 : U\, |_{\mathbb{Z}_{2\times 2}(i,j)} \in \mathcal{B}_c \text{ for } i,j \in \mathbb{Z}\right\}.$$

Clearly,

$$\text{if } \Sigma_{m\times n}(\mathcal{B}_c) = \emptyset \text{ for some } m,n \geq 2, \quad \text{then } \Sigma(\mathcal{B}_c) = \emptyset. \tag{2.1}$$

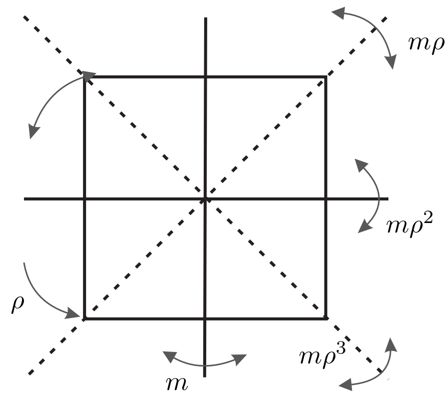
Then, the set $\mathcal{P}(\mathcal{B}_c)$ of all periodic patterns generated by $\mathcal{B}_c$ is defined by

$$\mathcal{P}(\mathcal{B}_c) = \{U = (u_{i,j}) \in \Sigma(\mathcal{B}_c) \mid u_{i,j} = u_{i+n,j} = u_{i,j+k} \text{ for all } i,j \in \mathbb{Z} \text{ and for some } n,k \geq 1\}.$$

For simplicity, a periodic pattern is also called a cycle.

Now, the symmetry of the unit square $\mathbb{Z}_{2\times 2}$ is introduced. The symmetry group of the rectangle $\mathbb{Z}_{2\times 2}$ is D_4, the dihedral group of order eight. The group D_4 is generated by the rotation ρ, through $\frac{\pi}{2}$, and the reflection m about the y-axis.

Denote the elements of D_4 by $D_4 = \{I, \rho, \rho^2, \rho^3, m, m\rho, m\rho^2, m\rho^3\}$.



Therefore, given a basic set $\mathcal{B}_c \subset \Sigma_{2\times 2}(p)$ and any element $\tau \in D_4$, another basic set $(\mathcal{B}_c)_\tau$ can be obtained by transforming the local patterns in $\mathcal{B}_c$ by τ.

Additionally, consider the permutation group S_p on $\mathcal{S}_p$. If $\eta \in S_p$ and $\eta(0) = i_0$, $\eta(1) = i_1, \ldots, \eta(p-1) = i_{p-1}$, we write

$$\eta = \begin{pmatrix} 0 & 1 & \cdots & p-1 \\ i_0 & i_1 & \cdots & i_{p-1} \end{pmatrix}.$$

For $\eta \in S_p$ and $\mathcal{B}_c \subset \Sigma_{2\times 2}(p)$, another basic set $(\mathcal{B}_c)_\eta$ can be obtained.

D_4 and S_p can be combined to define the equivalence classes of basic sets, as follows: given $\mathcal{B}_c \subset \Sigma_{2\times 2}(p)$, define the class $[\mathcal{B}_c]$ of $\mathcal{B}_c$ by

$$[\mathcal{B}_c] = \{\mathcal{B}_c' \subset \Sigma_{2\times 2}(p) : \mathcal{B}_c' = ((\mathcal{B}_c)_\tau)_\eta, \tau \in D_4, \eta \in S_p\}.$$

The nonemptiness of $\Sigma(\mathcal{B}_c)$ and $\mathcal{P}(\mathcal{B}_c)$ is clearly independent of the choice of elements in $[\mathcal{B}_c]$: for any $\mathcal{B}_c' \in [\mathcal{B}_c]$,

$$\Sigma(\mathcal{B}_c') \neq \emptyset \text{ (or } \mathcal{P}(\mathcal{B}_c') \neq \emptyset) \quad \text{if and only if} \quad \Sigma(\mathcal{B}_c) \neq \emptyset \text{ (or } \mathcal{P}(\mathcal{B}_c) \neq \emptyset).$$

More definitions are required.

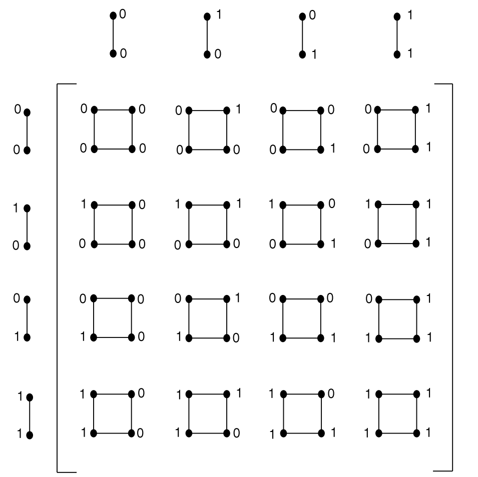
Definition 2.1. For $\mathcal{B}_c \subset \Sigma_{2\times 2}(p)$:

(i) $\mathcal{B}_c$ is called a cycle generator if $\mathcal{P}(\mathcal{B}_c) \neq \emptyset$.
(ii) $\mathcal{B}_c$ is called a minimal cycle generator if $\mathcal{P}(\mathcal{B}_c) \neq \emptyset$ and $\mathcal{P}(\mathcal{B}_c') = \emptyset$ for all $\mathcal{B}_c' \subsetneqq \mathcal{B}_c$.
(iii) $\mathcal{B}_c$ is called a noncycle generator if $\mathcal{P}(\mathcal{B}_c) = \emptyset$.
(iv) $\mathcal{B}_c$ is called a maximal noncycle generator if $\mathcal{P}(\mathcal{B}_c) = \emptyset$ and $\mathcal{P}(\mathcal{B}_c'') \neq \emptyset$ for all $\mathcal{B}_c'' \supsetneqq \mathcal{B}_c$.
(v) $\mathcal{C}_c(p)$ is the set of all minimal cycle generators that are subsets of $\Sigma_{2\times 2}(p)$.
(vi) $\mathcal{N}_c(p)$ is the set of all maximal noncycle generators that are subsets of $\Sigma_{2\times 2}(p)$.

Notably, if $\mathcal{B}_c$ is a cycle generator, then it has a subset of minimal cycle generators. In contrast, if $\mathcal{B}_c'$ is a noncycle generator, then $\mathcal{B}_c'$ is a subset of a maximal noncycle generator.

From now on, only the case $p = 2$ is considered: $\mathcal{S}_2 = \{0, 1\}$. In another work [1], the horizontal ordering matrix $\mathbf{X}_{2\times 2} = [x_{p,q}]_{4\times 4}$ for all local patterns in $\Sigma_{2\times 2}(2)$ is defined by

(2.2)



For any $n \geq 2$, the horizontal ordering matrix $\mathbf{X}_{2\times n}$ for all local patterns of $\Sigma_{2\times n}(2)$ can also be defined. The recursive formula for generating $\mathbf{X}_{2\times n}$ from $\mathbf{X}_{2\times 2}$ is as follows. Let

$$\mathbf{X}_{2\times n} = [x_{n;i,j}]_{2^n\times 2^n} = \begin{bmatrix} X_{2\times n;1} & X_{2\times n;2} \\ X_{2\times n;3} & X_{2\times n;4} \end{bmatrix}, \tag{2.3}$$

where $X_{2\times n;i}$ is a $2^{n-1}\times 2^{n-1}$ matrix of patterns. Then,

$$\mathbf{X}_{2\times(n+1)} = \begin{bmatrix} x_{1,1}X_{2\times n;1} & x_{1,2}X_{2\times n;2} & x_{1,3}X_{2\times n;1} & x_{1,4}X_{2\times n;2} \\ x_{2,1}X_{2\times n;3} & x_{2,2}X_{2\times n;4} & x_{2,3}X_{2\times n;3} & x_{2,4}X_{2\times n;4} \\ x_{3,1}X_{2\times n;1} & x_{3,2}X_{2\times n;2} & x_{3,3}X_{2\times n;1} & x_{3,4}X_{2\times n;2} \\ x_{4,1}X_{2\times n;3} & x_{4,2}X_{2\times n;4} & x_{4,3}X_{2\times n;3} & x_{4,4}X_{2\times n;4} \end{bmatrix} \tag{2.4}$$

is a $2^{n+1}\times 2^{n+1}$ matrix. Consequently, given a basic set $\mathcal{B}_c$, the associated horizontal transition matrix $\mathbf{H}_n(\mathcal{B}_c)$ is obtained from $\mathbf{X}_{2\times n}$. Indeed, $\mathbf{H}_2(\mathcal{B}_c) = [h_{i,j}]$, where $h_{i,j} = 1$ if and only if $x_{i,j} \in \mathcal{B}_c$. The recursive formula of $\mathbf{X}_{2\times n}$ can also be applied to $\mathbf{H}_n(\mathcal{B}_c)$. If

$$\mathbf{H}_n(\mathcal{B}_c) = \begin{bmatrix} H_{n;1} & H_{n;2} \\ H_{n;3} & H_{n;4} \end{bmatrix}_{2^n\times 2^n},$$

where $H_{n;j}$ is a $2^{n-1}\times 2^{n-1}$ matrix, then

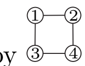
$$\mathbf{H}_{n+1}(\mathcal{B}_c) = \begin{bmatrix} h_{1,1}H_{n;1} & h_{1,2}H_{n;2} & h_{1,3}H_{n;1} & h_{1,4}H_{n;2} \\ h_{2,1}H_{n;3} & h_{2,2}H_{n;4} & h_{2,3}H_{n;3} & h_{2,4}H_{n;4} \\ h_{3,1}H_{n;1} & h_{3,2}H_{n;2} & h_{3,3}H_{n;1} & h_{3,4}H_{n;2} \\ h_{4,1}H_{n;3} & h_{4,2}H_{n;4} & h_{4,3}H_{n;3} & h_{4,4}H_{n;4} \end{bmatrix}. \tag{2.5}$$

Let $\Gamma_{m\times n}(\mathcal{B}_c)$ be the cardinal number of $\Sigma_{m\times n}(\mathcal{B}_c)$. Hence, $\Gamma_{m\times n}(\mathcal{B}_c)$ is equal to $\left|\mathbf{H}_n^{m-1}(\mathcal{B}_c)\right|$, where $|A|$ is the sum of all entries in matrix A.

For convenience, the name of each local pattern in $\Sigma_{2\times 2}(2)$, which is listed in (2.2), is given as follows.

Definition 2.2. Denote by

$$\mathbf{X}_{2\times 2} = \begin{bmatrix} O & e_2 & e_4 & r \\ e_1 & t & I & \overline{e}_3 \\ e_3 & J & b & \overline{e}_1 \\ l & \overline{e}_4 & \overline{e}_2 & E \end{bmatrix}. \tag{2.6}$$

Note that the ordering of four corners of $\mathbb{Z}_{2\times 2}$ is given by , which was used in [1, 2, 3]. For each $i \in \{1,2,3,4\}$, e_i ($\overline{e}_i$) is designed for one 1 (0) at the i-th corner and three 0s (1s) on the other corners. As two 1s (and two 0s), t stands for top, b for bottom, r for right, l for left, I for diagonal and J for anti-diagonal. Finally, O stands for four 0s, and the full matrix E stands for four 1s.

The following theorem groups all minimal cycle generators into seven classes and maximal noncycle generators into four classes. Table A.1 of Appendix A presents the symmetries of minimal cycle generators in $\mathcal{C}_c(2)$, and Table A.2 lists those of maximal noncycle generators in $\mathcal{N}_c(2)$.

Theorem 2.3.

(i) $\mathcal{C}_c(2)$ *contains* 17 *elements and is classified into seven classes of minimal cycle generators which are given by*
  (1) $[\{O\}] = \{\{O\},\{E\}\}$,
  (2) $[\{b,t\}] = \{\{b,t\},\{l,r\}\}$,
  (3) $[\{I,J\}] = \{\{I,J\}\}$,
  (4) $[\{e_1,e_2,e_3,e_4\}] = \{\{e_1,e_2,e_3,e_4\},\{\overline{e}_1,\overline{e}_2,\overline{e}_3,\overline{e}_4\}\}$,
  (5) $[\{e_1,e_2,\overline{e}_3,\overline{e}_4,b\}] = \{\{e_1,e_2,\overline{e}_3,\overline{e}_4,b\},\{e_3,e_4,\overline{e}_1,\overline{e}_2,t\},\{e_1,e_3,\overline{e}_2,\overline{e}_4,r\},$ $\{e_2,e_4,\overline{e}_1,\overline{e}_3,l\}\}$,
  (6) $[\{e_1,e_4,J\}] = \{\{e_1,e_4,J\},\{e_2,e_3,I\},\{\overline{e}_2,\overline{e}_3,J\},\{\overline{e}_1,\overline{e}_4,I\}\}$,
  (7) $[\{e_1,e_4,\overline{e}_1,\overline{e}_4\}] = \{\{e_1,e_4,\overline{e}_1,\overline{e}_4\},\{e_2,e_3,\overline{e}_2,\overline{e}_3\}\}$.

(ii) $\mathcal{N}_c(2)$ *contains* 56 *elements and is classified into four classes of maximal noncycle generators which are given by*
  (1) $[\{e_1,e_2,\overline{e}_1,\overline{e}_2,\overline{e}_4,J,r,b\}] \equiv [N_1]$,
  (2) $[\{e_1,e_2,\overline{e}_1,\overline{e}_2,\overline{e}_4,J,r,t\}] \equiv [N_2]$,
  (3) $[\{e_1,e_4,\overline{e}_1,\overline{e}_2,\overline{e}_3,I,l,t\}] \equiv [N_3]$,
  (4) $[\{e_1,e_2,e_3,\overline{e}_1,\overline{e}_2,\overline{e}_4,J,l,b\}] \equiv [N_4]$.

(iii) *If* $\mathcal{B}_c \in \mathcal{N}_c(2)$*, then* $\Sigma(\mathcal{B}_c) = \emptyset$.

Furthermore, (1.1) *holds for* $p = 2$.

Proof. The 17 basic sets in Theorem 2.3(i) are easily shown to be minimal cycle generators. The 56 basic sets that are listed in Table A.2 are obtained from the 17 minimal cycle generators in (i) by finding all maximal basic sets $\mathcal{B}_c \subset \Sigma_{2\times 2}(2)$ that do not contain any minimal cycle generator in (i). If the 56 basic sets cannot generate any periodic pattern, then the set of 17 minimal cycle generators is $\mathcal{C}_c(2)$ and the set of 56 basic sets, listed in Table A.2, is $\mathcal{N}_c(2)$.

Now, the stronger result that the 56 basic sets in Theorem 2.3(ii) cannot generate global patterns on $\mathbb{Z}^2$ is to be proven. Since (ii) specifies all four classes of the 56 basic sets, only the four cases N_i, $i = 1,2,3,4$ need to be considered. It is straightforward to check that $\mathbf{H}_4^4(N_i)$ is a zero matrix, $i = 1,2,3,4$; see Appendix A.3 for $\mathbf{H}_2(N_1)$ and $\mathbf{H}_4(N_1)$. The details are omitted here. Hence, $\Gamma_{5\times 4}(N_i) = 0$ such that $\Sigma_{5\times 4}(N_i) = \emptyset$, and then $\Sigma(N_i) = \emptyset$. The results (i), (ii) and (iii) follow.

Finally, from Theorem 2.3(iii), $\Sigma(\mathcal{B}_c) = \emptyset$ is easily seen for any $\mathcal{B}_c \in \Sigma_{2\times 2}(2)$ with $\mathcal{P}(\mathcal{B}_c) = \emptyset$. Therefore, (1.1) holds for $p = 2$. The proof is complete. □

3. Wang tiles

This section discusses edge coloring (Wang tiles). In this section, the unit square is still denoted by $\mathbb{Z}_{2\times 2}$. The left, right, bottom, and top edges of the unit square $\mathbb{Z}_{2\times 2}$ are given by $h_1(\mathbb{Z}_{2\times 2})$, $h_2(\mathbb{Z}_{2\times 2})$, $v_1(\mathbb{Z}_{2\times 2})$, and $v_2(\mathbb{Z}_{2\times 2})$, respectively. Denote the set of all local patterns with colored edges on $\mathbb{Z}_{2\times 2}$ (Wang tiles) over $\mathcal{S}_p$ by $\Sigma^w_{2\times 2}(p)$. Given $\mathcal{B} \subset \Sigma^w_{2\times 2}(p)$, let $\Sigma_{m\times n}(\mathcal{B})$ be the set of all local patterns on $\mathbb{Z}_{m\times n}$ generated by $\mathcal{B}$; let $\Sigma(\mathcal{B})$ be the set of all global patterns generated by $\mathcal{B}$, and $\mathcal{P}(\mathcal{B})$ be the set of all periodic patterns generated by $\mathcal{B}$. Clearly,

$$\text{if } \Sigma_{m\times n}(\mathcal{B}) = \emptyset \text{ for some } m, n \geq 2, \quad \text{then } \Sigma(\mathcal{B}) = \emptyset. \tag{3.1}$$

The ideas of corner coloring can be applied to edge coloring with some required modifications. The main difference between edge coloring and corner coloring is that the former is less rigid than the latter. More precisely, in edge coloring, every edge needs only be matched with other unit squares in the horizontal or vertical directions. Therefore, the horizontal and vertical matchings are mutually independent in the first stage. In corner coloring, the color of every corner must be matched with the colors of all four unit squares around the corner, and then the horizontal and vertical directions are closely related.

Since, in edge coloring, the permutations of colors in the horizontal and vertical directions are mutually independent, denote the permutations of colors in the horizontal and vertical edges by $\eta_h \in S_p$ and $\eta_v \in S_p$, respectively. Then, for any $\mathcal{B} \subset \Sigma^w_{2\times 2}(p)$, define the equivalence class $[\mathcal{B}]$ of $\mathcal{B}$ by

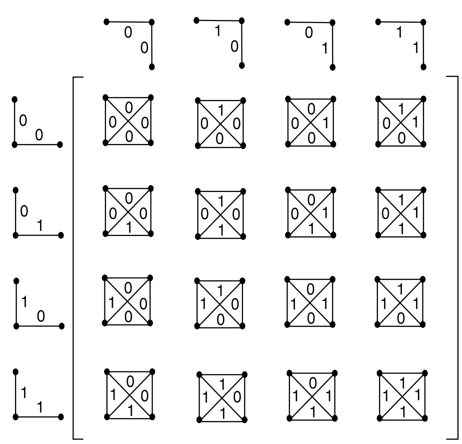
$$[\mathcal{B}] = \left\{ \mathcal{B}' \subset \Sigma^w_{2\times 2}(p) : \mathcal{B}' = \left(((\mathcal{B})_\tau)_{\eta_h}\right)_{\eta_v}, \tau \in D_4 \text{ and } \eta_h, \eta_v \in S_p \right\}.$$

As in corner coloring, the nonemptiness of $\Sigma(\mathcal{B})$ and $\mathcal{P}(\mathcal{B})$ is independent of the choice of elements in $[\mathcal{B}]$.

The minimal cycle generator and maximal noncycle generator are defined as in Definition 2.1. The sets of all minimal cycle generators and maximal noncycle generators contained in $\Sigma^w_{2\times 2}(p)$ are denoted by $\mathcal{C}(p)$ and $\mathcal{N}(p)$, respectively.

From now on, only the case $p = 2$ is considered. The vertical ordering matrix $\mathbf{Y}_{w;2\times 2} = [y_{w;i,j}]$ of all local patterns in $\Sigma^w_{2\times 2}(p)$ is denoted by

(3.2) $\mathbf{Y}_{w;2\times 2} =$

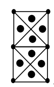


$$= \begin{bmatrix} y_{w;1,1} & y_{w;1,2} & y_{w;1,3} & y_{w;1,4} \\ y_{w;2,1} & y_{w;2,2} & y_{w;2,3} & y_{w;2,4} \\ y_{w;3,1} & y_{w;3,2} & y_{w;3,3} & y_{w;3,4} \\ y_{w;4,1} & y_{w;4,2} & y_{w;4,3} & y_{w;4,4} \end{bmatrix}$$

$$= \begin{bmatrix} O & E_2 & E_4 & R \\ E_3 & J & B & \overline{E}_1 \\ E_1 & T & I & \overline{E}_3 \\ L & \overline{E}_4 & \overline{E}_2 & E \end{bmatrix}. \tag{3.3}$$

To match colors in the vertical direction, the permutation matrix R_2 for four elements $\{1,2,3,4\}$ with $1 \to 3 \to 1$ and $2 \to 4 \to 2$ is introduced:

$$R_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{3.4}$$

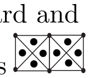
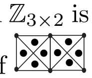
Now, the ordering matrix $\mathbf{Y}_{w;2\times 3}$ of local patterns , where $\bullet \in \{0,1\}$, on $\mathbb{Z}_{2\times 3}$ is arranged in

$$\mathbf{Y}_{w;2\times 3} = \mathbf{Y}_{w;2\times 2}^{[2]} \equiv \mathbf{Y}_{w;2\times 2}\left((I_4 + R_2)\mathbf{Y}_{w;2\times 2}\right), \tag{3.5}$$

where I_q is the $q \times q$ identity matrix. Furthermore, for $n \geq 3$, the ordering matrix $\mathbf{Y}_{w;2\times n}$ on $\mathbb{Z}_{2\times n}$ is obtained recursively by

$$\mathbf{Y}_{w;2\times n} = \mathbf{Y}_{w;2\times 2}^{[n-1]} \equiv \mathbf{Y}_{w;2\times 2}\left((I_4 + R_2)\mathbf{Y}_{w;2\times 2}^{[n-2]}\right) = \mathbf{Y}_{w;2\times 2}\left((I_4 + R_2)\mathbf{Y}_{w;2\times (n-1)}\right). \tag{3.6}$$

The proofs of (3.5) and (3.6) are straightforward and are omitted here.

The ordering matrix $\mathbf{Y}_{w;3\times 2}$ of local patterns , where $\bullet \in \{0,1\}$, on $\mathbb{Z}_{3\times 2}$ is quite different from that associated with corner coloring. Indeed, $\mathbf{Y}_{w;3\times 2}$ of  has 2^7 elements and is arranged in a $2^4 \times 2^4$ matrix, as presented in Appendix A.4.

Furthermore, to introduce the vertical ordering matrix $\mathbf{Y}_{w;m\times 2}$ on $\mathbb{Z}_{m\times 2}$, the following notation is required. For any two $q \times q$ matrices $A = (a_{i,j})$ and $B = (b_{i,j})$, where $a_{i,j}$ and $b_{i,j}$ are numbers or matrices, the Hadamard product of $A \circ B$ is defined by $A \circ B = (a_{i,j}b_{i,j})$. Then, for $m \geq 3$, if

$$\mathbf{Y}_{w;m\times 2} = \begin{bmatrix} Y_{w;m;1} & Y_{w;m;2} \\ Y_{w;m;3} & Y_{w;m;4} \end{bmatrix},$$

then

(3.7)

$$
\begin{aligned}
&\mathbf{Y}_{w;(m+1)\times 2} \\
=&\begin{bmatrix}
y_{w;1,1} & \times & y_{w;1,2} & \times & y_{w;1,1} & \times & y_{w;1,2} & \times \\
\times & y_{w;1,3} & \times & y_{w;1,4} & \times & y_{w;1,3} & \times & y_{w;1,4} \\
y_{w;2,1} & \times & y_{w;2,2} & \times & y_{w;2,1} & \times & y_{w;2,2} & \times \\
\times & y_{w;2,3} & \times & y_{w;2,4} & \times & y_{w;2,3} & \times & y_{w;2,4} \\
y_{w;3,1} & \times & y_{w;3,2} & \times & y_{w;3,1} & \times & y_{w;3,2} & \times \\
\times & y_{w;3,3} & \times & y_{w;3,4} & \times & y_{w;3,3} & \times & y_{w;3,4} \\
y_{w;4,1} & \times & y_{w;4,2} & \times & y_{w;4,1} & \times & y_{w;4,2} & \times \\
\times & y_{w;4,3} & \times & y_{w;4,4} & \times & y_{w;4,3} & \times & y_{w;4,4}
\end{bmatrix} \\
&\circ \begin{bmatrix}
Y_{w;m;1} & \times & Y_{w;m;1} & \times & Y_{w;m;2} & \times & Y_{w;m;2} & \times \\
\times & Y_{w;m;3} & \times & Y_{w;m;3} & \times & Y_{w;m;4} & \times & Y_{w;m;4} \\
Y_{w;m;1} & \times & Y_{w;m;1} & \times & Y_{w;m;2} & \times & Y_{w;m;2} & \times \\
\times & Y_{w;m;3} & \times & Y_{w;m;3} & \times & Y_{w;m;4} & \times & Y_{w;m;4} \\
Y_{w;m;1} & \times & Y_{w;m;1} & \times & Y_{w;m;2} & \times & Y_{w;m;2} & \times \\
\times & Y_{w;m;3} & \times & Y_{w;m;3} & \times & Y_{w;m;4} & \times & Y_{w;m;4} \\
Y_{w;m;1} & \times & Y_{w;m;1} & \times & Y_{w;m;2} & \times & Y_{w;m;2} & \times \\
\times & Y_{w;m;3} & \times & Y_{w;m;3} & \times & Y_{w;m;4} & \times & Y_{w;m;4}
\end{bmatrix},
\end{aligned}
$$

where $\times$ means no pattern.

As in $m = 2$, to match colors in the vertical direction for $\mathbf{Y}_{w;m\times 2}$, $m \geq 3$, the $2^{2m-2} \times 2^{2m-2}$ permutation matrix $R_m = [R_{m;i,j}]$ must be introduced. For any $1 \leq i \leq 2^{2m-2}$, the existence of a unique $\alpha_1, \alpha_2, \ldots, \alpha_{2m-2} \in \{0,1\}$ such that

$$i = 1 + \sum_{q=1}^{2m-2} \alpha_q 2^{2m-2-q}$$

is easy to verify. Let

$$\xi(i) = \sum_{q=1}^{m-1} \alpha_{2q-1} 2^{m-1-q}.$$

Hence, there exist unique $\alpha'_1, \alpha'_3, \ldots, \alpha'_{2m-3} \in \{0,1\}$ such that

$$\sum_{q=1}^{m-1} \alpha'_{2q-1} 2^{m-1-q} \equiv \xi(i) + 1 \pmod{2^{m-1}}.$$

Now, define

$$R_{m;i,j} = 1 \text{ if and only if } j = r(i), \tag{3.8}$$

where

$$r(i) = 1 + \sum_{q=1}^{m-1} \left(\alpha'_{2q-1} 2^{2m-2-(2q-1)} + \alpha_{2q} 2^{2m-2-2q} \right).$$

Furthermore, define

$$\mathbf{R}_m = \sum_{l=0}^{2^{m-1}-1} R_m^l. \tag{3.9}$$

Now, the ordering matrix $\mathbf{Y}_{w;m\times 3}$ on $\mathbb{Z}_{m\times 3}$ can be shown as

$$\mathbf{Y}_{w;m\times 3} = \mathbf{Y}^{[2]}_{w;m\times 2} \equiv \mathbf{Y}_{w;m\times 2}\left(\mathbf{R}_m \mathbf{Y}_{w;m\times 2}\right) \tag{3.10}$$

and

$$\mathbf{Y}_{w;m\times n} = \mathbf{Y}^{[n-1]}_{w;m\times 2} \equiv \mathbf{Y}_{w;m\times 2}\left(\mathbf{R}_m \mathbf{Y}^{[n-2]}_{w;m\times 2}\right) \tag{3.11}$$

for $n \geq 3$. The proofs of (3.10) and (3.11) are similar to those in corner coloring and are omitted here.

Given $\mathcal{B} \subset \Sigma^w_{2\times 2}(2)$, the associated transition matrix $\mathbf{V}_{w;m}(\mathcal{B})$ is obtained from $\mathbf{Y}_{w;m\times 2}$. Indeed, $\mathbf{V}_{w;2}(\mathcal{B}) = [v_{i,j}]$, where $v_{i,j} = 1$ if and only if $y_{w;i,j} \in \mathcal{B}$. As in corner coloring, the recursive formula of $\mathbf{Y}_{w;m\times 2}$ can also be applied to $\mathbf{V}_{w;m}(\mathcal{B})$ as follows. If

$$\mathbf{V}_{w;m}(\mathcal{B}) = \begin{bmatrix} V_{m;1} & V_{m;2} \\ V_{m;3} & V_{m;4} \end{bmatrix}_{2^{2m-2}\times 2^{2m-2}},$$

where $V_{m;j}$ is a $2^{2m-3} \times 2^{2m-3}$ matrix, then

$$\begin{aligned} &\mathbf{V}_{w;m+1}(\mathcal{B}) \\ &= \begin{bmatrix} v_{1,1} & 0 & v_{1,2} & 0 & v_{1,1} & 0 & v_{1,2} & 0 \\ 0 & v_{1,3} & 0 & v_{1,4} & 0 & v_{1,3} & 0 & v_{1,4} \\ v_{2,1} & 0 & v_{2,2} & 0 & v_{2,1} & 0 & v_{2,2} & 0 \\ 0 & v_{2,3} & 0 & v_{2,4} & 0 & v_{2,3} & 0 & v_{2,4} \\ v_{3,1} & 0 & v_{3,2} & 0 & v_{3,1} & 0 & v_{3,2} & 0 \\ 0 & v_{3,3} & 0 & v_{3,4} & 0 & v_{3,3} & 0 & v_{3,4} \\ v_{4,1} & 0 & v_{4,2} & 0 & v_{4,1} & 0 & v_{4,2} & 0 \\ 0 & v_{4,3} & 0 & v_{4,4} & 0 & v_{4,3} & 0 & v_{4,4} \end{bmatrix} \\ &\circ \begin{bmatrix} V_{m;1} & O_m & V_{m;1} & O_m & V_{m;2} & O_m & V_{m;2} & O_m \\ O_m & V_{m;3} & O_m & V_{m;3} & O_m & V_{m;4} & O_m & V_{m;4} \\ V_{m;1} & O_m & V_{m;1} & O_m & V_{m;2} & O_m & V_{m;2} & O_m \\ O_m & V_{m;3} & O_m & V_{m;3} & O_m & V_{m;4} & O_m & V_{m;4} \\ V_{m;1} & O_m & V_{m;1} & O_m & V_{m;2} & O_m & V_{m;2} & O_m \\ O_m & V_{m;3} & O_m & V_{m;3} & O_m & V_{m;4} & O_m & V_{m;4} \\ V_{m;1} & O_m & V_{m;1} & O_m & V_{m;2} & O_m & V_{m;2} & O_m \\ O_m & V_{m;3} & O_m & V_{m;3} & O_m & V_{m;4} & O_m & V_{m;4} \end{bmatrix}, \end{aligned} \tag{3.12}$$

where O_m is the $2^{2m-3} \times 2^{2m-3}$ zero matrix. Let $\Gamma_{m\times n}(\mathcal{B})$ be the cardinal number of $\Sigma_{m\times n}(\mathcal{B})$. Therefore,

$$\Gamma_{m\times n}(\mathcal{B}) = \left| \mathbf{V}_{w;m}(\mathcal{B})\left(\mathbf{R}_m \mathbf{V}_{w;m}(\mathcal{B})\right)^{n-2} \right|. \tag{3.13}$$

Now, the following theorem gives the six classes of 38 minimal cycle generators in $\mathcal{C}(2)$ and the one class of eight maximal noncycle generators in $\mathcal{N}(2)$. Tables A.5 and A.6 present the details of six equivalent classes of $\mathcal{C}(2)$ and the symmetries of eight maximal noncycle generators in $\mathcal{N}(2)$, respectively.

Theorem 3.1.

(i) *The six classes of minimal cycle generators in* $\mathcal{C}(2)$ *are given as follows:*
   (1) $[\{O\}]$,
   (2) $[\{E_1, E_4\}]$,
   (3) $[\{E_1, \overline{E}_1\}]$,

(4) $[\{B,T\}]$,
(5) $[\{E_1,B,R\}]$,
(6) $[\{E_1,E_2,B\}]$.

(ii) *The one class of maximal noncycle generators in* $\mathcal{N}(2)$ *is given by* $[\{E_1,E_2,\overline{E}_3,\overline{E}_4,T,R\}] \equiv [N_w]$.

(iii) *If* $\mathcal{B} \in \mathcal{N}(2)$*, then* $\Sigma(\mathcal{B}) = \emptyset$.

Furthermore, (1.1) *holds for* $p = 2$.

Proof. The idea of this proof is similar to that of corner coloring. The 38 basic sets in Table A.5 are easily seen to be minimal cycle generators. The eight basic sets in Table A.6 are obtained from the 38 minimal cycle generators in Table A.5 by finding all maximal basic sets $\mathcal{B} \subset \Sigma^w_{2\times 2}(2)$ that do not contain any minimal cycle generator in Table A.5.

Then, to prove (i), (ii) and (iii), only $\Sigma(N_w) = \emptyset$ need be proven. Form (3.8), (3.12) and (3.13), $\Gamma_{3\times 4}(N_w) = 0$ is straightforwardly proven; then, $\Sigma(N_w) = \emptyset$. Therefore, the results (i), (ii) and (iii) hold.

Finally, from (iii), $\Sigma(\mathcal{B}) = \emptyset$ is easily seen for any $\mathcal{B} \subset \Sigma^w_{2\times 2}(2)$ with $\mathcal{P}(\mathcal{B}) = \emptyset$. Therefore, (1.1) holds for $p = 2$ in corner coloring. The proof is complete. □

Appendix A

A.1. The symmetries of D_4 and S_2 of 17 minimal cycle generators in $\mathcal{C}_c(2)$ are listed in Table A.1.

Table A.1

Minimal cycle generator	ρ	ρ^2	ρ^3	m	$m\rho$	$m\rho^2$	$m\rho^3$	$0 \leftrightarrow 1$
(1) 1×1 $\{O\}$	•	•	•	•	•	•	•	(2)
(2) 1×1 $\{E\}$	•	•	•	•	•	•	•	(1)
(3) 1×2 $\{t,b\}$	(4)	•	(4)	•	(4)	•	(4)	•
(4) 2×1 $\{l,r\}$	(3)	•	(3)	•	(3)	•	(3)	•
(5) 2×2 $\{I,J\}$	•	•	•	•	•	•	•	•
(6) 2×2 $\{e_1,e_2,e_3,e_4\}$	•	•	•	•	•	•	•	(7)
(7) 2×2 $\{\overline{e}_1,\overline{e}_2,\overline{e}_3,\overline{e}_4\}$	•	•	•	•	•	•	•	(6)
(8) 3×2 $\{r,e_1,e_3,\overline{e}_2,\overline{e}_4\}$	(10)	(9)	(11)	(9)	(10)	•	(11)	(9)
(9) 3×2 $\{l,e_2,e_4,\overline{e}_1,\overline{e}_3\}$	(11)	(8)	(10)	(8)	(11)	•	(10)	(8)
(10) 2×3 $\{t,e_3,e_4,\overline{e}_1,\overline{e}_2\}$	(9)	(11)	(8)	•	(8)	(11)	(9)	(11)
(11) 2×3 $\{b,e_1,e_2,\overline{e}_3,\overline{e}_4\}$	(8)	(10)	(9)	•	(9)	(10)	(8)	(10)
(12) 3×3 $\{e_1,e_4,J\}$	(13)	•	(13)	(13)	•	(13)	•	(15)
(13) 3×3 $\{e_2,e_3,I\}$	(12)	•	(12)	(12)	•	(12)	•	(14)
(14) 3×3 $\{\overline{e}_2,\overline{e}_3,J\}$	(15)	•	(15)	(15)	•	(15)	•	(13)
(15) 3×3 $\{\overline{e}_1,\overline{e}_4,I\}$	(14)	•	(14)	(14)	•	(14)	•	(12)
(16) 4×4 $\{e_2,e_3,\overline{e}_2,\overline{e}_3\}$	(17)	•	(17)	(17)	•	(17)	•	•
(17) 4×4 $\{e_1,e_4,\overline{e}_1,\overline{e}_4\}$	(16)	•	(16)	(16)	•	(16)	•	•

A.2. Given $\mathcal{B}_c \in \mathcal{N}_c(2)$, denote by $\overline{\mathcal{B}}_c$, $\mathcal{B}_{c;1}$, $\mathcal{B}_{c;2}$, $\mathcal{B}_{c;3}$, $\mathcal{B}_{c;4}$, $\mathcal{B}_{c;5}$, $\mathcal{B}_{c;6}$, $\mathcal{B}_{c;7}$ the basic sets transformed by $\eta = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, ρ, ρ^2, ρ^3, m, $m\rho$, $m\rho^2$, $m\rho^3$, respectively. Table A.2 lists the 56 maximal noncycle generators in $\mathcal{N}_c(2)$.

TABLE A.2

$N_1 \equiv \{e_1, e_2, \overline{e}_1, \overline{e}_2, \overline{e}_4, J, r, b\}$	$\overline{N}_1 = \{\overline{e}_1, \overline{e}_2, e_1, e_2, e_4, I, l, t\}$
$N_{1;1} = \{e_1, e_3, \overline{e}_1, \overline{e}_2, \overline{e}_3, I, r, t\}$	$\overline{N}_{1;1} = \{\overline{e}_1, \overline{e}_3, e_1, e_2, e_3, J, l, b\}$
$N_{1;2} = \{e_3, e_4, \overline{e}_1, \overline{e}_3, \overline{e}_4, J, l, t\}$	$\overline{N}_{1;2} = \{\overline{e}_3, \overline{e}_4, e_1, e_3, e_4, I, r, b\}$
$N_{1;3} = \{e_2, e_4, \overline{e}_2, \overline{e}_3, \overline{e}_4, I, l, b\}$	$\overline{N}_{1;3} = \{\overline{e}_2, \overline{e}_4, e_2, e_3, e_4, J, r, t\}$
$N_{1;4} = \{e_1, e_2, \overline{e}_1, \overline{e}_2, \overline{e}_3, I, l, b\}$	$\overline{N}_{1;4} = \{\overline{e}_1, \overline{e}_2, e_1, e_2, e_3, J, r, t\}$
$N_{1;5} = \{e_2, e_4, \overline{e}_1, \overline{e}_2, \overline{e}_4, J, l, t\}$	$\overline{N}_{1;5} = \{\overline{e}_2, \overline{e}_4, e_1, e_2, e_4, I, r, b\}$
$N_{1;6} = \{e_3, e_4, \overline{e}_2, \overline{e}_3, \overline{e}_4, I, r, t\}$	$\overline{N}_{1;6} = \{\overline{e}_4, \overline{e}_4, e_2, e_3, e_4, J, l, b\}$
$N_{1;7} = \{e_1, e_3, \overline{e}_1, \overline{e}_3, \overline{e}_4, J, r, b\}$	$\overline{N}_{1;7} = \{\overline{e}_1, \overline{e}_3, e_1, e_3, e_4, I, l, t\}$

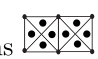
$N_2 \equiv \{e_1, e_2, \overline{e}_1, \overline{e}_2, \overline{e}_4, J, r, t\}$	$\overline{N}_2 = \{\overline{e}_1, \overline{e}_2, e_1, e_2, e_4, I, l, b\}$
$N_{2;1} = \{e_1, e_3, \overline{e}_1, \overline{e}_2, \overline{e}_3, I, l, t\}$	$\overline{N}_{2;1} = \{\overline{e}_1, \overline{e}_3, e_1, e_2, e_3, J, r, b\}$
$N_{2;2} = \{e_3, e_4, \overline{e}_1, \overline{e}_3, \overline{e}_4, J, l, b\}$	$\overline{N}_{2;2} = \{\overline{e}_3, \overline{e}_4, e_1, e_3, e_4, I, r, t\}$
$N_{2;3} = \{e_2, e_4, \overline{e}_2, \overline{e}_3, \overline{e}_4, I, r, b\}$	$\overline{N}_{2;3} = \{\overline{e}_2, \overline{e}_4, e_2, e_3, e_4, J, l, t\}$
$N_{2;4} = \{e_1, e_2, \overline{e}_1, \overline{e}_2, \overline{e}_3, I, l, t\}$	$\overline{N}_{2;4} = \{\overline{e}_1, \overline{e}_2, e_1, e_2, e_3, J, r, b\}$
$N_{2;5} = \{e_2, e_4, \overline{e}_1, \overline{e}_2, \overline{e}_4, J, r, t\}$	$\overline{N}_{2;5} = \{\overline{e}_2, \overline{e}_4, e_1, e_2, e_4, I, l, b\}$
$N_{2;6} = \{e_3, e_4, \overline{e}_2, \overline{e}_3, \overline{e}_4, I, r, b\}$	$\overline{N}_{2;6} = \{\overline{e}_3, \overline{e}_4, e_2, e_3, e_4, J, l, t\}$
$N_{2;7} = \{e_1, e_3, \overline{e}_1, \overline{e}_3, \overline{e}_4, J, l, b\}$	$\overline{N}_{2;7} = \{\overline{e}_1, \overline{e}_3, e_1, e_3, e_4, I, r, t\}$

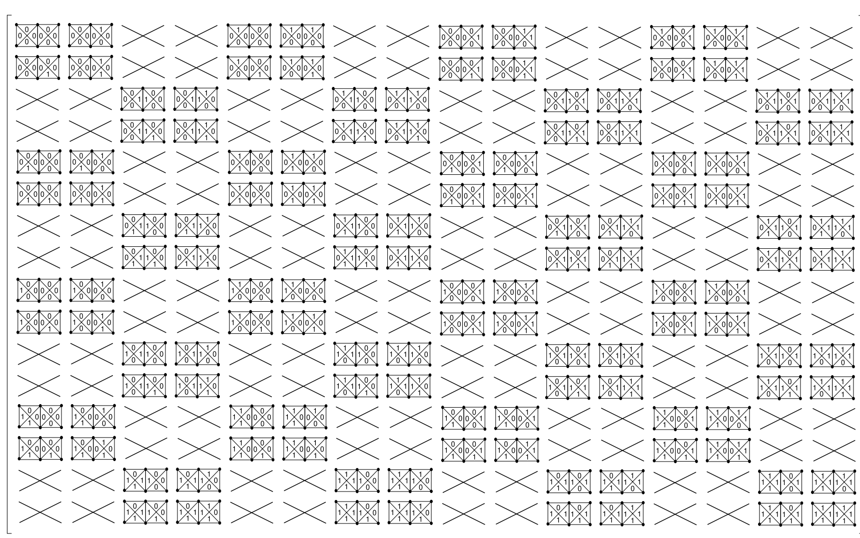
$N_3 \equiv \{e_1, e_4, \overline{e}_1, \overline{e}_2, \overline{e}_3, I, l, t\}$	$\overline{N}_3 = \{\overline{e}_1, \overline{e}_4, e_1, e_2, e_3, J, r, b\}$
$N_{3;1} = \{e_2, e_3, \overline{e}_1, \overline{e}_3, \overline{e}_4, J, l, b\}$	$\overline{N}_{3;1} = \{\overline{e}_2, \overline{e}_3, e_1, e_3, e_4, I, r, t\}$
$N_{3;2} = \{e_1, e_4, \overline{e}_2, \overline{e}_3, \overline{e}_4, I, r, b\}$	$\overline{N}_{3;2} = \{\overline{e}_1, \overline{e}_4, e_2, e_3, e_4, J, l, t\}$
$N_{3;3} = \{e_2, e_3, \overline{e}_1, \overline{e}_2, \overline{e}_4, J, r, t\}$	$\overline{N}_{3;3} = \{\overline{e}_2, \overline{e}_3, e_1, e_2, e_4, I, l, b\}$

$N_4 \equiv \{e_1, e_2, e_3, \overline{e}_1, \overline{e}_2, \overline{e}_4, J, l, b\}$	$\overline{N}_4 = \{\overline{e}_1, \overline{e}_2, \overline{e}_3, e_1, e_2, e_4, I, r, t\}$
$N_{4;1} = \{\overline{e}_1, \overline{e}_2, \overline{e}_3, e_1, e_3, e_4, I, r, b\}$	$\overline{N}_{4;1} = \{e_1, e_2, e_3, \overline{e}_1, \overline{e}_3, \overline{e}_4, J, l, t\}$
$N_{4;2} = \{\overline{e}_1, \overline{e}_3, \overline{e}_4, e_2, e_3, e_4, J, r, t\}$	$\overline{N}_{4;2} = \{e_1, e_3, e_4, \overline{e}_2, \overline{e}_3, \overline{e}_4, I, l, b\}$
$N_{4;3} = \{e_1, e_2, e_4, \overline{e}_2, \overline{e}_3, \overline{e}_4, I, l, t\}$	$\overline{N}_{4;3} = \{\overline{e}_1, \overline{e}_2, \overline{e}_4, e_2, e_3, e_4, J, r, b\}$
$N_{4;4} = \{\overline{e}_1, \overline{e}_2, \overline{e}_3, e_1, e_2, e_4, I, r, b\}$	$\overline{N}_{4;4} = \{e_1, e_2, e_3, \overline{e}_1, \overline{e}_2, \overline{e}_4, J, l, t\}$
$N_{4;5} = \{\overline{e}_1, \overline{e}_2, \overline{e}_4, e_2, e_3, e_4, J, l, b\}$	$\overline{N}_{4;5} = \{e_1, e_2, e_4, \overline{e}_2, \overline{e}_3, \overline{e}_4, I, r, t\}$
$N_{4;6} = \{e_1, e_3, e_4, \overline{e}_2, \overline{e}_3, \overline{e}_4, I, l, t\}$	$\overline{N}_{4;6} = \{\overline{e}_1, \overline{e}_3, \overline{e}_4, e_2, e_3, e_4, J, r, b\}$
$N_{4;7} = \{e_1, e_2, e_3, \overline{e}_1, \overline{e}_3, \overline{e}_4, J, r, t\}$	$\overline{N}_{4;7} = \{\overline{e}_1, \overline{e}_2, \overline{e}_3, e_1, e_3, e_4, I, l, b\}$

A.3. $\mathbf{H}_2(N_1) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ and

$$\mathbf{H}_4(N_1) = \begin{bmatrix}
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&1&0&1&0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0
\end{bmatrix}.$$

A.4. The ordering matrix of $\mathbf{Y}_{w;3\times 2}$ of local patterns  on $\mathbb{Z}_{3\times 2}$ is



A.5. The details of six equivalence classes of $\mathcal{C}(2)$ are listed in Table A.5.

TABLE A.5

$[\{O\}]$	$=$	$\{\{O\},\ \{I\},\ \{J\},\ \{E\}\}$
$[\{E_1,E_4\}]$	$=$	$\{\{E_1,E_4\},\ \{E_2,E_3\},\ \{\overline{E}_1,\overline{E}_4\},\ \{\overline{E}_2,\overline{E}_3\}\}$
$[\{E_1,\overline{E}_1\}]$	$=$	$\{\{E_1,\overline{E}_1\},\ \{E_2,\overline{E}_2\},\ \{E_3,\overline{E}_3\},\ \{E_4,\overline{E}_4\}\}$
$[\{B,T\}]$	$=$	$\{\{B,T\},\ \{L,R\}\}$
$[\{E_1,B,R\}]$	$=$	$\left\{\begin{matrix} \{E_1,B,R\}, & \{E_2,B,L\}, & \{E_3,T,R\}, & \{E_4,T,L\}, \\ \{\overline{E}_1,T,L\}, & \{\overline{E}_2,T,R\}, & \{\overline{E}_3,B,L\}, & \{\overline{E}_4,B,R\} \end{matrix}\right\}$
$[\{E_1,E_2,B\}]$	$=$	$\left\{\begin{matrix} \{E_1,E_2,B\}, & \{E_1,E_3,R\}, & \{E_2,E_4,L\}, & \{E_3,E_4,T\}, \\ \{E_1,\overline{E}_2,R\}, & \{E_1,\overline{E}_3,B\}, & \{E_2,\overline{E}_1,L\}, & \{E_2,\overline{E}_4,B\}, \\ \{E_3,\overline{E}_1,T\}, & \{E_3,\overline{E}_4,R\}, & \{E_4,\overline{E}_2,T\}, & \{E_4,\overline{E}_3,L\}, \\ \{\overline{E}_1,\overline{E}_2,T\}, & \{\overline{E}_1,\overline{E}_3,L\}, & \{\overline{E}_2,\overline{E}_4,R\}, & \{\overline{E}_3,\overline{E}_4,B\} \end{matrix}\right\}$

A.6. For the $N_w \in \mathcal{N}(2)$, denote by $N_{w;1}$, $N_{w;2}$, $N_{w;3}$, $N_{w;4}$, $N_{w;5}$, $N_{w;6}$, $N_{w;7}$ the other basic sets transformed by ρ, ρ^2, ρ^3, m, $m\rho$, $m\rho^2$, $m\rho^3$, respectively. The eight maximal noncycle generators in $\mathcal{N}_2$ are listed in Table A.6.

TABLE A.6

$N_w \equiv \{E_1,E_2,\overline{E}_3,\overline{E}_4,T,R\}$
$N_{w;1} = \{E_1,E_3,\overline{E}_2,\overline{E}_4,T,L\}$
$N_{w;2} = \{E_3,E_4,\overline{E}_1,\overline{E}_2,B,L\}$
$N_{w;3} = \{E_2,E_4,\overline{E}_1,\overline{E}_3,B,R\}$
$N_{w;4} = \{E_2,E_4,\overline{E}_1,\overline{E}_3,T,R\}$
$N_{w;5} = \{E_3,E_4,\overline{E}_1,\overline{E}_2,B,R\}$
$N_{w;6} = \{E_1,E_3,\overline{E}_2,\overline{E}_4,B,L\}$
$N_{w;7} = \{E_1,E_2,\overline{E}_3,\overline{E}_4,T,L\}$

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