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二階梁理論及其在側向-扭轉挫屈的應用(I)

A Second Order Beam Theory and Its Application In Lateral-Torsional Buckling (I)

計畫編號：NSC 91-2211-E-009-041
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計畫參與人員：陳弘虎

中文摘要

本研究提出一個二階梁理論共旋轉法以探討大旋轉但小應變之三維尤拉梁的非線性分析。將梁分割成若干個梁元素。每一個元素座標系統是建立在元素當前的變形位置上。梁元素的變形、平衡方程式及構成方程式，皆在元素座標系統中建立。以虛功原理及完整非線性梁理論之二階一致線性化推導梁元素的平衡方程式及構成方程式。平衡方程式及構成方程式保留至變形參數的二次項及部份的三次項所以能考慮撓曲、扭曲、與軸向變形間非線性的耦合效應。本研究將以數值例題探討提出之二階梁理論的正確性及實用性。

關鍵詞: 梁, 幾何非線性, 共旋轉法, 虛功原理

Abstract

A consistent co-rotational total Lagrangian formulation of second order beam theory is presented for the nonlinear analysis of three-dimensional elastic Euler beam with large rotations but small strains. The beam structure is divided into several segments. A set of segment coordinate system is constructed at the current configuration of the deformed beam segment. The deformations, equilibrium equations and constitutive equations of the beam segment are defined in the segment coordinates. The principle of virtual work and the consistent linearization of the fully geometrically nonlinear beam theory is used to derive the equilibrium equations and constitutive equation of the beam segment. The nonlinear coupling among bending, twisting, and stretching deformations is considered by retaining all terms up to the second order and some terms of third order of the deformation parameters in the equilibrium equations and constitutive equations of the second order beam theory. Numerical examples are presented to demonstrate the accuracy and effectiveness of the proposed second order beam theory.

Keywords: Beam, Geometrical Nonlinearity, Co-rotational Formulation, Virtual Work Principle.

1 INTRODUCTION

Three-dimensional beams are very important structural elements in all type of engineering systems. In many applications, these beams undergo finite rotations that require a nonlinear formulation to their structural analysis. In order to capture correctly all coupling among bending, twisting, and stretching deformations of the beam, the formulation of beam equations might be derived by the fully geometrically non-linear beam theory [1].

The beam structure is divided into several segments. A set of segment coordinate system is constructed at the current configuration of the deformed beam segment. The deformation, equilibrium equations and constitutive equations of the beam segment are defined in this segment coordinates. The deformations of the beam segment are determined by the unit extension of the centroid axis and the rotation of segment cross section coordinates, which are rigidly tied to segment cross section, relative to the segment coordinate system. Three rotation parameters proposed in [2] are used to describe the rotation of the segment cross section coordinates.

The principle of virtual work and the consistent linearization [2] of the fully geometrically nonlinear beam theory is used to derive the equilibrium equations and constitutive equation of the beam. In order to consider the nonlinear coupling among bending, twisting, and stretching deformations, the rotation parameters are retained up to the second order in the equilibrium equations and constitutive equation of the beam segment. Thus, the beam theory proposed here is called a second order beam theory. However, some third order terms, which may not be negligible for some problems, are also retained. Numerical examples are presented to demonstrate the accuracy and effectiveness of the proposed second order beam theory.

2 NONLINEAR FORMULATION

2.1 Basic assumptions

In this study, The beam structure is divided into several segments. The following assumptions are made in derivation of the beam segment behavior.

(1) The beam is prismatic and slender, and the Euler-Bernoulli hypothesis is valid.
(2) The cross section of the beam is doubly symmetric.
(3) The cross section of the beam does not deform in its own plane and strains within this cross section can be neglected.
(4) The out-of-plane warping of the cross section is the product of the twist rate of the beam element and the Saint Venant warping function for a prismatic beam of the same cross section.
(5) The deformation displacements and rotations of the beam segment are small.
(6) The strains of the beam are small.

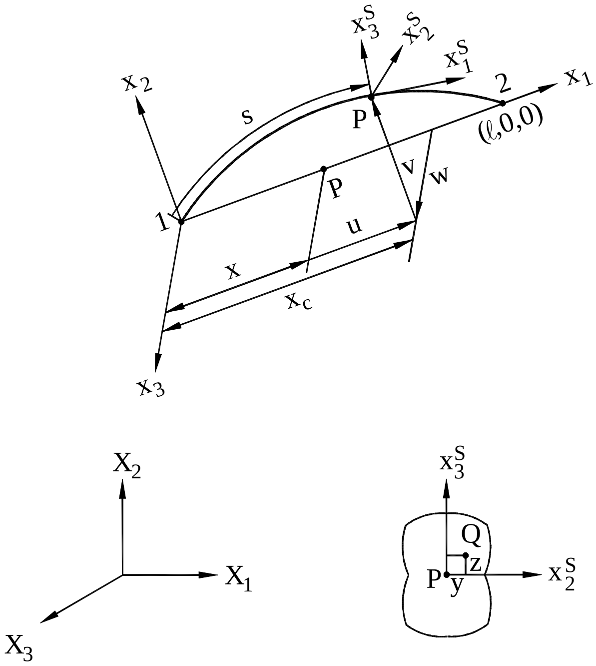
In conjunction with the co-rotational formulation, assumption (5) can always be satisfied, if the segment size is properly chosen.

2.2 Coordinate systems

In this paper, a co-rotational total Lagrangian formulation is adopted. In order to describe the system, we define three sets of right handed rectangular Cartesian coordinate systems:

(1) A fixed global set of coordinates, X_i (i = 1, 2, 3) (see Fig. 1); the nodal coordinates, displacements, rotations, the governing equations of the system are defined in this coordinates.
(2) Segment cross section coordinates, x_i^S (i = 1, 2, 3) (see Fig. 1).

(3) Segment coordinates, x_i (i = 1, 2, 3) (see Fig. 1); a set of segment coordinates is associated with each segment, which is constructed at the current configuration of the beam segment.



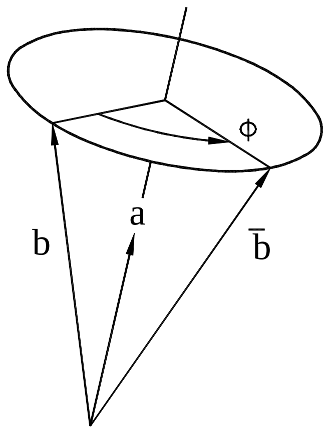


Fig.1 Coordinate systems. Fig.2 Rotation vector.

2.3 Rotation vector and rotation parameters

For convenience of the later discussion, the term 'rotation vector' is used to represent a finite rotation. Figure 2 shows that a vector $\mathbf{b}$ which as a result of the application of a rotation vector $\phi\mathbf{a}$ is transported to the new position $\bar{\mathbf{b}}$. The relation between $\bar{\mathbf{b}}$ and $\mathbf{b}$ may be expressed as [3]

$$\bar{\mathbf{b}} = \cos\phi\mathbf{b} + (1 - \cos\phi)(\mathbf{a}\cdot\mathbf{b}) + \sin\phi(\mathbf{a}\times\mathbf{b}) \tag{1}$$

where ϕ is the angle of counterclockwise rotation, and $\mathbf{a}$ is the unit vector along the axis of rotation.

Let $\mathbf{e}_i$ and $\mathbf{e}_i^S$ (i = 1, 2, 3) denote the unit vectors associated with the x_i and x_i^S axes, respectively. Here, the traid $\mathbf{e}_i^S$ in the deformed state is assumed to be achieved by the successive application of the following two rotation vectors to the traid $\mathbf{e}_i$ [2]:

$$\bar{\boldsymbol{\phi}}_n = \bar{\phi}_n\mathbf{n}, \qquad \bar{\boldsymbol{\phi}}_t = \bar{\phi}_t\mathbf{t} \tag{2,3}$$

where

$$\begin{aligned}\mathbf{n} &= \{0,\ \bar{\theta}_2\big/(\bar{\theta}_2^2 + \bar{\theta}_3^2)^{1/2},\ \bar{\theta}_2\big/(\bar{\theta}_2^2 + \bar{\theta}_3^2)^{1/2}\}\\ &= \{0, n_2, n_3\}\end{aligned} \tag{4}$$

$$\mathbf{t} = \{\cos\bar{\phi}_n, \bar{\theta}_2, \bar{\theta}_3\} \tag{5}$$

$$\cos\theta_n = (1 - \theta_2^2 - \theta_3^2)^{1/2} \tag{6}$$

$$\theta_2 = -\frac{dw(s)}{ds}, \qquad \theta_3 = \frac{dv(s)}{ds} \tag{7,8}$$

in which $\mathbf{n}$ is the unit vector perpendicular to the vectors $\mathbf{e}_1$ and $\mathbf{e}_1^S$, and $\mathbf{t}$ is the tangent unit vector of the deformed centroid axis of the beam segment. Note that $\mathbf{e}_1^S$ coincides with $\mathbf{t}$. θ_n is the inverse of $\cos\theta_n$. $v(s)$ and $w(s)$ are the lateral deflections of the centroid axis of the beam segment in the x_2 and x_3 directions, respectively, and s is the arc length of the deformed centroid axis.

Using Eqs. (2)-(8), the relation between the vectors $\mathbf{e}_i$ and $\mathbf{e}_i^S$ (i = 1, 2, 3) in the segment coordinate system may be obtained as

$$\mathbf{e}_i^S = [\mathbf{t}, \mathbf{R}_1, \mathbf{R}_2]\mathbf{e}_i = \mathbf{R}\mathbf{e}_i \tag{9}$$

where $\mathbf{R}$ is the so-called rotation matrix. The rotation matrix is determined by θ_i (i = 1, 2, 3). Thus, θ_i are called rotation parameters in this study.

Let $\theta = \{\theta_1, \theta_2, \theta_3\}$ be the column matrix of rotation parameters, $\delta\theta$ be the variation of θ. The traid $\mathbf{e}_i^S$ (i = 1, 2, 3) corresponding to θ may be rotated by a rotation vector $\delta\phi = \{\delta\phi_1, \delta\phi_1, \delta\phi_1\}$ to reach their new positions corresponding to $\theta + \delta\theta$ [2]. When θ_2 and θ_3 are much smaller than unity, the relationship between $\delta\theta$ and $\delta\phi$ may be approximated by [2]

$$\delta\theta = \begin{bmatrix} 1 & \theta_3/2 & -\theta_2/2 \\ -\theta_3 & 1 & 0 \\ \theta_2 & 0 & 1 \end{bmatrix} \delta\phi = \mathbf{T}^{-1}\delta\phi \tag{10}$$

2.4 Kinematics of beam segment

The deformations of the beam segment are described in the current segment coordinate system. From the kinematic assumptions made in this paper, the deformations of the beam segment may be determined by the displacements of the centroid axis of the beam segment, orientation of the cross section (segment cross section coordinates), and the out-of-plane warping of the cross section. Let Q (Fig. 1) be an arbitrary point in the beam segment and P be the point corresponding to Q on the centroid axis. The position vector of point Q in the undeformed and deformed configurations may be expressed as

$$\mathbf{r}_0 = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 \tag{11}$$

and

$$\mathbf{r} = x_c(s)\mathbf{e}_1 + v(s)\mathbf{e}_2 + w(s)\mathbf{e}_3 + \theta_{1,s}\mathbf{e}_1^s + y\mathbf{e}_2^s + z\mathbf{e}_3^s \tag{12}$$

where

$$x_c(s) = x + u(s) \tag{13}$$

is the x_1 coordinate of point P, $u(s)$ is the displacement of point P in the x_1 direction, $v(s)$ and $w(s)$ are the lateral deflections of point P in the x_2 and x_3 directions respectively as mentioned, s is the arc length of the deformed centroid axis measured from node 1 to point P.

In this study, the Green strains are used for the measure of strain. Using assumption 3, we only consider the strain components $\varepsilon_{11}, \varepsilon_{12}$ and ε_{13}. If x, y and z in Eq. (11) are regarded as the Lagrangian coordinates, the Green strain $\varepsilon_{11}, \varepsilon_{12}$ and ε_{13} are given by [4]

$$\varepsilon_{11} = \frac{1}{2}(\mathbf{r}_{,x}^t \mathbf{r}_{,x} - 1)$$
$$\varepsilon_{12} = \frac{1}{2}\mathbf{r}_{,x}^t \mathbf{r}_{,y}$$
$$\varepsilon_{13} = \frac{1}{2}\mathbf{r}_{,x}^t \mathbf{r}_{,z}. \tag{14}$$

Using the chain rule for differentiation, $\mathbf{r}_{,x}$ in Eq. (14) may be expressed as

$$\mathbf{r}_{,x} = \mathbf{r}_{,s}(1 + \varepsilon_0) \tag{15}$$

$$\varepsilon_0 = \frac{\partial s}{\partial x} - 1 \tag{16}$$

where ε_0 is the unit extension of the centroid axis.

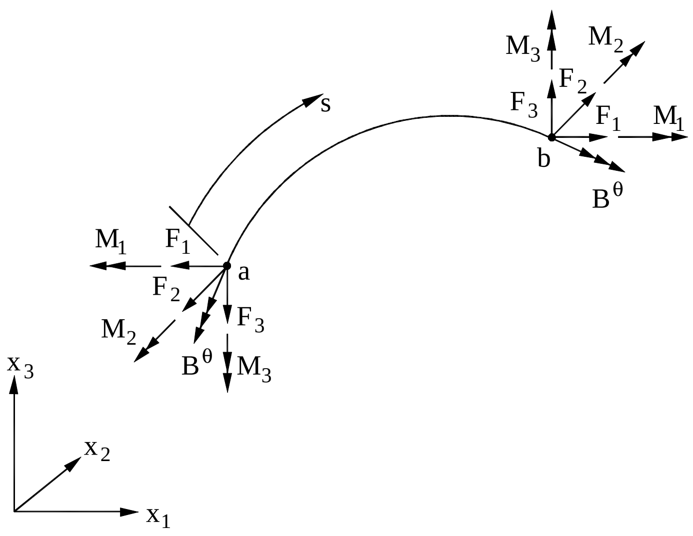


Fig.3 Free body of a portion of beam segment.

2.5 Equilibrium equations and constitutive equations

Here the equilibrium equations and constitutive equations of the beam segment are derived by the virtual work principle in the current segment coordinate. It is assumed that no external load is applied between the end points of the beam segment. The generalized displacements are chosen to be u_i (i = 1, 2, 3) ($u_1 = u, u_2 = v, u_3 = w$), the displacements of the centerline in the x_i directions,

$\bar{\ }_i$ (i = 1, 2, 3), the rotation parameters, and $\bar{\ }_{1,s}$, the twist rate of the centerline. The generalized forces corresponding to u_i, $\bar{\ }_i$, and $\bar{\ }_{1,s}$ are F_i (i = 1, 2, 3), the forces in the x_i directions, $\bar{M_i}$ (i = 1, 2, 3), the generalized moments, and $\bar{B}$, the generalized bimoment. Figure 3 shows a portion of the deformed centerline of the beam segment and the generalized forces at sections a and b. From Eq. (10) and the contragradient law [5], the relation between the conventional moment M_i (i = 1, 2, 3), the moments about the x_i axes, and the generalized moments $\bar{M_i}$ may be given by

$$\mathbf{M} = \mathbf{T}^{-t}\bar{\mathbf{M}} \tag{17}$$

where $\mathbf{M} = \{M_1 \quad M_2 \quad M_3\}$, $\bar{\mathbf{M}} = \{\bar{M_1} \quad \bar{M_2} \quad \bar{M_3}\}$, and $\mathbf{T}^{-t}$ is the transpose of $\mathbf{T}^{-1}$ given in Eq. (10).

Consider infinitesimal generalized virtual displacements from the equilibrium configuration, with components $¢u_i$, $¢\bar{\ }_i$ (i = 1, 2, 3), and $¢\bar{\ }_{1,s}$ as functions of the arc length of the centerline, s. The generalized forces at sections a and b are regarded as the external forces for the free body shown in Fig. 3. The virtual work of the external forces may be expressed by

$$¢W_{ext} = \left[\mathbf{F}^t ¢\mathbf{u} + \left(\bar{\mathbf{M}}\right)^t ¢\bar{\ } + \bar{B}\, ¢\bar{\ }_{1,s}\right]_a^b \tag{18}$$

$$\begin{aligned} \mathbf{F} &= \{F_1, F_2, F_3\} \\ ¢\mathbf{u} &= \{¢u_1, ¢u_2, ¢u_3\} = \{¢u, ¢v, ¢w\} \\ ¢\bar{\ } &= \{¢\bar{\ }_1, ¢\bar{\ }_2, ¢\bar{\ }_3\} \end{aligned} \tag{19}$$

where $[\ ]_a^b$ denotes that the value of the term in brackets at the upper limit subtracts the corresponding value at the lower limit.

The virtual work done by the internal stresses going through the virtual strains (that corresponding to the imposed virtual displacements) for the free body shown in Fig. 3 may be expressed by

$$\delta W_{int} = \int_a^b \left(\sigma_{11}\delta\varepsilon_{11} + \sigma_{12}\delta\varepsilon_{12} + \sigma_{13}\delta\varepsilon_{13}\right) dV \tag{20}$$

where V is the volume of the undeformed beam segment between section a and b, $dV = dAds/(1+\varepsilon_0)$, $\delta\varepsilon_{1j}$ (j = 1, 2, 3) are the variation of ε_{1j} with respective genearlized displacements. σ_{1j} (j = 1, 2, 3) are second Piola-Kirchhoff stresses. For linear elastic material, the stress-strain relations are given by

$$\sigma_{11} = E\varepsilon_{11}, \quad \sigma_{12} = 2G\varepsilon_{12}, \quad \sigma_{13} = 2G\varepsilon_{13} \tag{21}$$

where E is the Young's modulus and G is shear modulus.

The principle of virtual work requires that

$$\delta W_{int} = \delta W_{ext} \tag{22}$$

Substituting Eqs. (14), (18)-(20), and (21) into Eq. (22), and equating the terms on both sides of Eq. (22) corresponding to the same virtual generalized displacements and their derivatives, we may obtain

$$\bar{M}_{1,s} = E\left(I_y - I_z\right)\bar{\ }_{2,s}\bar{\ }_{3,s} \quad (23) \qquad \bar{M}_{2,s} - F_3$$

$$\bar{M}_{3,s} + F_2 = \underline{F_1\bar{\ }_3} + \frac{1}{2}C\bar{\ }_{1,s}\bar{\ }_{2,s} \tag{25}$$

$$F_{1,s} = 0, \qquad F_{2,s} = 0, \qquad F_{3,s} = 0 \tag{26-28}$$

$$\bar{M}_1 + \bar{B}_{,s} = E\varepsilon_0 I_p \bar{\ }_{1,s} + C[(1+\varepsilon_0)\bar{\ }_{1,s} + \frac{1}{2}\bar{\ }_{2,s}\bar{\ }_3 - \frac{1}{2}\bar{\ }_2\bar{\ }_{3,s}] + \underline{\frac{1}{2}EI_4\bar{\ }^3_{1,s}} \tag{29}$$

$$\bar{M}_2 = E[(1+4\varepsilon_0)I_y\bar{\ }_{2,s} + (I_y - I_z)\bar{\ }_1\bar{\ }_{3,s} - 3t_{yz}\bar{\ }_{1,ss}\bar{\ }_{3,s}] + \frac{1}{2}C\bar{\ }_{1,s}\bar{\ }_3 \tag{30}$$

$$\bar{M}_3 = E[(1+4\varepsilon_0)I_z\bar{\ }_{3,s} + (I_y - I_z)\bar{\ }_1\bar{\ }_{2,s} - 3t_{yz}\bar{\ }_{1,ss}\bar{\ }_{2,s}] - \frac{1}{2}C\bar{\ }_{1,s}\bar{\ }_2 \tag{31}$$

$$B^{-} = C_1(1+4£_0)^{-}_{1,ss} - 3Et_{yz}\,{}^{-}_{2,s}\,{}^{-}_{3,s} \tag{32}$$

$$F_1 = E[A£_0(1+\frac{5}{2}£_0)+\frac{1}{2}I_p\,{}^{-2}_{1,s}+\frac{5}{2}(I_y\,{}^{-2}_{2,s}+I_z\,{}^{-2}_{3,s}+{}^{-2}_{1,ss}A_{\mu\mu})]+C\,{}^{-2}_{1,s} \tag{33}$$

where $C = GJ$ is the torsional rigidity, and $C_1 = EA_{\mu\mu}$ is the warping rigidity.

Equations (23)-(28) are equilibrium equations and Eqs. (29)-(33) are constitutive equations of the beam segment in the current segment coordinates. Note that because all the terms up to the second order of the generalized displacements are retained in Eqs. (23)-(33), the beam theory proposed here is called second order beam theory. However, some third order terms (the underlined terms in Eqs. (24), (25), and (29)), which may not be negligible for some problems, are also retained.

3. NUMERICAL EXAMPLES

Consider a cantilever beam subjected to end torsion T as shown in Fig . 4. Because only the primary equilibrium path is considered, the ways of generating end torsion are rendered irrelevant here. It is assumed that warping is free at both end sections. Three cases of cross section are considered: (1) $b = 0.6\,mm$, $h = 10\,mm$, (2) $b = 0.6\,mm$, $h = 30\,mm$. The rest geometry and material properties of the beam are: $L = 240\,mm$, Young's modulus $E = 71240\,N/mm^2$, and shear modulus $G = 27190\,N/mm^2$.

It is observed that if $v(s) = w(s) = \;{}^{-}_{1,ss}(s) = 0$ are assumed in Eqs. (23)-(33), all force and displacement boundary conditions may be satisfied, and only Eqs. (26), (29), and (33) are not trivial.

From $F_1 = 0$ at the free end, Eqs. (26) and (33), and using the approximation $1+\frac{5}{2}£_0 \approx 1$, one may obtains

$$£_0 = \frac{-1}{A}(\frac{C}{E}+\frac{1}{2}I_p)\,{}^{-3}_{1,s} \tag{34}$$

Substituting Eq. (34) and $B^{-} = 0$ into Eq. (29), one may obtain

$$\frac{T}{C} = \theta_{1,s} + \frac{1}{C}\left[\underline{\frac{1}{2}EI_4} - \frac{1}{A}\left(\frac{C}{E} + \frac{1}{2}I_p\right)(EI_p + C)\right]\theta_{1,s}^3 \tag{35}$$

The results for cases (1)-(2)are shown in Figs. (5)-(6). As can be seen, the discrepancy between curves A, B, and C are remarked for case (2). Thus the underline term in Eq. (29) (or.Eq. (35))] may be required for reliable solutions.

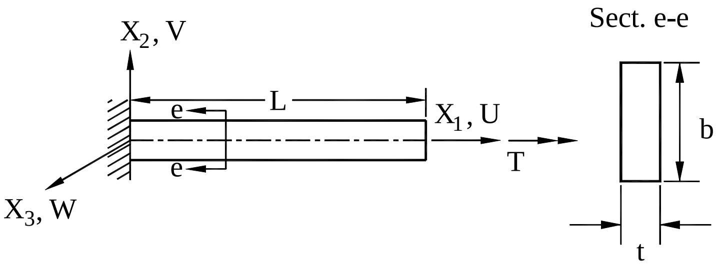


Fig. 4. Cantilever beam subjected to end torsion.

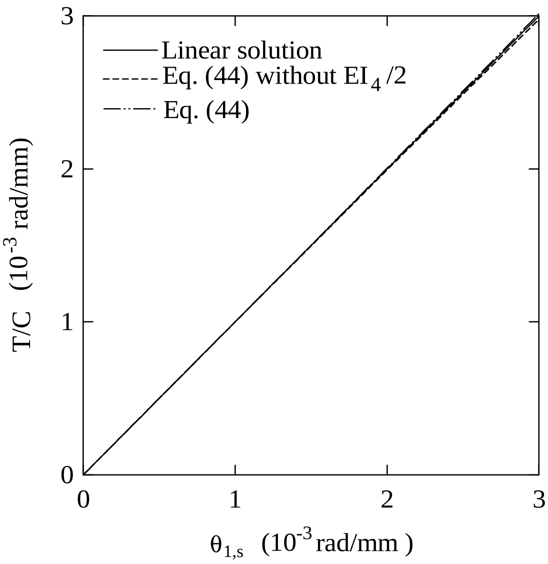


Fig. 5. Torsion-twist rate curve (case 1).

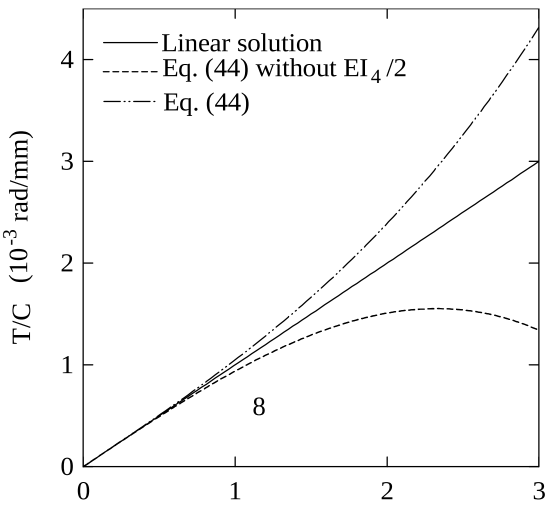


Fig. 6. Torsion-twist rate curve (case 2).

4 CONCLUSIONS

A consistent co-rotational total Lagrangian formulation of second order beam theory is presented for the nonlinear analysis of three-dimensional elastic Euler beam with large rotations but small strains.

The beam structure is divided into several segments. A set of segment coordinate system is constructed at the current configuration of the deformed beam segment. The deformation, equilibrium equations and constitutive equations of the beam segment are defined in the segment coordinates. In order to describe the orientation of the beam cross section, a set of segment cross section coordinates associated with each cross section of the beam segment is employed. Three rotation parameters are used to describe the rotation of the segment cross section coordinates. In this paper the deformations of the beam segment are determined by the unit extension of the centroid axis and rotations of segment cross section coordinate systems relative to segment coordinate system.

The principle of virtual work and the consistent linearization of the fully geometrically nonlinear beam theory is used to derive the equilibrium equations and constitutive equation of the beam segment. In order to consider the nonlinear coupling among bending, twisting, and stretching deformations, all terms up to the second order of the deformation parameters are retained in the equilibrium equations and constitutive equations of the second order beam theory. However, some third order terms, which may not be negligible for some problems, are also retained. Numerical examples are presented to demonstrate the accuracy and effectiveness of the proposed second order beam theory.

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